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Rudin extension theorems on product spaces, turning bands, and random fields on balls cross time

EMILIO PORCU[1,a], SAMUEL F. FENG[1,b], XAVIER EMERY[2,c] and ANA P. PERON[3,d]

[1]*Department of Mathematics & Research And Data Intelligence Support Center R-DISC, Khalifa University of Science and Technology, Abu Dhabi, United Arab Emirates,* [a]*emilio.porcu@ku.ac.ae,* [b]*samuel.feng@ku.ac.ae*
[2]*Department of Mining Engineering & Advanced Mining Technology Center, University of Chile, Santiago, Chile,* [c]*xemery@ing.uchile.cl*
[3]*Department of Mathematics, Institute of Mathematical and Computer Sciences, University of São Paulo, Brazil,* [d]*apperon@icmc.usp.br*

Characteristic functions that are radially symmetric have a dual interpretation, as they can be used as the isotropic correlation functions of spatial random fields. Extensions of isotropic correlation functions from balls into d-dimensional Euclidean spaces, $\mathbb{R}^d$, have been understood after Rudin. Yet, extension theorems on product spaces are elusive, and a counterexample provided by Rudin on rectangles suggests that the problem is challenging. This paper provides extension theorems for multiradial characteristic functions that are defined in balls embedded in $\mathbb{R}^d$ cross, either $\mathbb{R}^{d'}$ or the unit sphere $\mathbb{S}^{d'}$ embedded in $\mathbb{R}^{d'+1}$, for any two positive integers d and d'. We then examine Turning Bands operators that provide bijections between the class of multiradial correlation functions in given product spaces, and multiradial correlations in product spaces having different dimensions. The combination of extension theorems with Turning Bands provides a connection with random fields that are defined in balls cross linear or circular time.

Keywords: Characteristic functions; Rudin's extensions; random fields; turning bands

1. Introduction

1.1. Extension problems

The study of positive definite functions traces back to Hilbert (1888) and Carathéodory (1907). The extension of such functions from an original space to a wider space has preoccupied mathematicians since the early 1940s, and we refer to Calderon and Pepinsky (1952), Krein (1940) as well to the *tour de force* by Rudin (1963, 1970).

While Rudin (1970)'s extension theorem refers to positive definite functions that are defined over some compact interval on the real line, subsequent research has been devoted to attain extension theorems for multidimensional spaces, the radial case being of special interest to both probability theory and spatial statistics. Indeed, a *normalized* positive definite radial function defined in the d-dimensional Euclidean space, $\mathbb{R}^d$, is the characteristic function of a random vector in $\mathbb{R}^d$, as well as the correlation function of a Gaussian random field that is stationary and isotropic (*i.e.*, radially symmetric) in $\mathbb{R}^d$ (Matheron, 1973, Schoenberg, 1938). For a comprehensive review on the extension problem, the reader is referred to Sasvári (2006).

Gneiting and Sasvári (1999) proved that any positive definite radial function defined in a ball embedded in $\mathbb{R}^d$, $d > 1$, which is not necessarily continuous at the origin, admits an extension to a positive definite radial function in $\mathbb{R}^d$. To the best of our knowledge, generalizations to product spaces under radiality have not been considered so far. The difficulty of the problem is actually confirmed by the counterexample produced by Rudin (1963), who proves that positive definite functions defined on rectangles in $\mathbb{R}^2$ might not have a positive definite extension to the whole plane. Theorem 4.3.6 in Sasvári (1994) shows that extensions in product spaces are possible, but the radiality of the extension is not necessarily preserved. Example 4.2.9(b) in Sasvári (1994) for a strip embedded in $\mathbb{R}^2$ suggests that such extensions might be possible under suitable regularity assumptions on the function to be extended.

1.2. Turning bands operator

Matheron (1965, 1971) proposed the illustrative terms *montée* (upgrading) and *descente* (downgrading) to describe operators which, when applied to suitable radially symmetric characteristic functions in $\mathbb{R}^d$, yield radially symmetric characteristic functions in $\mathbb{R}^{d'}$, for d' being larger or smaller than d. Wendland (1995) adopted the name *walk through dimensions* to describe the role of these operators and showed their effect on radial characteristic functions in terms of smoothness.

Related to the montée, Matheron (1972, 1973) defined the so-called *Turning Bands* operator allowing for a bijection between the class of symmetric characteristic functions on the real line, and the class of radially symmetric characteristic functions in $\mathbb{R}^d$, for $d > 1$. The impact of Turning Bands can be appreciated in subsequent developments in probability theory (Cambanis, Keener and Simons, 1983, Daley and Porcu, 2014, Eaton, 1981), spatial statistics (Gneiting, 1999, Gneiting and Sasvári, 1999, Porcu, Gregori and Mateu, 2006), and geostatistical simulations (Emery and Lantuéjoul, 2006, Emery and Porcu, 2019, Lantuéjoul, 2002), to mention a few.

1.3. Our contribution

We consider multiradial characteristic functions (*i.e.*, componentwise isotropic correlation functions) that are defined in product spaces. Specifically, we consider the case of the product of a d-dimensional ball with radius $1/2$, $\mathbb{B}_d$, with the d'-dimensional Euclidean space, $\mathbb{R}^{d'}$, and a positive definite function that is radial in $\mathbb{B}_d$ and in $\mathbb{R}^{d'}$. This case is of interest in application fields such as meteorology, climatology, geodesy and geophysics, where there is a need to model atmospheric, gravimetric, seismic or magnetic data indexed by latitude, longitude, altitude/depth and time, e.g., satellite data, data located in the Earth's mantle or on the Earth's surface (Ern et al., 2018, Finlay, 2020, Gillet et al., 2013, Hofmann-Wellenhof and Moritz, 2006, Meschede and Romanowicz, 2015, Xu and Wang, 2021). We also consider the product space $\mathbb{B}_d \times \mathbb{S}^{d'}$, for $\mathbb{S}^{d'}$ the d'-dimensional unit sphere embedded in $\mathbb{R}^{d'+1}$; here, by multiradiality we mean that the positive definite function is radial in $\mathbb{B}_d$, and depends on the inner product on $\mathbb{S}^{d'}$. For these two cases, we prove that Rudin's extensions are indeed possible. Our proof is based on results of independent interest: we prove characterizations of these types of positive definite functions in terms of partial Fourier transforms.

