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A moment map for the variety of Jordan algebras

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We study the variety of complex n-dimensional Jordan algebras using techniques from Geometric Invariant Theory. More specifically, we use the Kirwan–Ness theorem to construct a Morse-type stratification of the variety of Jordan algebras into finitely many invariant locally closed subsets, with respect to the energy functional associated to the canonical moment map. In particular we obtain a new, cohomology-free proof of the well-known rigidity of semisimple Jordan algebras in the context of the variety of Jordan algebras.

Keywords: Moment map; Jordan algebras; geometric invariant theory; Kirwan–Ness theorem.

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1. Introduction

The main goal of this paper is to introduce techniques from Geometric Invariant Theory (GIT) in the study of finite-dimensional complex Jordan algebras. In particular, we consider a naturally defined moment map for the affine variety of n-dimensional Jordan algebras and the associated energy functional, and use it to define a notion of soliton Jordan algebra. Somehow in a dual sense, the search for a soliton Jordan algebra in an isomorphism class of Jordan algebras can also be thought of as the search for a 'best' (Hermitian) metric in a Jordan algebra. Herein we also make an effort to relate the new geometric invariants to the 'old' algebraic invariants of Jordan algebras (with partial success).

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1.1. *Basic setup*

In this work we only consider finite-dimensional algebras over $\mathbb{C}$. A Jordan algebra is a commutative algebra $\mathfrak{A}$ satisfying the following Jordan identity:

$$[L_a, L_{a^2}] = 0$$

for all $a \in \mathfrak{A}$, where $L_a : \mathfrak{A} \to \mathfrak{A}$ denotes the left-multiplication by a, and $[,]$ denotes the commutator. Here we only note that Jordan algebras are, in general, non-associative, and refer the reader to [14] for more about Jordan algebras and their importance.

We can view a commutative algebra of dimension n as a commutative multiplication in $\mathbb{C}^n$, namely, as a tensor $\mu \in S^2(\mathbb{C}^n)^* \otimes \mathbb{C}^n \cong S^2(\mathbb{C}^{n*}) \otimes \mathbb{C}^n =: V_n$ (symmetric in the first two arguments). Further, the Jordan identity is equivalent to its linearized form

$$(ab, c, d) + (bd, c, a) + (da, c, b) = 0$$

for all a, b, c, $d \in \mathfrak{A}$, where $(a, b, c) = (ab)c - a(bc)$ is the associator of a, b, c. These are polynomial equations in V_n, therefore the space $\mathcal{J}_n$ of Jordan algebras of a dimension n can be naturally identified with an affine variety in V_n. It is useful to note that these polynomials are homogeneous, so we could also view the space of Jordan algebras modulo complex scaling as a projective variety in $\mathbb{P}V_n$; however, we prefer to do calculations in the vector space V_n.

There is a natural action of the group $G := GL(n, \mathbb{C})$ on the space V_n, which corresponds to 'change of basis'. Namely, for $g \in G$ and $\mu \in V_n$, one puts

$$g \cdot \mu(a, b) = g(\mu(g^{-1}a, g^{-1}b)) \tag{1}$$

for all a, $b \in \mathbb{C}^n$. It is clear that the orbit $G \cdot \mu$ yields the isomorphism class of μ. In particular, $\mathcal{J}_n$ is an invariant subvariety.

In order to use methods from GIT, we need to fix a background Hermitian product Q on $\mathbb{C}^n$ (whereas in the case of Lie algebras Q would correspond to a left-invariant Hermitian metric on the corresponding simply-connected complex Lie group, in our case there is no obvious global object attached to Q). However, we will see that the results that we shall obtain for isomorphism classes of Jordan algebra will not depend on Q in an essential way. The group of unitary transformations of $(\mathbb{C}^n, Q)$ is a maximal compact subgroup $K \cong U(n)$ of G, which also acts by unitary transformations on V_n, if this space is equipped with the induced Hermitian product (which we denote with the same symbol). Note that fixing Q is equivalent to fixing K. Now, it is standard that the action of the *compact* group K on the projective variety $\mathbb{P}V_n$ yields a moment map $\mathbb{P}V_n \to \mathfrak{k}^*$, where $\mathfrak{k}$ is the Lie algebra of K and the star refers to the dual space (see [9, Sec. 2] or [18, 6.14]); we prefer to do calculations in the vector space V_n and we use a different normalization, so we shall equivalently consider a scale invariant moment map $m : V_n \backslash \{0\} \to i\mathfrak{k}$ (see Sec. 2.2 for precise definitions); note that $i\mathfrak{k}$ is the space of Hermitian transformations of $(\mathbb{C}^n, Q)$. The square norm $E_n = ||m||^2$, which we call *energy*, has the very interesting property that its gradient flow lines are contained in the orbits of G, so the limits of those lines define degenerations of the initial algebra. This brings us to the study of critical points of E_n, which we call *solitons* (for reasons that shall become clear later).

The Kempf–Ness theorem [8] provides a beautiful characterization of closed orbits of a rational representation V of a complex reductive linear algebraic group G, namely, these are precisely the orbits that contain a *minimal* vector, i.e. a vector realizing the distance of the orbit to the origin (with respect to Q). The minimal vectors in any closed G-orbit form a unique K-orbit. Since the moment map associated to Q controls the infinitesimal change of the norm of vectors, it turns out that minimal vectors precisely correspond to zeros of the moment map (or the energy functional). The Kempf–Ness theorem also implies that a non-closed orbit with positive distance to the origin contains a unique non-trivial closed orbit in its closure. In particular, this gives a description of the categorical quotient $V//G$ (which is by definition the affine variety defined by the G-invariant polynomials in V).

On the other hand, the union of the orbits containing zero in its closure is called the *null cone* N of the representation, and its elements are called *unstable* vectors (in our case of $GL(n,\mathbb{C})$ acting on V_n, every vector is unstable since we can always rescale the multiplication that defines the algebra, which is to say that $N = V_n$). Kirwan [9] and Ness [16] showed how to use the moment map to study the orbit space of N, for a general representation V of a complex reductive linear algebraic group G; herein we are not so much interested in this orbit space, but in the related stratification of N. Consider a Hermitian product on V, the associated maximal compact subgroup K of G, the associated moment map, energy and solitons. Kirwan and Ness proved that all non-minimal solitons, i.e. those solitons with positive energy, do occur in N. Further, there is only a finite set of distinguished G-orbits in N which contain solitons; moreover, for each such orbit $G \cdot v$, the set of solitons comprise a single K-orbit (up to scaling) and they are minima of the energy on $G \cdot v$. Finally, although the energy is not a Morse(–Bott) function, the Kirwan–Ness theorem yields a Morse-type stratification of N into finitely many invariant smooth subvarieties, each of which being the stable manifold of the set of solitons of a certain *type*.

In this paper, we adapt such GIT methods to the setting of Jordan algebras, in parallel with some of the work of Lauret for Lie algebras [10], and use them to prove a number of results which we explain in the sequel. We also formulate some conjectures and open problems.

In the following, in case there is no danger of confusion, we denote Q simply by $\langle \cdot, \cdot \rangle$ and the associated norm by $|| \cdot ||$.

1.2. *Main results*

Recall that a (*geometric*) *deformation* of a Jordan algebra $\mu \in \mathcal{J}_n$ is a Jordan algebra $\nu \in \mathcal{J}_n$ whose G-orbit contains μ in its Zariski closure (which is the same as closure in the Hausdorff topology), that is $\mu \in \overline{G \cdot \nu}$. In this case, we also say that μ is a *degeneration* of ν and we write $\nu \to \mu$. The degeneration $\nu \to \mu$ is called *trivial* if μ is isomorphic to ν. A Jordan algebra $\mu \in \mathcal{J}_n$ is called (*geometrically*) *rigid* if its G-orbit is Zariski-open; in our context (finite-dimension, base field $\mathbb{C}$), this is equivalent to the non-existence of non-trivial deformations of μ or, yet, to having the closure of the orbit of μ in $\mathcal{J}_n$ coincide with an irreducible component of $\mathcal{J}_n$.

A commutative algebra $\mu \in V_n$ will be called a *soliton* if μ is a critical point of E_n. A G-orbit in V_n (or isomorphism class of commutative algebras) is called *distinguished* if it contains a soliton. For $\mu \in V_n \setminus \{0\}$, we write $M_\mu = ||\mu||^2 m(\mu)$ and call it the *moment matrix* of μ. We have a structure result for Jordan solitons.

Proposition 1.1. *Let $\mu \in V_n$ and write $\mathfrak{A} = (\mathbb{C}^n, \mu)$ for the corresponding commutative algebra. Then*

(a) *μ is a soliton if and only if its moment matrix $M_\mu = c_\mu I + D_\mu$, where $c_\mu < 0$, I denotes the identity matrix, and D_μ is a derivation of μ and a Hermitian matrix with respect to Q; further, in this case there is a positive multiple of D_μ which has rational eigenvalues.*
(b) *If μ is a Jordan soliton then there is a maximal semisimple subalgebra $\mathfrak{S}$ of $\mathfrak{A}$ such that $\mathfrak{A} = \mathfrak{S} + \mathfrak{N}$ is a D_μ-invariant decomposition, where $\mathfrak{N}$ denotes the radical of $\mathfrak{A}$. Further, $D_\mu|\mathfrak{S} = 0$.*

Using this structure result and the classic theory of semisimple Jordan algebras (see [2]), we prove the following theorem.

Theorem 1.2. *Every isomorphism class of complex semisimple Jordan algebras is distinguished. Further, in each dimension the semisimple Jordan solitons are characterized among Jordan algebras as having the lowest possible value of the energy for that dimension.*

The relation $\mu \to \nu$ defines a partial order on $\mathcal{J}_n$, and it follows from properties of the stratification that the functional E_n behaves well with regard to this partial order:

Proposition 1.3. *Let $\mu, \nu \in \mathcal{J}_n$. If $G \cdot \mu$ is a distinguished orbit and the energy of a soliton in $G \cdot \mu$ is larger than the energy of ν, then μ cannot degenerate to ν.*

As is well known and traditionally proved via cohomological methods, every n-dimensional semisimple Jordan algebra is rigid in the *scheme* of n-dimensional Jordan algebras [4, 5]. We apply Theorem 1.2 to give a simple, cohomology-free proof of this result in the context of the variety $\mathcal{J}_n$ of Jordan algebras.

Theorem 1.4 ([4, 5]). *Every finite-dimensional complex semisimple Jordan algebra is rigid.*

The class of nilpotent Jordan algebras of dimension n forms a G-invariant closed subset $\mathcal{N}_n$ of $\mathcal{J}_n$. It is known that there is no non-trivial n-dimensional complex

Table 1. Complex one-dimensional Jordan algebras and their solitons.

Isomorphism class	Multiplication table	Properties
$\mathfrak{A}_{1,1}$	$e_1^2 = e_1$	A, S

Table 2. The stratification of $\mathcal{J}_1$.

G-orbits	Soliton type	β	E_1
$\mathfrak{A}_{1,1}$	$(0; 1)$	$\mathrm{diag}(-1)$	1

Table 3. Complex two-dimensional Jordan algebras and their solitons.

Isomorphism class	Multiplication table	Properties
$\mathfrak{A}_{2,1}$	$e_1^2 = e_1$, $e_1n_1 = n_1$	A, U
$\mathfrak{A}_{2,2}$	$e_1^2 = e_1$, $e_1n_1 = \frac{1}{2}n_1$	—
$\mathfrak{A}_{2,3}$	$n_1^2 = n_2$	A, N
$\mathfrak{A}_{2,4} = (\mathfrak{A}_{1,1})^2$	$e_1^2 = e_1$, $e_2^2 = e_2$	A, SS, D
$\mathfrak{A}_{2,5} = \mathfrak{A}_{1,1} \times \mathfrak{T}$	$e_1^2 = e_1$, $n_1^2 = 0$	A, D

Jordan algebra lying in all irreducible components of $\mathcal{J}_n$ (see [13, Theorem 4.65]). On the other hand, by studying some specific degenerations, we prove the following theorem.

