JOURNAL OF DIFFERENTIAL EQUATIONS **123**, 437–479 (1995)

Reaction–Diffusion Systems on Domains with Thin Channels*

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Received October 28, 1993; revised June 13, 1994

The purpose of this work is to consider a weakly coupled system of semilinear parabolic equations; namely, a reaction–diffusion system, which is typically a model of chemical reaction, population biology, morphogenesis, etc. This system, considered in a bounded spatial domain, $\Omega \subset \mathbb{R}^2$, is written as

$$u_t = D\Delta u + f(u), \qquad (t, x) \in (0, \infty) \times \Omega, \tag{1}$$

with Newmann boundary conditions, $u = (u^1, ..., u^m)$ and D is an $m \times m$ positive diagonal matrix. We suppose that Ω_ε is smooth and that there exists a bounded and smooth set, Ω_0, such that $R_\varepsilon = \Omega_\varepsilon \backslash \Omega_0$ approaches portions of a curve as $\varepsilon \to 0$. In fact we assume that R_ε is a finite union of thin domains over a curve. We prove that the dynamic behavior of (1) is determined by a finite dimensional ODE $dy/dt = h_\varepsilon(y)$, and $h_\varepsilon \to h_0$ in the C^1 sense, where h_0 is given. In fact we prove existence and convergence of inertial manifolds for (1).

1. INTRODUCTION

The purpose of this work is to consider a weakly coupled system of semilinear parabolic equations; namely, a reaction–diffusion system, which is typically a model of chemical reaction, popultion biology, morphogenis, etc. This system, considered in a bounded spatial domain, $\Omega \subset \mathbb{R}^2$, is written as

$$\frac{\partial u}{\partial t} = D\Delta u + f(u), \qquad (t, x) \in (0, \infty) \times \Omega, \tag{I.2}$$

$$\frac{\partial u}{\partial n} = 0, \qquad (t, x) \in (0, \infty) \times (\partial \Omega) \tag{I.3}$$

* Partially supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP).

0022-0396/95 $12.00

where $f\colon \mathbb{R}^m \to \mathbb{R}^m$, $\partial/\partial n$ denotes the outer normal derivative on the boundary $\partial\Omega$ of Ω, and

$$u = \begin{pmatrix} u^1 \\ \vdots \\ u^m \end{pmatrix}, \qquad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \qquad (x_1, x_2) \in \mathbb{R}^2,$$

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_m \end{pmatrix}, \qquad d^* \geqslant d_i \geqslant d_* > 0, \qquad \text{for all} \quad i = 1, ..., m.$$

These systems have been study extensively, and the dynamics depends particularly on the type of domain considered. In this work we will suppose that Ω is a finite union of fixed domains joined by thin channels. We will consider channels that are thin domains over curves (according to Hale and Raugel (1991, 1992)).

The main contribution of this work will be to show the existence of inertial manifolds for (I.2)–(I.3) and how they behave as we make the channels disappear, that is, as the channels converge to lower dimensional curves. This will be done by taking into account information coming from the channels.

Loosely speaking, an inertial manifold for a dissipative system is a finite dimensional manifold in the phase space of the dynamical system (which is infinite dimensional for PDEs), which is invariant, contains the global attractor and attracts at exponential rate the trajectories starting in any bounded set of the phase space. A global attractor is a set $\mathscr{A}$ of the phase space which is invariant (the forward flow restricted to $\mathscr{A}$ is onto), and for any bounded set B of the phase space, the distance of the forward flow acting on B from $\mathscr{A}$ approaches zero, as time goes to infinity.

The existence of these inertial manifolds will follow easily from the usual theorems available in the literature (we mainly will use the results of Chow and Lu (1988) and Rodriguez-Bernal (1990)). The contribution will be on the behavior of these manifolds; more precisely, we will show that they converge to the inertial manifold of a PDE on the fixed domain and an ODE in the lower dimensional curves.

In the past years, much research has been devoted to the behavior of solutions of reaction diffusion systems. We mention only some lore directly related to or work. Conway *et al.* (1978) have shown that, under certain hypothesis, given any domain Ω, the large time behavior of solutions of (I.2)–(I.3) are such that the solutions converge to their averages, as time goes to infinity, where these averages will satisfy $\dot{x} = f(x) + O(e^{-\sigma t})$. This shows that the asymptotic behavior is determined by f alone. The basic consumption is that the value $d_* \lambda_2(\Omega)$ is dominant over the derivative of f, where d_* is the minimum of the diffusion coefficients d_i, $i = 1, ..., m$, and $\lambda_2(\Omega)$ is the second eigenvalue of $-\Delta$ on the domain Ω under (I.3). Later, Hale (1985) proved that, under this same basic hypothesis, the large time behavior of solutions of (I.2)–(I.3) is qualitatively determined by an m-dimensional ODE system

$$\frac{du}{dt} = f(u). \tag{I.4}$$

If Ω is a dumbbell-shaped domain, Morita (1990) extended these results, supposing that the value $d_* \lambda_3(\Omega)$ is dominant over the derivative of f in some sense, where $\lambda_3(\Omega)$ is the third eigenvalue of $-\Delta$ with (I.3), and obtained an ODE system which "determined" the asymptotic behavior of the solutions to the original system in Ω. This ODE is of higher dimension than (I.4), and contains (I.4) as its subsystem. However, hypotheses were imposed which implied that the qualitative properties of the dynamics were independent of the part of the domain whose measure is going to zero (that is, channels). Hale and Raugel (1991, 1992) solve the problem completely if the fixed domain is empty; that is, if Ω is a thin domain, and also obtained an ODE that dictates the asymptotic behavior of the original system in Ω.

Our objective is to extend both works and allow the influence of the channels and the fix domain over the dynamics. The basic hypothesis will still be a gap condition, as before, that is the distance between two consecutive eigenvalues dominates the derivative of f. Briefly, the main tools that we use in this work are the results on eigenvalues and eigenfunction of $-\Delta$ on domains with channels, due to Arrieta (1991) and (1993), and the theorems of existence of invariant manifolds due to Chow and Lu (1988) and Rodriguez-Bernal (1990). But by Far, what makes the results of this work possible are the ones by Arrieta.

1.1. *Setting of the Problem*

1.1.1. *Domains.* To introduce our domains, let $\varepsilon_0 > 0$ be fixed, and let I_0 be a subset of $(0, \varepsilon_0)$ with the property that 0 is an accumulation point of I_0.

Let $\Omega_0 \subset \mathbb{R}^2$ be a bounded, smooth open set with finitely many connected components, that will be denoted by Ω_0^i, and denote by $\{\Omega_\varepsilon\}_{\varepsilon \in I_0}$ a family of bounded, smooth, connected domains such that $\Omega_0 \subset \Omega_\varepsilon$. Denote by $R_\varepsilon = \Omega_\varepsilon \backslash \bar{\Omega}_0$ and by $\Gamma_\varepsilon = \partial R_\varepsilon \cap \partial\Omega_0$ (see Fig. 1). We will assume further, that each connected component of R_ε, that will be denoted by R_ε^i, is a thin domain over a curve, according to Hale and Raugel (1991), that approaches a portion of a curve as $\varepsilon \to 0$, and will denote by n_R and n_Γ

(independent of ε) the number of connected components of R_ε and Γ_ε, respectively. More precisely, we will assume the following:

(H0). *For each $1 \leqslant i \leqslant n_R$, let R_0^i be a segment of a smooth nonself-intersecting curve p^i in $\mathbb{R}^2$. We let*

$$R_0^i = \{p^i(s), a^i < s < b^i\}, \qquad R_0 = \bigcup_{i=1}^{n_R} R_0^i,$$

and we will denote by $v^i(s)$ the normal vector at $p^i(s)$, taken in such a way that they vary continuously with s. Let g_1^i, g_2^i be positive and bounded C^1-functions in $\mathbb{R}$, such that, for some $k^i > 0$,

$$\frac{1}{(g_1^i + g_2^i)^2} \geqslant k^i.$$

We will suppose that (see Fig. 2)

$$R_\varepsilon^i = \{p^i(s) + \alpha_\varepsilon^i v^i(s), \text{ where } -\varepsilon g_1^i(s) < \alpha_\varepsilon^i < \varepsilon g_2^i(s), \text{for all } a^i < s < b^i\}.$$

The next step is to fix the interaction between Ω_0 and R_ε, and this will be done with the hypothesis below. This hypothesis is the key point in the work and was established by Arrieta (1993) in his work on convergence of eigenvalues. Its importance will be pointed out later. We assume the following

(H1). *If $u_\varepsilon \in H^1(\Omega_\varepsilon)$ with $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqslant R$ for some positive R independent of ε, then there exists $\bar{u}_\varepsilon \in H^1_{\Gamma_\varepsilon}(R_\varepsilon) = \{u \in H^1(R_\varepsilon) \text{ such that } u = 0 \text{ in } \Gamma_\varepsilon\}$ satisfying,*

$$\|u_\varepsilon - \bar{u}_\varepsilon\|_{L^2(R_\varepsilon)} \xrightarrow{\varepsilon \to 0} 0 \tag{I.5}$$

$$\|\nabla \bar{u}_\varepsilon\|_{L^2(R_\varepsilon)} \leqslant \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + o(1). \tag{I.6}$$

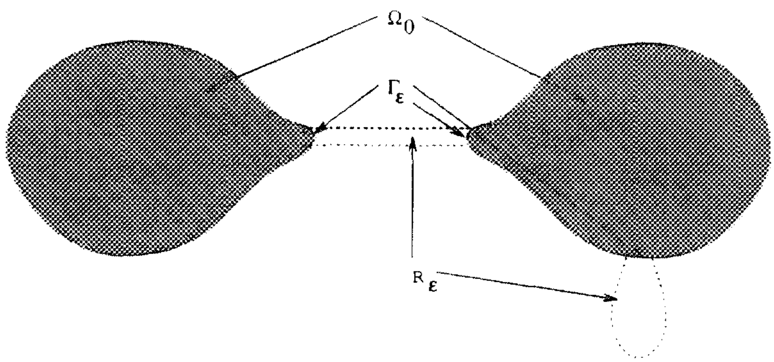

FIG. 1. Domains.

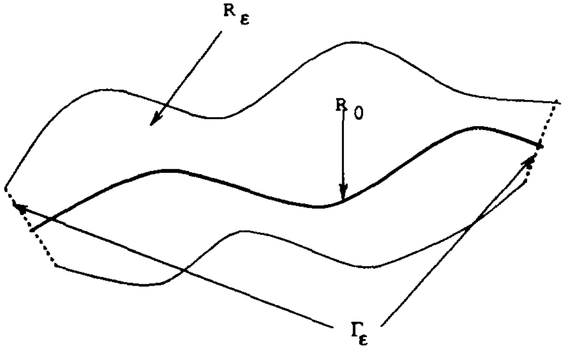

FIG. 2. Channel.

Whenever we have the situation described above we will say that $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon \in I_0}$ form a triple, (see Fig. 3, the shadowed region is Ω_ε).

1.1.2. *Spaces and Notation.* Now we are in position to define the spaces that will be used in this work. Given a positive integer $m > 1$, let $X_\varepsilon = (L^2(\Omega_\varepsilon))^m$, $X_\varepsilon^1 = (H^1(\Omega_\varepsilon))^m$, $Y_\varepsilon = (L^2(R_\varepsilon))^m$ and $Y_\varepsilon^1 = (H^1_{\Gamma_\varepsilon}(R_\varepsilon))^m$, where $(\cdot)^m$ denotes the product space. The notations X_ε, X_ε^1 are meaningful for $\varepsilon = 0$, namely replace Ω_ε by Ω_0. On the other hand, we will denote by $Y_0 = (L^2(R_0))^m$ and $Y_0^1 = (H_0^1(R_0))^m$, where $H_0^1(R_0)$ denotes the usual Sobolev space. The corresponding norms are

$$\|u\|_\varepsilon = \left(\sum_{i=1}^m \|u^i\|^2_{L^2(\Omega_\varepsilon)}\right)^{1/2}, \qquad \|u\|_{1,\varepsilon} = \left(\sum_{i=1}^m \|u^i\|^2_{H^1(\Omega_\varepsilon)}\right)^{1/2}.$$

Also denote,

$$|||u|||_\varepsilon = \left(\sum_{i=1}^m \|u^i\|^2_{L^2(R_\varepsilon)}\right)^{1/2}, \qquad |||u|||_{1,\varepsilon} = \left(\sum_{i=1}^m \|u^i\|^2_{H^1(R_\varepsilon)}\right)^{1/2}.$$

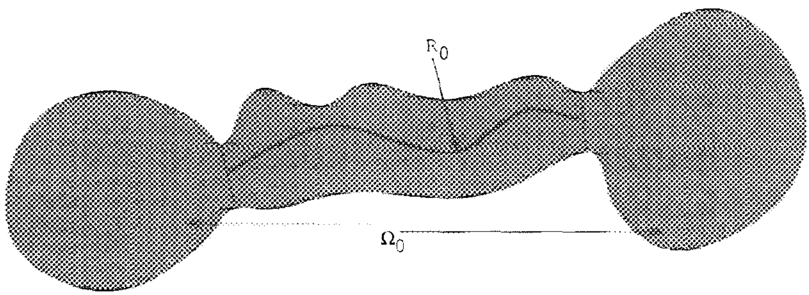

FIG. 3. A possible triple.

The inner product in X_ε will be denoted by

$$(u, v) \equiv \sum_{i=1}^{m} \int_{\Omega_\varepsilon} u^i(x)\, v^i(x)\, dx.$$

We will also denote norms of vectors in $\mathbb{R}^m$ by

$$|y|_m = \left(\sum_{i=1}^{m} |y^i|^2 \right)^{1/2}, \qquad y = \begin{pmatrix} y^1 \\ \vdots \\ y^m \end{pmatrix} \in \mathbb{R}^m$$

and, for $y_k \in \mathbb{R}^m$, $k = 1, ..., n$ we define the norm of $Y = (y_1, ..., y_n) \in \mathbb{R}^{nm}$ by

$$|Y|_{nm} = \left(\sum_{k=1}^{n} |y_k|_m^2 \right)^{1/2}.$$

Let F_1, F_2 be Banach spaces and U an open subset of F_1. For any $k \geqslant 0$, let

$$C^k(U, F_2) = \{f\colon U \to F_2 : f \text{ is } k\text{-times differentiable and } \sup_{x \in U} |D_x^i f(x)| < \infty, \text{ for } 0 \leqslant i \leqslant k\}$$

and

$$|f|_k = \sum_{i=0}^{k} \sup_{x \in U} |D_x^i f(x)|,$$

where D_x^i is the ith differentiation operator. Let

$$\text{Lip}(D_x^k f) = \sup_{x \neq y,\ x, y \in U} \frac{|D_x^k f(x) - D_x^k f(y)|}{|x - y|}.$$

Then we will denote by

$$C^{k,1}(U, F_2) = \{f \in C^k(U, F_2) \colon \text{Lip}(D_x^k f) < \infty\}$$

and $|f|_{k,1} = |f|_k + \text{Lip}(D_x^k f)$.

Clearly $C^k(U, F_2)$ and $C^{k,1}(U, F_2)$ are Banach spaces with the norms $|\cdot|_k$ and $|\cdot|_{k,1}$.

Let $\mathscr{L}^k(F_1, F_2)$ be the Banach space of all k-multilinear continuous maps from F_1 into F_2. For $\Phi \in \mathscr{L}^k(F_1, F_2)$, $\|\Phi\|_{\mathscr{L}}^k$ denotes the norm of Φ.