Additionally, we extend Matheron's Turning Bands operators to the aforementioned product spaces. While the first case $\mathbb{B}_d \times \mathbb{R}^{d'}$ can be proved through a direct inspection, the case $\mathbb{B}_d \times \mathbb{S}^{d'}$ comes to us as a surprise. Merging extensions with Turning Bands over product spaces, we find a connection related to Gaussian random fields that are defined over balls cross linear or circular time. Specifically, we provide new classes of nonseparable correlation functions that are isotropic over balls and symmetric on the real line, or circularly symmetric on the circle.

The outline of the paper is the following. Section 2 provides background material. Section 3 challenges the extension problem for characteristic functions. Results related to new Turning Bands-type operators over product spaces are reported in Section 4. Section 5 connects Sections 3 and 4 to illustrate a method for constructing correlation functions that are isotropic over the ball $\mathbb{B}_d$, and symmetric over linear time, $\mathbb{R}$, or circularly symmetric over circular time, $\mathbb{S}^1$.

As a result, our contribution and innovation can be sketched as follows:

1. We prove extension theorems for multiradial characteristic functions over product spaces. This subject is almost unexplored, with an example in Sasvári (2006) being a notable exception.
2. We extend Matheron's turning bands operator to product spaces involving the d-dimensional sphere. To date, turning bands operators over product spaces have not been considered, and the literature on turning bands operators over spheres has been elusive.
3. We merge the contributions in the previous points to provide useful constructions for nonseparable correlation functions for random fields that are continuously defined over balls cross linear or circular time. So far, only correlation functions over balls (without the time dimension) have been proposed by earlier literature.

2. Background and notation

2.1. Schoenberg classes

We consider the class $\mathcal{P}(\mathbb{R}^d, \|\cdot\|_d)$ of real-valued continuous mappings $f : [0,\infty) \to \mathbb{R}$ with $f(0) = 1$ such that $f(\|\cdot\|_d)$ is positive definite and radial in $\mathbb{R}^d$, with $\|\cdot\|_d$ denoting the Euclidean norm in $\mathbb{R}^d$. The reason to use the subindex d in the definition of the norm $\|\cdot\|_d$ will become apparent subsequently. Specifically, we have

$$\sum_{j=1}^{N}\sum_{k=1}^{N} a_j a_k f\left(\|\boldsymbol{x}_j - \boldsymbol{x}_k\|_d\right) \geq 0, \tag{1}$$

for any arbitrary choice of real constants $\{a_k\}_{k=1}^N$ and sequence of points $\{\boldsymbol{x}_k\}_{k=1}^N \subset \mathbb{R}^d$. Such functions are termed isotropic in spatial statistics, and the reader is referred to Daley and Porcu (2014), with the references therein, for a comprehensive treatment. A characterization of the class $\mathcal{P}(\mathbb{R}^d, \|\cdot\|_d)$ is available thanks to Schoenberg (1938): a continuous mapping f defined on the positive real line belongs to $\mathcal{P}(\mathbb{R}^d, \|\cdot\|_d)$ if and only if

$$f(x) = \int_{[0,\infty)} \Omega_d(xu)\mathrm{d}F(u), \qquad x \geq 0, \tag{2}$$

where F is a probability measure on the positive real line and $\Omega_d(t) = \mathbb{E}\left(\mathrm{e}^{\mathrm{i}t\langle \mathbf{e}_1, \boldsymbol{\eta}\rangle_d}\right)$, with i the imaginary unit, $\mathbf{e}_1$ a unit vector in $\mathbb{R}^d$, and $\boldsymbol{\eta}$ a random vector that is uniformly distributed on the spherical shell $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. Here, $\langle\cdot,\cdot\rangle_d$ stands for the inner product between two elements of $\mathbb{R}^d$, and clearly $\langle \boldsymbol{x}, \boldsymbol{x}\rangle_d = \|\boldsymbol{x}\|_d^2$, for $\boldsymbol{x} \in \mathbb{R}^d$.

The kernel Ω_d has several representations. Here, we invoke direct inspection to write Ω_d as

$$\Omega_d(t) = \frac{\Gamma(d/2)J_{d/2-1}(t)}{\left(\frac{t}{2}\right)^{d/2-1}} = \sum_{n=0}^{\infty} \frac{\Gamma(d/2)\left(-\frac{t^2}{4}\right)^n}{\Gamma(d/2+n)n!}, \qquad t \geq 0, \tag{3}$$

where J_ν stands for the Bessel function of the first kind of order ν. Since $|\Omega_d(t)| \leq \Omega_d(0) = 1$ (Daley and Porcu, 2014), one has $f(0) = 1$. Clearly, $f \in \mathcal{P}(\mathbb{R}^d, \|\cdot\|_d)$ is the radial part of the characteristic function of a random vector, $\boldsymbol{X}$, that is equal (in distribution) to the product between the random vector $\boldsymbol{\eta}$ and a random variable, Z, independent of $\boldsymbol{\eta}$ and distributed according to F.

We define $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$ as the class of real-valued continuous mappings $\varphi : [0,\infty)^2 \to \mathbb{R}$ with $\varphi(0,0) = 1$, such that $\varphi(\|\cdot\|_d, \|\cdot\|_{d'})$ is positive definite in $\mathbb{R}^d \times \mathbb{R}^{d'}$, i.e., $\sum_{j=1}^N \sum_{k=1}^N a_j a_k \times \varphi\left(\|\boldsymbol{x}_j - \boldsymbol{x}_k\|_d, \|\boldsymbol{y}_j - \boldsymbol{y}_k\|_{d'}\right) \geq 0$ for any arbitrary choice of real constants $\{a_k\}_{k=1}^N$ and sequences of points $\{\boldsymbol{x}_k\}_{k=1}^N \subset \mathbb{R}^d$ and $\{\boldsymbol{y}_k\}_{k=1}^N \subset \mathbb{R}^{d'}$. Such functions φ are called *multiradial* in Porcu, Mateu and Christakos (2009), and arguments therein show that $\varphi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$ for given $d, d' \in \mathbb{N}$ if and only if

$$\varphi(x,t) = \int_{[0,\infty)^2} \Omega_d(xu)\Omega_{d'}(tv)\mathrm{d}H(u,v), \qquad x,t \geq 0, \tag{4}$$

with H being a probability measure on the positive quadrant of $\mathbb{R}^2$. Clearly, $\varphi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$ implies $\varphi(\cdot,0)$ and $\varphi(0,\cdot)$ to belong to $\mathcal{P}(\mathbb{R}^d, \|\cdot\|_d)$ and $\mathcal{P}(\mathbb{R}^{d'}, \|\cdot\|_{d'})$, respectively.