Theorem 1.5. *For all $n \geq 2$ there exists a non-trivial n-dimensional complex nilpotent Jordan algebra lying in all irreducible components of $\mathcal{N}_n$; further, the orbit of this algebra is characterized as realizing the highest possible value of the energy E_n. It follows that the projectivization $\mathbb{P}\mathcal{N}_n$ is connected for $n \geq 2$.*

Finally, we give a complete explicit description of the case $n \leq 4$, see Tables 1–4. The *type* of a soliton μ is $(d_1 < \cdots < d_r; m_1, \ldots, m_r)$, where $d_1, \ldots, d_r$ is the sequence of coprime integers which is proportional to the eigenvalues of $m(\mu) + E_n(\mu)I = \frac{1}{||\mu||^2}D_\mu$, and $m_1, \ldots, m_r$ are the respective multiplicities. The stratification is finite and the strata are parametrized by $\beta = m(\mu)$ or, equivalently, by the soliton type. We deduce the following theorem.

Theorem 1.6. *Every isomorphism class of complex Jordan algebras of dimension at most 4 is distinguished, except in the case of the nilpotent Jordan algebra $\mathfrak{A}_{4,63}$ (see Table 8) which is not.*

In Tables 1, 3, 5, 7 and 8 we list the complex Jordan solitons of dimension at most 4. The coefficients in the multiplication table are chosen so that the algebra in each isomorphism type is a soliton, except $\mathfrak{A}_{4,63}$ for which there is no soliton. In the column of properties we indicate whether the algebra is associative (A), simple (S), semisimple (SS), nilpotent (N), unitary (U), or decomposable (D); the absence of such a qualification means its negation, except that a simple Jordan algebra is

Table 4. The stratification of $\mathcal{J}_2$.

G-orbits	Soliton type	β	E_2
$\mathfrak{A}_{2,4}$	$(0;2)$	$\mathrm{diag}(-\frac{1}{2},-\frac{1}{2})$	$1/2$
$\mathfrak{A}_{2,1}$, $\mathfrak{A}_{2,5}$, $\mathfrak{A}_{2,2}$	$(0<1;1,1)$	$\mathrm{diag}(-1,0)$	1
$\mathfrak{A}_{2,3}$	$(1<2;1,1)$	$\mathrm{diag}(-2,1)$	5

Table 5. Complex three-dimensional Jordan algebras and their solitons.

Isom class	Multiplication table	Properties
$\mathfrak{A}_{3,1}=(\mathfrak{A}_{1,1})^3$	$e_1^2=e_1,\ e_2^2=e_2,\ e_3^2=e_3$	SS, A, D
$\mathfrak{A}_{3,2}$	$e_1^2=e_1,\ e_2^2=e_3^2=\frac{\sqrt{5}}{2}e_1,\ e_1e_2=e_2,\ e_1e_3=e_3$	S
$\mathfrak{A}_{3,3}=\mathfrak{A}_{2,1}\times\mathfrak{A}_{1,1}$	$e_1^2=e_1,\ e_2^2=\sqrt{3}e_2,\ e_1n_1=n_1$	A, U, D
$\mathfrak{A}_{3,4}$	$e_1^2=e_1,\ e_2^2=\sqrt{\frac{5}{3}}e_1,\ e_1e_2=e_2,\ e_1n_1=n_1$	U
$\mathfrak{A}_{3,5}=\mathfrak{A}_{2,2}\times\mathfrak{A}_{1,1}$	$e_1^2=e_1,\ e_2^2=\sqrt{\frac{3}{2}}e_2,\ e_1n_1=\frac{1}{2}n_1$	D
$\mathfrak{A}_{3,6}=(\mathfrak{A}_{1,1})^2\times\mathfrak{T}$	$e_1^2=e_1,\ e_2^2=e_2,\ n_1^2=0$	A, D
$\mathfrak{A}_{3,7}$	$e_1^2=e_1,\ e_1n_1=n_1,\ e_1n_2=n_2,\ n_1^2=n_2$	A, U
$\mathfrak{A}_{3,8}$	$e_1^2=e_1,\ e_1n_1=n_1,\ e_1n_2=n_2$	A, U
$\mathfrak{A}_{3,9}=\mathfrak{A}_{2,1}\times\mathfrak{T}$	$e_1^2=e_1,\ e_1n_1=n_1,\ n_2^2=0$	A, D
$\mathfrak{A}_{3,10}$	$e_1^2=e_1,\ e_1n_1=\frac{1}{2}n_1,\ e_1n_2=n_2,\ n_1^2=\sqrt{\frac{7}{10}}n_2$	—
$\mathfrak{A}_{3,11}$	$e_1^2=e_1,\ e_1n_1=\frac{1}{2}n_1,\ e_1n_2=n_2$	—
$\mathfrak{A}_{3,12}$	$e_1^2=e_1,\ e_1n_1=\frac{1}{2}n_1,\ e_1n_2=\frac{1}{2}n_2$	—
$\mathfrak{A}_{3,13}$	$e_1^2=e_1,\ e_1n_1=\frac{1}{2}n_1,\ n_1^2=\sqrt{\frac{3}{10}}n_2$	—
$\mathfrak{A}_{3,14}=\mathfrak{A}_{2,2}\times\mathfrak{T}$	$e_1^2=e_1,\ e_1n_1=\frac{1}{2}n_1,\ n_2^2=0$	D
$\mathfrak{A}_{3,15}=\mathfrak{A}_{2,3}\times\mathfrak{A}_{1,1}$	$e_1^2=\sqrt{5}e_1,\ n_1^2=n_2$	A, D
$\mathfrak{A}_{3,16}=\mathfrak{A}_{1,1}\times(\mathfrak{T})^2$	$e_1^2=e_1,\ n_1^2=n_2^2=0$	A, D
$\mathfrak{A}_{3,17}$	$n_1^2=n_2,\ n_1n_2=n_3$	A, N
$\mathfrak{A}_{3,18}$	$n_1n_2=n_3$	A, N
$\mathfrak{A}_{3,19}=\mathfrak{A}_{2,3}\times\mathfrak{T}$	$n_1^2=n_2,\ n_3^2=0$	A, N, D

Table 6. The stratification of $\mathcal{J}_3$.

G-orbits	Soliton type	β	E_3
$\mathfrak{A}_{3,1}$, $\mathfrak{A}_{3,2}$	$(0;3)$	$\mathrm{diag}(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$	$1/3$
$\mathfrak{A}_{3,3}$, $\mathfrak{A}_{3,4}$, $\mathfrak{A}_{3,5}$, $\mathfrak{A}_{3,6}$	$(0<1;2,1)$	$\mathrm{diag}(-\frac{1}{2},-\frac{1}{2},0)$	$1/2$
$\mathfrak{A}_{3,7}$, $\mathfrak{A}_{3,10}$, $\mathfrak{A}_{3,13}$, $\mathfrak{A}_{3,15}$	$(0<1<2;1,1,1)$	$\mathrm{diag}(-\frac{5}{6},-\frac{1}{3},\frac{1}{6})$	$5/6$
$\mathfrak{A}_{3,8}$, $\mathfrak{A}_{3,9}$, $\mathfrak{A}_{3,11}$, $\mathfrak{A}_{3,12}$, $\mathfrak{A}_{3,14}$, $\mathfrak{A}_{3,16}$	$(0<1;1,2)$	$\mathrm{diag}(-1,0,0)$	1
$\mathfrak{A}_{3,17}$	$(1<2<3;1,1,1)$	$\mathrm{diag}(-\frac{4}{3},-\frac{1}{3},\frac{2}{3})$	$7/3$
$\mathfrak{A}_{3,18}$	$(1<2;2,1)$	$\mathrm{diag}(-1,-1,1)$	3
$\mathfrak{A}_{3,19}$	$(3<5<6;1,1,1)$	$\mathrm{diag}(-2,1,0)$	5

semisimple, and a semisimple Jordan algebra is unitary. In the decomposable case we exhibit the minimal decomposition; the notation $\mathfrak{T}$ refers to the trivial (one-dimensional) algebra. In Tables 2, 4, 6 and 9 we describe the strata of $\mathcal{J}_n$ for $n\leq 4$: the orbits they contain, the soliton type, the parameter (β), and the value of the energy ($E_n=||\beta||^2$).

The property of a G-orbit to be distinguished is independent of the choice of Hermitian product Q. Indeed there is an action of G on the space of Hermitian

Table 7. Complex four-dimensional Jordan algebras and their solitons, first part.

Isom class	Multiplication table	Properties
$\mathfrak{A}_{4,1}$	$e_1^2 = e_1,\ e_2^2 = \frac{\sqrt{5}}{2}e_1,\ e_3^2 = \frac{\sqrt{5}}{2}e_1,\ e_1e_2 = e_2,\ e_1e_3 = e_3,\ e_4^2 = \sqrt{\frac{5}{2}}e_4$	SS, D
$\mathfrak{A}_{4,2}$	$e_1^2 = e_1,\ e_1e_2 = e_2,\ e_1e_3 = e_3,\ e_1e_4 = e_4,\ e_2e_3 = \sqrt{\frac{7}{5}}e_1,\ e_4^2 = \sqrt{\frac{7}{5}}e_1$	S
$\mathfrak{A}_{4,3} = (\mathfrak{A}_{1,1})^4$	$e_1^2 = e_1,\ e_2^2 = e_2,\ e_3^2 = e_3,\ e_4^2 = e_4$	SS, A, D
$\mathfrak{A}_{4,4} = \mathfrak{A}_{2,1} \times (\mathfrak{A}_{1,1})^2$	$e_1^2 = e_1,\ e_1n_1 = n_1,\ e_2^2 = \sqrt{3}e_2,\ e_3^2 = \sqrt{3}e_3$	U, A, D
$\mathfrak{A}_{4,5} = (\mathfrak{A}_{1,1})^3 \times \mathfrak{T}$	$e_1^2 = e_1,\ e_2^2 = e_2,\ e_3^2 = e_3,\ n_1^2 = 0$	A, D
$\mathfrak{A}_{4,6} = \mathfrak{A}_{2,2} \times (\mathfrak{A}_{1,1})^2$	$e_1^2 = e_1,\ e_2^2 = \sqrt{\frac{3}{2}}e_2,\ e_3^2 = \sqrt{\frac{3}{2}}e_3,\ e_1n_1 = \frac{1}{2}n_1$	D
$\mathfrak{A}_{4,7} = \mathfrak{A}_{3,4} \times \mathfrak{A}_{1,1}$	$e_1^2 = e_1,\ e_2^2 = \sqrt{\frac{5}{3}}e_1,\ e_1e_2 = e_2,\ e_1n_1 = n_1,\ e_3^2 = \sqrt{\frac{10}{3}}e_3$	U, D
$\mathfrak{A}_{4,8} = \mathfrak{A}_{3,2} \times \mathfrak{T}$	$e_1^2 = e_1,\ e_2^2 = \frac{\sqrt{5}}{2}e_1,\ e_3^2 = \frac{\sqrt{5}}{2}e_1,\ e_1e_2 = e_2,\ e_1e_3 = e_3,\ n_1^2 = 0$	D
$\mathfrak{A}_{4,9}$	$e_1^2 = e_1,\ e_2^2 = \frac{\sqrt{7}}{2}e_1,\ e_3^2 = \frac{\sqrt{7}}{2}e_1,\ e_1e_2 = e_2,\ e_1e_3 = e_3,\ e_1n_1 = n_1$	U
$\mathfrak{A}_{4,10} = \mathfrak{A}_{2,2} \times \mathfrak{A}_{1,1} \times \mathfrak{T}$	$e_1^2 = e_1,\ e_1n_1 = \frac{1}{2}n_1,\ e_2^2 = \sqrt{\frac{3}{2}}e_2,\ n_2^2 = 0$	D
$\mathfrak{A}_{4,11} = \mathfrak{A}_{3,4} \times \mathfrak{T}$	$e_1^2 = e_1,\ e_2^2 = \sqrt{\frac{5}{3}}e_1,\ e_1e_2 = e_2,\ e_1n_1 = n_1,\ n_2^2 = 0$	D
$\mathfrak{A}_{4,12} = \mathfrak{A}_{3,12} \times \mathfrak{A}_{1,1}$	$e_1^2 = e_1,\ e_2^2 = \sqrt{2}e_2,\ e_1n_1 = \frac{1}{2}n_1,\ e_1n_2 = \frac{1}{2}n_2$	D
$\mathfrak{A}_{4,13} = (\mathfrak{A}_{2,2})^2$	$e_1^2 = e_1,\ e_2^2 = e_2,\ e_1n_1 = \frac{1}{2}n_1,\ e_2n_2 = \frac{1}{2}n_2$	D
$\mathfrak{A}_{4,14} = \mathfrak{A}_{3,11} \times \mathfrak{A}_{1,1}$	$e_1^2 = e_1,\ e_1n_1 = \frac{1}{2}n_1,\ e_1n_2 = n_2,\ e_2^2 = \sqrt{\frac{7}{2}}e_2$	D
$\mathfrak{A}_{4,15} = \mathfrak{A}_{2,1} \times \mathfrak{A}_{2,2}$	$e_1^2 = e_1,\ e_2^2 = \sqrt{2}e_2,\ e_1n_1 = n_1,\ e_2n_2 = \frac{1}{\sqrt{2}}n_2$	D
$\mathfrak{A}_{4,16}$	$e_1^2 = k(\cos^3 t - \sin^3 t)e_1 + k^2 \cos t \sin t(\cos t + \sin t)e_2,$ $e_2^2 = k^{-1} \cos t \sin t(-\cos t + \sin t)e_1 + (\cos^3 t + \sin^3 t)e_2,$ $e_1e_2 = \cos t \sin t((\cos t + \sin t)e_1 + k(-\cos t + \sin t)e_2),$ $e_1n_1 = \frac{k}{2}(\cos t - \sin t)n_1,\ e_1n_2 = \frac{k}{2}\cos t n_2,$ $e_2n_1 = \frac{1}{2}(\cos t + \sin t)n_1,\ e_2n_2 = \frac{1}{2}\sin t n_2$	$k \approx 1.20577,$ $t \approx 1.22166$
$\mathfrak{A}_{4,17}$	$e_1^2 = k(\cos^3 t - \sin^3 t)e_1 + k^2 \cos t \sin t(\cos t + \sin t)e_2,$ $e_2^2 = k^{-1} \cos t \sin t(-\cos t + \sin t)e_1 + (\cos^3 t + \sin^3 t)e_2,$ $e_1e_2 = \cos t \sin t(\cos t + \sin t)e_1 + k \cos t \sin t(-\cos t + \sin t)e_2,$ $e_1n_1 = \frac{k}{2}(\cos t - \sin t)n_1,\ e_1n_2 = k \cos t n_2,$ $e_2n_1 = \frac{1}{2}(\cos t + \sin t)n_1,\ e_2n_2 = \sin t n_2$	$k \approx 1.54492,$ $t \approx 1.45358$, U