Now let $J \subset \mathbb{R}$ be an interval (in most cases, we will let $J = \mathbb{R}^- = (-\infty, 0]$). For any $\eta \in \mathbb{R}$ and any Banach space F, we denote by $C_\eta(J, F)$ the following Banach space

$$C_\eta(J, F) = \{f : J \to F : f \text{ is continuous and } \sup_{t \in J} e^{-\eta t} |f(t)|_F < \infty\}$$

with norm

$$|f|_{C_\eta(J, F)} = \sup_{t \in J} e^{-\eta t} |f(t)|_F.$$

1.1.3. *Equation.* As mentioned before, the objective of this work is to prove a convergence result for (I.2). In order to do this we need to define (I.2) in an abstract form and define what will be the limit equation. To "guess" what the limit equation should be, we will make use of the results of Arrieta (1993). Let $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon \in I_0}$ be a given triple, and consider the following eigenvalue problems:

$$\left.\begin{aligned} -\Delta u &= \lambda u, && \text{in } \Omega_\varepsilon \\ \frac{\partial u}{\partial n} &= 0, && \text{in } \partial\Omega_\varepsilon \end{aligned}\right\} \tag{I.7}$$

$$\left.\begin{aligned} -\Delta u &= \lambda u, && \text{in } R_\varepsilon \\ \frac{\partial u}{\partial n} &= 0, && \text{in } \partial R_\varepsilon \backslash \Gamma_\varepsilon \\ u &= 0, && \text{in } \Gamma_\varepsilon \end{aligned}\right\} \varepsilon \in I_0. \tag{I.8}$$

We will denote by $\{\lambda_n(\Omega_\varepsilon)\}$ and $\{\tau_n(R_\varepsilon)\}$ the eigenvalues of (I.7) and (I.8) respectively, always increasingly ordered and counting mulitplicity, and we will also denote $\{\varphi_{n, \Omega_\varepsilon}\}$ and $\{\gamma_{n, R_\varepsilon}\}$ the respective set of orthonormal eigenfunctions. We will consider

$$\{\lambda_n^\varepsilon\}_{n=1}^{\infty} = \{\lambda_n(\Omega_0)\}_{n=1}^{\infty} \cup \{\tau_n(R_\varepsilon)\}_{n=1}^{\infty}$$

always arranged in an increasing order and counting multiplicity. We will state Arrieta's results precisingly later on, but loosely speaking, he proves that, given a triple $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon = I_0}$ (thus (H1) is satisfied), then $\{\lambda_n(\Omega_\varepsilon) - \lambda_n^\varepsilon\}_{n=1}^{\infty} \to 0$, as $\varepsilon \to 0$, and the same for the eigenfunctions. If we consider now the following eigenvalue problem

$$\left.\begin{aligned} -\frac{1}{(g_1^i + g_2^i)}((g_1^i + g_2^i)\, u_s)_s &= \mu u, \qquad \text{in } R_0^i \\ u(s) = 0, \qquad &\text{for} \quad s = a^i, b^i. \end{aligned}\right\}, \qquad 1 \leqslant i \leqslant n_R, \tag{I.9}$$

and denote by $\{\mu_n(i)\}$, $\{\chi_{n,i}\}$ the eigenvalues and eigenfunctions of (I.9), and

$$\{\mu_n\}_{n=1}^{\infty} = \bigcup_{i=1}^{n_R} \{\mu_j(i)\}_{j=1}^{\infty}, \qquad \{\lambda_n^0\}_{n=1}^{\infty} = \{\lambda_n(\Omega_0)\}_{n=1}^{\infty} \cup \{\mu_n\}_{n=1}^{\infty},$$

always arranged in an increasing order, counting multiplicity. Then from Hale and Raugel (1992) we have that $\{\lambda_n^{\varepsilon}\}_{n=1}^{\infty} \to \{\lambda_n^0\}_{n=1}^{\infty}$, as $\varepsilon \to 0$, and similarly for the eigenfunctions.

With these results, one can "a priori" guess that the "limit equation", if any, will consist of an evolution problem in Ω_0, decoupled with an evolution problem on R_0.

Now we are in position to define (I.2) and the "limit equation" in abstract forms. Let A_ε be a closed operator in X_ε with domain $\mathscr{D}(A_\varepsilon) = \{u \in (H^2(\Omega_\varepsilon))^m$ such that u satisfies $\partial u/\partial n = 0$ in $\partial\Omega_\varepsilon\}$, satisfying $A_\varepsilon u = -D\Delta u$, $u \in \mathscr{D}(A_\varepsilon)$; A_0 be a closed operator in X_0 with domain $\mathscr{D}(A_0) = \{u \in (H^2(\Omega_0))^m$ such that u satisfies (I.3) in $\partial\Omega_0\}$, satisfying $A_0 = -D\Delta$ in $\mathscr{D}(A_0)$; $\tilde{A}_0$ be a closed operator in Y_0 with domain $\mathscr{D}(\tilde{A}_0) = \{u \in (H^2(R_0))^m$ such that $u = 0$ for $s = a^i, b^i$, for all $1 \leqslant i \leqslant n_R\}$, satisfying $\tilde{A}_0 = -D(1/(g_1 + g_2))(\partial/\partial s)((g_1 + g_2)(\partial/\partial s))$ in $\mathscr{D}(\tilde{A}_0)$, where g_1, g_2 represents g_1^i, g_2^i in each R_0^i; and $\tilde{A}_\varepsilon$ be a closed operator in Y_ε with domain $\mathscr{D}(\tilde{A}_\varepsilon) = \{u \in (H^2(R_\varepsilon))^m$ such that u satisfies (I.3) in $\partial R_\varepsilon \backslash \Gamma_\varepsilon$ and $u = 0$ in $\Gamma_\varepsilon\}$, satisfying $\tilde{A}_\varepsilon = -D\Delta$ in $\mathscr{D}(\tilde{A}_\varepsilon)$.

With these operators in mind, we can define, for each $\varepsilon \in \bar{I}_0$, the closed operator C_ε in $X_0 \times Y_\varepsilon$ with domain $\mathscr{D}(C_\varepsilon) = \mathscr{D}(A_0) \times \mathscr{D}(\tilde{A}_\varepsilon)$, satisfying

$$C_\varepsilon = \begin{pmatrix} A_0 & 0 \\ 0 & \tilde{A}_\varepsilon \end{pmatrix}.$$

We will denote by $e^{-A_\varepsilon t}$, $e^{-\tilde{A}_\varepsilon t}$ and $e^{-C_\varepsilon t}$ the semigroups generated by $-A_\varepsilon$, $-\tilde{A}_\varepsilon$ and $-C_\varepsilon$ respectively.

The next step is to assume that we can extend the nonlinearity of f to the abstract spaces, that is if we define $\hat{f}$ and $\tilde{f}_\varepsilon$, by

$$\hat{f}(u)(x) = f(u(x)), \qquad \text{for all} \quad u \in X_\varepsilon^1;$$

$$\tilde{f}_\varepsilon(u)(x) = (f(u_1(x)), f(u_2(x))), \qquad \text{for all} \quad u = (u_1, u_2) \in X_0^1 \times Y_\varepsilon^1, \tag{I.10}$$

We will assume that

(H2). 1. $\hat{f}\colon X_\varepsilon^1 \to X_\varepsilon$;

2. $\tilde{f}_\varepsilon\colon X_0^1 \times Y_0^1 \to X_0 \times Y_\varepsilon$;

3. *Given any bounded set* $\mathscr{B} \subset X_\varepsilon^1$ *and* $\tilde{\mathscr{B}} \subset X_0^1 \times Y_\varepsilon^1$, *we will suppose that* $\hat{f} = (\hat{f}^1, ..., \hat{f}^m) \in C^{1,1}(\mathscr{B}, X_\varepsilon)$ *and* $\tilde{f}_\varepsilon = (\tilde{f}_\varepsilon^1, ..., \tilde{f}_\varepsilon^m) \in C^{1,1}(\tilde{\mathscr{B}}, X_0 \times Y_\varepsilon)$ *are such that there exist constants* $N(\mathscr{B})$ *and* $\tilde{N}(\tilde{\mathscr{B}})$, *with*

$$|\hat{f}|_{1,1} \leqslant N(\mathscr{B}), \qquad |\tilde{f}_\varepsilon|_{1,1} \leqslant \tilde{N}(\tilde{\mathscr{B}}).$$

4. *There exists a sequence of real numbers* $\{r_j\}_{j=1}^m$, $r_j > 1$, *such that if we define, for each* $1 \leqslant j \leqslant m$, $\check{f}^j : (L^{r_j}(R_\varepsilon))^m \to L^2(R_\varepsilon)$, *by* $\check{f}^j(u)(x) = f^j(u(x))$, *then, given any bounded set* $\mathscr{B} \subset (L^{r_j}(R_\varepsilon))^m$, *there exists constant* $\check{N}(\mathscr{B})$, *with*

$$\text{Lip}(\check{f}^j) \leqslant \check{N}(\mathscr{B}),$$

in $\mathscr{B}$.

5. *There exists* $r > 1$, *such that* $\check{f} = (\check{f}^1, \cdot, \check{f}^m) \in C^{1,1}((L^r(R_\varepsilon))^m, (L^2(R_\varepsilon))^m)$, *and*

$$\text{Lip}(D_x \check{f}) \leqslant \check{N}_x(\mathscr{B}).$$

Remark 1.1. Hypothesis (H2)-4. just establishes some polynomial growth for each one of the components f^j.

Therefore, we may write the system (I.2) in the abstract form as follows:

$$\begin{aligned} \frac{du}{dt} &= -A_\varepsilon u + \hat{f}(u), \qquad t > 0 \\ u(0) &= u_0 \end{aligned} \tag{I.11}$$

for all $u_0 \in X_\varepsilon^1$. By a solution of Eq. (I.11), we mean a solution of the integral equation

$$u(t) = e^{-A_\varepsilon t} u_0 + \int_0^t e^{-A_\varepsilon(t-s)} \hat{f}(u(s))\, ds, \qquad t \geqslant 0, \quad u_0 \in X_\varepsilon^1. \tag{I.12}$$

Moreover, we can define the following system

$$\begin{aligned} \frac{du}{dt} &= -C_\varepsilon u + \tilde{f}_\varepsilon(u), \qquad t > 0 \\ u(0) &= u_0 \end{aligned} \tag{I.13}$$

for all $u_0 \in X_0^1 \times Y_\varepsilon^1$, and $\varepsilon \in \bar{I}_0$. By a solution of Eq. (I.11), we mean a solution of the integral equation

$$u(t) = e^{-C_\varepsilon t} u_0 + \int_0^t e^{-C_\varepsilon(t-s)} \tilde{f}_\varepsilon(u(s))\, ds, \qquad t \geqslant 0, \qquad u_0 \in X_0^1 \times Y_\varepsilon^1. \tag{I.14}$$

We will suppose the following:

(H3). *Suppose that* f *is such that* (I.11) *and* (I.13) (*or equivalently* (I.12) *and* (I.14)) *have a unique solution in* $C([0, \infty); X_\varepsilon^1) \cap C^1((0, \infty); X_\varepsilon)$ *and* $C([0, \infty); X_0^1 \times Y_\varepsilon^1) \cap C^1((0, \infty); X_0 \times Y_\varepsilon)$, *respectively*.

Remark 1.2. Note that $\mathscr{D}(A_\varepsilon + \delta I)^{1/2}) = X_\varepsilon^1$ holds for any constant $\delta > 0$. Hence, we can define the semiflow in X_ε^1, $\{T_f^\varepsilon(t)\}_{t \geq 0}$, by

$$T_f^\varepsilon(t)\, u_0 \equiv u(t; u_0), \tag{I.15}$$

where $u(t; u_0)$ is a solution of (I.12), satisfying $u(0; u_0) = u_0$.

Remark 1.3. Observe that if, for each $\varepsilon \in I_0$, we define

$$\begin{aligned} \xi_n^\varepsilon &= (\varphi_{i, \Omega_0}, 0), \qquad \text{if } \lambda_n^\varepsilon = \lambda_i(\Omega_0) \\ \xi_n^\varepsilon &= (0, \gamma_{j, R_\varepsilon}), \qquad \text{if } \lambda_n^\varepsilon = \tau_j(R_\varepsilon) \end{aligned} \tag{I.16}$$

and by,

$$\begin{aligned} \xi_n^0 &= (\varphi_{i, \Omega_0}, 0), \qquad \text{if } \lambda_n^0 = \lambda_i(\Omega_0) \\ \xi_n^0 &= (0, \chi_{j, i}), \qquad \text{if } \lambda_n^0 = \mu_j(i) \end{aligned} \tag{I.17}$$

then, for each $\varepsilon \in \bar{I}_0$, $\{\lambda_n^\varepsilon\}$, $\{\xi_n^\varepsilon\}$ for a complete set of eigenvalues and eigenvectors for C_ε in $X_0 \times Y_\varepsilon$. Moreover, C_ε is a sectorial operator in $X_0 \times Y_\varepsilon$ and the $\frac{1}{2}$-fractional power space is $X_0^1 \times Y_\varepsilon^1$. With these in mind, we will denote by S_f^ε the semiflow of (I.13).

Remark 1.4. Observe that, since we are assuming (H2), (H3) just states that the semiflows are defined for all time, since local existence is straight forward (Henry 1981).

We will need to assume some kind of dissipation for our equations. In order to do this we will need the following notion,

DEFINITION 1.5. Let X be a Banach space. For all bounded sets $A, B \subset X$ we can define the following notion of distance

$$\operatorname{dist}_X(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|_X.$$

We will impose the following dissipative assumption

(H4). *Suppose that* f *is such that there exists a constant* $\mathcal{N} > 0$ *independent of* ε, *such that, for each* ε *and* d^*, d_*, *there exist positively invariant* *sets* $\mathscr{B}_\varepsilon$ *and* $\tilde{\mathscr{B}}_\varepsilon$, *relative to* T_f^ε *and* S_f^ε *respectively* (*that is* $T_f^\varepsilon(t)\,\mathscr{B}_\varepsilon \subset \mathscr{B}_\varepsilon$ *and* $S_f^\varepsilon(t)\,\tilde{\mathscr{B}}_\varepsilon \subset \tilde{\mathscr{B}}_\varepsilon$) *such that*

$$\mathscr{B}_\varepsilon \subset \{u \in X_\varepsilon^1 : \|u\|_{1,\varepsilon} \leqslant \mathscr{N}\}$$

$$\tilde{\mathscr{B}}_\varepsilon \subset \{u \in X_0^1 \times Y_\varepsilon^1 : \|u\|_{X_0^1 \times Y_\varepsilon^1} \leqslant \mathscr{N}\}.$$

and

1. $\mathscr{B}_\varepsilon$ *attracts bounded sets of* X_ε^1 *under the flow defined by* (I.11); *that is*,

$$\mathrm{dist}_{X_\varepsilon}(T_f^\varepsilon(t)\,U, \mathscr{B}_\varepsilon) \to 0$$

as $t \to \infty$, *for each bounded set* $U \subset X_\varepsilon^1$;

2. $\tilde{\mathscr{B}}_\varepsilon$ *attracts bounded sets of* $X_0^1 \times Y_\varepsilon^1$ *under the flow defined by* (I.13); *that is*,

$$\mathrm{dist}_{X_0 \times Y_\varepsilon}(S_f^\varepsilon(t)\,U, \tilde{\mathscr{B}}_\varepsilon) \to 0$$

as $t \to \infty$, *for each bounded set* $U \subset X_0^1 \times Y_\varepsilon^1$.

Remark 1.6. Usually, hypothesis (H4) is not easy to be verified for a general system. On the other hand, if our system comes from a physical system we expect that (H4) should hold.

Remark 1.7. One could also consider nonlinearities with exponential growth as in Carvalho (1992, Chap. I, Sect. 3 and still satisfy (H2–H4).

1.1.4. *Spectral Gap Condition.* As mentioned before, the aim of this work is to study the asymptotic behavior of solutions of the reaction–diffusion system (I.2)–(I.3) in the parameterized domain $\Omega_\varepsilon \subset \mathbb{R}^2$, $\varepsilon \in I_0$, or equivalently (I.11).

The idea is to prove that (I.11) has an inertial manifold, for each $\varepsilon > 0$, and then show that these inertial manifolds converge to an inertial manifold of a "limit equation" (that will be (I.13) with $\varepsilon = 0$). Therefore, to prove the existence of these inertial manifolds we will need to assume a *Spectral Gap Condition*, not only for our equation but for what will be our limit equation. It turns out that, due to Arrieta's results, the condition will be the same for (I.11) and the limit equation, since we will be able to state a condition independent of ε. Furthermore, since Hale and Raugel (1992) already proved such convergence just for thin domains, we will need to compare just (I.11) and (I.13) for $\varepsilon \in I_0$.