The seminal paper by Schoenberg (1942) characterizes the class $\mathcal{P}(\mathbb{S}^d, \vartheta_d)$ of real-valued continuous mappings $\psi : [0,\pi] \to \mathbb{R}$ with $\psi(0) = 1$, such that $\psi(\vartheta_d(\cdot,\cdot))$ is positive definite on the unit sphere, $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, and where ϑ_d denotes the geodesic distance over $\mathbb{S}^d$, defined as $\vartheta_d(\boldsymbol{x},\boldsymbol{y}) = \arccos(\langle \boldsymbol{x},\boldsymbol{y}\rangle_{d+1})$, $\boldsymbol{x},\boldsymbol{y} \in \mathbb{S}^d$. The definition of positive definiteness for this case can be adapted *mutatis mutandis* by substituting the geodesic distance ϑ_d for the Euclidean norm $\|\cdot\|_d$ in Equation (1). Specifically, Schoenberg proved that $\psi \in \mathcal{P}(\mathbb{S}^d, \vartheta_d)$ for a given $d \in \mathbb{N}$, $d > 1$, if and only if

$$\psi(\theta) = \sum_{n=0}^{\infty} b_{n,d} \frac{\mathcal{G}_n^{(d-1)/2}(\cos\theta)}{\mathcal{G}_n^{(d-1)/2}(1)}, \qquad \theta \in [0,\pi], \tag{5}$$

where $\mathcal{G}_n^{\lambda}$ is the nth Gegenbauer polynomial of order $\lambda > 0$ (Szegö, 1939), and where $\{b_{n,d}\}_{n=0}^{\infty}$ is a uniquely determined sequence of nonnegative coefficients summing up to one, *i.e.*, a probability mass system. The decomposition (5) remains valid for $d = 1$ provided that the normalized Gegenbauer polynomials are replaced by the Chebyshev polynomials of the first kind.

Finally, the class $\mathcal{P}(\mathbb{R}^d \times \mathbb{S}^{d'}, \|\cdot\|_d, \vartheta_{d'})$ of continuous real-valued mappings $\psi : [0,\infty) \times [0,\pi] \to \mathbb{R}$ such that $\psi(0,0) = 1$ and $\psi(\|\cdot\|_d, \vartheta_{d'})$ is positive definite has been characterized by Berg and Porcu (2017) through uniquely determined expansions of the type

$$\psi(x,\theta) = \sum_{n=0}^{\infty} b_{n,d'}(x) \frac{\mathcal{G}_n^{(d'-1)/2}(\cos\theta)}{\mathcal{G}_n^{(d'-1)/2}(1)}, \qquad x \geq 0,\ \theta \in [0,\pi], \tag{6}$$

where $\{b_{n,d'}(\cdot)\}_{n=0}^{\infty}$ is a uniquely determined sequence of members of $\mathcal{P}(\mathbb{R}^d, \|\cdot\|_d)$ with the additional requirement that $\sum_{n=0}^{\infty} b_{n,d'}(0) = 1$ (again, for $d' = 1$, one has to replace the normalized Gegenbauer polynomials in (6) by the Chebyshev polynomials of the first kind). We follow Daley and Porcu (2014) and Berg and Porcu (2017) to refer the sequences $\{b_{n,d}\}_{n=0}^{\infty}$ in (5) and $\{b_{n,d'}(\cdot)\}_{n=0}^{\infty}$ in (6), as d-Schoenberg sequences of coefficients, and d'-Schoenberg sequences of functions, respectively.

2.2. Rudin's extension

Let $\mathbb{B}_d = \{\boldsymbol{x} \in \mathbb{R}^d,\ \|\boldsymbol{x}\|_d < 1/2\}$ denote the open ball embedded in $\mathbb{R}^d$. Rudin (1970) considers the class $\mathcal{P}(\mathbb{B}_d, \|\cdot\|_d)$ of continuous functions $g : [0,1) \to \mathbb{R}$ with $g(0) = 1$ such that the composition $g(\|\cdot\|_d)$ is positive definite, in the sense of Equation (1) with the points $\{\boldsymbol{x}_k\}_{k=1}^N$ now belonging to $\mathbb{B}_d$ instead of $\mathbb{R}^d$. Clearly, $f \in \mathcal{P}(\mathbb{R}^d, \|\cdot\|_d)$ implies the corresponding restriction to $[0,1)$ to belong to $\mathcal{P}(\mathbb{B}_d, \|\cdot\|_d)$.

The opposite is not obvious and has been shown by Rudin (1970). We state the result formally for the convenience of the reader.

Theorem 2.1. *Let d be a positive integer. Let g belong to the class $\mathcal{P}(\mathbb{B}_d, \|\cdot\|_d)$. Then, there exists a continuous mapping $f : [0,\infty) \to \mathbb{R}$ belonging to the class $\mathcal{P}(\mathbb{R}^d, \|\cdot\|_d)$ such that $f(t) = g(t)$ for all $t \in [0,1)$.*

Rudin's beautiful result provides an important message to the spatial statistics community. The class of isotropic correlation functions in $\mathbb{B}_d$ is not larger than the Schoenberg class $\mathcal{P}(\mathbb{R}^d, \|\cdot\|_d)$ restricted to $\mathbb{B}_d$. Hence, the function belonging to the class $\mathcal{P}(\mathbb{B}_d, \|\cdot\|_d)$ enjoys the scale mixture representation in Equation (2).

Surprisingly, analogues of Rudin's extensions for the case of product spaces are rare. Sasvári (1994) provides a generalization by considering the product space $V_2 \times G_1$, with G_1 being a commutative group, and V_2 a subgroup of an arbitrary group, G_2. Unfortunately, the extension to $G_2 \times G_1$ is not necessarily radial, and until now no extensions are available for the classes introduced in Section 2.1. Throughout, we use the notation $\mathcal{P}(\mathbb{B}_d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$ and $\mathcal{P}(\mathbb{B}_d \times \mathbb{S}^{d'}, \|\cdot\|_d, \vartheta_{d'})$ in analogy with the classes that have been previously defined.