Table 7. (*Continued*)

Isom class	Multiplication table	Properties
$\mathfrak{A}_{4,18}$	$e_1^2=e_1,\ e_2^2=\sqrt{\frac{7}{3}}e_1,\ e_1e_2=e_2,\ e_1n_1=n_1,\ e_1n_2=n_2$	U
$\mathfrak{A}_{4,19}=(\mathfrak{A}_{1,1})^2\times(\mathfrak{T})^2$	$e_1^2=e_1,\ e_2^2=e_2,\ n_1^2=n_2^2=0$	A, D
$\mathfrak{A}_{4,20}=\mathfrak{A}_{2,1}\times\mathfrak{A}_{1,1}\times\mathfrak{T}$	$e_1^2=e_1,\ e_2^2=\sqrt{3}e_2,\ e_1n_1=n_1,\ n_2^2=0$	A, D
$\mathfrak{A}_{4,21}=\mathfrak{A}_{3,8}\times\mathfrak{A}_{1,1}$	$e_1^2=e_1,\ e_2^2=\sqrt{5}e_2,\ e_1n_1=n_1,\ e_1n_2=n_2$	A, U, D
$\mathfrak{A}_{4,22}=(\mathfrak{A}_{2,1})^2$	$e_1^2=e_1,\ e_2^2=e_2,\ e_1n_1=n_1,\ e_2n_2=n_2$	A, U, D
$\mathfrak{A}_{4,23}=\mathfrak{A}_{3,13}\times\mathfrak{A}_{1,1}$	$e_1^2=e_1,\ e_1n_1=\frac{1}{2}n_1,\ n_1^2=\sqrt{\frac{3}{10}}n_2,\ e_2^2=\sqrt{\frac{3}{2}}e_2$	D
$\mathfrak{A}_{4,24}=\mathfrak{A}_{3,10}\times\mathfrak{A}_{1,1}$	$e_1^2=e_1,\ e_1n_1=\frac{1}{2}n_1,\ e_1n_2=n_2,\ n_1^2=\sqrt{\frac{7}{10}}n_2,\ e_2^2=\sqrt{\frac{7}{2}}e_2$	D
$\mathfrak{A}_{4,25}$	$e_1^2=k(\cos^3 t-\sin^3 t)e_1+k^2\cos t\sin t(\cos t+\sin t)e_2,$ $e_2^2=k^{-1}\cos t\sin t(-\cos t+\sin t)e_1+(\cos^3 t+\sin^3 t)e_2,$ $e_1e_2=\cos t\sin t(\cos t+\sin t)e_1+k\cos t\sin t(-\cos t+\sin t)e_2,$ $e_1n_1=\frac{k}{2}(\cos t-\sin t)n_1,\ e_1n_2=k\cos tn_2,$ $e_2n_1=\frac{1}{2}(\cos t+\sin t)n_1,\ e_2n_2=\sin tn_2,\ n_1^2=\ell n_2$	$k\approx 1.54492,$ $t\approx 1.45358,$ $\ell\approx 0.836502,$ U
$\mathfrak{A}_{4,26}=\mathfrak{A}_{2,3}\times(\mathfrak{A}_{1,1})^2$	$e_1^2=e_1,\ e_2^2=e_2,\ n_1^2=\frac{1}{\sqrt{5}}n_2,\ n_2^2=0$	A, D
$\mathfrak{A}_{4,27}=\mathfrak{A}_{3,7}\times\mathfrak{A}_{1,1}$	$e_1^2=e_1,\ e_1n_1=n_1,\ e_1n_2=n_2,\ n_1^2=n_2,\ e_2^2=\sqrt{5}e_2$	U, A, D
$\mathfrak{A}_{4,28}=\mathfrak{A}_{2,2}\times(\mathfrak{T})^2$	$e_1^2=e_1,\ e_1n_1=\frac{1}{2}n_1,\ n_2^2=n_3^2=0$	D
$\mathfrak{A}_{4,29}=\mathfrak{A}_{3,11}\times\mathfrak{T}$	$e_1^2=e_1,\ e_1n_1=\frac{1}{2}n_1,\ e_1n_2=n_2,\ n_3^2=0$	D
$\mathfrak{A}_{4,30}=\mathfrak{A}_{3,12}\times\mathfrak{T}$	$e_1^2=e_1,\ e_1n_1=\frac{1}{2}n_1,\ e_1n_2=\frac{1}{2}n_2,\ n_3^2=0$	D
$\mathfrak{A}_{4,31}$	$e_1^2=e_1,\ e_1n_1=n_1,\ e_1n_2=n_2,\ e_1n_3=\frac{1}{2}n_3$	-
$\mathfrak{A}_{4,32}$	$e_1^2=e_1,\ e_1n_1=\frac{1}{2}n_1,\ e_1n_2=\frac{1}{2}n_2,\ e_1n_3=n_3$	-
$\mathfrak{A}_{4,33}$	$e_1^2=e_1,\ e_1n_1=\frac{1}{2}n_1,\ e_1n_2=\frac{1}{2}n_2,\ e_1n_3=\frac{1}{2}n_3$	-
$\mathfrak{A}_{4,34}=\mathfrak{A}_{1,1}\times(\mathfrak{T})^3$	$e_1^2=e_1,\ n_1^2=n_2^2=n_3^2=0$	A, D
$\mathfrak{A}_{4,35}=\mathfrak{A}_{2,1}\times(\mathfrak{T})^2$	$e_1^2=e_1,\ e_1n_1=n_1,\ n_2^2=n_3^2=0$	A, D
$\mathfrak{A}_{4,36}$	$e_1^2=e_1,\ e_1n_1=n_1,\ e_1n_2=n_2,\ e_1n_3=n_3$	U, A
$\mathfrak{A}_{4,37}=\mathfrak{A}_{3,8}\times\mathfrak{T}$	$e_1^2=e_1,\ e_1n_1=n_1,\ e_1n_2=n_2,\ n_3^2=0$	A, D
$\mathfrak{A}_{4,38}=\mathfrak{A}_{3,17}\times\mathfrak{A}_{1,1}$	$e_1^2=\sqrt{7}e_1,\ n_1^2=n_2,\ n_1n_2=n_3$	A, D
$\mathfrak{A}_{4,39}$	$e_1^2=e_1,\ e_1n_1=n_1,\ e_1n_2=n_2,\ e_1n_3=n_3,\ n_1^2=n_2,\ n_1n_2=n_3$	U, A
$\mathfrak{A}_{4,40}=\mathfrak{A}_{2,3}\times\mathfrak{A}_{1,1}\times\mathfrak{T}$	$n_1^2=n_2,\ e_1^2=\sqrt{5}e_1,\ n_3^2=0$	D

Table 8. Complex four-dimensional Jordan algebras and their solitons, second part.

Isom class	Multiplication table	Properties
$\mathfrak{A}_{4,41} = \mathfrak{A}_{3,18} \times \mathfrak{A}_{1,1}$	$e_1^2 = \sqrt{6}e_1,\, n_1n_2 = n_3$	A, D
$\mathfrak{A}_{4,42}$	$e_1^2 = e_1,\, e_1n_1 = n_1,\, e_1n_2 = n_2,\, e_1n_3 = n_3,\, n_1^2 = \sqrt{\frac{7}{5}}n_2$	U, A
$\mathfrak{A}_{4,43}$	$e_1^2 = e_1,\, e_1n_1 = n_1,\, e_1n_2 = n_2,\, e_1n_3 = n_3,\, n_1^2 = n_2^2 = \sqrt{\frac{7}{6}}n_3$	U, A
$\mathfrak{A}_{4,44} = \mathfrak{A}_{3,13} \times \mathfrak{T}$	$e_1^2 = \sqrt{\frac{10}{3}}e_1,\, e_1n_1 = \sqrt{\frac{5}{6}}n_1,\, n_1^2 = n_2,\, n_3^2 = 0$	D
$\mathfrak{A}_{4,45}$	$e_1^2 = 2e_1,\, e_1n_1 = n_1,\, n_1^2 = n_2^2 = n_3$	—
$\mathfrak{A}_{4,46} = \mathfrak{A}_{2,2} \times \mathfrak{A}_{2,3}$	$e_1^2 = e_1,\, e_1n_1 = \frac{1}{2}n_1,\, n_2^2 = \sqrt{\frac{3}{10}}n_3$	D
$\mathfrak{A}_{4,47} = \mathfrak{A}_{2,1} \times \mathfrak{A}_{2,3}$	$e_1^2 = e_1,\, e_1n_1 = n_1,\, n_2^2 = \sqrt{\frac{3}{5}}n_3$	A, D
$\mathfrak{A}_{4,48}$	$e_1^2 = e_1,\, e_1n_1 = \frac{1}{2}n_1,\, e_1n_2 = \frac{1}{2}n_2,\, n_1^2 = \sqrt{\frac{2}{5}}n_3$	—
$\mathfrak{A}_{4,49}$	$e_1^2 = e_1,\, e_1n_1 = \frac{1}{2}n_1,\, e_1n_2 = \frac{1}{2}n_2,\, n_1^2 = n_2^2 = \frac{1}{\sqrt{3}}n_3$	—
$\mathfrak{A}_{4,50}$	$e_1^2 = e_1,\, e_1n_2 = \frac{1}{2}n_2,\, e_1n_3 = \frac{1}{2}n_3,\, n_1n_2 = \frac{1}{\sqrt{3}}n_3$	—
$\mathfrak{A}_{4,51} = \mathfrak{A}_{3,10} \times \mathfrak{T}$	$e_1^2 = e_1,\, e_1n_1 = \frac{1}{2}n_1,\, e_1n_2 = n_2,\, n_1^2 = \sqrt{\frac{7}{10}}n_2,\, n_3^2 = 0$	D
$\mathfrak{A}_{4,52}$	$e_1^2 = e_1,\, e_1n_1 = \frac{1}{2}n_1,\, e_1n_2 = n_2,\, n_1^2 = \sqrt{\frac{7}{10}}n_3$	—
$\mathfrak{A}_{4,53}$	$e_1^2 = e_1,\, e_1n_1 = \frac{1}{2}n_1,\, e_1n_2 = \frac{1}{2}(n_2 + \alpha n_3) = \alpha e_1n_3,\, n_1^2 = \beta n_3$	$\alpha^2 = \frac{\sqrt{345}-5}{20},\, \beta^2 = \frac{\sqrt{345}+45}{80}$
$\mathfrak{A}_{4,54} = \mathfrak{A}_{3,7} \times \mathfrak{T}$	$e_1^2 = e_1,\, e_1n_1 = n_1,\, e_1n_2 = n_2,\, n_1^2 = n_2,\, n_3^2 = 0$	A, D