In order to compare the equations we will need some extension operators, since the solutions live in different spaces. Consider first any linear extension operator $e_1 : H^1(\Omega_0) \to H^1(\Omega_\varepsilon)$ in such a way that there exists a constant $c(\Omega_0) > 0$ with the property that, for all $u \in H^1(\Omega_0)$, we have that

$$\|u\|_{H^1(\Omega_0)} \leqslant \|e_1 u\|_{H^1(\Omega_\varepsilon)} \leqslant c(\Omega_0)\, \|u\|_{H^1(\Omega_0)}.$$

For a proof of the existence of such an extension, see Brezis (1983). Then we can define the linear extension operator $E_1: X_0^1 \to X_\varepsilon^1$ by

$$E_1 = e_1 I(m),$$

where $I(m)$ is the identityh matrix in $\mathbb{R}^m$. Thus we still have the estimate above. Similarly we can define the extension operator $E_2: Y1_\varepsilon \to X_\varepsilon^1$, being just the extension to zero outside R_ε. Therefore, it is clear that, for any $u = (u^1, u^2) \in X_0^1 \times Y_\varepsilon^1$, we have that, restricting to get a map to $X_0^1 \times Y_\varepsilon^1$,

$$D_x \tilde{f}_\varepsilon(u) = \begin{pmatrix} D_x \hat{f}(E_1 u^1)\, E_1 & 0 \\ 0 & D_x \hat{f}(E_2 u^2)\, E_2 \end{pmatrix}. \tag{I.18}$$

We will also fix the extension $E: X_0^1 \times Y_\varepsilon^1 \to X_\varepsilon^1$, by

$$E(u^1, u^2) = E_1 u^1 + E_2 u^2 \in X_\varepsilon^1. \tag{I.19}$$

Remark 1.8. Let N and $\tilde{N}$ be given by (H2) and $\mathcal{N}$ be given by (H4). If we take B and $\tilde{B}$ to be the balls of radius $2\mathcal{N}$ in X_ε^1 and $X_0^1 \times Y_\varepsilon^1$ respectively, then we can assume without loss of generality that

$$L \equiv \mathrm{Lip}(\hat{f}) = \mathrm{Lip}(\tilde{f}_0) = \mathrm{Lip}(\tilde{f}_\varepsilon)$$

and

$$N \equiv N(B) = \tilde{N}(\tilde{B}).$$

Since (H4) holds, the asymptotics of the equations (I.11) and I.13) are confined in the sets $\mathscr{B}_\varepsilon$ and $\tilde{\mathscr{B}}_\varepsilon$, respectively, by cutting off the nonlinearities outside these sets we will have new systems, whose nonlinear terms are globally Lipschitz and bounded, so they are easy to treat. As shown later the asymptotic behavior of both, truncated and original systems, are the same. To truncate the nonlinearity we will use a cutoff function that will restrict our nonlinearities to a neighborhood of the ball of radius $\mathcal{N}$. Let $\zeta: [0, \infty] \to [0, 1]$ be a smooth function satisfying $\zeta(s) = 1$ for $0 \leqslant s \leqslant 1$ and $\zeta(s) = 0$ for $s \geqslant 4$, and denote by $\aleph: X_\varepsilon^1 \to [0, 1]$ and $\tilde{\aleph}^\varepsilon: X_0^1 \times Y_\varepsilon^1 \to [0, 1]$ the functions

$$\aleph(u) = \left(\frac{\|u\|_{1,\varepsilon}^2}{\mathcal{N}^2}\right)$$

$$\tilde{\aleph}^\varepsilon(u) = \zeta\left(\frac{\|u\|_{X_0^1 \times Y_\varepsilon^1}^2}{\mathcal{N}^2}\right).$$

From the smoothness of the norm, we have that $\aleph$ and $\tilde{\aleph}^{\varepsilon}$ are $C^{1,1}$ and we can assume without loss of generality that $|\aleph|_{1,1}, |\tilde{\aleph}^{\varepsilon}|_{1,1} \leqslant L_{\aleph}$.

Remark 1.9. If we define $f_{\aleph} = \aleph \hat{f}$, $\tilde{f}^{\varepsilon}_{\aleph} = \tilde{\aleph}^{\varepsilon} \tilde{f}_{\varepsilon}$, then it is clear that $f_{\aleph}$ and $\tilde{f}^{\varepsilon}_{\aleph}$ are $C^{1,1}$ and have $C^{1,1}$-norm uniformly bounded by $\tilde{L} = NL_{\aleph} + L$ (similarly we can cut $\check{f}$).

The last hypothesis that we need to assume is the Spectral Gap Condition, and we will be as follows

(H5). *Given* $d_* \leqslant d^*$, *we will assume that there exist* n, p, q, *with* $n = p + q$ *such that*

$$\{\lambda_j^0\}_{j=1}^{n} = \{\lambda_j(\Omega_0)\}_{j=1}^{p} \cup \{\mu_j\}_{j=1}^{q}; \qquad \lambda_{n+1}^0 d_* - \lambda_n^0 d^* > 0;$$
$$\frac{2 + 3\sqrt{2}\,(\lambda_{n+1}^0 d_* - \lambda_n^0 d^*)^{1/2}}{\lambda_{n+1}^0 d_* - \lambda_n^0 d^*} (NL_{\aleph} + L) < 0.5.$$

Notice that these conditions do not depend on ε, but only on f, Ω_0 and R_0. Later we will show some examples where this hypothesis will always be satisfied.

Remark 1.10. To illustrate hypothesis (H5), let us supposet that $m = 1$ (we have a scalar reaction diffusion equation) and that $d_* = d^* = D$. Then (see Section 2) we have that the eigenvalues coming from the channels, μ_n, satisfy (H5) (since they grow like n^2). Therefore, the only restriction is on the eigenvalues comming from Ω_0, and how they relate with the ones from the channel.

For instance, if we assume that the eigenvalues $\lambda_n(\Omega_0)$ also grow like n^2 then, since the union will grow like n^2 (see Section 2), we have that (H5) is satisfied.

1.2. *Results*

1.2.1. *Definition and Existence of Inertial Manifolds.*

Now we will state precisely what we mean by an inertial manifold and show its existence. First we need the definition of the so-called global attractors.

DEFINITION 1.11. If $T(t)$ is a C^0 semigroup on a Banach space X generated by an evolution equation, then a set $\mathscr{A} \subset X$ is said to be a global attractor if $\mathscr{A}$ is compact, invariant ($T(t)\mathscr{A} = \mathscr{A}$ for $t \geqslant 0$) and, for any bounded set $B \subset X$, $\operatorname{dist}_X(T(t)B, \mathscr{A}) \to 0$ as $t \to +\infty$.

The existence of such attractors follows easily from Hale (1988), and can be stated as follows, THEOREM 1.1 (Hale (1988)). *Let* $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon \in I_0}$ *be a triple. Suppose that* (H2), (H3), *and* (H4) *hold, then for each* $\varepsilon \in I_0$ *the problems* (I.11) *and* (I.13) *have global attractors* $\mathcal{A}_\varepsilon$ *and* $\tilde{\mathcal{A}}_\varepsilon$, *respectively. Moreover, the attractors are uniformly bounded for* $\varepsilon \in \bar{I}_0$.

This theorem establishes the existence of global attractors for our equations, for each $\varepsilon \in I_0$. The next step is to prove the relationship between them. And for this we will make use of the concept of inertial manifolds.

DEFINITION 1.12. Given an evolutionary equation in a Banach space X which defines a semigroup $T(t)$ on X and has a global attractor $\mathcal{A}$. We say that M is an m-dimensional inertial manifold of the equation if M is a m-dimensional submanifold of X which contains the attractor $\mathcal{A}$, is positively invariant under $T(t)$; that is, $T(t)\,M \subset M$, and is exponentially attracting.

The existence of inertial manifolds will be deduced from the results of Chow and Lu (1988) and Rodriguez-Bernal (1990), and is summarized as follows.

THEOREM 1.2. *Let* $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon \in I_0}$ *be a triple, and* d_*, d^* *be fix. Suppose that* (H2), (H3), (H4), *and* (H5) *hold. Then there exists a positive number* $\varepsilon_0 > 0$, *such that, for* $\varepsilon \in \bar{I}_0 \cap [0, \varepsilon_0]$, *there exist inertial manifolds* $\mathcal{M}_\varepsilon \subset X^1_\varepsilon$ *and* $\tilde{\mathcal{M}}_\varepsilon \subset X^1_0 \times Y^1_\varepsilon$ *of* (I.11) *and* (I.13), *respectively, of dimension* $m(p+q)$. *Furthermore, for* $\varepsilon \in \bar{I}_0 \cap [0, \varepsilon_0]$, *the flows of* (I.11) *and* (I.13) *on the inertial manifolds* $\mathcal{M}_\varepsilon$ *and* $\tilde{\mathcal{M}}_\varepsilon$ *are given respectively by the following* $m(p+q)$*-dimensional ordinary differential equations*

$$\frac{dy}{dt} = h_\varepsilon(y), \qquad y \in \mathbb{R}^{m(p+q)}$$

and

$$\frac{dy}{dt} = \tilde{h}_\varepsilon(y), \qquad y \in \mathbb{R}^{m(p+q)}.$$

The proof is a standard application of the existence theorem for inertial manifolds. First, since we know that the attractors are contained in the ball radius $\mathcal{N}$, we will truncate the nonlinearities in a neighborhood of these balls, in such a way that the resulting systems are more tractable, these will be done by making use of Remark 1.9. Thus, for each ε, we have the truncated systems

$$u_t + A_\varepsilon u = f_{\aleph}, \qquad \text{in} \quad X^1_\varepsilon \tag{I.20}$$

$$u_t + C_\varepsilon u = \tilde{f}^\varepsilon_{\aleph}, \qquad \text{in} \quad X^1_0 \times Y^1_\varepsilon. \tag{I.21}$$

Note that the flow of (I.11) and (I.20) are the same in the ball of radius $\mathcal{N}$ (similar for (I.13) and (I.21)).

Now let us state some results due to Rodriguez-Bernal (1990) which will assure us that it is sufficient to prove the existence of inertial manifolds for the truncated systems, since this will imply the existence of inertial manifolds in the full systems. But first, consider the following definition.

DEFINITION 1.13. A semigroup S in a Banach space E is said to be uniformly bounded, if and only if, for all $T>0$, $B\subset E$, bounded, we have that $\bigcup_{t=0}^{T} S(t)\, B$ is bounded in E.

Remark 1.14. The uniform bounded principle holds in our case, since the dissipative condition (H4) was taken uniformly in bounded sets.

THEOREM 1.3 (Rodriguez-Bernal (1990)). *Let $\mathcal{N}>0$, and let us consider* (I.20) (*or*, (I.21)). *Then the following are true,*

(i) *Let $\mathcal{M}$ be an inertial manifold for* (I.20) (*or*, (I.21)), *closed in X_ε^1* (*or*, $X_0^1\times Y_\varepsilon^1$), *then $\mathcal{A}_\varepsilon\subset\mathcal{M}$* (*or*, $\tilde{\mathcal{A}}_\varepsilon\subset\mathcal{M}$).

(ii) *Assume that T_f^ε* (*or S_f^ε*) *is uniformly bounded, let $\mathcal{M}_*\subset\mathcal{M}$ be open in $\mathcal{M}$ such that $\mathcal{A}_\varepsilon\subset\mathcal{M}_*$* (*or* $\tilde{\mathcal{A}}_\varepsilon\subset\mathcal{M}_*$). *Then for all $B\subset X_\varepsilon^1$* (*or*, $B\subset X_0^1\times Y_\varepsilon^1$), *bounded, there exists $C(B)\geqslant 0$ such that*

$$\operatorname{dist}_{X_\varepsilon^1}(T_f^\varepsilon(t)\, B, \mathcal{M}_*)\leqslant C(B)\exp(-Mt), \qquad t\geqslant 0.$$

Similarly, for (I.21).

THEOREM 1.4 (Rodriguez-Bernal (1990)). *Suppose that T_f^ε is uniformly bounded. Let $\mathcal{N}>0$ and $\mathcal{M}$ be an inertial manifold, closed in X_ε^1, for $T_{f_\aleph}^\varepsilon$, such that $T_{f_\aleph}^\varepsilon$, such that $T_{f_\aleph}^\varepsilon|_{\mathcal{M}}$ is open.*

Let $B_0\subset X_\varepsilon^1$ be open and bounded, such that $\mathcal{A}_\varepsilon\subset B_0$, then B_0 is absorbing, and $B_*=\bigcup_{t\geqslant 0} T_f^\varepsilon(t)\, B_0$ is absorbing and bounded (*since T_f^ε is uniformly bounded*). *Moreover, suppose that we have chosen $\mathcal{N}$ such that $B_*\subset\overline{B(\mathcal{N})}$, where $\overline{B(\mathcal{N})}$ is the ball of radius $\mathcal{N}$ in X_ε^1. If we denote by $\mathcal{M}_*$ the* (*nonempty*) *interior* (*in $\mathcal{M}$*) *of $\mathcal{M}\cap B_*$, then $\mathcal{M}_*$ is an inertial manifold for T_f^ε. In particular if B_0 is positively invariant for T_f^ε then $B_*=B_0$ and $\mathcal{M}_*=\mathcal{M}\cap B_0$ is an inertial manifold for T_f^ε.*

Remark 1.15. Loosely speaking, the previous theorems say that the inertial manifold for the truncated system contains the attractor of the original system, and that from an inertial manifold of the truncated problem one can construct an inertial manifold for the original one.

Remark 1.16. We are going to prove the existence of an inertial manifold for (I.20) (or, (I.21)), and this will be constructed as a graph of a fixed point of an operator. This will berify all hypothesis of the theorem above since: $\mathcal{M}$ will be closed in X^1_ε, because is a graph of a continuous function globally defined in a finite dimensional space, and, furthermore, $T^\varepsilon_{f_N}|_{\mathcal{M}}$ is an homeomorphism of $\mathcal{M}$ and so is open.

With this, from now on we are going to use the truncated systems, instead of the original ones, having in mind that the theorems above are applied.

The proof of existence of inertial manifolds uses projections over finitely many eigenvalues, namely we define the spectral projections $\mathscr{P}^\varepsilon$ and $\mathscr{I}^\varepsilon$ by

$$\begin{aligned} \mathscr{P}^\varepsilon u &= \sum_{k=1}^{n} \langle u, \varphi_{k,\Omega_\varepsilon}\rangle_\varepsilon \varphi_{k,\Omega_\varepsilon} \\ \mathscr{I}^\varepsilon u &= u - \mathscr{P}^\varepsilon u, \end{aligned} \tag{I.22}$$

for all $u \in X_\varepsilon$, where n is given by (H5) and, for all $u = (u^1, ..., u^m) \in X_\varepsilon$ and $\varphi \in L^2(\Omega_\varepsilon)$, we have

$$\langle u, \varphi\rangle_\varepsilon = \begin{pmatrix} \int_{\Omega_\varepsilon} u^1 \varphi \\ \vdots \\ \int_{\Omega_\varepsilon} u^m \varphi \end{pmatrix}.$$

We shall use the notation $u^{\mathscr{P}} = \mathscr{P}^\varepsilon u$ and $u^{\mathscr{I}} = \mathscr{I}^\varepsilon u$. Using $\mathscr{P}^\varepsilon$ we can decompose X_ε as

$$\begin{aligned} X_\varepsilon &= X^{\mathscr{P}}_\varepsilon \oplus X^{\mathscr{I}}_\varepsilon \\ X^{\mathscr{P}}_\varepsilon &= \{u \in X_\varepsilon : \mathscr{P}^\varepsilon u = u\} \\ X^{\mathscr{I}}_\varepsilon &= \{u \in X_\varepsilon : \mathscr{P}^\varepsilon u = 0\} \\ X^{\mathscr{P}}_{1,\varepsilon} &= \{u \in X^1_\varepsilon : \mathscr{P}^\varepsilon u = u\} \\ X^{\mathscr{I}}_{1,\varepsilon} &= \{u \in X^1_\varepsilon : \mathscr{P}^\varepsilon u = 0\}. \end{aligned} \tag{I.23}$$

Similarly, we can define the spectral projections $\tilde{\mathscr{P}}^\varepsilon$ and $\tilde{\mathscr{I}}^\varepsilon$ by

$$\begin{aligned} \tilde{\mathscr{P}}^\varepsilon u &= \sum_{k=1}^{n} \langle u, \xi^\varepsilon_k\rangle^\varepsilon \xi^\varepsilon_k \\ \tilde{\mathscr{I}}^\varepsilon u &= u - \tilde{\mathscr{P}}^\varepsilon u, \end{aligned} \tag{I.24}$$

for all $u \in X_0 \times Y_\varepsilon$, where n is given by (H5) and, for all $u = (u_1, u_2) \in X_0 \times Y_\varepsilon$, with $u_i = (u_i^1, ..., u_i^m)$, and $\varphi = (\varphi_1, \varphi_2) \in L^2(\Omega_0) \times L^2(R_\varepsilon)$, we have

$$\langle u, \varphi \rangle^\varepsilon = \begin{pmatrix} \int_{\Omega_0} u_1^1 \varphi_1 + \int_{R_\varepsilon} u_2^1 \varphi_2 \\ \vdots \\ \int_{\Omega_0} u_1^m \varphi_1 + \int_{R_\varepsilon} u_2^m \varphi_2 \end{pmatrix}.$$