2.3. The turning bands operator

We now revisit the Turning Bands operator introduced in Section 1.2, which gives a bijection between $\mathcal{P}(\mathbb{R}, \|\cdot\|_1)$ and $\mathcal{P}(\mathbb{R}^d, \|\cdot\|_1)$. We use the subindex d to indicate a function φ_d belonging to the class $\mathcal{P}(\mathbb{R}^d, \|\cdot\|_d)$, $d > 1$. Matheron (1973) proved that

$$\varphi_d(t) = \frac{2\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \frac{1}{t} \int_0^t \varphi_1(u) \left(1 - \frac{u^2}{t^2}\right)^{(d-3)/2} \mathrm{d}u, \tag{7}$$

for $\varphi_1 \in \mathcal{P}(\mathbb{R}, \|\cdot\|_1)$. Gneiting (1999) uses Equation (7) in concert with Theorem 2.1 to prove that the Turning Bands operator also provides a bijection between the classes $\mathcal{P}(\mathbb{B}_1, \|\cdot\|_1)$ and $\mathcal{P}(\mathbb{B}_d, \|\cdot\|_d)$.

3. The extension problem in product spaces

We start by illustrating a result that provides the basis for a constructive proof of our extension theorems.

Theorem 3.1. *(a). Let d, d' be two positive integers. Let $\varphi : [0,1) \times [0,\infty) \to \mathbb{R}$ be continuous with $\varphi(0,0) = 1$, and such that $\varphi(x, \|\cdot\|_{d'})$ is absolutely integrable in $\mathbb{R}^{d'}$ for every $x \in [0,1)$. Then, φ belongs to $\mathcal{P}(\mathbb{B}_d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$ if and only if the mapping φ_{ω}, defined through*

$$\varphi_{\omega}(x) := \int_{\mathbb{R}^{d'}} \mathrm{e}^{-\mathrm{i}\langle \boldsymbol{\omega}, \boldsymbol{y}\rangle_{d'}} \varphi(x, \|\boldsymbol{y}\|_{d'}) \mathrm{d}\boldsymbol{y}, \quad x \in [0,1), \tag{8}$$

is such that $\varphi_{\omega}/\varphi_{\omega}(0)$ belongs to the class $\mathcal{P}(\mathbb{B}_d, \|\cdot\|_d)$ for every $\boldsymbol{\omega} \in \mathbb{R}^{d'}$.

(b). Let $\psi : [0,1) \times [0,\pi] \to \mathbb{R}$ be continuous with $\psi(0,0) = 1$. Then, $\psi \in \mathcal{P}(\mathbb{B}_d \times \mathbb{S}^{d'}, \|\cdot\|_d, \vartheta_{d'})$ if and only if the functions $b_{n,d'}(\cdot)$, defined through

$$b_{n,d'}(x) := c_{d'} \int_0^{\pi} \psi(x,\theta)(\sin\theta)^{d'-1} \mathcal{G}_n^{(d'-1)/2}(\cos\theta)\mathrm{d}\theta, \qquad x \in [0,1), \tag{9}$$

form a sequence of members of $\mathcal{P}(\mathbb{B}_d, \|\cdot\|_d)$ for all $n \in \mathbb{N}_0$ with the additional requirement that $\sum_{n=0}^{\infty} b_{n,d'}(0) = 1$. Here, $c_{d'}$ is a strictly positive normalization constant that depends on d'. When $d' = 1$, the Gegenbauer polynomials in (9) need to be replaced by Chebyshev polynomials.

Proof. (a). To prove the necessity, first note that $\varphi_{\boldsymbol{\omega}}$ is well-defined thanks to the integrability condition on φ in concert with the fact that the complex exponential is uniformly bounded by one. If $\varphi \in \mathcal{P}(\mathbb{B}_d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$, then a direct application of the Schur product theorem (that is, positive definite functions are closed under product, see Berg, Christensen and Ressel, 1984, Theorem 3.1.12) shows that the complex-valued mapping

$$(\boldsymbol{x},\boldsymbol{y}) \mapsto \mathrm{e}^{-\mathrm{i}\langle\boldsymbol{\omega},\boldsymbol{y}\rangle_{d'}} \varphi(\|\boldsymbol{x}\|_d, \|\boldsymbol{y}\|_{d'}), \qquad (\boldsymbol{x},\boldsymbol{y}) \in \mathbb{B}_d \times \mathbb{R}^{d'},$$

is positive definite in $\mathbb{B}_d \times \mathbb{R}^{d'}$ for every $\boldsymbol{\omega} \in \mathbb{R}^{d'}$. Accordingly, the mapping

$$\boldsymbol{x} \mapsto \mathrm{e}^{-\mathrm{i}\langle\boldsymbol{\omega},\boldsymbol{y}\rangle_{d'}} \varphi(\|\boldsymbol{x}\|_d, \|\boldsymbol{y}\|_{d'}), \qquad \boldsymbol{x} \in \mathbb{B}_d,$$

is positive definite in $\mathbb{B}_d$ for every $\boldsymbol{\omega} \in \mathbb{R}^{d'}$ and $\boldsymbol{y} \in \mathbb{R}^{d'}$ and is not identically zero. Since positive definite functions are a convex cone that is closed under scale mixtures, we get that $\varphi_{\boldsymbol{\omega}}$ as defined in (8) is the radial part of a positive definite function in $\mathbb{B}_d$ and is not identically zero, which ensures that $\varphi_{\boldsymbol{\omega}}(0)$ is strictly positive and that $\varphi_{\boldsymbol{\omega}}/\varphi_{\boldsymbol{\omega}}(0)$ is well-defined.