Table 8. (*Continued*)

Isom class	Multiplication table	Properties
$\mathfrak{A}_{4,55}$	$e_1^2 = e_1,\ e_1n_1 = n_1,\ e_1n_2 = n_2,\ e_1n_3 = \frac{1}{2}n_3,\ n_3^2 = \sqrt{\frac{11}{10}}n_2$	—
$\mathfrak{A}_{4,56}$	$e_1^2 = e_1,\ e_1n_1 = n_1,\ e_1n_2 = n_2,\ e_1n_3 = \frac{1}{2}n_3,\ n_1^2 = \sqrt{\frac{11}{10}}n_2$	—
$\mathfrak{A}_{4,57}$	$e_1^2 = e_1,\ e_1n_1 = n_1,\ e_1n_2 = n_2,\ e_1n_3 = \frac{1}{2}n_3,\ n_1^2 = n_3^2 = \sqrt{\frac{11}{12}}n_2$	—
$\mathfrak{A}_{4,58}$	$e_1^2 = e_1,\ e_1n_1 = n_1,\ e_1n_2 = \frac{1}{2}n_2,\ e_1n_3 = \frac{1}{2}n_3,\ n_3^2 = \frac{2}{\sqrt{5}}n_1$	—
$\mathfrak{A}_{4,59}$	$e_1^2 = e_1,\ e_1n_1 = n_1,\ e_1n_2 = \frac{1}{2}n_2,\ e_1n_3 = \frac{1}{2}n_3,\ n_2^2 = n_3^2 = \sqrt{\frac{2}{3}}n_1$	—
$\mathfrak{A}_{4,60}$	$e_1^2 = e_1,\ e_1n_1 = n_1,\ e_1n_2 = \frac{1}{2}n_2,\ e_1n_3 = \frac{1}{2}n_3,\ n_1n_2 = \sqrt{\frac{2}{3}}n_3$	—
$\mathfrak{A}_{4,61}$	$n_1^2 = n_2,\ n_2^2 = n_4,\ n_1n_2 = n_3,\ n_1n_3 = n_4$	A, N
$\mathfrak{A}_{4,62}$	$n_1^2 = n_2,\ n_4^2 = 2n_2,\ n_1n_2 = \sqrt{3}n_3$	N
$\mathfrak{A}_{4,63}$	$n_1n_2 = n_3,\ n_1n_3 = n_4,\ n_2^2 = n_4$	N, not distinguished
$\mathfrak{A}_{4,64}$	$n_1n_2 = n_3,\ n_1n_3 = n_4$	N
$\mathfrak{A}_{4,65}$	$n_1^2 = \frac{2}{\sqrt{3}}n_2,\ n_2n_3 = n_4$	N
$\mathfrak{A}_{4,66}$	$n_1^2 = n_2,\ n_3^2 = n_4,\ n_1n_2 = \frac{\sqrt{3}}{2}n_4$	A, N
$\mathfrak{A}_{4,67} = \mathfrak{A}_{3,17} \times \mathfrak{T}$	$n_1^2 = n_2,\ n_1n_2 = n_3,\ n_4^2 = 0$	A, N, D
$\mathfrak{A}_{4,68} = (\mathfrak{A}_{2,3})^2$	$n_1^2 = n_2,\ n_3^2 = n_4$	A, N, D
$\mathfrak{A}_{4,69}$	$n_1^2 = n_2,\ n_1n_3 = \sqrt{\frac{3}{2}}n_4$	A, N
$\mathfrak{A}_{4,70}$	$n_1^2 = n_3n_4 = n_2$	A, N
$\mathfrak{A}_{4,71} = \mathfrak{A}_{3,18} \times \mathfrak{T}$	$n_1n_2 = n_3,\ n_4^2 = 0$	A, N
$\mathfrak{A}_{4,72} = \mathfrak{A}_{2,3} \times (\mathfrak{T})^2$	$n_1^2 = n_2,\ n_3^2 = n_4^2 = 0$	A, N

Table 9. The stratification of $\mathcal{J}_4$.

G-orbits $\mathfrak{A}_{4,k}$	Soliton type	β	E_4
$1 \leq k \leq 3$	$(0; 4)$	$\operatorname{diag}(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})$	$1/4$
$4 \leq k \leq 9$	$(0 < 1; 3, 1)$	$\operatorname{diag}(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 0)$	$1/3$
$23 \leq k \leq 27$	$(0 < 1 < 2; 2, 1, 1)$	$\operatorname{diag}(-\frac{5}{11}, -\frac{5}{11}, -\frac{2}{11}, \frac{1}{11})$	$5/11$
$10 \leq k \leq 22$	$(0 < 1; 2, 2)$	$\operatorname{diag}(-\frac{1}{2}, -\frac{1}{2}, 0, 0)$	$1/2$
$38 \leq k \leq 39$	$(0 < 1 < 2 < 3; 1, 1, 1, 1)$	$\operatorname{diag}(-\frac{7}{10}, -\frac{4}{10}, -\frac{1}{10}, \frac{2}{10})$	$7/10$
$k \in \{41, 43, 45, 49, 50, 57, 59, 60\}$	$(0 < 1 < 2; 1, 2, 1)$	$\operatorname{diag}(-\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4})$	$3/4$
$k = 53$	$(0 < 1 < 2; 1, 1, 2)$	$\operatorname{diag}(-\frac{9}{11}, -\frac{4}{11}, \frac{1}{11}, \frac{1}{11})$	$9/11$
$k \in \{40, 42, 44, 46,$ $47, 48, 51, 52, 54, 55, 56, 58\}$	$(0 < 3 < 5 < 6; 1, 1, 1, 1)$	$\operatorname{diag}(-\frac{5}{6}, -\frac{1}{3}, 0, \frac{1}{6})$	$5/6$
$28 \leq k \leq 37$	$(0 < 1; 1, 3)$	$\operatorname{diag}(-1, 0, 0, 0)$	1
$k = 62$	$(1 < 2 < 3; 2, 1, 1)$	$\operatorname{diag}(-\frac{8}{11}, -\frac{8}{11}, -\frac{1}{11}, \frac{6}{11})$	$15/11$
$k = 65$	$(3 < 4 < 6 < 10; 1, 1, 1, 1)$	$\operatorname{diag}(-\frac{4}{5}, -\frac{3}{5}, -\frac{1}{5}, \frac{3}{5})$	$7/5$
$k = 61,\ 63,\ 64$	$(1 < 2 < 3 < 4; 1, 1, 1, 1)$	$\operatorname{diag}(-1, -1/2, 0, 1/2)$	$3/2$
$k = 66$	$(2 < 3 < 4 < 6; 1, 1, 1, 1)$	$\operatorname{diag}(-1, -\frac{1}{7}, -\frac{4}{7}, \frac{5}{7})$	$13/7$
$k = 70$	$(1 < 2; 3, 1)$	$\operatorname{diag}(-\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}, 1)$	$7/3$
$k = 67$	$(3 < 6 < 7 < 9; 1, 1, 1, 1)$	$\operatorname{diag}(-\frac{4}{3}, -\frac{1}{3}, 0, \frac{2}{3})$	$7/3$
$k = 68$	$(1 < 2; 2, 2)$	$\operatorname{diag}(-1, -1, \frac{1}{2}, \frac{1}{2})$	$5/2$
$k = 69$	$(3 < 4 < 6 < 7; 1, 1, 1, 1)$	$\operatorname{diag}(-\frac{5}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{3}{4})$	$11/4$
$k = 71$	$(2 < 3 < 4; 2, 1, 1)$	$\operatorname{diag}(-1, -1, 0, 1)$	3
$k = 72$	$(3 < 5 < 6; 1, 2, 1)$	$\operatorname{diag}(-2, 0, 0, 1)$	5

products on $\mathbb{C}^n$ given by

$$g \cdot Q(a, b) = Q(g^{-1}a, g^{-1}b)$$

for $g \in G$ and a, $b \in \mathbb{C}^n$. This action is transitive, so, if we fix Q, any other Hermitian product on $\mathbb{C}^n$ is of the form $g \cdot Q$ for some $g \in G$. Using a superscript to denote the moment matrices associated to different Hermitian products, it is easy to see that

$$M_\mu^{g\cdot Q} = gM_{g^{-1}\cdot\mu}^{Q}g^{-1}. \tag{2}$$

Hence the claim follows from the structure result for solitons (Proposition 1.1(a)).

Thinking in a dual way, fix Q and suppose the isomorphism class of a Jordan algebra $\mathfrak{A} = (\mathbb{C}^n, \mu)$ is distinguished. Then $g \cdot \mu$ is a soliton for some $g \in G$, and the relation (2) can be used to see that μ is a soliton with respect to $g^{-1} \cdot Q$, that is $g^{-1} \cdot Q$ is a 'best' metric on $\mathfrak{A}$. It follows from Kirwan–Ness theory that the critical set of E_n on a G-orbit, if non-empty, is a single K-orbit, so the best metric on $\mathfrak{A}$, if existing, is unique.

We point out that several formulae and results herein are similar to those for Lie algebras [10]. An important difference of Jordan algebras from Lie algebras is that, in general, left multiplications are not derivations. So arguments for Lie algebras that depend on this property will not work in the setting of Jordan algebras. Of course this should not be seen as a problem, but as a central feature of Jordan algebras; and the real problem is knowing how to use this feature to our advantage.[a]

2. Preliminaries

2.1. *Semisimplicity and the radical of Jordan algebras*

Most of the results in this subsection are due to Albert. A good reference is [19, Chap. IV].

A complex Jordan algebra $\mathfrak{A}$ is called *simple* if $\mathfrak{A}^2 \neq 0$ and $\mathfrak{A}$ has no non-trivial ideals, and it is called *semisimple* if it is a direct product of simple Jordan algebras. A semisimple Jordan algebra has an identity element.

Let $\mathfrak{A}$ be a complex Jordan algebra. An element $x \in \mathfrak{A}$ is called *nilpotent* if $x^k = 0$ for some integer $k \geq 2$; this is equivalent to the left-multiplication operator L_x being a nilpotent operator. There is an ideal $\mathrm{Rad}(\mathfrak{A})$, called the *radical* of $\mathfrak{A}$, which is the unique maximal nilideal of $\mathfrak{A}$ (that is the maximal ideal consisting entirely of nilpotent elements). Furthermore, $\mathrm{Rad}(\mathfrak{A})$ is nilpotent in the sense that there is an integer t with the property that any product $x_1 \cdots x_t$ of t elements from $\mathrm{Rad}(\mathfrak{A})$ is zero; hence $\mathrm{Rad}(\mathfrak{A})$ is also the unique maximal nilpotent ideal of $\mathfrak{A}$. With this definition of radical, the quotient algebra $\mathfrak{A}/\mathrm{Rad}(\mathfrak{A})$ is always semisimple, and $\mathfrak{A}$ is semisimple if and only if $\mathrm{Rad}(\mathfrak{A}) = 0$.

A result of Albert characterizes the radical $\mathrm{Rad}(\mathfrak{A})$ as the kernel of the symmetric bilinear form $\tau : \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}$ defined by $\tau(x, y) = \mathrm{Tr}(L_{xy})$ for x, $y \in \mathfrak{A}$.

The Wedderburn Principal Theorem for Jordan algebras, proved by Albert and Penico [17], states that any complex Jordan algebra $\mathfrak{A}$ can be written as a vector space direct sum $\mathfrak{A} = \mathfrak{S} + \mathfrak{N}$ for some maximal semisimple subalgebra $\mathfrak{S}$ of $\mathfrak{A}$ isomorphic to $\mathfrak{A}/\mathfrak{N}$, where $\mathfrak{N} = \mathrm{Rad}(\mathfrak{A})$.