We shall use the notation $u^{\tilde{\mathscr{P}}} = \tilde{\mathscr{P}}^\varepsilon u$ and $u^{\tilde{\mathscr{I}}} = \tilde{\mathscr{I}}^\varepsilon u$. Using $\tilde{\mathscr{P}}^\varepsilon$ we can decompose $X_0 \times Y_\varepsilon$ as

$$\begin{aligned} X_0 \times Y_\varepsilon &= \tilde{X}_\varepsilon^{\tilde{\mathscr{P}}} \oplus \tilde{X}_\varepsilon^{\tilde{\mathscr{I}}} \\ \tilde{X}_\varepsilon^{\tilde{\mathscr{P}}} &= \{u \in X_0 \times Y_\varepsilon : \tilde{\mathscr{P}}^\varepsilon u = u\} \\ \tilde{X}_\varepsilon^{\tilde{\mathscr{I}}} &= \{u \in X_0 \times Y_\varepsilon : \tilde{\mathscr{P}}^\varepsilon u = 0\} \\ \tilde{X}_{1,\varepsilon}^{\tilde{\mathscr{P}}} &= \{u \in X_0^1 \times Y_\varepsilon^1 : \tilde{\mathscr{P}}^\varepsilon u = u\} \\ \tilde{X}_{1,\varepsilon}^{\tilde{\mathscr{I}}} &= \{u \in X_0^1 \times Y_\varepsilon^1 : \tilde{\mathscr{P}}^\varepsilon = 0\}. \end{aligned} \tag{I.25}$$

DEFINITION 1.17. *Given* $Y = (y_1, ..., y_n) \in \mathbb{R}^{mn}$ *and an orthonormal set* $\{\varphi_i\}$, *in some Banach space* Z, *we set*

$$[Y]_\varphi^n = \sum_{i=1}^n y_i \varphi_i \in Z^m. \tag{I.26}$$

In particular, for each $\varepsilon \in \bar{I}_0$, if we take as orthonormal sets $\{\varphi_{i,\Omega_\varepsilon}\}_{i=1}^n \subset X_\varepsilon$ and $\{\xi_i^\varepsilon\}_{i=1}^n \subset X_0 \times Y_\varepsilon$, the correspondences $Y \mapsto [Y]_{\varphi_{\cdot,\Omega_\varepsilon}}^n$ and $Y \mapsto [Y]_{\xi^\varepsilon}^n$ define linear isomorphisms $\mathbb{R}^{mn} \to X_\varepsilon^{\mathscr{P}}$ and $\mathbb{R}^{mn} \to \tilde{X}_\varepsilon^{\tilde{\mathscr{P}}}$, respectively. Then we have

$$\|[Y]_{\varphi_{\cdot,\Omega_\varepsilon}}^n\|_\varepsilon = |Y|_{nm}, \qquad \|[Y]_{\varphi_{\cdot,\Omega_\varepsilon}}^n\|_{1,\varepsilon} \leqslant (1 + \lambda_n^{1/2}(\Omega_\varepsilon))\, |Y|_{nm}$$

$$\|[Y]_{\xi^\varepsilon}^n\|_{X_0 \times Y_\varepsilon} = |Y|_{nm}, \qquad \|[Y]_{\xi^\varepsilon}^n\|_{X_0 \times Y_\varepsilon}\|_{X_0^1 \times Y_\varepsilon^1} \leqslant (1 + (\lambda_n^\varepsilon)^{1/2})\, |Y|_{nm}.$$

Remark 1.18. Notice that, since we will have convergence of the eigenvalues and eigenvectors, the spectral projections will also converge. This will be enough to apply Chow and Lu's theorem on the existence of inertial manifolds, since the gap condition (H5) can be imposed on the limit eigenvalues. This will also show that the manifolds are a graph of a Lipschitz function, since in these standard theorems the manifold is seek of the form $\{u = v + w$, with $w = \phi(v)\}$, where v is in $X_\varepsilon^{\mathscr{P}}$. Therefore lower order modes (v) will determine the higher ones (w).

1.2.2. *Convergence of Inertial Manifolds and Its Applications to Attractors.* Theorem 1.2 proves the existence of inertial manifolds for each ε fixed, but still does not tell us the behavior at the limit as ε goes to zero. The main contribution of this work will be to show such behavior.

THEOREM 1.5. *Let* $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon \in I_0}$ *be a triple, and* d_*, d^* *be fixed. Suppose that* (H2), (H3), (H4), *and* (H5) *hold. Let* $h_\varepsilon, \tilde{h}_\varepsilon$ *be given by Theorem* 1.2, *then*

1. $h_\varepsilon - \tilde{h}_\varepsilon$ *converge to* 0 *in* $C^1(U; \mathbb{R}^{m(p+q)})$ *as* $\varepsilon \to 0$ *for every bounded open set* U *in* $\mathbb{R}^{m(p+q)}$;

2. h_ε *converge to* h_0 *in* $C^1(U; \mathbb{R}^{m(p+q)})$ *as* $\varepsilon \to 0$ *for every bounded open set* U *in* $\mathbb{R}^{m(p+q)}$.

With this theorem we are able to prove upper semicontinuity of the attractors. Moreover, if $m = 1$ (scalar case), using the transversality of the stable and unstable manifolds of hyperbolic equilibria of (I.13) (Henry 1985), we see that the flow of (I.11) on the attractor $\mathscr{A}_\varepsilon$ is topologically equivalent to the flow of (I.13) on the attractor $\tilde{\mathscr{A}}_0$ if the equilibria are all hyperbolic.

The proof is accomplished by proving the convergence of the functions which define the inertial manifold as a graph.

1.3. *Examples*

We want to mention some examples to illustrate our work and at the same time convince the reader that the results presented here are a generalization of the results by Morita (1990) and Hale and Raugel]1991, 1992). The difference in the examples will be only on the distribution of the eigenvalues. For the sake of simplicity, let us suppose that Ω_ε are the so called "dumbbell-shaped" domains, as defined precisely by Hale and Vegas (1984) and Vegas (1983). We have that (see Section 2) the eigenvalues μ_i behave like i^2 and thus, there exists an i_0 such that (H5) is true, if applied just for μ_i. Now what differs from the previous works is the placement of the eigenvalues λ_{i,Ω_0}, and that is what will differentiate these examples.

1.3.1. *Thin Domains.* We want to emphasize with this example that if we consider Ω_0 to be empty, we recover the results of Hale and Raugel (1991), (1992), simply because our proof is based on the fact that we broke the system (I.11) into two systems (I.13) which are completely decoupled, and one of them being exactly a thin domain problem. In this sense we generalized Hale and Raugel (1991, 1992).

1.3.2. *Morita's Results.* Morita's result is based upon the fact that if we increase the diffusion (or, equivalently decrease our domain Ω_0) then the corresponding eigenvalues increase (except the zero eigenvalue). Thus what he assumes is that the diffusion is large enough such that the gap between the second eigenvalue $\lambda_{1,\Omega_0} = \lambda_{2,\Omega_0}$, that is the zero eigenvalues, and the next eigenvalue satisfies (H5). In other words, he assumes (H5) with $p = 2$ and $q = 0$.

His result also apply just for thin domains that are straight channels (that is g_1, g_2 almost constants), on the other hand we can allow more general thin domains.

Thus showing that our results includes his case.

Remark 1.19. Actually Morita's result is about existence of local invariant manifolds. That is, he does not assume at first that the system has a global attractor, only a local attractor. We also can generalize our results to cover this case, but the proof is exactly the same. The assumption of existence of global attractors (that is (H4)) is very realistic in many physical systems.

1.3.3. *A More General Example.* Let us suppose that there exists a subsequence of eigenvalues of the fixed domain λ_{i_k,Ω_0}, such that λ_{i_k,Ω_0} and λ_{i_k+1,Ω_0} grows like i_k^2 (and $(i_k+1)^2$), then the union of the eigenvalues will have a subsequence with this same properties and therefore, given any diffusion and nonlinearity with Lipschtz constant small enough (in relation to the size of the largest channel), will satisfy (H5) (this will follows from Section 2).

This shows that we can have examples that (H5) is verified with any p, q. Thus showing that we are actually allowing the channel to influence the dynamics.

1.3.4. *A Final Remark.* We want to mention that, when we have more than one channel, the number of eigenvalues that will be sed in the limit equation will not be the same for each one, but in fact, this number will depend only on the length of each channel. We will have more eigenvalues from the larger channels. Notice that the number of eigenvalues can be chosen (that is, the dimension of the inertial manifold) independent of g^i. This follows from Lemma 2.3 and Section 2).

2. Eigenvalue Problem

In this section, we are going to describe the results of Arrieta (1991, 1993). These results will prove the relationship between the following eigenpairs (see Section 1.1.3) $\{\lambda_n(\Omega_\varepsilon), \varphi_n, \Omega_\varepsilon\}$, $\{\lambda_n(\Omega_0), \varphi_{n,\Omega_0}\}$, $\{\tau_n(R_\varepsilon), \gamma_{n,R_\varepsilon}\}$

and $\{\mu_n(i), \chi_{n,i}\}$. Let us define a family of orthonormal eigenfunctions in $L^2(\Omega_\varepsilon)$ by

$$\Phi_n^\varepsilon = \begin{Bmatrix} \varphi_{i,\Omega_0}, \text{ in } \Omega_0 \\ 0, \text{ in } R_\varepsilon \end{Bmatrix}, \qquad \text{if } \lambda_n^\varepsilon = \lambda_i(\Omega_0)$$
$$\Phi_n^\varepsilon = \begin{Bmatrix} 0, \text{ in } \Omega_0 \\ \gamma_{j,R_\varepsilon}, \text{ in } R_\varepsilon \end{Bmatrix}, \qquad \text{if } \lambda_n^\varepsilon = \tau_j(R_\varepsilon) \tag{II.1}$$

thus, since $\Omega_0 \cap R_\varepsilon = \varnothing$, $\Phi_n^\varepsilon \in H^1(\Omega_0) \cup H^1_{\Gamma_\varepsilon}(R_\varepsilon)$, meaning that the correspondent restrictions of Φ_n^ε to Ω_0 and R_ε are in these spaces.

DEFINITION 2.1. We say that a point $x_\varepsilon \in \mathbb{R}^+$ divides the spectrum if there exist three positive constants δ, M, N, such that

$$[x_\varepsilon - \delta, x_\varepsilon + \delta] \cap \{\lambda_n^\varepsilon\}_{n=1}^\infty = \varnothing \qquad \varepsilon \in I_0 \tag{II.2}$$

$$0 \leqslant x_\varepsilon \leqslant M \qquad \varepsilon \in I_0 \tag{II.3}$$

$$N(x_\varepsilon) = \#\{\lambda_i^\varepsilon : \lambda_i^\varepsilon \leqslant x_\varepsilon\} \leqslant N < \infty. \tag{II.4}$$

Moreover, if x_ε divides the spectrum, we can define the projection operator:

$$P_{x_\varepsilon} : L^2(\Omega_\varepsilon) \to [\Phi_1^\varepsilon, ..., \Phi_{N(x_\varepsilon)}^\varepsilon]$$
$$g \to \sum_{i=1}^{N(x_\varepsilon)} (g, \Phi_i^\varepsilon)_{X_\varepsilon} \Phi_i^\varepsilon$$

From Arrieta (1991), (1993), we have the following theorem.

THEOREM 2.1. *If $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon \in I_0}$ is a triple (and so (H1) is verified), then*

(i) $\lim_{\varepsilon \to 0} (\lambda_n(\Omega_\varepsilon) - \lambda_n^\varepsilon) = 0, \qquad n \in Z^+.$

(ii) $\begin{cases} \|\varphi_{r_\varepsilon,\Omega_\varepsilon} - P_{x_\varepsilon}\varphi_{r_\varepsilon,\Omega_\varepsilon}\|_{H^1(\Omega_0 \cup R_\varepsilon)} \xrightarrow{\varepsilon \to 0} 0 \qquad r_\varepsilon = 1, ..., N(x_\varepsilon), \\ \textit{for any } x_\varepsilon \textit{ which divides the spectrum.} \end{cases}$

We need some known properties of the eigenvalue problem on the thin domain R_ε, the proofs can be found in Arrieta (1991, 1993). The following result is true, THEOREM 2.2. *Suppose* (H0) *holds and let* $n \in Z^+$, *fixed. Denote by,*

$$\eta = \sup\{|g_1' + g_2'| : x \in (0, 1)\}$$

$$\zeta = \sup\{|g_2'| : x \in (0, 1)\}.$$

There exists $\varepsilon_n \in I_0 \cap (0, \varepsilon_0)$, *and* $c_n \in \mathbb{R}^+$, *such that, for* $0 < \varepsilon < \varepsilon_n$,

$$0 \leqslant \mu_n - \tau_n^\varepsilon \leqslant c_n \varepsilon^2 (\eta + \zeta)^2 \tag{II.5}$$

$$\|\gamma_n^\varepsilon - (\gamma_n^\varepsilon, \varepsilon^{-1/2} \chi_n)_{L^2(R_\varepsilon)} \varepsilon^{-1/2} \chi_n\|^2_{H^1(R_\varepsilon)} = O(\varepsilon^2 (\eta - \zeta)^2) \tag{II.6}$$

Moreover, from Sturm–Liouville theory, we know that the eigenvalues $\{\mu_n(i)\}$ are simple, and in fact we know the following (see Courant and Hilbert (1955), p. 415).

LEMMA 2.3. *There exist* $C > 0$ *and an integer* N_0 *such that, for all* $n \geqslant N_0$

$$\left| \mu_n^i - \frac{\pi^2 n^2}{(b^i - a^i)^2} \right| \leqslant C.$$

For more references for these results with application to thin domains, see Hale and Raugel (1991, 1992).

With this, we can prove the following corollary.

COROLLARY 2.4. *If* (H0) *holds, then there exists a subsequence* $\{n_k\} \subset \mathbb{N}$, *with* ∞ *as an accumulation point, such that there exist* $C_1, C_2 > 0$ *such that,*

$$|\mu_j - C_1 j^2| \leqslant C_2, \tag{II.7}$$

where $j = n_k$, $n_k + 1$, *for all k. And, in particular, there exists* C_3 *such that*

$$|\mu_{n_k+1} - \mu_{n_k}| \geqslant C_3 n_k. \tag{II.8}$$

The interesting thing to observe is that μ_{n_k} and μ_{n_k+1} will be eigenvalues coming from the longest channel. The proof of this corollary follows from the following proposition, which is straight forward.

PROPOSITION 2.5. *Let* $N \geqslant 1$ *be a given integer. Suppose that, for each* $j = 1, ..., N$, *we have a sequence* $\{\lambda_i^j\}_{i=1}^\infty \subset \mathbb{R}^+$, *such that there exist constants* v, N_0, a^j *with*

$$|\lambda_i^j - a^j i^2| \leqslant b, \quad \textit{for all} \quad i \geqslant N_0, \quad j = 1, ...N.$$

Then, if $\{\mu_k\}_{k=1}^{\infty} = \{\lambda_i^1\}_{i=1}^{\infty} \cup \cdots \cup \{\lambda_i^N\}_{i=1}^{\infty}$, *there exists* $J \subset \mathbb{N}$, *with* ∞ *as accumulation point, and a constant* $c > 0$, *such that for all* $k \in J$,

$$\mu_{k+1} - \mu_k \geqslant ck.$$

3. Existence of Inertial Manifolds

The objective of this Section is to prove Theorem 1.2. As mentioned before, we are going to use Chow and Lu (1988) to show the existence of invariant manifolds. Therefore, we will state this result precisely.

3.1. *Chow and Lu's Existence Theorem.*

In order to state the theorem precisely, we will need to fix some notation. Let X, Y and Z be Banach spaces. Suppose that $X \subseteq Y \subseteq Z$, X is continuously imbedded in Y and Y is continuously imbedded in Z. Let $S(t)$ $(t \geqslant 0)$ be a strongly continuous semigroup of bounded linear operators on Z. Consider the following assumptions

(CL1). $Z = Z_1 \oplus Z_2$, *where* Z_1 *and* Z_2 *are invariant linear subspaces under* $S(t)$.

(CL2). $P_i S(t) = S(t) P_i$, $i = 1, 2$, *where* P_i *is a projection from* Z *to* Z_i.

(CL3). $P_i X$ *and* $P_i Y$ $(i = 1, 2)$ *are invariant under* $S(t)$ *and* $S(t)\, Y \subseteq X$ *for* $t \geqslant 0$.