To prove the sufficiency, we let $\varphi_{\boldsymbol{\omega}}$ be defined as in (8), and let $\varphi_{\boldsymbol{\omega}}/\varphi_{\boldsymbol{\omega}}(0)$ be a member of the class $\mathcal{P}(\mathbb{B}_d, \|\cdot\|_d)$ for every $\boldsymbol{\omega} \in \mathbb{R}^{d'}$. Equation (8) in concert with the Fourier inversion theorem allows us to write

$$\varphi(x,y) = \frac{1}{(2\pi)^{d'}} \int_{\mathbb{R}^{d'}} \mathrm{e}^{\mathrm{i}\langle\boldsymbol{\omega},y\boldsymbol{e}'_1\rangle_{d'}} \varphi_{\boldsymbol{\omega}}(x)\mathrm{d}\boldsymbol{\omega}, \qquad x \in [0,1), y \in [0,\infty), \tag{10}$$

with $\boldsymbol{e}'_1$ a unit vector in $\mathbb{R}^{d'}$. Note that the integral in (10) does not depend on the particular choice of $\boldsymbol{e}'_1$. In fact, let $T : \mathbb{R}^{d'} \to \mathbb{R}^{d'}$ be an orthogonal operator. Then, one has

$$\begin{aligned} \varphi_{T\boldsymbol{\omega}}(x) &= \frac{1}{(2\pi)^{d'}} \int_{\mathbb{R}^{d'}} \mathrm{e}^{-\mathrm{i}\langle T\boldsymbol{\omega},\boldsymbol{y}\rangle_{d'}} \varphi(x, \|\boldsymbol{y}\|_{d'})\mathrm{d}\boldsymbol{y} \\ &= \frac{1}{(2\pi)^{d'}} \int_{\mathbb{R}^{d'}} \mathrm{e}^{-\mathrm{i}\langle\boldsymbol{\omega},T^{-1}\boldsymbol{y}\rangle_{d'}} \varphi(x, \|\boldsymbol{y}\|_{d'})\mathrm{d}\big(T^{-1}\boldsymbol{y}\big) = \varphi_{\boldsymbol{\omega}}(x), \end{aligned} \tag{11}$$

for $\boldsymbol{\omega} \in \mathbb{R}^{d'}$, $x \in [0,1)$. Because $\varphi_{\boldsymbol{\omega}}(\|\cdot\|_d)$ is positive definite in $\mathbb{B}_d$ and the complex exponential $\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\omega},\|\cdot\|_{d'}\boldsymbol{e}'_1\rangle_{d'}}$ is positive definite in $\mathbb{R}^{d'}$, their product is positive definite in $\mathbb{B}_d \times \mathbb{R}^{d'}$ owing to the Schur product theorem. Accordingly, φ, as defined in (10), belongs to $\mathcal{P}(\mathbb{B}_d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$ since the class of positive definite functions is a convex cone closed under scale mixtures.

(b). The proof of this part of the statement is a straight application of Theorem 3.4 in Berg and Porcu (2017) and is thus omitted. □

We are now ready to provide our Rudin-type extensions for the product spaces.

Theorem 3.2. *(a). Let $\varphi : [0,1) \times [0,\infty) \to \mathbb{R}$ be a member of the class $\mathcal{P}(\mathbb{B}_d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$ for some $d, d' \in \mathbb{N}$, with the additional requirement that $\varphi(x, \|\cdot\|_{d'})$ is absolutely integrable in $\mathbb{R}^{d'}$ for all $x \in [0,1)$. Then, there exists a mapping $\widetilde{\varphi} : [0,\infty)^2 \to \mathbb{R}$ belonging to the class $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$ such that $\widetilde{\varphi} = \varphi$ on $[0,1) \times [0,\infty)$.*

(b). Let $\psi : [0,1) \times [0,\pi] \to \mathbb{R}$ be a member of the class $\mathcal{P}(\mathbb{B}_d \times \mathbb{S}^{d'}, \|\cdot\|_d, \vartheta_{d'})$. Then, there exists a mapping $\widetilde{\psi} : [0,\infty) \times [0,\pi] \to \mathbb{R}$ belonging to the class $\mathcal{P}(\mathbb{R}^d \times \mathbb{S}^{d'}, \|\cdot\|_d, \vartheta_{d'})$ such that $\widetilde{\psi} = \psi$ on $[0,1) \times [0,\pi]$.

Proof. Let φ be as asserted. We use part (a) of Theorem 3.1 to claim that the mapping $\varphi_{\omega}/\varphi_{\omega}(0)$ as defined in (8) belongs to the class $\mathcal{P}(\mathbb{B}_d, \|\cdot\|_d)$ for all $\omega \in \mathbb{R}^{d'}$. Rudin's extension theorem (Theorem 2.1) then implies that there exists a mapping $\widetilde{\varphi}_{\omega} : [0,\infty) \to \mathbb{R}$ that is identical to φ_{ω} on $[0,1)$ and such that $\widetilde{\varphi}_{\omega}/\widetilde{\varphi}_{\omega}(0)$ belongs to the class $\mathcal{P}(\mathbb{R}^d, \|\cdot\|_d)$ for every $\omega \in \mathbb{R}^{d'}$. Hence, we make use of (10) in concert with the Fourier inversion theorem to claim that the mapping $\widetilde{\varphi}$ defined through

$$\widetilde{\varphi}(x,y) := \frac{1}{(2\pi)^{d'}} \int_{\mathbb{R}^{d'}} \mathrm{e}^{\mathrm{i}\langle \omega, y\boldsymbol{e}_1' \rangle_{d'}} \widetilde{\varphi}_{\omega}(x)\mathrm{d}\omega, \quad x,y \geq 0, \tag{12}$$

is identical to φ on $[0,1)\times[0,\infty)$. Note that $\widetilde{\varphi}$ is well defined since $\widetilde{\varphi}_{\omega}/\widetilde{\varphi}_{\omega}(0) \in \mathcal{P}(\mathbb{R}^d, \|\cdot\|_d)$ implies $|\widetilde{\varphi}_{\omega}(x)| \leq \widetilde{\varphi}_{\omega}(0) = \varphi_{\omega}(0)$ for all $x \geq 0$. Hence, the existence of the integral above is directly deduced from the existence of $\varphi(0,y)$ for any $y \geq 0$. Based on the Schur product theorem and the fact that the class of positive definite functions is closed under scale mixtures, we conclude that $\widetilde{\varphi}$ belongs to the class $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$.