2.2. *A review of GIT*

Let $\pi : G \to GL(V)$ be a rational representation of a connected complex reductive group G on a finite-dimensional complex vector space V. Fix a maximal compact subgroup K of G (necessarily connected and unique up to inner automorphism of G), and a K-invariant Hermitian inner product $\langle , \rangle$ on V. Then the Lie algebra $\mathfrak{k}$ of K is a real form of the Lie algebra $\mathfrak{g}$ of G, namely, $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$. We also fix an Ad_K-invariant Hermitian inner product on $\mathfrak{g}$, denoted $(,)$, which is positive definite (respectively, negative definite) on $i\mathfrak{k}$ (respectively, $\mathfrak{k}$).

The *moment map* is the map $m : V \setminus \{0\} \to \mathfrak{g}$ defined by

$$(m(v), X) = \frac{1}{2||v||^2} \left.\frac{d}{dt}\right|_{t=0} ||e^{tX} \cdot v||^2 = \frac{\langle X \cdot v, v\rangle}{\langle v, v\rangle}, \tag{3}$$

[a] After finishing this paper, we became aware of the recent preprint [20] concerning a moment map for the variety of associative algebras.

where we write $g \cdot v := \pi(g)v$ and $X \cdot v := d\pi(X)v$, for all $g \in G$, $X \in \mathfrak{g}$, $v \in V \setminus \{0\}$. We have that $m(v)$ controls the norm of vectors in a neighborhood of v in the orbit $G \cdot v$ orthogonal to v, and $-m(v) \cdot v$ is the direction of fastest decrease in the norm. Note that $m(v) \in i\mathfrak{k}$ and m is K-invariant by K-invariance of $\langle , \rangle$. A *minimal* vector is vector of minimal length in its orbit. The Kempf–Ness theorem says that minimal vectors correspond to the zeros of the moment map, and an orbit is closed if and only it contains a minimal vector; further the closure of any G-orbit in V contains exactly one K-orbit of minimal vectors [8]. The square norm of m yields the energy functional

$$E : V \setminus \{0\} \to \mathbb{R}, \quad E(v) = ||m(v)||^2.$$

Of course E is K-invariant and scale-invariant. It follows from the above that a G-orbit in V is closed if and only if it meets $E^{-1}(0)$.

On the other hand, the union of all the orbits containing the origin in its closure comprise the so-called *null cone* N of V. The Kirwan–Ness theorem says that the non-minimal critical points of E, i.e. those critical points v of E with $E(v) > 0$, all occur in the null cone N and determine a stratification of $N \setminus \{0\}$ [9, 16]. So there is a set of distinguished orbits in N, namely those containing a critical point of E, which play a role similar to the closed orbits in $V \setminus N$.

Since $(dm_v(\xi), X) = 2\Re\langle X \cdot v, \xi\rangle / ||v||^2$ for $X \in i\mathfrak{k}$, $\xi \in T_vV = V$, $\xi \perp v$, we easily obtain for the differential of E at a point $v \in V \setminus \{0\}$,

$$dE_v(\xi) = 2\Re(dm_v(\xi), m(v)) = \frac{4\Re\langle m(v) \cdot v, \xi\rangle}{||v||^2}.$$

Now E is constant in the complex radial direction, and the component of $m(v) \cdot v$ in the direction of v is $\frac{\langle m(v) \cdot v, v\rangle}{||v||^2} v = ||m(v)||^2 v$, so the gradient of E at v is given by

$$\nabla E(v) = \frac{4}{||v||^2}(m(v) \cdot v - ||m(v)||^2 v). \tag{4}$$

Therefore $v \in V \setminus \{0\}$ is a critical point of E if and only if

$$m(v) \cdot v \in \mathbb{R}v. \tag{5}$$

Because of this self-similarity characteristic, a critical point of E is called a *soliton* (following [11]). We will also say a G-orbit containing a soliton is *distinguished*. Of course every minimal vector is a soliton. For the convenience of the reader, we next collect the results of Kirwan–Ness theory related to solitons that we shall need.

Theorem 2.1 (Kirwan–Ness). *With the above notation, we have*

(a) *The subset of solitons in a given G-orbit is either empty or consists of precisely one K-orbit, up to scaling.*
(b) *Every soliton v is a minimum of E on $G \cdot v$. Thus solitons are the vectors closest to being minimal in their G-orbits.*
(c) *The solitons which are not minimal vectors all occur in the null cone N of V.*

(d) *The flow of* $-\nabla E$ *starting at* v *stays in* $G\cdot v$ *and converges to a soliton* $w \in \overline{G\cdot v}$ *as* $t \to \pm\infty$. *There is precisely one* K*-orbit up to scaling of solitons* $z \in \overline{G\cdot v}$ *such that* $m(z) \in Ad_K m(w)$, *which consists of the limit-set of* $G\cdot v$.
(e) *The critical set of* E *is a finite disjoint union of closed subsets* $\{C_\beta\}_{\beta\in\mathcal{B}}$ *indexed by a finite set* $\mathcal{B}$ *of adjoint* K*-orbits in* $\mathfrak{k}$ *(or points in a positive Weyl chamber). The corresponding stable manifolds* $\{S_\beta\}_{\beta\in\mathcal{B}}$ *form a finite* G*-invariant stratification of* $N\backslash\{0\}$ *by locally closed irreducible non-singular subvarieties.*

3. The Moment Map for Commutative Algebras

In this section we specialize to the case of n-dimensional commutative algebras and, in particular, Jordan algebras. Let $V_n = S^2(\mathbb{C}^{n*}) \otimes \mathbb{C}^n$ be the space of symmetric bilinear maps $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$, which we identify with (non-necessarily associative) commutative algebras of dimension n. There is a natural action of $G = GL(n,\mathbb{C})$ on V_n given by (1). The action of the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$ of G on V_n, obtained from differentiation of (1), is given by

$$A\cdot\mu(x,y) = A(\mu(x,y)) - \mu(Ax,y) - \mu(x,Ay) \tag{6}$$

for all $A \in \mathfrak{g}$, $\mu \in V_n$ and x, $y \in \mathbb{C}^n$. Note that the isotropy algebra of μ, namely, the subalgebra of $\mathfrak{g}$ consisting of elements $A \in \mathfrak{g}$ satisfying $A\cdot\mu = 0$, is isomorphic to the Lie algebra $\mathrm{Der}(\mu)$ of derivations of μ.

Consider the canonical Hermitian product $\langle,\rangle$ of $\mathbb{C}^n$. The unitary group $K = U(n)$ is a maximal compact subgroup of G and its Lie algebra $\mathfrak{k}$ is a real form of $\mathfrak{g}$. The Hermitian product canonically extends to a Hermitian product on V_n, denoted by the same symbol, namely,

$$\langle\mu,\nu\rangle = \sum_{ijk}\langle\mu(x_i,x_j),x_k\rangle\overline{\langle\nu(x_i,x_j),x_k\rangle},$$

where $x_1,\ldots,x_n$ is any fixed orthonormal basis of $\mathbb{C}^n$. The elements of K act on V_n by unitary transformations, and those of $\mathfrak{k}$ (respectively, $i\mathfrak{k}$) act on V_n by skew-Hermitian (respectively, Hermitian) endomorphisms. We also consider the Ad_K-invariant Hermitian product on $\mathfrak{g}$ given by $(A,B) = \mathrm{Tr}(AB^*)$, where A, $B \in \mathfrak{g}$.

The moment map $m : V_n \setminus \{0\} \to i\mathfrak{k}$ is defined as in (3) and has the form $m(\mu) = \frac{1}{||\mu||^2}M_\mu$, where M_μ is the moment matrix of μ. We will obtain an explicit formula for M_μ in terms of the algebra structure of μ and the Hermitian product in a moment. It turns out to be the same formula as that in [10] for skew-symmetric algebras.

Recall that $(M_\mu, A) = \langle A\cdot\mu,\mu\rangle$ for all $A \in i\mathfrak{k}$.

Lemma 3.1. *For all* $\mu \in V_n$ *and* $D \in \mathrm{Der}(\mu)$, *we have*

(a) $\mathrm{Tr}(M_\mu) = -||\mu||^2$;
(b) $(M_\mu, D) = 0$.

Proof. (a) Note that $\mathrm{Tr}(M_\mu) = (M_\mu, I) = \langle I \cdot \mu, \mu \rangle = -||\mu||^2$, as $I \cdot \mu = -\mu$.

(b) We have $(M_\mu, D) = \langle D \cdot \mu, \mu \rangle = 0$, since $D \cdot \mu = 0$ by (6) and the derivation property. □

For $\mu \in V_n$, let $L_x^\mu : \mathbb{C}^n \to \mathbb{C}^n$ denote the left multiplication by $x \in \mathbb{C}^n$ in the algebra μ, that is $L_x^\mu y = \mu(x, y)$ for all $y \in \mathbb{C}^n$. We also use a superscript $()^*$ to denote the adjoint of a map $\mathbb{C}^n \to \mathbb{C}^n$ with respect to the fixed Hermitian product.

Proposition 3.2. *For all $\mu \in V_n \setminus \{0\}$ and x, $y \in \mathbb{C}^n$ we have*

$$\langle M_\mu x, y \rangle = -2 \sum_{ij} \langle L_x x_i, x_j \rangle \overline{\langle L_y x_i, x_j \rangle} + \sum_{ij} \overline{\langle L_{x_i} x_j, x \rangle} \langle L_{x_i} x_j, y \rangle,$$

where $(x_1, \ldots, x_n)$ is an orthonormal basis of $\mathbb{C}^n$. It follows that

$$M_\mu = -2 \sum_{i=1}^{n} L_{x_i}^{\mu *} L_{x_i}^{\mu} + \sum_{i=1}^{n} L_{x_i}^{\mu} L_{x_i}^{\mu *}.$$

Proof. Let $\mu \in V_n$ and consider $A \in i\mathfrak{k}$ such that $Ax_1 = x_1$ and $Ax_i = 0$ for $i = 2, \ldots, n$. We have

$$\begin{aligned}
(M_\mu, A) &= \langle A \cdot \mu, \mu \rangle \\
&= \sum_{ijs} \langle (A\mu)(x_s, x_i), x_j \rangle \overline{\langle \mu(x_s, x_i), x_j \rangle} \\
&= \sum_{ijs} \langle A(\mu(x_s, x_i)) - \mu(Ax_s, x_i) - \mu(x_s, Ax_i), x_j \rangle \overline{\langle \mu(x_s, x_i), x_j \rangle} \\
&= \sum_{ijs} (\langle \mu(x_s, x_i), Ax_j \rangle - \langle \mu(Ax_s, x_i), x_j \rangle - \langle \mu(x_s, Ax_i), x_j \rangle) \overline{\langle \mu(x_s, x_i), x_j \rangle} \\
&= \sum_{si} \langle \mu(x_s, x_i), x_1 \rangle \overline{\langle \mu(x_s, x_i), x_1 \rangle} - \sum_{ij} \langle \mu(x_1, x_i), x_j \rangle \overline{\langle \mu(x_1, x_i), x_j \rangle} \\
&\quad - \sum_{sj} \langle \mu(x_s, x_1), x_j \rangle \overline{\langle \mu(x_s, x_1), x_j \rangle} \\
&= \sum_{si} |\langle x_s, L_{x_i}^{\mu *} x_1 \rangle|^2 - 2 \sum_{ij} |\langle L_{x_i}^{\mu} x_1, x_j \rangle|^2
\end{aligned}$$

and, on the other hand,

$$(M_\mu, A) = \sum_i \langle M_\mu x_i, Ax_i \rangle = \langle M_\mu x_1, x_1 \rangle.$$

Since $x_1 \in \mathbb{C}^n$ is arbitrary, a simple polarization argument proves the first identity in the statement, and the second one is a reformulation. □

3.1. *The structure of solitons*

Next, we prove Proposition 1.1. Let $\mathfrak{A} = (\mathbb{C}^n, \mu)$ be a commutative algebra, where $\mu \in V_n$. We have seen in (5) that μ is a soliton if and only if $D_\mu := M_\mu - c_\mu I$ kills μ, for $c_\mu = -||M_\mu||^2/||\mu||^2 \in \mathbb{R}$ (recall that $I \cdot \mu = -\mu$); owing to (6), this is equivalent to D_μ being a derivation of μ. Since $M_\mu^* = M_\mu$ and $c_\mu \in \mathbb{R}$, also $D_\mu^* = D_\mu$. For part (a), it remains only to show that if μ is a soliton then some positive multiple of D_μ has rational eigenvalues.