(CL4). $S(t)$ *can be extended to a group on* Z_1.

(CL5). *There exist constants* $\alpha, \beta, \gamma, \eta, M$, *and* M^* *such that* $\alpha > 0$, $\beta > 0$, $0 \leqslant \gamma < 1$, $M \geqslant 1$, $M^* \geqslant 0$.

$$\|e^{-\eta t} S(t) P_1 y\|_X \leqslant M e^{\alpha t} \|y\|_Y, \qquad \textit{for} \quad t \leqslant 0, \quad y \in Y, \tag{III.1}$$

$$\|e^{-\eta t} S(t) P_2 x\|_X \leqslant M e^{-\beta t} \|x\|_X, \qquad \textit{for} \quad t \geqslant 0, \quad x \in X, \tag{III.2}$$

$$\|e^{-\eta t} S(t) P_2 y\|_X \leqslant (M t^{-\gamma} + M^*)\, e^{-\beta t} \|y\|_Y, \qquad \textit{for} \quad t > 0, \quad y \in Y. \tag{III.3}$$

Let us also define the constant

$$K(\alpha, \beta, \gamma) = M\left(\frac{1}{\alpha} + \frac{2-\gamma}{1-\gamma}\beta^{-1+\gamma}\right) + M^* \frac{1}{\beta}. \tag{III.4}$$

Finally, if $F \in C^k(X, Y)$, let us consider the following integral equation

$$x(t) = S(t-t_0)\, x(t_0) + \int_{t_0}^{t} S(t-t_0)\, F(x(s))\, ds, \tag{III.5}$$

where $x(t)$ is a map from an interval $J \subseteq \mathbb{R}$ to X.

DEFINITION 3.1. If $x\colon J \to X$ is continuous and satisfies (III.5) for all t_0, $t \in J$, $t_0 \leqslant t$, then we call $x(t)$ a solution of (III.5) on J. For $x_0 \in X$, we denote by $x(t, t_0)$ a solution of (III.5) which equals x_0 at $t = 0$.

With this we have the following theorem due to Chow and Lu (1988).

THEOREM 3.1 (Chow and Lu 1988). *Let $\eta < 0$. Assume that (CL1)–(CL5) are satisfied. If $F \in C^k(X, Y)$, $\beta + (k-1)\eta > 0$, and*

$$K(\alpha, \beta + (k-1)\eta, \gamma)(\operatorname{Lip} F) < 1, \tag{III.6}$$

then there exists a C^k invariant manifold $\mathcal{M}$ for the flow defined by (III.5) and $\mathcal{M}$ satisfies

(i) $\mathcal{M} = \{x_0 \mid x(t, x_0)$ *is defined for all* $t \in \mathbb{R}^-$ *and* $P_2 x(t, x_0) \in C^0(\mathbb{R}^-, X)\}$, *and*

(ii) $\mathcal{M} = \{\xi + h(\xi) \mid \xi \in P_1 X\}$,

where $h\colon P_1 X \to P_2 X$ is C^k.

Moreover, if we assume further that $k \geqslant 1$, $K(\alpha, \beta, \gamma)(\operatorname{Lip} F) < 1$, and

$$\frac{MK(\alpha, \beta, \gamma)\operatorname{Lip}(F)}{1 - K(\alpha, \beta, \gamma)\operatorname{Lip}(F)} < 1, \tag{III.7}$$

then for any solution $y(t, x_0)$ of (III.5) on $[0, \infty)$, there exists a unique $x_0^* \in \mathcal{M}$ such that

$$\sup_{t \geqslant 0} e^{-\eta t} \| y(t, x_0) - x(t, x_0^*) \|_X < +\infty. \tag{III.8}$$

Remark 3.2. Let B be any bounded set of $P_1 X$, and consider the spaces $F_1 = C^0(B, C_\eta(\mathbb{R}^-, X))$ and $F_2 = C^0(B, \mathcal{L}^1(P_1 X, C_\eta(\mathbb{R}^-, X)))$. For $\psi \in F_1$ and $\Psi \in F_2$, define

$$\mathcal{K}(\psi)(t, \xi) = S(t)\, \xi + \int_0^t S(t-s)\, P_1 F(\psi(\xi))\, ds$$
$$+ \int_{-\infty}^{t} S(t-s)\, P_2 F(\psi(\xi))\, ds \tag{III.9}$$

and

$$\mathscr{F}_\psi(\Psi) = S(t)\,\xi + \int_0^t S(t-s)\,P_1 D_x F(\psi(s,\xi))\,\Psi\,ds$$

$$+ \int_{-\infty}^t S(t-s)\,P_2 D_x F(\psi(s,\xi))\,\Psi\,ds. \tag{III.10}$$

In the definition of $\mathscr{F}_\psi(\Psi)$, we assume that, for every $\zeta \in P_1 X$, $\mathscr{F}_\psi(\Psi)\cdot\zeta \in C^0(B, C_\eta(\mathbb{R}^-, X))$ and is defined by

$$(\mathscr{F}_\psi(\Psi)\cdot\zeta)(s,\xi) = S(t)\,\xi + \int_0^t S(t-s)\,P_1 D_x F(\psi(s,\xi))(\Psi\cdot\zeta)(s,\xi)\,ds$$

$$+ \int_{-\infty}^t S(t-s)\,P_2 D_x F(\psi(s,\xi))(\Psi\cdot\zeta)(s,\xi)\,ds.$$

Then $\mathscr{K}$ and $\mathscr{F}_\psi$ are contractions with constant $K(\alpha, \beta, \gamma)(\operatorname{Lip} F) < 1$, for every $\psi \in F_1$. The proof of Theorem 3.1, implies that, if φ is the fixed point of $\mathscr{K}$, then $h(\xi) = \varphi(\xi)(0) - S(0)\,\xi$ and $D_\xi\varphi$ is the unique fixed point of $\mathscr{F}_\varphi$. This characterization of h and $D_\xi h$ will be very useful to prove the convergence results.

3.2. *Proof of Existence Theorem*

Now to prove Theorem 1.2 we need only to verify the hypotheses (CL1)–(CL5). But first, as mentioned before, the results of Rodriguez-Bernal (1990) allow us restrict the study to the truncated systems (I.20)–(I.21). Thus if we prove the existence results for these systems we are proving Theorem 1.2. Therefore from now on we will drop the index $\aleph$ in the nonlinearities. We will start by proving such existence for (I.20).

Let us begin by defining the spaces, X, Y, Z, Z_1 and Z_2, in the following way

$$Z = Y = X_\varepsilon; \qquad Z_1 = X_\varepsilon^{\mathscr{P}}; \qquad Z_2 = X_\varepsilon^{\mathscr{I}}; \qquad X = X_\varepsilon^{\mathscr{P}} \oplus X_{1,\varepsilon}^{\mathscr{I}},$$

where $X_\varepsilon^{\mathscr{P}}$, $X_\varepsilon^{\mathscr{I}}$ are given by (I.23). Moreover, we have that $P_1 = \mathscr{P}^\varepsilon$ and $P_2 = \mathscr{I}^\varepsilon$. Finally $S(t) = e^{-A_\varepsilon t}$. Therefore it is clear that (CL1)–(CL4) hold, and it follows from Henry (1981) and (H5) that, (CL5) holds with

$$\eta_\varepsilon = -\frac{\lambda_{n+1}(\Omega_\varepsilon)\,d_* + \lambda_n(\Omega_\varepsilon)\,d^*}{2} < 0;$$

$$\alpha = \beta = \frac{\lambda_{n+1}(\Omega_\varepsilon)\,d_* - \lambda_n(\Omega_\varepsilon)\,d^*}{2} > 0;$$

$$M = 1; \qquad \gamma = 1/2; \qquad M^* = 0.$$

Thus we have

$$K(\alpha, \beta, \gamma) = \frac{2 + 3\sqrt{2}\,(\lambda_{n+1}(\Omega_\varepsilon)\, d_* - \lambda_n(\Omega_\varepsilon)\, d^*)^{1/2}}{\lambda_{n+1}(\Omega_\varepsilon)\, d_* - \lambda_n(\Omega_\varepsilon)\, d^*}.$$

Thus using (H5) and the convergence of the eigenvalues, it follows that for all $\varepsilon \in I_0$, with $\varepsilon \leqslant \varepsilon_1$, $K(\alpha, \beta, \gamma)\,\mathrm{Lip}(\hat{f}) < 0.5$, all of the hypotheses of Theorem 3.1 hold and we have proved the existence of the exponential attracting invariant manifold, with rate η_ε, given by a graph of *a* C^1 function, which will be denoted by $\sigma^\varepsilon\colon \mathbb{R}^{nm} \to X^1_\varepsilon$.

Remark 3.3. Observe that, for all $u \in X$, if we denote by

$$y_k = \langle u, \varphi_{k, \Omega_\varepsilon} \rangle_\varepsilon, \qquad y = (y_1, ..., y_n),$$

and using the notation in (I.26), we have that the flow on the inertial manifold is given by the ODE

$$\begin{aligned} &\frac{d}{dt} y_i(t) = \langle f([y]^n_{\varphi, \Omega_\varepsilon} + \sigma^\varepsilon(y)), \varphi_{i, \Omega_\varepsilon} \rangle_\varepsilon - \lambda_i(\Omega_\varepsilon)\, D y_i \\ &y_i(0) = \mathscr{Y}_i(0), \qquad \text{for} \quad i = 1, ..., n. \end{aligned} \tag{III.11}$$

Therefore, in Theorem 1.2, we have that $h_\varepsilon(y) = (h^1_\varepsilon(y), ..., h^n_\varepsilon(y))$, where

$$h^i_\varepsilon(y) = \langle f([y]^n_{\varphi, \Omega_\varepsilon} + \sigma^\varepsilon(y)), \varphi_{i, \Omega_\varepsilon} \rangle_\varepsilon - \lambda_i(\Omega_\varepsilon)\, D y_i.$$

And this finishes the proof of existence for (I.20).

For (I.21) we proceed similarly. Using the notation in (I.24) and (I.25), we can define the spaces, X, Y, Z, Z_1 and Z_2, in the following way

$$Z = Y = X_0 \times Y_\varepsilon; \qquad Z_1 = \tilde{X}^{\mathscr{P}}_\varepsilon; \qquad Z_2 = \tilde{X}^{\mathscr{T}}_\varepsilon; \qquad X = \tilde{X}^{\mathscr{P}}_\varepsilon \oplus \tilde{X}^{\mathscr{T}}_{1,\varepsilon}.$$

Moreover, we have that $P_1 = \tilde{\mathscr{P}}^\varepsilon$ and $P_2 = \tilde{\mathscr{T}}^\varepsilon$. Finally $S(t) = e^{-C_\varepsilon t}$. Therefore it is clear that (CL1)–(CL4) hold, and it follows from Henry (1981) that, (CL5) holds with

$$\tilde{\eta}_\varepsilon = -\frac{\lambda^\varepsilon_{n+1} d_* + \lambda^\varepsilon_n d^*}{2} < 0; \qquad \alpha = \beta = \left(\frac{\lambda^\varepsilon_{n+1} d_* - \lambda^\varepsilon_n d^*}{2}\right) > 0;$$

$$M = 1; \qquad \gamma = 1/2; \qquad M^* = 0.$$

Thus we have that

$$K(\alpha, \beta, \gamma) = \frac{2 + 3\sqrt{2}\,(\lambda^{\varepsilon}_{n+1} d_* - \lambda^{\varepsilon}_n d^*)^{1/2}}{\lambda^{\varepsilon}_{n+1} d_* - \lambda^{\varepsilon}_n d^*}.$$

Thus using (H5) and the convergence of the eigenvalues, it follows that for all $\varepsilon \in I_0$, with $\varepsilon \leqslant \varepsilon_1$, $K(\alpha, \beta, \gamma)\,\mathrm{Lip}(\tilde{f}_\varepsilon) < 0.5$, all hypotheses of Theorem 3.1 hold and we have proved the existence of the exponential attracting invariant manifold, with attracting rate $\tilde{\eta}_\varepsilon$, given by a graph of a C^1 function, which will be denoted by $\kappa^\varepsilon\colon \mathbb{R}^{mn} \to X^1_0 \times Y^1_\varepsilon$; thus, proving Theorem 1.2.

Remark 3.4. Observe that, for all $\mu \in X$, if we denote by

$$y_k = \langle u, \xi^\varepsilon_k \rangle_\varepsilon, \qquad y = (y_1, ..., y_n),$$

and using the notation in (I.26), we have that the flow on the inertial manifold is given by the ODE

$$\begin{aligned} \frac{d}{dt} y_i(t) &= \langle \tilde{f}_\varepsilon([y]^n_{\xi^\varepsilon} + \kappa^\varepsilon(y)), \xi^\varepsilon_i \rangle_\varepsilon - \lambda^\varepsilon_i D y_i \\ y_i(0) &= \mathscr{Y}_i, \qquad \text{for} \quad i = 1, ..., n. \end{aligned} \tag{III.12}$$

Therefore, in Theorem 1.2, we have that $\tilde{h}^1_\varepsilon(y), ..., \tilde{h}^n_\varepsilon(y))$, where

$$\tilde{h}^i_\varepsilon(y) = \langle \tilde{f}_\varepsilon([y]^n_{\xi^\varepsilon} + \kappa^\varepsilon(y)), \xi^\varepsilon_i \rangle_\varepsilon - \lambda^\varepsilon_i D y_i.$$

And this finishes the proof of existence for (I.21).

To distinguish the operators that give us σ, κ as fixed points, all the operators related to κ will have a $\tilde{}$ (for example $\tilde{\mathscr{K}}$ and $\tilde{\mathscr{F}}_\varphi$).

4. Convergence of Invariant Manifolds

In this section we will prove the convergence of the flows on the invariant manifold, for the truncated system, having in mind that this will automatically prove our result, therefore we are going to drop the index $\aleph$ in the nonlinearities. First let us point out the following.

Remark 4.1. Since the system (I.21) is defined in a product space, it is decoupled, we have that $\kappa^\varepsilon = (\kappa^\varepsilon_1, \kappa^\varepsilon_2)$, and, using the results of Hale and Raugel (1991, 1992), if we consider the equation

$$\frac{du}{dt} = -D \frac{1}{g_1 + g_2} ((g_1 + g_2) u_s)_s + \tilde{f}_0(u), \qquad \text{in} \quad R_0, \tag{IV.1}$$

and project this equation in the first q-eigenfunctions $\{\chi_j\}$ we have that this equation has an invariant manifold given by a graph of a Lipschtz function, $\kappa_2^0\colon \mathbb{R}^q \to Y_0^1$, such that the flow on the invariant manifold is given by the ODE

$$\dot{y}_i = (f([y]_\chi^q + \kappa_2^0(y)), \chi_i)_{R_0} - \mu_i D y_i, \qquad \text{for} \quad i = 1, ..., q. \tag{IV.2}$$

Moreover,

$$\sup_{y \in \mathbb{R}^q} |||\kappa_2^\varepsilon(y) - \kappa_2^0(y) \circ {}^\varepsilon L_1^{-1}|||_{1,\varepsilon} \to 0; \tag{IV.3}$$

$$|||\kappa_2^\varepsilon(y) - \kappa_2^0(y) \circ {}^\varepsilon L_1^{-1} - \kappa_2^\varepsilon(y') + \kappa_2^0(y') \circ {}^\varepsilon L_1^{-1}|||_{1,\varepsilon} \leqslant \alpha(\varepsilon)\, |y - y'|, \tag{IV.4}$$

where $\alpha(\varepsilon) \to 0$ and ${}^\varepsilon L_1$ is the operator that takes R_ε into the tubular domain, see Hale and Raugel (1991).