(b). We provide a constructive proof. By Theorem 3.1, part (b), $\psi \in \mathcal{P}(\mathbb{B}_d \times \mathbb{S}^{d'}, \|\cdot\|_d, \vartheta_{d'})$ implies the sequence of functions $\{b_{n,d'}(\cdot)\}_{n=0}^{\infty}$ to be contained in $\mathcal{P}(\mathbb{B}_d, \|\cdot\|_d)$. Hence, we can invoke again Theorem 2.1 to claim that there exists a sequence $\{B_{n,d'}(\cdot)\}_{n=0}^{\infty}$ of functions in $\mathcal{P}(\mathbb{R}^d, \|\cdot\|_d)$ that is identical to $\{b_{n,d'}(\cdot)\}_{n=0}^{\infty}$ on $[0,1)$. Additionally, the summability of $\{B_{n,d'}(\cdot)\}_{n=0}^{\infty}$ at zero is inherited from that of $\{b_{n,d'}(\cdot)\}_{n=0}^{\infty}$ at zero. Thus, a direct application of classical inversion formulae and Theorem 3.4 in Berg and Porcu (2017) shows that the mapping

$$(x,\theta) \mapsto \widetilde{\psi}(x,\theta) = \sum_{n=0}^{\infty} B_{n,d'}(x) \frac{\mathcal{G}_n^{(d'-1)/2}(\cos\theta)}{\mathcal{G}_n^{(d'-1)/2}(1)}, \qquad x \geq 0, \theta \in [0,\pi],$$

belongs to the class $\mathcal{P}(\mathbb{R}^d \times \mathbb{S}^{d'}, \|\cdot\|_d, \vartheta_{d'})$ and is identical to ψ on $[0,1)\times[0,\pi)$. □

Some comments are in order. A direct inspection into Theorem 4.3.2 of Sasvári (1994) in concert with Theorem 3.1 shows that radiality over the second argument is actually not necessary, so that our extension theorem works *mutatis mutandis* in product spaces with radiality in $\mathbb{B}_d$ only. We also note that our result generalizes Theorem 4.1.9 in Sasvári (1994), corresponding to the case of a function in $\mathcal{P}(\mathbb{B}_1 \times \mathbb{R}, \|\cdot\|_1, \|\cdot\|_1)$.

Theorem 3.1 does not help finding multiradial extensions for the case $\mathbb{B}_d \times \mathbb{B}_{d'}$. Again, Theorem 4.3.2 in Sasvári (1994) shows that such extensions are possible, but these extensions may or may not be multiradial. We consider this nontrivial case an open problem.

We also note that Theorem 3.2 does not contradict Rudin (1963), who proved that positive definite functions defined in hyperrectangles of $\mathbb{R}^d$ might not be extended to positive definite functions in $\mathbb{R}^d$, for $d > 1$. Yet, some cases allow for a positive answer. Let $\varphi \in \mathcal{P}(\mathbb{B}_d \times \mathbb{B}_{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$ such that $\varphi(x,t) = \varphi_1(x)\varphi_2(t)$, for $\varphi_1 \in \mathcal{P}(\mathbb{B}_d, \|\cdot\|_d)$ and $\varphi_2 \in \mathcal{P}(\mathbb{B}_{d'}, \|\cdot\|_{d'})$. Then, φ_1 extends to $\widetilde{\varphi}_1 \in \mathcal{P}(\mathbb{R}^d, \|\cdot\|_d)$ and φ_2 extends to $\widetilde{\varphi}_2 \in \mathcal{P}(\mathbb{R}^{d'}, \|\cdot\|_{d'})$, i.e., $\varphi_i = \widetilde{\varphi}_i$ on $[0,1)$, $i = 1,2$. Hence, the product $\widetilde{\varphi}(x,t) = \widetilde{\varphi}_1(x)\widetilde{\varphi}_2(t)$ is an extension of φ in the sense of Rudin. A similar comment applies to the function

$$\varphi(x,t) := \int_{[0,\infty)} \int_{[0,\infty)} \Omega_d(xu)\Omega_{d'}(tv)\mathrm{d}F(u,v),$$

for F a probability measure on the positive quadrant of $\mathbb{R}^2$, with Ω_d as defined at (3). This does not mean that the classes $\mathcal{P}(\mathbb{B}_d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$ and $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$ are bijective. This is certainly true for the classes $\mathcal{P}(\mathbb{B}_d, \|\cdot\|_d)$ and $\mathcal{P}(\mathbb{R}^d, \|\cdot\|_d)$, as proved by Gneiting and Sasvári (1999).

Besides, Theorem 3.1 in concert with Krein's work (Krein, 1940) show that, if φ_ω in (8) is analytic, or if $\varphi_\omega(t_o) = 1$ for some $t_o \in [-1,1]$, then the extension of $\varphi \in \mathcal{P}(\mathbb{B}_1 \times \mathbb{R}^{d'}, \|\cdot\|_1, \|\cdot\|_{d'})$ to $\mathcal{P}(\mathbb{R} \times \mathbb{R}^{d'}, \|\cdot\|_1, \|\cdot\|_{d'})$ is unique. We are not aware of any extension for higher dimensional spaces. The alternative to uniqueness is indeterminacy, which happens when there is a countable number of extensions. This would be definitely worthy of a thorough investigation.

A beautiful result in Crum (1956) shows that, if φ is a measurable mapping that is isotropic and positive definite in $\mathbb{R}^d$, with $d > 1$, then φ is continuous except possibly at zero, which proves a conjecture by Schoenberg (1938). The implications of such a result are illustrated by Gneiting and Sasvári (1999): from a geostatistical perspective, the restriction to measurable functions is immaterial, and practically we can write any isotropic covariance function on $\mathbb{R}^d$, $d > 1$, as the sum of a pure nugget effect (*i.e.*, a covariance function that is identically equal to zero except at the origin) and a continuous covariance function. Gneiting and Sasvári (1999) then couple Crum's result with Rudin's extension to show that every measurable isotropic correlation on $\mathbb{B}_d$ admits a measurable extension in the sense of Rudin. In the case of product spaces, however, measurability and Crum's decomposition might be an issue. To illustrate, consider a *space-time* correlation function $\varphi \in \mathcal{P}(\mathbb{B}_d \times \mathbb{B}_1, \|\cdot\|_d, \|\cdot\|_1)$ defined as $\varphi(x,t) = \varphi_1(x)\varphi_2(t)$, for $\varphi_1 \in \mathcal{P}(\mathbb{B}_d, \|\cdot\|_d)$ and $\varphi_2 \in \mathcal{P}(\mathbb{B}_1, \|\cdot\|_1)$. Clearly, the measurability theorem is not valid for φ_2 – see Crum (1956) for a counterexample. Hence, it seems that the product space case inherits the same problem emphasized by Crum (1956).