Without loss of generality, we may assume $D_\mu \neq 0$. Let $x_1, \ldots, x_n$ be an orthonormal basis of eigenvectors of D_μ; then it is also a basis of eigenvectors of M_μ. Consider the 'structure constants' μ_{ij}^k given by $\mu(x_i, x_j) = \sum_k \mu_{ij}^k x_k$ for all i, j. Let $\mathfrak{h}$ be the subspace of $\mathfrak{g}$ consisting of endomorphisms of $\mathbb{C}^n$ that are diagonal on that basis, let $\alpha_{ij}^k \in \mathfrak{h}$ have matrix $-E_{ii} - E_{jj} + E_{kk}$ in that basis, where E_{ab} has a 1 in the (a, b)-entry and 0 elsewhere, and consider the subspace F of $\mathfrak{h}$ spanned by all α_{ij}^k with $\mu_{ij}^k \neq 0$.

Note that $A \in \mathfrak{h}$ is a derivation of μ if and only if $\mu(x_i, x_j)$ lies in the $(a_i + a_j)$-eigenspace of A, for all i, j; this is equivalent to $a_k = a_i + a_j$ whenever $\mu_{ij}^k \neq 0$. Therefore $\mathrm{Der}(\mu) \cap \mathfrak{h} = F^\perp \cap \mathfrak{h}$. Owing to $M_\mu \in \mathfrak{h}$ and Lemma 3.1(b), we have $M_\mu \in F$. Now let $P : \mathfrak{h} \to F$ be orthogonal projection. Applying P throughout the equation $M_\mu = c_\mu I + D_\mu$ yields

$$M_\mu = c_\mu P(I).$$

Therefore

$$-\frac{1}{c_\mu} D_\mu = I - \frac{1}{c_\mu} M_\mu = I - P(I).$$

Since F is spanned by matrices with integer coefficients, $P(I)$ has rational coefficients, and this finishes the proof of part (a).

We next address part (b) of Proposition 1.1. Due to [6, p. 869], the radical $\mathfrak{N}$ of $\mathfrak{A}$ is a characteristic ideal of $\mathfrak{A}$, thus D_μ-invariant. Next, using results of Mostow and Auslander–Brezin, we show there is a maximal semisimple subalgebra $\mathfrak{S}$ of $\mathfrak{A}$ which is D_μ-invariant. First we note that D_μ is semisimple, since it is Hermitian. Therefore the one-dimensional Lie algebra $\mathfrak{d}$ of derivations of $\mathfrak{A}$ generated by D_μ is completely reducible. Its algebraic hull $\mathfrak{d}^\#$ consists of derivations [1, (1.2)], and it is also completely reducible [1, (1.5)]. Now the associated connected algebraic subgroup of $GL(\mathfrak{A})$ is a completely reducible group of automorphisms of $\mathfrak{A}$ [1, (1.6)]. But a completely reducible group of automorphisms of a Jordan algebra preserves some maximal semisimple subalgebra $\mathfrak{S}$ [15, p. 215], and hence the same is true of D_μ.

Finally, we prove that $D_\mu|\mathfrak{S} = 0$. Since D_μ is a Hermitian endomorphism of $\mathbb{C}^n$, it is semisimple and has real eigenvalues. Let $x \in \mathbb{C}^n$ be an eigenvector of D_μ with associated eigenvalue $d \in \mathbb{R}$. The derivation property implies that x^k is an eigenvector of D_μ with eigenvalue kd for every integer $k \geq 2$. By finite-dimensionality, if $d \neq 0$ then x is nilpotent; it thus follows that $x \in \mathfrak{N}$. This shows that the only eigenvalues of D_μ that can occur for eigenvectors in $\mathfrak{S}$ are zero, and finishes the proof of the proposition.

4. Semisimple Jordan Algebras

In this section we show that the semisimple Jordan algebras realize the minimal value of the energy E_n for all n, up to a change of basis (Proposition 4.5); Theorem 1.2 is a consequence. A good reference for Jordan-related results in this section is [2, Kap. VIII].

4.1. *Peirce decomposition*

Let $\mathfrak{A}$ be a complex semisimple Jordan algebra. Then there exists a *complete orthogonal system of idempotents*, or *Jordan frame*, that is a maximal set $\{e_1, \dots, e_r\}$ of primitive idempotents of $\mathfrak{A}$ such that $e_1 + \cdots + e_r$ is the identity element of $\mathfrak{A}$, and $e_i e_j = 0$ if $i \neq j$. A Jordan frame is unique up to an automorphism of $\mathfrak{A}$. In the case of a simple Jordan algebra $\mathfrak{A}$, the number $r \geq 2$ of elements in a Jordan frame is called the *degree* of $\mathfrak{A}$.

A Jordan frame as above gives rise to a canonical decomposition of $\mathfrak{A}$ into a vector space direct sum, as follows. The eigenvalues of an idempotent e_k acting by multiplication on the algebra can only be 0, $\frac{1}{2}$ and 1, and we denote the corresponding eigenspaces by $\mathfrak{A}_0(e_k)$, $\mathfrak{A}_{\frac{1}{2}}(e_k)$ and $\mathfrak{A}_1(e_k)$, respectively. The *Peirce decomposition* of $\mathfrak{A}$ is

$$\mathfrak{A} = \oplus_{i \leq j} \mathfrak{A}_{ij},$$

where

$$\mathfrak{A}_{ii} = \mathfrak{A}_1(e_i)$$

for all i and

$$\mathfrak{A}_{ij} = \mathfrak{A}_{\frac{1}{2}}(e_i) \cap \mathfrak{A}_{\frac{1}{2}}(e_j)$$

for $i \neq j$.

4.2. *Killing form*

Next, assume in addition to the above that $\mathfrak{A}$ is simple algebra of dimension n. Fix a Jordan frame and the associated Peirce decomposition. It is known that if $r \geq 3$, then $\dim \mathfrak{A}_{ij} = d \in \{1, 2, 4, 8\}$ for all i, j. Also, in case $r = 2$ we put $d = n - 2$. Now we have

$$n = r + \frac{r(r-1)}{2} d. \tag{7}$$

An element $a \in \mathfrak{A}$ is called *regular* if $\mathbb{C}[a] := \operatorname{span}_{\mathbb{C}}\{1, a, \dots, a^{r-1}\}$ is r-dimensional (equivalently, $\dim \mathbb{C}[a]$ is maximal among $\dim \mathbb{C}[b]$ for all $b \in \mathfrak{A}$).

The *reduced trace* of a regular element $a \in \mathfrak{A}$ is

$$\mathrm{tr}(x) = \mathrm{Tr}(L|_{\mathbb{C}[x]}),$$

and the function 'reduced trace' can be uniquely extended to a linear map $\mathrm{tr} : \mathfrak{A} \to \mathbb{C}$ (compare [3, Chap. 2, Sec. 2]). Since $\mathfrak{A}$ is simple, we have

$$\mathrm{tr}(ab) = \frac{r}{n}\tau(a, b). \tag{8}$$

The *Killing form* of $\mathfrak{A}$ is the symmetric bilinear form $K : \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}$ defined by

$$K(a, b) = \mathrm{Tr}(L_a L_b)$$

for a, $b \in \mathfrak{A}$. It follows from the trace formula that [2, Satz 9.4, Kap. VIII]

$$K(a, b) = \left(1 + (r-2)\frac{d}{4}\right)\mathrm{tr}(ab) + \frac{d}{4}\mathrm{tr}(a)\mathrm{tr}(b). \tag{9}$$

4.3. *Moment map*

Suppose $\mathfrak{A}$ is a complex semisimple Jordan algebra given by $\mu \in \mathcal{J}_n$. Then the trace form τ is a non-degenerate symmetric bilinear form. Also, there exists an Euclidean real form $\mathfrak{A}_0$ of $\mathfrak{A}$, that is $\mathfrak{A}_0$ is a formally real Jordan algebra such that $\mathfrak{A}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{A}$ ([2, Satz 5.6, Kap. IX] or [3, Theorem 8.5.2]). The restriction of τ to $\mathfrak{A}_0$ is a positive-definite Euclidean inner product ([2, Satz 3.4, Kap. IX] or [3, Proposition 3.1.5]); we extend it to a Hermitian inner product on $\mathfrak{A}$, denoted by $\langle , \rangle$.

Lemma 4.1. *Let $a \in \mathfrak{A}$. Then the adjoint L_a^* of $L_a : \mathfrak{A} \to \mathfrak{A}$ with respect to $\langle , \rangle$ is $L_{\bar{a}}$, where $\bar{a} \in \mathfrak{A}$ is the complex conjugate of a over $\mathfrak{A}_0$. In particular, L_a is a Hermitian operator for $a \in \mathfrak{A}_0$.*

Proof. We compute for b, $c \in \mathfrak{A}$ that

$$\langle L_a b, c\rangle = \tau(ab, \bar{c}) = \tau(b, a\bar{c}) = \tau(b, \overline{L_{\bar{a}}c}) = \langle b, L_{\bar{a}}c\rangle,$$

since L_a is self-adjoint for τ, which proves the statement. □

The next result shows that the moment matrix of a complex semisimple Jordan algebra $\mathfrak{A}$ is essentially given by its Killing form.

Proposition 4.2. *Let M_μ be the Hermitian matrix which is the moment matrix of $\mathfrak{A} = (\mathbb{C}^n, \mu)$, and let K_μ be the Killing form of $\mathfrak{A}$. Then*

$$\langle M_\mu a, b\rangle = -K_\mu(a, \bar{b})$$

for all a, $b \in \mathfrak{A}$.

Proof. Let $\{x_1,\ldots,x_n\}$ be an orthonormal basis of $\mathbb{C}^n$ with respect to $\langle,\rangle$ which is contained in $\mathfrak{A}_0$. Using Proposition 3.2 and Lemma 4.1, we can write

$$\begin{aligned}\langle M_\mu a,b\rangle &= -\sum_i \langle (L^\mu_{x_i})^2 a,b\rangle\\ &= -\sum_i \langle L^\mu_{x_i} a, L^\mu_{x_i} b\rangle\\ &= -\sum_i \langle L^\mu_a x_i, L^\mu_b x_i\rangle\\ &= -\sum_i \langle L^\mu_{\bar b} L^\mu_a x_i, x_i\rangle\\ &= -\mathrm{Tr}(L^\mu_{\bar b} L^\mu_a)\\ &= -K_\mu(a,\bar b),\end{aligned}$$

as desired. □

Lemma 4.3. *Let $\mathfrak{A}$ be a complex simple Jordan algebra represented by $\mu\in\mathcal{J}_n$. Denote by r and d, respectively, the degree of $\mathfrak{A}$ and the dimension of the off-diagonal Peirce components $\mathfrak{A}_{ij}$ $(i\neq j)$, where we fix a Jordan frame $(e_1,\ldots,e_r)$ contained in an Euclidean real form. Fix also a Hermitian product on $\mathbb{C}^n$ by putting* (*see* (8))

$$\langle x,y\rangle := \frac{r}{n}\mathrm{Tr}(L^\mu_{x\bar y}) = \mathrm{tr}^\mu(x\bar y)$$

for $x,y\in\mathbb{C}^n$. Then the Jordan frame is an orthonormal set and can be completed to an orthonormal basis of $\mathbb{C}^n$ by adding elements $e^k_{ij}\in\mathfrak{A}_{ij}$, $k=1,\ldots,d$, for all $i\neq j$. In this basis, the moment matrix of $\mathfrak{A}=(\mathbb{C}^n,\mu)$ is given by

$$M_\mu = -\alpha I - \frac{d}{4}N,$$

where $\alpha = 1+(r-2)\frac{d}{4}$ and

$$N = \left(\begin{array}{ccc|c} 1 & \cdots & 1 & \\ \vdots & & \vdots & \mathbf{0} \\ 1 & \cdots & 1 & \\ \hline & \mathbf{0} & & \mathbf{0} \end{array}\right)_{\substack{\\ r \qquad n-r}}.$$