Therefore, from the remark above and from Remark 3.3 and Remark 3.4, in order to prove Theorem 1.5 it is suffices to prove that $\|\sigma - E\kappa\| \to 0$ in the C^1 sense. The first step is to fix some notation. Let us suppose that $B \subset \mathbb{R}^{nm}$ and let $\mathscr{Y} \in B$, consider the isomorphisms $\Pi\colon \mathbb{R}^{nm} \to X_\varepsilon^{\mathscr{P}}$, $\tilde{\Pi}\colon \mathbb{R}^{nm} \to \tilde{X}_\varepsilon^{\mathscr{P}}$, defined by

$$\Pi(\mathscr{Y}) = [\mathscr{Y}]^n_{\varphi, \Omega_\varepsilon}; \qquad \tilde{\Pi}(\mathscr{Y}) = [\mathscr{Y}]^n_{\xi}. \tag{IV.5}$$

Then, if we denote by $\eta = \tilde{\eta}_0$ (defined in Section 3.2), and let $\varphi \in C^0(B, C_\eta(\mathbb{R}^-, X))$, $\tilde{\varphi} \in C^0(B, C_\eta(\mathbb{R}^-, \tilde{X}))$ be the fixed points of $\mathscr{K}$ and $\tilde{\mathscr{K}}$, respectively, then

$$\varphi(0, \chi) = \chi + \sigma(\Pi^{-1}\chi), \qquad \tilde{\varphi}(0, \tilde{\chi}) = \tilde{\chi} + \tilde{\kappa}(\tilde{\Pi}^{-1}\tilde{\chi}),$$

for all $\chi \in P_1 X, \tilde{\chi} \in \tilde{P}_1 \tilde{X}$, where $X, Y, \tilde{X}, \tilde{Y}, P_1$, and $\tilde{P}_1$ are defined in section 3.2.

Therefore to prove Theorem 1.5 it is enough to prove that

$$\sup_{\mathscr{Y} \in B} |\varphi(t, \Pi\mathscr{Y}) - E\tilde{\varphi}(t, \tilde{\Pi}\mathscr{Y})|_{C_\eta((-\infty, 0], X)} \to 0 \tag{IV.6}$$

and

$$\sup_{\mathscr{Y}, \mathscr{Z} \in B} |D_\xi \varphi(t, \Pi\mathscr{Y})\, \Pi\mathscr{Z} - E D_\xi \tilde{\varphi}(t, \tilde{\Pi}\mathscr{Y})\, \Pi\mathscr{Z}|_{C_\eta((-\infty, 0], X)} \to 0, \tag{IV.7}$$

as $\varepsilon \to 0$.

The proof of (IV.6) will be done by comparing the operators $\mathscr{K}^\varepsilon$ and $\tilde{\mathscr{K}}^\varepsilon$, this comparison is very technical and to clarify the proof we will break it in several lemmas. The first step is to define both operators in the same space. To do this let E be the extension defined by (I.19) and consider the following Lemmas,

LEMMA 4.1. *Let* $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon \in I_0}$ *be a triple. Suppose that* (H2), (H3), (H4), *and* (H5) *hold. Then,* $E\tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1}\cdot) \in F_1$, *where* F_1 *is given by Remark* 3.2 *and* Π, $\tilde{\Pi}$ *are given by* (IV.5).

Proof. Observe that, since $\tilde{\varphi}$ is continuous with respect to the initial condition $\mathscr{Y} \in B$, and Π, $\tilde{\Pi}$ are isomorphisms, the only thing left to prove is that,

$$\sup_{t \leqslant 0} e^{-\eta t} \|E\tilde{\varphi}(t, \tilde{\Pi}\mathscr{Y})\|_{1,\varepsilon} < \infty.$$

And this is true since,

$$\begin{aligned}
&\sup_{t \leqslant 0} \varepsilon^{-\eta t} \|E\tilde{\varphi}(t, \tilde{\Pi}\mathscr{Y})\|_{1,\varepsilon} \\
&\quad \leqslant \sup_{t \leqslant 0} e^{-\eta t}(\|\tilde{\varphi}(t, \tilde{\Pi}\mathscr{Y})\|_{X_0^1 \times Y_\varepsilon^1} + |||E_1(\varphi(t, \tilde{\Pi}\mathscr{Y}))^1|||_{1,\varepsilon}) < \infty. \quad \blacksquare
\end{aligned}$$

LEMMA 4.2. *Let* $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon \in I_0}$ *be a triple. Suppose that* (H5) *holds. Then, given any* $1 \leqslant k \leqslant n$ *we have that*

$$\sup_{t \leqslant 0} \int_t^0 e^{-\eta(t-s)} |e^{-\lambda_k(\Omega_\varepsilon)(t-s)} - e^{-\lambda_k^\varepsilon(t-s)}| \, ds \to 0$$

as ε *goes to zero.*

Proof. The proof is a simple application of the mean value theorem and will be ommited. ∎

With this we can compare the operators $\mathscr{K}$ and $\tilde{\mathscr{K}}$ in the following way,

PROPOSITION 4.3. *Let* $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon \in I_0}$ *be a triple. Suppose that* (H2), (H3), (H4), *and* (H5) *hold. For any given bounded set* $B \subset \mathbb{R}^{nm}$ *we have that*

$$\sup_{\mathscr{Y} \in B} |(\mathscr{K}(E\tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1}\cdot)) - E\tilde{\mathscr{K}}(\tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1}\cdot)))(t, \Pi\mathscr{Y})|_{C_\eta(\mathbb{R}^-, X)} \to 0; \tag{IV.8}$$

as ε *goes to zero.*

Proof. From the definition of the operators (see (III.10)), we have that

$$\sup_{\mathcal{Y}\in B} |(\mathcal{H}(E\tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1}\cdot)) - E\tilde{\mathcal{H}}(\tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1})))(t, \Pi\mathcal{Y})|_{C_\eta((-\infty,0], X)}$$

$$= \sup_{\mathcal{Y}\in B} \Bigg| e^{-A_\varepsilon t}\Pi\mathcal{Y} - Ee^{-C_\varepsilon t}\tilde{\Pi}\mathcal{Y} + \int_0^t e^{-A_\varepsilon(t-s)}\mathcal{P}^\varepsilon\hat{f}(E\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y}))$$

$$- Ee^{-C_\varepsilon(t-s)}\tilde{\mathcal{P}}^\varepsilon\tilde{f}_\varepsilon(\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y}))\, ds + \int_{-\infty}^t e^{-A_\varepsilon(t-s)}\mathcal{J}^\varepsilon\tilde{f}(E\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y}))$$

$$- Ee^{-C_\varepsilon(t-s)}\tilde{\mathcal{J}}^\varepsilon\tilde{f}_\varepsilon(\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y}))\, ds \Bigg|_{C_\eta((-\infty,0], X)}. \tag{IV.9}$$

The following lemmas will prove that each term in (IV.9) goes to zero, thus proving the proposition.

LEMMA 4.4. *Let* $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon\in I_0}$ *be a triple. Suppose that* (H2), (H3), (H4), *and* (H5) *hold. For any given bounded set* $B \subset \mathbb{R}^{nm'}$ *we have that*

$$\sup_{\mathcal{Y}\in B} |e^{-A_\varepsilon t}\Pi\mathcal{Y} - Ee^{-C_\varepsilon t}\tilde{\Pi}\mathcal{Y}|_{C_\eta((-\infty,0], X)} \to 0,$$

as ε *goes to zero.*

Proof. Using the definition of Π, $\tilde{\Pi}$, we get that, for each $\mathcal{Y} \in B$,

$$|e^{-A_\varepsilon t}\Pi\mathcal{Y} - Ee^{-C_\varepsilon t}\tilde{\Pi}\mathcal{Y}|_{C_\eta((-\infty,0], X)} = |[\mathcal{Y}]_{e^{-\lambda(\Omega_\varepsilon)t}\varphi_{\Omega_\varepsilon}} - e^{-\lambda_\varepsilon^{g} t}E\xi_\varepsilon^{g}|_{C_\eta((-\infty,0], X)}.$$

Thus, from Lemma 4.2 and the convergence of the eigenvalues and eigenfunctions, it follows the result. ∎

LEMMA 4.5. *Let* $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon\in I_0}$ *be a triple. Suppose that* (H2), (H3), (H4), *and* (H5) *hold. For any given bounded set* $B \subset \mathbb{R}^{nm'}$ *we have that*

$$\sup_{\mathcal{Y}\in B} \Bigg| \int_0^t e^{-A_\varepsilon(t-s)}\mathcal{P}^\varepsilon\hat{f}(E\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y}))$$

$$- Ee^{-C_\varepsilon(t-s)}\tilde{\mathcal{P}}^\varepsilon\tilde{f}_\varepsilon(\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y}))\, ds \Bigg|_{C_\eta((-\infty,0], X)} \to 0,$$

as ε *goes to zero.*

Proof. Using the definition of the norm, we have to prove the convergence to zero of

$$\leqslant \sup_{\mathcal{Y} \in B} \sup_{t \leqslant 0} e^{-\eta t} \int_t^0 \| e^{-A_\varepsilon(t-s)} \mathcal{P}^\varepsilon \hat{f}(E\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y})) - E e^{-C_\varepsilon(t-s)} \tilde{\mathcal{P}}^\varepsilon \tilde{f}_\varepsilon(\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y})) \|_{1,\varepsilon} \, ds. \tag{IV.10}$$

Now, observe that, summing and subtracting some terms and using the definition of the projections, we get

$$\begin{aligned} &\| e^{-A_\varepsilon(t-s)} \mathcal{P}^\varepsilon \hat{f}(E\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y})) - E e^{-C_\varepsilon(t-s)} \tilde{\mathcal{P}}^\varepsilon \tilde{f}_\varepsilon(\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y})) \|_{1,\varepsilon} \\ &\quad \leqslant \sum_{k=1}^{n} \| (e^{-\lambda_k(\Omega_\varepsilon)(t-s)} - e^{-\lambda_k^\varepsilon(t-s)}) \langle \hat{f}(E\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y})), \varphi_{k,\Omega_\varepsilon} \rangle_\varepsilon \varphi_{k,\Omega_\varepsilon} \\ &\qquad + e^{-\lambda_k^\varepsilon(t-s)} \langle \hat{f}(E\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y})), \varphi_{k,\Omega_\varepsilon} \rangle_\varepsilon (\varphi_{k,\Omega_\varepsilon} - E\xi_k^\varepsilon) \\ &\qquad + e^{-\lambda_k^\varepsilon(t-s)} (\langle \hat{f}(E\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y})), \varphi_{k,\Omega_\varepsilon} \rangle_\varepsilon - \langle \tilde{f}_\varepsilon(\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y})), \xi_k^\varepsilon \rangle^\varepsilon) E\xi_k^\varepsilon \|_{1,\varepsilon}. \end{aligned} \tag{IV.11}$$

Now observe that, from the convergence of the eigenvalues and eigenfunctions, and the uniform bound on $\tilde{\varphi}$, we get that

$$\begin{aligned} &\sup_{\mathcal{Y} \in B} \sup_{t \leqslant 0} e^{-\eta t} \int_t^0 \| (e^{-\lambda_k(\Omega_\varepsilon)(t-s)} - e^{-\lambda_k^\varepsilon(t-s)}) \langle \hat{f}(E\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y})), \varphi_{k,\Omega_\varepsilon} \|_{1,\varepsilon} \, ds \\ &\quad \leqslant K \sup_{\mathcal{Y} \in B} \sup_{t \leqslant 0} e^{-\eta t} \int_t^0 | e^{-\lambda_k(\Omega_\varepsilon)(t-s)} - e^{-\lambda_k^\varepsilon(t-s)} | \, \| \hat{f}(E\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y})) \|_\varepsilon \, ds \\ &\quad \leqslant K \sup_{\mathcal{Y} \in B} \sup_{t \leqslant 0} \int_t^0 e^{-\eta(t-s)} | e^{-\lambda_k(\Omega_\varepsilon)(t-s)} - e^{-\lambda_k^\varepsilon(t-s)} | \\ &\qquad \times | \hat{f}(E\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y})) |_{C_\eta(\mathbb{R}^-, X)} \, ds \\ &\quad \leqslant KN \sup_{t \leqslant 0} \int_t^0 e^{-\eta(t-s)} | e^{-\lambda_k(\Omega_\varepsilon)(t-s)} - e^{-\lambda_k^\varepsilon(t-s)} | \, ds \to 0; \end{aligned} \tag{IV.12}$$

we refer to Lemma 4.2 for the proof that these exponentials go to zero. Moreover, the following convergence follows with similar arguments,

$$\sup_{\mathcal{Y} \in B} \sup_{t \leqslant 0} e^{-\eta t} \int_t^0 \| e^{-\lambda_k^\varepsilon(t-s)} \langle \hat{f}(E\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y})), \varphi_{k,\Omega_\varepsilon} \rangle_\varepsilon \times (\varphi_{k,\Omega_\varepsilon} - E\xi_k^\varepsilon) \|_{1,\varepsilon} \, ds \to 0. \tag{IV.13}$$

Therefore, it remains to prove that the last term in (IV.11) goes to zero. To do this, observe that it is enough to prove that

$$\sup_{\mathscr{Y}\in B}\sup_{t\leq 0} e^{-\eta t}\int_t^0 e^{-\lambda_k^\varepsilon(t-s)}\,|\langle \hat{f}(E\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})),\varphi_{k,\Omega_\varepsilon}\rangle_\varepsilon - \langle \tilde{f}_\varepsilon(\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})),\xi_k^\varepsilon\rangle^\varepsilon|\,ds\to 0.$$

But, using the definition of the inner products on page 16, we get that

$$\langle \hat{f}(E\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})),\varphi_{k,\Omega_\varepsilon}\rangle_\varepsilon - \langle \tilde{f}_\varepsilon(\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})),\xi_k^\varepsilon\rangle^\varepsilon$$

$$=\begin{pmatrix}\int_{\Omega_\varepsilon}(\hat{f}(E\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})))^1\,\varphi_{k,\Omega_\varepsilon}\\ \vdots\\ \int_{\Omega_\varepsilon}(\hat{f}(E\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})))^m\,\varphi_{k,\Omega_\varepsilon}\end{pmatrix}$$

$$-\begin{pmatrix}\int_{\Omega_0}(E_1(\tilde{f}_\varepsilon(\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})))_1)^1\,(\xi_k^\varepsilon)_1+\int_{R_\varepsilon}(E_2(\tilde{f}_\varepsilon(\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})))_2)^1\,(\xi_k^\varepsilon)_2\\ \vdots\\ \int_{\Omega_0}(E_1(\tilde{f}_\varepsilon(\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})))_1)^m\,(\xi_k^\varepsilon)_1+\int_{R_\varepsilon}(E_2(\tilde{f}_\varepsilon(\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})))_2)^m\,(\xi_k^\varepsilon)_2\end{pmatrix}$$

Having in mind the definitions of $\hat{f}$ and $\tilde{f}_\varepsilon$ (see (I.10)), we can write

$$\hat{f}(E\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})),\varphi_{k,\Omega_\varepsilon}\rangle_\varepsilon - \langle \tilde{f}_\varepsilon(\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})),\xi_k^\varepsilon\rangle^\varepsilon$$

$$=\begin{pmatrix}\int_{\Omega_0}(f(E\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})))^1\,\varphi_{k,\Omega_\varepsilon}-(E_1(f(\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})))_1)^1\,(\xi_k^\varepsilon)_1\\ \vdots\\ \int_{\Omega_0}(f(E\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})))^m\,\varphi_{k,\Omega_\varepsilon}-(E_1(f(\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})))_1)^m\,(\xi_k^\varepsilon)_1\end{pmatrix}$$

$$+\begin{pmatrix}\int_{R_\varepsilon}(f(E\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})))^1\,\varphi_{k,\Omega_\varepsilon}-(E_2(f(\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})))_2)^1\,(\xi_k^\varepsilon)_2\\ \vdots\\ \int_{R_\varepsilon}(f(E\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})))^m\,\varphi_{k,\Omega_\varepsilon}-(E_2(\tilde{f}_\varepsilon(\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})))_2)^m\,(\xi_k^\varepsilon)_2\end{pmatrix}$$

Now, from the definition of the extensions, we have that E_2 is zero in Ω_0 and E_1 is the identity in Ω_0, thus we get

$$\begin{pmatrix} \int_{\Omega_0} (f(E\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})))^1 \varphi_{k,\Omega_\varepsilon} - (E_1(f(\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y}))_1)^1 (\xi_k^\varepsilon)_1 \\ \vdots \\ \int_{\Omega_0} (f(E\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})))^m \varphi_{k,\Omega_\varepsilon} - (E_1(f(\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y}))_1)^m (\xi_k^\varepsilon)_1 \end{pmatrix}$$

$$= \begin{pmatrix} \int_{\Omega_0} ((f(\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})))_1)^1 (\varphi_{k,\Omega_\varepsilon} - (\xi_k^\varepsilon)_1) \\ \vdots \\ \int_{\Omega_0} ((f(\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})))_1)^m (\varphi_{k,\Omega_\varepsilon} - (\xi_k^\varepsilon)_1) \end{pmatrix}$$

thus, using once more the convergence of eigenvalues and eigenvectors and the uniform bounds on $\tilde{\varphi}$, we get that

$$\sup_{\mathscr{Y} \in B} \sup_{t \leq 0} e^{-\eta t} \int_t^0 e^{-\lambda_k^\varepsilon (t-s)} \times \left| \begin{pmatrix} \int_{\Omega_0} (E_1(f(\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})))_1)^1 (\varphi_{k,\Omega_\varepsilon} - (\xi_k^\varepsilon)_1) \\ \vdots \\ \int_{\Omega_0} (E_1(f(\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})))_1)^m (\varphi_{k,\Omega_\varepsilon} - (\xi_k^\varepsilon)_1) \end{pmatrix} \right| ds \to 0. \qquad \text{(IV.14)}$$