4. Matheron's turning bands operator in product spaces

Turning Bands operators are largely unexplored in product spaces.

For the class $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$, we invoke the arguments in Porcu, Mateu and Christakos (2009) to assert that $\varphi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$ if and only if it can be written as in Equation (4), for a probability distribution H defined on the positive quadrant of $\mathbb{R}^2$. Direct inspection allows to rewrite (4) as

$$\varphi(x,t) = \int_{[0,\infty)} \Omega_d(xu) \mathrm{d}\widetilde{H}_{t,d'}(u), \qquad x,t \ge 0, \tag{13}$$

with $\widetilde{H}_{t,d'}(u) := \int_{[0,\infty)} \Omega_{d'}(tv)\mathrm{d}H(u,v)$. One has $|\Omega_{d'}(t)| \le \Omega_{d'}(0) = 1$, hence $|\widetilde{H}_{t,d'}(u)| \le \widetilde{H}_{0,d'}(u) \le 1$, since the total mass of H over $\mathbb{R}^2$ is identically equal to one. A similar argument implies $|\widetilde{H}_{t,d'}(u)| \le \widetilde{H}_{t,d'}(0) \le 1$. Thus, we can use Equation (13) in concert with Fubini's theorem, as well as an integral representation for Bessel functions and Formula 9.1.20 of Abramowitz and Stegun (1966), to get the following relation: for $\varphi_{d,d'} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$, one has

$$\varphi_{d,d'}(x,t) = \frac{2\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \frac{1}{x} \int_0^x \varphi_{1,d'}(u,t) \left(1 - \frac{u^2}{x^2}\right)^{(d-3)/2} \mathrm{d}u, \tag{14}$$

for $\varphi_{1,d'} \in \mathcal{P}(\mathbb{R} \times \mathbb{R}^{d'}, \|\cdot\|_1, \|\cdot\|_{d'})$. Clearly, such an operator is in bijection between the classes $\mathcal{P}(\mathbb{R} \times \mathbb{R}^{d'}, \|\cdot\|_1, \|\cdot\|_{d'})$ and $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$. Hence, we can use Theorem 3.2, part (a), while invoking the arguments in Gneiting (1999), to claim that the Turning Bands operator provides a bijection between the classes $\mathcal{P}(\mathbb{B}_1 \times \mathbb{R}^{d'}, \|\cdot\|_1, \|\cdot\|_{d'})$ and $\mathcal{P}(\mathbb{B}_d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$.

A less obvious result is to prove that such bijections hold for the class $\mathcal{P}(\mathbb{R}^d \times \mathbb{S}^{d'}, \|\cdot\|_d, \vartheta_{d'})$ as well.

Theorem 4.1. *Let d, d' be positive integers. Let $\psi_{1,d'}$ be a member of the class $\mathcal{P}(\mathbb{R} \times \mathbb{S}^{d'}, \|\cdot\|_1, \vartheta_{d'})$. Then, the Turning Bands operator in Equation (14) provides a function $\psi_{d,d'}$ being a member of the class $\mathcal{P}(\mathbb{R}^d \times \mathbb{S}^{d'}, \|\cdot\|_d, \vartheta_{d'})$.*

Proof. By assumption, $\psi_{1,d'} \in \mathcal{P}(\mathbb{R} \times \mathbb{S}^{d'}, \|\cdot\|_1, \vartheta_{d'})$. This implies that there exists a sequence of functions $\{b_{n,d'}(\cdot)\}_{n=0}^{\infty}$ such that $b_{n,d'}(\cdot)/b_{n,d'}(0) \in \mathcal{P}(\mathbb{R}, \|\cdot\|_1)$ with $\sum_n b_{n,d'}(0) = 1$. Hence, we can apply Matheron's Turning Bands as in Equation (7) to obtain a sequence $\{\widetilde{b}_{n,d'}(\cdot)\}_{n=0}^{\infty}$ in $\mathcal{P}(\mathbb{R}^d, \|\cdot\|_d)$, with

$$\widetilde{b}_{n,d'}(x) = \frac{2\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \frac{1}{x} \int_0^x b_{n,d'}(u) \left(1 - \frac{u^2}{x^2}\right)^{(d-3)/2} \mathrm{d}u. \tag{15}$$

In view of Equation (15), there exists a function $\widetilde{\psi}_{d,d'} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{S}^{d'}, \|\cdot\|_d, \vartheta_{d'})$ such that

$$\begin{aligned}
\widetilde{\psi}_{d,d'}(x,\theta) &= \sum_{n=0}^{\infty} \widetilde{b}_{n,d'}(x) \frac{\mathcal{G}_n^{(d'-1)/2}(\cos\theta)}{\mathcal{G}_n^{(d'-1)/2}(1)} \\
&= \sum_{n=0}^{\infty} \left(\frac{2\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \frac{1}{x} \int_0^x b_{n,d'}(u) \left(1 - \frac{u^2}{x^2}\right)^{(d-3)/2} \mathrm{d}u \right) \frac{\mathcal{G}_n^{(d'-1)/2}(\cos\theta)}{\mathcal{G}_n^{(d'-1)/2}(1)} \\
&= \frac{2\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \frac{1}{x} \int_0^x \left(1 - \frac{u^2}{x^2}\right)^{(d-3)/2} \left(\sum_{n=0}^{\infty} b_{n,d'}(u) \frac{\mathcal{G}_n^{(d'-1)/2}(\cos\theta)}{\mathcal{G}_n^{(d'-1)/2}(1)} \right) \mathrm{d}u \\
&= \frac{2\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \frac{1}{x} \int_0^x \left(1 - \frac{u^2}{x^2}\right)^{(d-3)/2} \psi_{1,d'}(u,\theta) \mathrm{d}u, \quad x \geq 0, \theta \in [0,\pi],
\end{aligned}$$