Proof. Consider the Peirce decomposition $\mathfrak{A}=\oplus_{i\le j}\mathfrak{A}_{ij}$ with respect to a Jordan frame $(e_1,\ldots,e_r)$. Then

$$\mathfrak{A}_0(e_k)=\oplus_{i,j\neq k}\mathfrak{A}_{ij},\quad \mathfrak{A}_{\frac12}(e_k)=\oplus_{i\neq k}\mathfrak{A}_{ik}\quad\text{and}\quad \mathfrak{A}_1(e_k)=\mathfrak{A}_{kk},$$

where $\dim \mathfrak{A}_{ij} = d$ for all $i \neq j$, and $\mathfrak{A}_{kk} = \mathbb{C} \cdot e_k$ for all k. Using $e_k^2 = e_k$, (7) and (8), we see immediately that

$$\langle e_k, e_k \rangle = \mathrm{tr}^\mu(e_k) = \frac{r}{n}\mathrm{Tr}(L^\mu_{e_k}) = \frac{r}{n}\left(\frac{1}{2}(r-1)d + 1\right) = 1$$

for all k. Moreover, using

$$\mathfrak{A}_{ij} \cdot (\mathfrak{A}_{ii} + \mathfrak{A}_{jj}) \subset \mathfrak{A}_{ij}, \quad \mathfrak{A}_{ij} \cdot \mathfrak{A}_{ij} \subset \mathfrak{A}_{ii} + \mathfrak{A}_{jj}$$

$$\mathfrak{A}_{ij} \cdot \mathfrak{A}_{jk} \subset \mathfrak{A}_{ik}, \quad \mathfrak{A}_{ij} \cdot \mathfrak{A}_{ik} \subset \mathfrak{A}_{jk},$$

for mutually different i, j, k, we see that, for $e_{ij}^k \in \mathfrak{A}_{ij}$ $(i \neq j)$,

$$\mathrm{tr}^\mu(e_{ij}^k) = \frac{r}{n}\mathrm{Tr}(L^\mu_{e_{ij}^k}) = 0.$$

Finally, formula (9) and Proposition 4.2 yield

$$\langle M_\mu x, y \rangle = -\alpha \langle x, y \rangle - \frac{d}{4}\mathrm{tr}^\mu(x)\mathrm{tr}^\mu(\bar{y}).$$

Therefore

$$\langle M_\mu e_i, e_j \rangle = -\alpha \delta_{ij} - \frac{d}{4},$$

$$\langle M_\mu e_\ell, e_{ij}^k \rangle = 0$$

and

$$\langle M_\mu e_{ij}^k, e_{rs}^t \rangle = -\alpha \delta_{ir}\delta_{js}\delta_{kt},$$

as wished. □

Remark 4.4. Using the calculation in the proof of Lemma 4.3, it is easy to see that

$$E_n(\mu) = \frac{\mathrm{Tr} M_\mu^2}{(\mathrm{Tr} M_\mu)^2} = \frac{n\alpha^2 + \frac{r^2 d^2}{16} + \frac{r\alpha d}{2}}{(n\alpha + \frac{rd}{4})^2} = \frac{n\alpha^2 + \frac{r\alpha d}{2} + \frac{r^2 d^2}{16}}{n\alpha^2 + \frac{r\alpha d}{2} + \frac{r^2 d^2}{16n}} \cdot \frac{1}{n} > \frac{1}{n},$$

where $n \geq 2$. In Proposition 4.5, we will see that the Peirce basis can be slightly changed to lower the value of E_n.

Theorem 1.2 is derived from the following result.

Proposition 4.5. *For all n, the minimum value of $E_n : \mathcal{J}_n \to \mathbb{R}$ is $1/n$, and this value is attained only at semisimple Jordan algebras. Conversely, if μ is semisimple then $E_n(g \cdot \mu) = 1/n$ for some $g \in G$.*

Proof. Note that $E_n(\mu) = \frac{1}{n}$ if and only if $M_\mu = c_\mu I$. Indeed if $c_1, \ldots, c_n$ are the (real) eigenvalues of M_μ, and we want to minimize the value of $c_1^2 + \cdots + c_n^2$ subject to the condition that $c_1 + \cdots + c_n = 1$, we immediately obtain that $c_1 = \cdots = c_n = 1/n$.

We first prove that the moment matrix of a complex semisimple Jordan algebras is scalar, up to a change of basis. It is enough to consider a complex simple Jordan algebra μ. We start with the basis $\{e_i\} \cup \{e_{ij}^k\}$ of the Euclidean real form as in Lemma 4.3 and the associated moment matrix $M_\mu = -\alpha I - \frac{d}{4}N$. Let $\{f_1, f_2, \ldots, f_r\}$ be a (positively oriented) real orthonormal basis of $\mathrm{span}_{\mathbb{R}}(e_1 \ldots, e_r)$ such that its first element is parallel to the identity element $1 = e_1 + \cdots + e_r$. Then $\{f_i\} \cup \{e_{ij}^k\}$ is an orthonormal basis of $\mathbb{R}^n$. Let $h \in SO(n)$, $h = \left(\begin{smallmatrix} h' & 0 \\ 0 & I \end{smallmatrix}\right)$, where $h' \in SO(r)$, be such that h maps $f_i \mapsto e_i$, $e_{ij}^k \mapsto e_{ij}^k$, and put $\nu = h \cdot \mu$. Since the moment map is $U(n)$-equivariant, we have

$$M_\nu = hM_\mu h^{-1} = -\alpha I - \frac{d}{4}\left(\begin{array}{cccc|c} r & & & & \\ & 0 & & & \mathbf{0} \\ & & \ddots & & \\ & & & 0 & \\ \hline & & \mathbf{0} & & \mathbf{0} \end{array}\right). \tag{10}$$

We next show that it is possible to rescale the basis (in fact, only f_1) so that the moment matrix becomes scalar.

We need to make some remarks about the ν_{ij}^k. Denote by $\{f_1, \ldots, f_n\}$ the orthonormal basis of $\mathbb{R}^n$ above constructed so that f_1 is parallel to 1, $\{f_1, \ldots, f_r\}$ spans $\mathfrak{A}_{11} + \ldots + \mathfrak{A}_{rr}$ and $\{f_{r+1}, \ldots, f_n\}$ spans $\oplus_{ij}\mathfrak{A}_{ij}$. Let $x \in \mathbb{R}^n$ be a unit vector. Then $(x^2 = \mu(x, x))$

$$x^2 = \frac{\langle x^2, 1\rangle}{||1||^2} 1 + y, \quad y \perp x.$$

Note that $\langle x^2, 1\rangle = ||x||^2 = 1$ and $||1||^2 = \frac{r}{n}\mathrm{Tr}(I) = r$. It follows that

$$x^2 = \frac{1}{\sqrt{r}} f_1 + y, \quad y \perp f_1. \tag{11}$$

We also have $\langle \mu(f_i, f_j), 1\rangle = \langle f_i, f_j\rangle = 0$ if $i \neq j$. Since $\nu(e_i, e_j) = h(\mu(f_i, f_j))$, it follows that

$$\nu = \frac{1}{\sqrt{r}} \sum_{i=1}^{n} e_1' e_i' \otimes e_i + \frac{1}{\sqrt{r}} \sum_{i=2}^{n} e_i'^2 \otimes e_1 + \text{terms not involving } e_1,\, e_1'; \tag{12}$$

here we denote by $e_1', \ldots, e_n'$ the dual basis of $e_1, \ldots, e_n$.

Next we consider the following one-parameter deformation of ν, and show that the moment matrix is scalar for some value of the parameter. Set $\nu_t = g_t^{-1} \cdot \nu$, where

$$g_t = \exp(-tE_{11}) = \begin{pmatrix} e^{-t} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

We can write $\nu = \sum_{ijk} \nu_{ij}^k v_k^{ij}$, where $v_k^{ij} = e_i' e_j' \otimes e_k$ and the sum runs through $i \leq j$, and then

$$||v_k^{ij}||^2 = 2(i \neq j) \quad \text{and} \quad ||v_k^{ii}||^2 = 1.$$

Also, v_k^{ij} is a weight vector of $S^2(\mathbb{C}^{n*}) \otimes \mathbb{C}^n$ of weight $\alpha_{ij}^k = -\theta_i - \theta_j + \theta_k$, with respect to the Cartan subalgebra of $\mathfrak{g}$ consisting of diagonal matrices, where we denote by θ_i the projection onto the ith-diagonal entry. Now

$$\nu_t = \sum_{ijk} e^{t\alpha_{ij}^k(E_{11})} \nu_{ij}^k v_k^{ij}$$

and

$$E_{\ell\ell} \cdot \nu_t = \sum_{ijk} e^{t\alpha_{ij}^k(E_{11})} \alpha_{ij}^k(E_{\ell\ell}) \nu_{ij}^k v_k^{ij}.$$

It follows that, for $\ell \geq 2$,

$$\begin{aligned} (M_{\nu_t}, E_{\ell\ell}) &= \langle E_{\ell\ell} \cdot \nu_t, \nu_t \rangle \\ &= \sum_{ijk} e^{2t\alpha_{ij}^k(E_{11})} \alpha_{ij}^k(E_{\ell\ell}) |\nu_{ij}^k|^2 ||v_k^{ij}||^2 \\ &= 0 + \frac{1}{r} e^{2t}(-2) \cdot 1 + \underbrace{\sum_{ijk \neq 1} \alpha_{ij}^k(E_{\ell\ell}) |\nu_{ij}^k|^2 ||v_k^{ij}||^2}_{=: c_\ell} \\ &= -\frac{2}{r} e^{2t} + c_\ell, \end{aligned} \tag{13}$$

where we have used the form (12). Since $(M_{\nu_0}, E_{\ell\ell}) = -\frac{2}{r} + c_\ell$ is independent of $\ell \geq 2$, owing to (10) we deduce that $c_2 = \cdots = c_n = c$ for some $c \in \mathbb{R}$.

We next describe the first entry of M_{ν_t}, again by using (12):

$$\begin{aligned} (M_{\nu_t}, E_{11}) &= \sum_{ijk} e^{2t\alpha_{ij}^k(E_{11})} \alpha_{ij}^k(E_{11}) |\nu_{ij}^k|^2 ||v_k^{ij}||^2 \\ &= \frac{1}{r} e^{-2t}(-1)(1 + (n-1)2) + \frac{1}{r} e^{2t} \cdot 1 \cdot ((n-1) \cdot 1) \\ &= -\frac{2n-1}{r} e^{-2t} + \frac{n-1}{r} e^{2t}. \end{aligned} \tag{14}$$

Use (13) and (14) to investigate ($\ell \geq 2$)

$$\gamma(t) = (M_{\nu_t}, E_{11}) - (M_{\nu_t}, E_{\ell\ell}) = \frac{n+1}{r} e^{2t} - \frac{2n-1}{r} e^{-2t} - c.$$

Since $\lim_{t \to -\infty} \gamma(t) = -\infty$ and $\lim_{t \to +\infty} \gamma(t) = +\infty$, there is $t_0 \in \mathbb{R}$ such that $\gamma(t_0) = 0$. We have shown that all diagonal entries of $M_{\nu_{t_0}}$ are equal. The last step is to show that the off-diagonal entries of $M_{\nu_{t_0}}$ (indeed of M_{ν_t} for all t) vanish.

Since M_ν is a diagonal matrix, it suffices to show that ($1 \le p < q \le n$):

$$\frac{d}{dt}(M_{\nu_t}, E_{pq} + E_{qp}) = 2\Re\langle (E_{pq} + E_{qp}) \cdot \nu_t, E_{11} \cdot \nu_t\rangle \tag{15}$$

vanishes.

Note that

$$E_{11} \cdot \nu_t = -\frac{e^{-t}}{\sqrt{r}} \sum_{i=1}^{n} v_i^{1i} + \frac{e^t}{\sqrt{r}} \sum_{i=2}^{n} v_1^{ii},$$

so we need only to know the v_i^{1i}- and v_1^{ii}-components of $(E_{pq} + E_{qp}) \cdot \nu_t$.