In the channels we do not have E_1 zero, but we have E_2 being the identity. First, observe that we can proceed as above to get

$$\sup_{\mathscr{Y} \in B} \sup_{t \leq 0} e^{-\eta t} \int_t^0 e^{-\lambda_k^\varepsilon (t-s)} \times \left| \begin{pmatrix} \int_{R_\varepsilon} (f(E\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})))^1 (\varphi_{k,\Omega_\varepsilon} - (\xi_k^\varepsilon)_2) \\ \vdots \\ \int_{R_\varepsilon} (f(E\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})))^m (\varphi_{k,\Omega_\varepsilon} - (\xi_k^\varepsilon)_2) \end{pmatrix} \right| ds \to 0. \qquad \text{(IV.15)}$$

So we are left to prove that

$$\sup_{\mathscr{Y}\in B}\sup_{t\leqslant 0} e^{-\eta t}\int_t^0 e^{-\lambda_k^\varepsilon(t-s)} \times \left|\begin{pmatrix} \int_{R_\varepsilon} (f(E\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})))^1 (\xi_k^\varepsilon)_2 - (E_2(f(\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y}))_2)^1 (\xi_k^\varepsilon)_2 \\ \vdots \\ \int_{R_\varepsilon} (f(E\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})))^m (\xi_k^\varepsilon)_2 - (E_2(\tilde{f}_\varepsilon(\tilde{\varphi}(s,\Pi\mathscr{Y}))_2)^m (\xi_k^\varepsilon)_2 \end{pmatrix}\right| ds \to 0. \tag{IV.16}$$

So, in order to prove this, we need to make use of $\check{f}$ defined in page 9. Let $1\leqslant j\leqslant m$, we get that

$$\sup_{\mathscr{Y}\in B}\sup_{t\leqslant 0} e^{-\eta t}\int_t^0 e^{-\lambda_k^\varepsilon(t-s)}\left|\int_{R_\varepsilon} ((f(E\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})))^j - (E_2(f(\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})_2)))^j)(\xi_k^\varepsilon)_2\right| ds$$

$$\leqslant \sup_{\mathscr{Y}\in B}\sup_{t\leqslant 0} e^{-\eta t}\int_t^0 e^{-\lambda_k^\varepsilon(t-s)}\,\|\check{f}^j(E\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})) - \check{f}^j((\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y}))_2)\|_{L^2(R_\varepsilon)}$$

Now using (H2)–4. we get the result since $(E_1(\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y}))_1)^j$ doesn't depend on ε, is uniformly bounded in $C_\eta(\mathbb{R}^-, H^1(R_\varepsilon))$ and the measure of R_ε is going to zero. So we have proved (IV.16).

And this proves the Lemma. ∎

LEMMA 4.6. *Let* $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon\in I_0}$ *be a triple. Suppose that* (H2), (H3), (H4), *and* (H5) *hold. For any given bounded set* $B\subset\mathbb{R}^{nm}$ *we have that*

$$\sup_{\mathscr{Y}\in B}\left|\int_{-\infty}^t e^{-A_\varepsilon(t-s)}\mathscr{I}^\varepsilon\hat{f}(E\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})) - Ee^{-C_\varepsilon(t-s)}\tilde{\mathscr{J}}^\varepsilon\tilde{f}_\varepsilon(\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y}))\,ds\right|_{C_\eta((-\infty,0],X)} \to 0,$$

as ε *goes to zero.*

Proof. We need to show that given $\delta>0$, there exists $\varepsilon_1=\varepsilon_1(B,\delta)> <0$, such that for all $\varepsilon\in I_0\cap(0,\varepsilon_1)$ we have that

$$\sup_{\mathscr{Y}\in B}\left|\int_{-\infty}^t e^{-A_\varepsilon(t-s)}\mathscr{I}^\varepsilon\hat{f}(E\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y})) - Ee^{-C_\varepsilon(t-s)}\tilde{\mathscr{J}}^\varepsilon\tilde{f}_\varepsilon(\tilde{\varphi}(s,\tilde{\Pi}\mathscr{Y}))\,ds\right|_{C_\eta((-\infty,0],X)} <\delta.$$

So let us fix $\delta > 0$, and let us define $k = k(B, \delta) > 0$ such that

$$\frac{N\pi^{1/2}}{d_*(\lambda_k^0 - \eta)} \leqslant \delta/2, \tag{IV.17}$$

notice that this is verified since the eigenvalues blow up (see Section 2).

With this, let $\mathscr{P}^{*\varepsilon}$ be the projection on $(\varphi_{1,\Omega_\varepsilon}, ..., \varphi_{k,\Omega_\varepsilon})$, $\mathscr{I}^{*\varepsilon} = I - P^{*\varepsilon}$, and $\tilde{\mathscr{P}}^{*\varepsilon}$ be the projection on $(\xi_1^\varepsilon, ..., \xi_k^\varepsilon)$, $\tilde{\mathscr{I}}^{*\varepsilon} = I - \tilde{P}^{*\varepsilon}$. Therefore we can write

$$\left\| \int_{-\infty}^{t} e^{-A_\varepsilon(t-s)} \mathscr{I}^\varepsilon \hat{f}(E\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})) - Ee^{-C_\varepsilon(t-s)} \tilde{\mathscr{I}}^\varepsilon \tilde{f}_\varepsilon(\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y}))\, ds \right\|_{1,\varepsilon}$$

$$\leqslant \left\| \int_{-\infty}^{t} Ee^{-C_\varepsilon(t-s)} \tilde{\mathscr{I}}^{*\varepsilon} \tilde{\mathscr{I}} \tilde{f}_\varepsilon(\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y}))\, ds \right\|_{1,\varepsilon}$$

$$+ \left\| \int_{-\infty}^{t} e^{-A_\varepsilon(t-s)} \mathscr{I}^{*\varepsilon} \mathscr{I} \hat{f}(E\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y}))\, ds \right\|_{1,\varepsilon}$$

$$+ \int_{-\infty}^{t} \| e^{-A_\varepsilon(t-s)} \mathscr{P}^{*\varepsilon} \mathscr{I} \hat{f}(E\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y}))$$

$$- Ee^{-C_\varepsilon(t-s)} \tilde{\mathscr{P}}^{*\varepsilon} \tilde{\mathscr{I}} \tilde{f}_\varepsilon(\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})) \|_{1,\varepsilon}\, ds$$

$$\leqslant \int_{-\infty}^{t} \| e^{-A_\varepsilon(t-s)} \mathscr{P}^{*\varepsilon} \mathscr{I} \hat{f}(E\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y}))$$

$$- Ee^{-C_\varepsilon(t-s)} \tilde{\mathscr{P}}^{*\varepsilon} \tilde{\mathscr{I}} \tilde{f}_\varepsilon(\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})) \|_{1,\varepsilon}\, ds + \delta.$$

Now we can proceed as in the lemma (the proof is the same, just the projections have changed) to prove that the remaining terms also go to zero. Thus proving the lemma. ∎

Therefore applying the last lemmas to (IV.9) we have proved Proposition IV.8. ∎

With this proposition, we have the following theorem.

THEOREM 4.7. *Let* $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon \in I_0}$ *be a triple. If* (H2), (H3), (H4), *and* (H5) *hold. For any given bounded set* $B \subset \mathbb{R}^{nm}$ *we have that*

$$\sup_{\mathscr{Y} \in B} |\varphi(t, \Pi\mathscr{Y}) - E\tilde{\varphi}(t, \tilde{\Pi}\mathscr{Y})|_{C_\eta((-\infty, 0], X)} \to 0 \quad as \quad \varepsilon \to 0. \tag{IV.18}$$

Proof. By Remark 3.2, we have that,

$$
\begin{aligned}
&\sup_{\mathscr{Y} \in B} |\varphi(t, \Pi\mathscr{Y}) - E\tilde{\varphi}(t, \tilde{\Pi}\mathscr{Y})|_{C_\eta((-\infty, 0], X)} \\
&\quad = \sup_{\mathscr{Y} \in B} |\mathscr{K}(\varphi)(t, \Pi\mathscr{Y}) - E\tilde{\mathscr{K}}(\tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1}\cdot))(t, \Pi\mathscr{Y})|_{C_\eta(\mathbb{R}^-, X)} \\
&\quad \leqslant \sup_{\mathscr{Y} \in B} (|\mathscr{K}(\varphi(\cdot, \cdot))(t, \Pi\mathscr{Y}) - \mathscr{K}(E\tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1}\cdot))|_{C_\eta(\mathbb{R}^-, X)} \\
&\qquad + |\mathscr{K}(E\tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1}\cdot)) - E\tilde{\mathscr{K}}(\tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1}\cdot))(t, \Pi\mathscr{Y})|_{C_\eta(\mathbb{R}^-, X)})
\end{aligned}
\tag{IV.19}
$$

From Remark 3.2 we know that $\mathscr{K}$ is a contraction, with constant $C < 1$, independent of ε. Thus applying this to (IV.19), we get

$$
\begin{aligned}
&(1 - C) \sup_{\mathscr{Y} \in B} |\varphi(t, \Pi\mathscr{Y}) - E\tilde{\varphi}(t, \tilde{\Pi}\mathscr{Y})|_{C_\eta((-\infty, 0], X)} \\
&\quad \leqslant \sup_{\mathscr{Y} \in B} |\mathscr{K}(E\tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1}\cdot)) - E\tilde{\mathscr{K}}(\tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1}\cdot))(t, \Pi\mathscr{Y})|_{C_\eta(\mathbb{R}^-, X)},
\end{aligned}
$$

thus, applying Proposition 4.3, we finish the proof. ∎

The next step is to prove a stronger type of convergence; namely, C^1 convergence. As observed before, we need only to prove (IV.7). The proof of (IV.7) is very similar to the proof of (IV.6), the only difference is that, instead of comparing the operators $\mathscr{K}$ and $\tilde{\mathscr{K}}$, we are going to compare the operators $\mathscr{F}_\varphi$ and $\tilde{\mathscr{F}}_{\tilde{\varphi}}$. As before, we need some lemmas that will help us to establish the desired convergence.

LEMMA 4.8. *Let* $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon \in I_0}$ *be a triple. Suppose that* (H2), (H3), (H4), *and* (H5) *hold. Then,* $ED_\xi\tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1}\cdot)\, \tilde{\Pi}\Pi^{-1} \in F_2$, *where* F_2 *is given by Remark* 3.2 *in page* 22 *and* $\Pi, \tilde{\Pi}$ *are given by* (IV.5).

Proof. As in Lemma 4.1, if $\mathscr{Z}, \mathscr{Y} \in B$, the only thing left to prove is that,

$$
\sup_{t \leqslant 0} e^{-\eta t} \|ED_\xi\tilde{\varphi}(t, \tilde{\Pi}\mathscr{Y})\, \tilde{\Pi}\mathscr{Z}\|_{1, \varepsilon} < \infty.
$$

And this is true since, $D_\xi\tilde{\varphi} \in \tilde{F}_2$ and E is bounded. ∎

With this we can compare the operators $\mathscr{F}_\varphi$ and $\tilde{\mathscr{F}}_{\tilde{\varphi}}$ in the following way, PROPOSITION 4.9. *Let* $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon \in I_0}$ *be a triple. Suppose that* (H2), (H3), (H4), *and* (H5) *hold. For any given bounded set* $B \subset \mathbb{R}^{nm}$ *we have that*

$$\sup_{\mathscr{Y}, \mathscr{Z} \in B} |((\mathscr{F}_\varphi(ED_\xi\tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1}\cdot)) - E\tilde{\mathscr{F}}_{\tilde{\varphi}}(D_\xi\tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1}\cdot))) \cdot (E\tilde{\Pi}\mathscr{Z}) \cdot (t, \Pi\mathscr{Y})|_{C_\eta(\mathbb{R}^-, X)} \to 0; \tag{IV.20}$$

as ε *goes to zero.*

Proof. From the definition of the operators (see (III.10)), we have that

$$\begin{aligned}
&\sup_{\mathscr{Y}, \mathscr{Z} \in B} |((\mathscr{F}_\varphi(ED_\xi\tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1}\cdot)) - E\tilde{\mathscr{F}}_{\tilde{\varphi}}(D_\xi\tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1}\cdot))) \cdot (E\tilde{\Pi}\mathscr{Z}) \cdot (t, \Pi\mathscr{Y})|_{C_\eta(\mathbb{R}^-, X)} \\
&= \sup_{\mathscr{Y}, \mathscr{Z} \in B} \Big| e^{-A_\varepsilon t}\Pi\mathscr{Y} - Ee^{-C_\varepsilon t}\tilde{\Pi}\mathscr{Y} + \int_0^t e^{-A_\varepsilon(t-s)}\mathscr{P}^\varepsilon D_x\hat{f}(\varphi(s, \Pi\mathscr{Y})) \\
&\quad \times ED_x\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})\,\tilde{\Pi}\mathscr{Z} - E_e^{-C_\varepsilon(t-s)}\tilde{\mathscr{P}}^\varepsilon D_x\tilde{f}_\varepsilon(\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y}))\,D_x \\
&\quad \times \tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})\,\tilde{\Pi}\mathscr{Z}\,ds + \int_{-\infty}^t e^{-A_\varepsilon(t-s)}\mathscr{G}^\varepsilon D_x\hat{f}(\varphi(s, \Pi\mathscr{Y})) \\
&\quad \times ED_\xi\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})\,\tilde{\Pi}\mathscr{Z} - Ee^{-C_\varepsilon(t-s)}\tilde{\mathscr{G}}^\varepsilon D_x\tilde{f}_\varepsilon(\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})) \\
&\quad \times D_x\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})\,\tilde{\Pi}\mathscr{Z}\,ds \Big|_{C_\eta(\mathbb{R}^-, X)}
\end{aligned} \tag{IV.21}$$

The following lemmas will prove that each term in (IV.21) goes to zero, thus proving the proposition.

LEMMA 4.10. *Let* $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon \in I_0}$ *be a triple. Suppose that* (H2), (H3), (H4), *and* (H5) *hold. For any given bounded set* $B \subset \mathbb{R}^{nm}$ *we have that*

$$\begin{aligned}
\sup_{\mathscr{Y} \in B} \Big| &\int_0^t e^{-A_\varepsilon(t-s)}\mathscr{P}^\varepsilon D_x\hat{f}(\varphi(s, \Pi\mathscr{Y}))\,ED_x\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})\,\tilde{\Pi}\mathscr{Z} \\
&- Ee^{-C_\varepsilon(t-s)}\tilde{\mathscr{P}}^\varepsilon D_x\tilde{f}_\varepsilon(\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y}))\,D_x\tilde{\varphi}(s, \tilde{\Pi}\mathscr{Y})\,\tilde{\Pi}\mathscr{Z}\,ds \Big|_{C_\eta((-\infty, 0], X)} \to 0,
\end{aligned}$$

as ε *goes to zero.*

Proof. The proof follows exactly the same steps as the proof of Lemma 4.5, we just need to use (H2)-5. instead of (H2)-4. and (I.18). Therefore for the sake of simplicity we will omit it. ∎

LEMMA 4.11. *Let* $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon \in I_0}$ *be a triple. Suppose that* (H2), (H3), (H4), *and* (H5) *hold. For any given bounded set* $B \subset \mathbb{R}^{nm}$ *we have that*

$$\sup_{\mathcal{Y} \in B} \left| \int_{-\infty}^{t} e^{-A_\varepsilon(t-s)} \mathcal{J}^\varepsilon D_x \hat{f}(\varphi(s, \Pi\mathcal{Y}))\, ED_\xi \tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y})\, \tilde{\Pi}\mathcal{Z} \right.$$

$$\left. - Ee^{-C_\varepsilon(t-s)} \tilde{\mathcal{J}}^\varepsilon D_x \tilde{f}_\varepsilon(\tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y}))\, D_x \tilde{\varphi}(s, \tilde{\Pi}\mathcal{Y})\, \tilde{\Pi}\mathcal{Z}\, ds \right|_{C_\eta((-\infty, 0], X)} \to 0,$$

as ε *goes to zero.*

Proof. Once again, the proof follows the same steps as the proof of Lemma 4.6 and will be omitted. ∎

Therefore applying the last lemmas to (IV.21) we have proved Proposition IV.20. ∎

With this we have the following theorem.