where series and definite integral can be swapped because the series is absolutely convergent (Berg and Porcu, 2017). The proof is completed by noting that

$$\begin{aligned}
\left| \sum_{n=0}^{\infty} \widetilde{b}_{n,d'}(x) \right| &= \left| \widetilde{\psi}_{d,d'}(x,0) \right| \\
&= \left| \frac{2\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \frac{1}{x} \sum_{n=0}^{\infty} \int_0^x b_{n,d'}(u) \left(1 - \frac{u^2}{x^2}\right)^{(d-3)/2} \mathrm{d}u \right| \\
&\leq \frac{2\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \frac{1}{x} \sum_{n=0}^{\infty} b_{n,d'}(0) \int_0^x \left(1 - \frac{u^2}{x^2}\right)^{(d-3)/2} \mathrm{d}u \\
&= \frac{x\sqrt{\pi}\Gamma\left((d-1)/2\right)}{2\Gamma\left(d/2\right)} \frac{2\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \frac{1}{x} \sum_{n=0}^{\infty} b_{n,d'}(0) \\
&= 1,
\end{aligned}$$

where the equality in the third line has been obtained by using formula 3.249.5 of Gradshteyn and Ryzhik (2014). □

5. Random fields over balls cross linear or circular time

Gneiting (1999) considers the function

$$\varphi_1(x) = \begin{cases} 1-\alpha x & 0 \le x < 1 \\ 1-\alpha(2-x) & 1 \le x < 2, \end{cases}$$

continued periodically to $[0,\infty)$ with period 2. One has

$$\varphi_1(x) = 1 - \frac{\alpha}{2} + \sum_{n=1}^{\infty} \frac{4\alpha \cos\big((2n-1)\pi x\big)}{\pi^2(2n-1)^2},$$

which shows that $\varphi_1 \in \mathcal{P}(\mathbb{R}, \|\cdot\|_1)$ for $0 < \alpha \le 2$. Clearly, the restriction of φ_1 to $[0,1)$, that we denote $\widetilde{\varphi}_1(x) = 1-\alpha x$, belongs to the class $\mathcal{P}(\mathbb{B}_1, \|\cdot\|_1)$. Using the fact that the Turning Bands provides a bijection for this class, a direct computation shows that

$$\varphi_d(x) = 1 - \frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma((d+1)/2)}\alpha x,$$

belongs to the class $\mathcal{P}(\mathbb{B}_d, \|\cdot\|_d)$. This provides an upper bound for the function $\varphi(x) := 1-\alpha_d x$, $0 \le x < 1$, to belong to the class $\mathcal{P}(\mathbb{B}_d, \|\cdot\|_d)$.

The developments in previous sections allow to elaborate similar strategies for positive definite functions that are isotropic in d-dimensional balls and, either, symmetric over linear time ($\mathbb{R}$), or isotropic over circular time ($\mathbb{S}^1$). For instance, one can consider the function

$$\varphi_{1,d'}(x,t) = \begin{cases} 1+\alpha(t)/2-\alpha(t)x & 0 \le x < 1 \\ 1+\alpha(t)-\alpha(t)(2-x) & 1 \le x < 2, \end{cases}$$

continued periodically to $[0,\infty)$ with period 2, and where $\alpha \in \mathcal{P}(\mathbb{R}^{d'}, \|\cdot\|_{d'})$ or $\alpha \in \mathcal{P}(\mathbb{S}^{d'}, \vartheta_{d'})$. Clearly,

$$\varphi_{1,d'}(x,t) = 1 + \sum_{n=1}^{\infty} \frac{4\alpha(t)\cos\big((2n-1)\pi x\big)}{\pi^2(2n-1)^2},$$

which proves that $\varphi_{1,d'} \in \mathcal{P}(\mathbb{R}\times\mathbb{R}^{d'}, \|\cdot\|_1, \|\cdot\|_{d'})$ (resp. $\mathcal{P}(\mathbb{R}\times\mathbb{S}^{d'}, \|\cdot\|_1, \vartheta_{d'})$). Hence, we can mimic the arguments in Gneiting (1999) in concert with Theorem 4.1 to show that the function

$$\varphi_{d,d'}(x,t) = \frac{1+\alpha(t)\left(\frac{1}{2}-\alpha_d x\right)}{1+\frac{\alpha(0)}{2}},$$

belongs to the class $\mathcal{P}(\mathbb{B}_d \times \mathbb{R}^{d'}, \|\cdot\|_d, \|\cdot\|_{d'})$ or $\mathcal{P}(\mathbb{B}_d \times \mathbb{S}^{d'}, \|\cdot\|_d, \vartheta_{d'})$ provided $\alpha \in \mathcal{P}(\mathbb{R}^{d'}, \|\cdot\|_{d'})$ or $\alpha \in \mathcal{P}(\mathbb{S}^{d'}, \vartheta_{d'})$, respectively. Here, $\alpha_d = \Gamma(d/2)/\big(\sqrt{\pi}\Gamma((d+1)/2)\big)$.

A worked example

Define $p : [0,\infty) \to \mathbb{R}$ as $p(t) = \exp\left(-\frac{t^2}{5}\right)$ and $\bar{h} : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ as

$$\bar{h}(x,t) = \frac{1}{\mathrm{e}} \exp\left(p(t)\cos(2\pi x)\right)\cos(p(t)\sin(2\pi x)) \tag{16}$$

Since $p^2(t) \le 1$ for all t, $\bar{h}$ admits the following expansion (Gradshteyn and Ryzhik, 2014, formula 1.449.2):

$$\bar{h}(x,t) = \sum_{k=0}^{\infty} \frac{p^k(t)\cos(2\pi k x)}{k!\mathrm{e}}.$$

Note that $\bar{h}(0,0) = 1$, and furthermore that p is positive definite on $[0,\infty)$. Given the results above, we may compute the Turning Bands operator, thereby extending $\bar{h}$ to the radial part of a positive definite function on $\mathbb{R}^d \times \mathbb{R}$:

$$h_d(x,t) = \frac{2\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \frac{1}{x} \int_0^x \bar{h}(u,t)\left(1 - \frac{u^2}{x^2}\right)^{(d-3)/2} \mathrm{d}u, \quad x,t \in [0,\infty).$$

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