THEOREM 4.12. *Let* $\{(\Omega_\varepsilon, \Omega_0, R_0)\}_{\varepsilon \in I_0}$ *be a triple. If* (H2), (H3), (H4) *and* (H5) *hold. For any given bounded set* $B \subset \mathbb{R}^{nm}$ *we have that*

$$\sup_{\mathcal{Y}, \mathcal{Z} \in B} |D_\xi \varphi(t, \Pi\mathcal{Y})\, \Pi\mathcal{Z} - E\tilde{D}_\xi \varphi(t, \tilde{\Pi}\mathcal{Y})\, \tilde{\Pi}\mathcal{Z}|_{C_\eta((-\infty, 0], X)} \to 0 \qquad as \quad \varepsilon \to 0. \tag{IV.22}$$

Proof. By Remark 3.2, we have that,

$$\sup_{\mathcal{Y}, \mathcal{Z} \in B} |D_\xi \varphi(t, \Pi\mathcal{Y})\, \Pi\mathcal{Z} - E\tilde{D}_\xi \varphi(t, \tilde{\Pi}\mathcal{Y})\, \tilde{\Pi}\mathcal{Z}|_{C_\eta((-\infty, 0], X)}$$

$$= \sup_{\mathcal{Y}, \mathcal{Z} \in B} |((\mathcal{F}_\varphi(D_\xi \varphi(\cdot, \cdot)) - E\tilde{\mathcal{F}}_{\tilde{\varphi}}(D_\xi \tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1} \cdot)\, E\tilde{\Pi}\Pi^{-1}))$$

$$\cdot (\Pi\mathcal{Z})) \cdot (t, \Pi\mathcal{Y})|_{C_\eta(\mathbb{R}^-, X)}$$

$$\leqslant \sup_{\mathcal{Y}, \mathcal{Z} \in B} |((\mathcal{F}_\varphi(D_\xi \varphi(\cdot, \cdot)) - \mathcal{F}_\varphi(ED_\xi \tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1} \cdot)\, E\tilde{\Pi}\Pi^{-1}))$$

$$\cdot (\Pi\mathcal{Z})) \cdot (t, \Pi\mathcal{Y})|_{C_\eta(\mathbb{R}^-, X)}$$

$$+ \sup_{\mathcal{Y}, \mathcal{Z} \in B} |((\mathcal{F}_\varphi(ED_\xi \tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1} \cdot)) - E\tilde{\mathcal{F}}_{\tilde{\varphi}}(D_\xi \tilde{\varphi}(\cdot, \tilde{\Pi}\Pi^{-1} \cdot)))$$

$$\cdot (E\tilde{\Pi}\mathcal{Z})) \cdot (t, \Pi\mathcal{Y})|_{C_\eta(\mathbb{R}^-, X)} \tag{IV.23}$$

From Remark 3.2 we know that $\mathscr{F}_\varphi$ is a contraction, with constant $C<1$, independent of ε. Thus applying this to (IV.23), we get

$$(1-C)\sup_{\mathscr{Y},\,\mathscr{Z}\in B}|D_\xi\varphi(t,\Pi\mathscr{Y})\,\Pi\mathscr{Z}-E\tilde{D}_\xi\varphi(t,\tilde{\Pi}\mathscr{Y})\,\tilde{\Pi}\mathscr{Z}|_{C_\eta((-\infty,0],X)}$$
$$\leqslant \sup_{\mathscr{Y},\,\mathscr{Z}\in B}|((\mathscr{F}_\varphi(ED_\xi\tilde{\varphi}(\cdot,\tilde{\Pi}\Pi^{-1}\cdot))-E\tilde{\mathscr{F}}_{\tilde{\varphi}}(D_\xi\tilde{\varphi}(\cdot,\tilde{\Pi}\Pi^{-1}\cdot)))$$
$$\cdot(E\tilde{\Pi}\mathscr{Z}))\cdot(t,\Pi\mathscr{Y})|_{C_\eta(\mathbb{R}^-,X)}$$

thus, applying Proposition 4.9, we finish the proof. ∎

5. Continuity of Attractors

From Theorem 1.1, we know that (I.11) and (I.13) have global attractors $\mathscr{A}_\varepsilon$ and $\tilde{\mathscr{A}}_\varepsilon$, respectively. Moreover, if we denote by $\mathscr{M}_\varepsilon$ and $\tilde{\mathscr{M}}_\varepsilon$ the inertial manifolds of (I.11) and (I.13), respectively (defined by Theorem 1.2), then we have that $\mathscr{A}_\varepsilon\subset\mathscr{M}_\varepsilon$ and $\tilde{\mathscr{A}}_\varepsilon\subset\tilde{\mathscr{M}}_\varepsilon$.

Consider the spaces X^1_ε as defined before and denote, for any $\delta>0$ and any subset B of X^1_ε, by $\mathscr{N}_{1,\varepsilon}(B,\delta)$ the δ-neighborhood of B in X^1_ε.

THEOREM 5.1. *Let* $\{(\Omega_\varepsilon,\Omega_0,R_0)\}_{\varepsilon\in I_0}$ *be a triple. Suppose that* (H2), (H3), (H4), *and* (H5) *hold, then the attractors* $\mathscr{A}_\varepsilon$ *and* $\tilde{A}_\varepsilon$ *are upper-semi-continuous at* $\varepsilon=0$, *in the sense that, for any* $\delta>0$, *there exists a positive number* $\varepsilon_1=\varepsilon_1(\delta)$ *such that, for* $0<\varepsilon\leqslant\varepsilon_1$,

$$\mathscr{A}_\varepsilon\subset E\tilde{\mathscr{A}}_\varepsilon+\mathscr{N}_{1,\varepsilon}(\{0\},\delta) \tag{V.1}$$

Proof. Let t_0 be a fixed positive number.
By (H4),

$$\|E\varphi\|_{1,\varepsilon}\leqslant\mathscr{N},\qquad\text{for all}\quad\varphi\in\tilde{\mathscr{A}}_\varepsilon.$$

Since $\mathscr{A}_\varepsilon$ is an attractor of T^ε_f, there exists a time $\tau_0=(\varepsilon,\delta/2,\mathscr{N})\geqslant t_0$ such that, for $t\geqslant\tau_0$,

$$\inf_{\psi_0\in\mathscr{A}_\varepsilon}\|T^\varepsilon_f(t)\,v-\psi_0\|_{1,\varepsilon}\leqslant\frac{\delta}{2}, \tag{V.2}$$

for any $v\in X^1_\varepsilon$ with $\|v\|_{1,\varepsilon}\leqslant\mathscr{N}$.

Suppose that φ, $\tilde{\varphi}$ are the fixed points of $\mathscr{K}$, $\tilde{\mathscr{K}}$ respectively. Then for $t_0\leqslant t\leqslant\tau_0$,

$$\|T^\varepsilon_f(t)\,\varphi(0,\Pi\mathscr{Y})-\tilde{E}S^\varepsilon_f(t)\,\tilde{\varphi}(0,\tilde{\Pi}\mathscr{Y})\|_{1,\varepsilon}$$
$$=\|\varphi(t,\Pi\mathscr{Y})-E\tilde{\varphi}(t,\tilde{\Pi}\mathscr{Y})\|_{1,\varepsilon}\to 0. \tag{V.3}$$

Thus

$$\inf_{\psi_0 \in \mathscr{A}_\varepsilon} \|\psi_0 - ES_f^\varepsilon(t)\,\tilde{\varphi}(0, \tilde{\Pi}\mathscr{Y})\|_{1,\varepsilon}$$

$$\leqslant \inf_{\psi_0 \in \mathscr{A}_\varepsilon} \|T_f^\varepsilon(t)\,\varphi(0, \Pi\mathscr{Y}) - ES_f^\varepsilon(t)\,\tilde{\varphi}(0, \tilde{\Pi}\mathscr{Y})\|_{1,\varepsilon}$$

$$+ \inf_{\psi_0 \in \mathscr{A}_\varepsilon} \|T_f^\varepsilon(t)\,\varphi(0, \Pi\mathscr{Y}) - \psi_0\|_{1,\varepsilon}$$

$$\leqslant \|T_f^\varepsilon(t)\,\varphi(0, \Pi\mathscr{Y}) - ES_f^\varepsilon(t)\,\tilde{\varphi}(0, \tilde{\Pi}\mathscr{Y})\|_{1,\varepsilon} + \frac{\delta}{2} \leqslant \delta,$$

if ε is small enough. Now, since $S_f^\varepsilon(\tau_0)\,\tilde{\mathscr{A}}_\varepsilon = \tilde{\mathscr{A}}_\varepsilon$ have the result. ∎

6. Application of Convergence Theorem

In this setion, we give a simple application of the C^1 convergence, just proved in the previous sections. But first we need to make some restrictions. Let $B \subset \partial\Omega_0 \backslash \Gamma_\varepsilon$, we will replace the boundary condition (I.3) by

$$\frac{\partial u}{\partial n} = 0, \qquad (t, x) \in (0, \infty) \times (\partial\Omega_\varepsilon \backslash B)$$
$$u = 0, \qquad (t, x) \in B. \tag{VI.1}$$

With these modifications (see Section 8) all the results in the previous Sections remain true, if we incorporate these modifications in the spaces. Let us assume also that (H5) is verified with'$p = 0$ and some $n = q$, that is, in the splitting of the spectrum, we will take q modes from the channels and no modes from the fixed domain.

Our objective is give more complete comparison of the orbit structure, namely we say that the semigroup $T_f^\varepsilon(t)$ on X_ε^1 is topologically equivalent to the flow defined by $S_f^0(t)$ on $X_0^1 \times Y_0^1$ if there is a homeomorphism $h\colon \mathscr{A}_\varepsilon \to \tilde{\mathscr{A}}_0$ which preserves orbits and the sense of direction in time.

Theorem 6.1. *Suppose that we incorporate the conditions* (VI.1) *into our equations* (I.11) *and* (I.13) *and all the hypotheses of the previous sections hold.*

Let us assume that $m = 1$, *that is, we have a scalar reaction diffusion equation, and that we have* (H5) *verified with* $p = 0$, *are hyperbolic, then the flows defined by* (I.11) *and* (I.13) (*with* $\varepsilon = 0$) *are topologically equivalent.*

Proof. From the existence of the inertial manifolds follows that the flow on the attractors are governed by an ordinary differential equation. We show that the vector fields on these manifolds converge, in the C^1 sense, as $\varepsilon \to 0$. Now observe that since $p=0$ the flow on the inertial manifold is the same as the flow on the inertial manifold of the following one dimensional scalar reaction-diffusion equation

$$\begin{aligned} \frac{du}{dt} &= -\tilde{A}_0 u + \tilde{f}_0(u), \qquad t>0 \\ u(0) &= u_0 \end{aligned} \tag{VI.2}$$

for all $u_0 \in Y_0^1$. Since this reduced equation is given in a one dimensional open set R_0, the stable and unstable manifolds are transversal (see Henry (1985)). Therefore, the finite dimensional ODEs for $\varepsilon = 0$ are Morse–Smale and the topological equivalence of $T_f^\varepsilon(t)$ follows from the classical results (see, for example, Palis and Melo (1982) and Peixoto (1977)).

Remark 6.1. *For a good introduction to Morse-Smale systems, see Henry* (1985), *Palis* (1969) *and Palis and Melo* (1982). ∎

7. Example

In this section we will illustrate our results with a simple example. For simplicity, we will suppose that $m=1$ (that is, we have a scalar reaction diffusion equation). The advantage is that (H2), (H3) and (H4) hold if we just assume that $f\colon \mathbb{R} \to \mathbb{R}$ is C^2, and satisfies the following condition,

$$\limsup_{|x| \to +\infty} \frac{f(x)}{x} \leqslant -\delta \tag{VII.1}$$

$$|f''(x)| \leqslant c(1+|x|^{\tilde{\gamma}}) \qquad \text{for} \quad x \in \mathbb{R} \tag{VII.2}$$

where $\delta > 0$, $0 \leqslant \tilde{\gamma} < +\infty$.

Next, let us suppose that Ω_ε are the so called "dumbbell-shaped" domains, as defined precisely by Hale and Vegas (1984) and Vegas (1983). As mentioned before, if we consider just the eigenvalues comming from the channel, hypothesis (H5) is satisfied. In this simple example, we will show that we really get a generalization of Morita (1990). The example will be such that when we increase the diffusion the channel also increases. That is, let $l, d > 0$ be given, and suppose that our channel is defined by

$$R_\varepsilon = \{(x, y) \in \mathbb{R}^2 \text{ such that } 0 < x < l\sqrt{d},\ 0 < y < l\sqrt{d}\,\varepsilon\}.$$

So, let us consider the equation

$$\begin{aligned} &u_t = d\Delta u + f(u), && (t, (x, y)) \in (0, \infty) \times \Omega_\varepsilon, \\ &\frac{\partial u}{\partial n} = 0, && (t, (x, y)) \in (0, \infty) \times \partial\Omega_\varepsilon. \end{aligned} \tag{VII.3}$$

where f satisfies (VII.1) and (VII.2).

In order to apply Morita's results we need the following quantities,

$$\gamma_0 = \inf_{\varepsilon \in I_0} \lambda_3(\Omega_\varepsilon);$$

$$\gamma_1 = d\gamma_0;$$

$$\kappa_1 = \sup_{u \in \mathbb{R}} |f_u(u)|_{op} < \infty.$$

Then, the basic hypothesis in Morita's Theorem is

$$\gamma_1 > \kappa_1. \tag{VII.4}$$

Now observe that $\lambda_1(\Omega_\varepsilon) = 0$ and $\lambda_2(\Omega_\varepsilon) \to 0$, and from Theorem 2.1 we have that, if Ω_0 is small enough, $\lambda_3(\Omega_\varepsilon)$ converges to μ_1, but

$$\mu_1 = \frac{\pi^2}{l^2 d},$$

therefore,

$$\gamma_1 \to \frac{\pi^2}{l^2},$$

as $\varepsilon \to 0$. Therefore, if we fix the nonlinearity (and thus fix κ_1), we have that if l is big enough, then (VII.4) is not satisfied and we cannot apply his existence theorem.

We cannot apply Morita's result because he never consider the influence of the channel in the dynamics. And in this example this is crucial.

On the other hand, we can apply our theorem to (VII.3). To do this let us change variables in such a way that, if we denote by $h = l^2 f$ then (VII.3) can be rewritten as

$$\begin{aligned} &u_t = \Delta u + h(u), && (t, (x, y)) \in (0, \infty) \times \Omega_\varepsilon, \\ &\frac{\partial u}{\partial n} = 0, && (t, (x, y)) \in (0, \infty) \times \partial\Omega_\varepsilon. \end{aligned} \tag{VII.5}$$

Where our channel takes the following form

$$R_\varepsilon = \{(x, y) \in \mathbb{R}^2 \text{ such that } 0 < x < 1, 0 < y < \varepsilon\}.$$

Moreover, if we increase d then we actually are decreasing Ω_0. Therefore fixed l we can find a d sufficiently large such that all hypothesis of our theorem are satisfied, since we fixed the eigenvalues of the channel now. And observe that the larger l is the greater is the dimension of our invariant manifold since if we increase l we are increasing the lipchitz constant of h and thus needing a larger gap.

8. General Remarks

In this section we give some general remarks concerning our work. We start with hypothesis (H0). As seen in the example we do not need to assume that R_ε has this special form, but actually we can take a general thin domain (as long as (H1) is satisfied), the disadvantage is that (H5) gets much more difficult to be verified.

As already used in Section 6, we can use mixed boundary conditions, namely, if we consider $B \subset \partial\Omega_0 \setminus \Gamma_\varepsilon$, we will replace the boundary condition (I.3) by

$$\frac{\partial u}{\partial n} = 0, \qquad (t, x) \in (0, \infty) \times (\partial\Omega_\varepsilon \setminus B)$$

$$u = 0, \qquad (t, x) \in B.$$

With this all results in the previous work if we include this extra condition in the spaces, for example $X_\varepsilon^1 = (H_B^1(\Omega_\varepsilon))^m$, where $H_B^1(\Omega_\varepsilon)$ follows the same definition as $H_{\Gamma_\varepsilon}^1(R_\varepsilon)$, in page 6.

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