We compute

$$\langle ((E_{pq} + E_{qp}) \cdot \nu_t)(e_1, e_i), e_i\rangle = -\delta_{p1}\langle \nu_t(e_q, e_i), e_i\rangle,$$

$$\langle (E_{11} \cdot \nu_t)(e_1, e_i), e_i\rangle = -\langle \nu_t(e_1, e_i), e_i\rangle,$$

and, for $i = 2, \dots, n$,

$$\langle ((E_{pq} + E_{qp}) \cdot \nu_t)(e_i, e_i), e_1\rangle = \delta_{p1}\langle \nu_t(e_i, e_i), e_q\rangle,$$

$$\langle (E_{11} \cdot \nu_t)(e_i, e_i), e_1\rangle = \langle \nu_t(e_i, e_i), e_1\rangle.$$

Plugging these formulae into (15) already yields zero, unless $p = 1$, for which

$$\frac{1}{2}\frac{d}{dt}(M_{\nu_t}, E_{1q} + E_{q1}) = \Re \sum_{i=2}^{n} \langle \nu_t(e_q, e_i), e_i\rangle \overline{\langle \nu_t(e_1, e_i), e_i\rangle} + \langle \nu_t(e_i, e_i), e_q\rangle \overline{\langle \nu_t(e_i, e_i), e_1\rangle}. \tag{16}$$

Note that, using i, $q \ge 2$,

$$\langle \nu_t(e_i, e_i), e_q\rangle = \langle \nu(e_i, e_i), e_q\rangle,$$

$$\langle \nu_t(e_i, e_i), e_1\rangle = e^t\langle \nu(e_i, e_i), e_1\rangle,$$

$$\langle \nu_t(e_q, e_i), e_i\rangle = \langle \nu(e_i, e_i), e_q\rangle,$$

$$\langle \nu_t(e_1, e_i), e_i\rangle = e^{-t}\langle \nu(e_i, e_i), e_1\rangle,$$

so

$$\begin{aligned}\frac{d}{dt}(M_{\nu_t}, E_{1q} + E_{q1}) &= \sum_{i=2}^{n} e^{-t}\langle \nu(e_i, e_i), e_q\rangle\langle \nu(e_i, e_i), e_1\rangle \\ &\quad + e^t\langle \nu(e_i, e_i), e_q\rangle\langle \nu(e_i, e_i), e_1\rangle \\ &= \sum_{i=2}^{n} \frac{1}{\sqrt{r}}(e^t + e^{-t})\langle \nu(e_i, e_i), e_q\rangle \\ &= \frac{1}{\sqrt{r}}(e^t + e^{-t}) \left\langle \sum_{i=2}^{n} \nu(e_i, e_i), e_q \right\rangle.\end{aligned}$$

Finally, we claim that the basis $f_1, \dots, f_n$ can be chosen so that $\sum_{i=2}^{n} \mu(f_i, f_i) \in \mathbb{R} \cdot 1$. This will imply $\sum_{i=2}^{n} \nu(e_i, e_i) \in \mathbb{R} \cdot e_1 \perp e_q$, and hence $\frac{d}{dt}(M_{\nu_t}, E_{1q} + E_{q1}) = 0$.

The claim is proved in two steps. In the first step, recall the idempotents $e_1 \dots, e_r$ with $e_i e_j = 0$ for $i \neq j$ and $e_1 + \cdots + e_r = 1$. Set $f_1 = \frac{1}{\sqrt{r}}$ and

$$f_i = \frac{1}{\sqrt{i(i-1)}}(e_1 + \cdots + e_{i-1} - (i-1)e_i)$$

for $i = 2, \dots, r$. Then a quick calculation yields

$$\sum_{i=2}^{r} f_i^2 = \left(1 - \frac{1}{r}\right) \cdot 1.$$

In the second step, we invoke [2, Satz 9.1, Kap. VIII]. It says that $u^2 = \frac{1}{2}(e_i + e_j)$ for all $u \in \mathfrak{A}_{ij}$ $(i \neq j)$ with $||u|| = 1$. Finally,

$$\sum_{i=r+1}^{n} f_i^2 = \sum_{1 \le i < j \le r} \frac{d}{2}(e_i + e_j) = \frac{(r-1)d}{2} \sum_{i=1}^{r} e_i = \frac{(r-1)d}{2} \cdot 1$$

and hence

$$\sum_{i=1}^{n} f_i^2 = (r-1)\left(\frac{1}{r} + \frac{d}{2}\right) \cdot 1,$$

as wished.

In the remainder of the proof, we show that, conversely, if M_μ is a scalar matrix then $\mathfrak{A} = (\mathbb{C}^n, \mu)$ is semisimple.

Write $\mathfrak{A} = \mathfrak{N} + \mathfrak{S}$ (direct sum of vector spaces), where $\mathfrak{N}$ is the radical of $\mathfrak{A}$ and $\mathfrak{S}$ is a semisimple Jordan algebra. Suppose, by contradiction, that $\mathfrak{N} \neq 0$. Then

$$\mathfrak{N}^{[0]} = \mathfrak{N}, \ \mathfrak{N}^{[k+1]} = (\mathfrak{N}^{[k]})^3$$

for $k \geq 0$ defines a decreasing sequence of ideals of $\mathfrak{A}$ ([17, Lemma 2.2]; see also [21, Lemma 3, Sec. 3, Chap. 4]). Nilpotency of $\mathfrak{N}$ yields a minimal $k_0 \geq 0$ such that $\mathfrak{N}^{[k_0+1]} = 0$. Let $\mathfrak{B} = \mathfrak{N}^{[k_0]} \neq 0$. Then

$$\mathfrak{B}_0 = \mathfrak{B}, \ \mathfrak{B}_{k+1} = \mathfrak{A}\mathfrak{B}_k^2 + \mathfrak{B}_k^2,$$

for $k \geq 0$, is a decreasing sequence of ideals of $\mathfrak{A}$ [17, Lemma 2.2]. By [17, Theorem 2.5], there is a minimal $k_1 \geq 1$ such that $\mathfrak{B}_{k_1} \subset \mathfrak{B}^2$. Now there are two cases.

If $\mathfrak{B}_{k_1} \neq 0$, then $\mathfrak{C} = \mathfrak{B}_{k_1}$ is a non-zero ideal of $\mathfrak{A}$ with

$$\mathfrak{C}^2 = \mathfrak{B}_{k_1}^2 \subset \mathfrak{B}^2\mathfrak{B} = \mathfrak{B}^3 = (\mathfrak{N}^{[k_0]})^3 = \mathfrak{N}^{[k_0+1]} = 0.$$

If $\mathfrak{B}_{k_1} = 0$, then $\mathfrak{C} = \mathfrak{B}_{k_1-1}$ is a non-zero ideal of $\mathfrak{A}$ with

$$\mathfrak{C}^2 = \mathfrak{B}_{k_1-1}^2 = 0$$

since $0 = \mathfrak{B}_{k_1} = \mathfrak{A}\mathfrak{B}_{k_1-1}^2 + \mathfrak{B}_{k_1-1}^2$.

In any case, we have found a non-zero ideal $\mathfrak{C}$ of $\mathfrak{A}$ with $\mathfrak{C}^2 = 0$. We can now finish the proof. Let $x_1, \dots, x_m$ be an orthonormal basis of $\mathfrak{C}$, and let $y_1, \dots, y_{n-m}$

be an orthonormal basis of $\mathfrak{C}^\perp$. For any $x \in \mathfrak{C}$, owing to Proposition 3.2 and the facts that $L^\mu_x x_i = 0$ for all i and that the image of L^μ_x is contained in $\mathfrak{C}$, we have

$$\begin{aligned}0 &> \langle M_\mu x, x\rangle \\ &= -2\sum_{ij} |\langle L^\mu_x y_i, x_j\rangle|^2 + \sum_{ij} |\langle L^\mu_{x_i} y_j, x\rangle|^2 + \sum_{ij} |\langle \underbrace{L^\mu_{y_i} x_j}_{=L^\mu_{x_j} y_i}, x\rangle|^2 + \sum_{ij} |\langle L^\mu_{y_i} y_j, x\rangle|^2.\end{aligned}$$

We make $x = x_k$ and sum over $k = 1, \ldots, m$ to obtain

$$\begin{aligned}0 &> \sum_k \langle M_\mu x_k, x_k\rangle \\ &= -2\sum_{ijk} |\langle L^\mu_{x_k} y_i, x_j\rangle|^2 + 2\sum_{ijk} |\langle L^\mu_{x_i} y_j, x_k\rangle|^2 + \sum_{ijk} |\langle L^\mu_{y_i} y_j, x_k\rangle|^2 \\ &= \sum_{ijk} |\langle L^\mu_{y_i} y_j, x_k\rangle|^2 \\ &\geq 0,\end{aligned}$$

a contradiction. Hence $\mathfrak{N} = 0$, as desired. □

5. The Maximal Value of E_n

Since the energy E_n is constant along rays in $V_n \setminus \{0\}$, it attains a maximum value. In this section we determine those points of maxima and prove Theorem 1.5.

We introduce two important complex Jordan algebras. We give them names following the analogy with Lie algebras:

(a) The *Heisenberguian Jordan algebra* μ_{Heis} has a basis $\{n_1, \ldots, n_n\}$ $(n \geq 2)$ satisfying $n_1^2 = n_2$, and the other products equal to zero. This is a nilpotent Jordan algebra.
(b) The *hyperbolic Jordan algebra* μ_{hyp} has a basis $\{e, n_1, \ldots, n_{n-1}\}$ $(n \geq 2)$ satisfying $e^2 = e$, $en_i = \frac{1}{2}n_i$ for all i, and the other products equal to zero. This algebra has been considered in [13, Teorema 4.65], where it was shown that it is rigid.

Proposition 5.1. *Every Jordan algebra of dimension at least two which is not isomorphic to* μ_{hyp} *degenerates to* μ_{Heis}. *Further, the only Jordan algebras in* $\mathcal{J}_n$ $(n \geq 2)$ *for which the only non-trivial degeneration is to the trivial Jordan algebra* (*all products zero*) *are* μ_{Heis} *and* μ_{hyp}.

Proof. Let $\mu \in \mathcal{J}_n$. Suppose there is $x_1 \in \mathbb{C}^n$ such that x_1, $x_2 := \mu(x_1, x_1)$ are linearly independent; complete this set to a basis $x_1, \ldots, x_n$. Define $g_t \in G$ by setting $g_t x_1 = t x_1$, $g_t x_i = t^2 x_i$ for $i = 2, \ldots, n$, and put $\mu_t = g_t^{-1} \cdot \mu$. Then $\mu_t(x_1, x_1) = x_2$ for all $t \neq 0$. Moreover, for $(i, j) \neq (1, 1)$ there is $m \in \{3, 4\}$ such that

$$\mu_t(x_i, x_j) = t^{m-1}\mu_{ij}^1 x_1 + t^{m-2}(\mu_{ij}^2 x_2 + \cdots + \mu_{ij}^n x_n) \to 0$$

as $t \to 0$. In other words, $\mu \to \mu_{\mathrm{Heis}}$.

In case there is no x_1 as a above, there is a non-zero linear map $\ell : \mathbb{C}^n \to \mathbb{C}$ such that $\mu(x, x) = \ell(x)x$. By polarization, $\mu(x, y) = \frac{1}{2}(\ell(x)y + \ell(y)x)$. Let $n_1, \ldots, n_{n-1}$ be a basis of $\ker \ell$ and choose e such that $\ell(e) = 1$. Then $\mu(e, e) = e$, $\mu(e, n_i) = \frac{1}{2}n_i$ and $\mu(n_i, n_j) = 0$ for all i, j, which shows that $\mu \cong \mu_{\mathrm{hyp}}$.

Further, $\dim \mathrm{Der}(\mu_{\mathrm{hyp}}) = n^2 - n$ and $\dim \mathrm{Der}(\mu_{\mathrm{Heis}}) = n + (n-2)(n-1) = n^2 - 2n + 2$; in fact, if $d \in \mathrm{Der}(\mu_{\mathrm{Heis}})$, then $d(n_1)$ is arbitrary, $d(n_i) \in \mathrm{span}(n_2, \ldots, n_n)$ for $i \geq 2$, and $d(n_2) = 2n_1 d(n_1)$. Since $\dim \mathrm{Der}(\mu_{\mathrm{Heis}}) \leq \dim \mathrm{Der}(\mu_{\mathrm{hyp}})$ for $n \geq 2$, it follows that $\mu_{\mathrm{hyp}} \not\to \mu_{\mathrm{Heis}}$ [7, p. 284]. Also, $\mu_{\mathrm{Heis}} \not\to \mu_{\mathrm{hyp}}$ because $\dim \mu_{\mathrm{hyp}}(\mathbb{C}^n, \mathbb{C}^n) = n > 1 = \dim \mu_{\mathrm{Heis}}(\mathbb{C}^n, \mathbb{C}^n)$ (alternatively, μ_{Heis} is associative, but μ_{hyp} is not). □

Since μ_{hyp} is not a nilpotent Jordan algebra, it follows from Proposition 5.1 that μ_{Heis} lies in the closure of every G-orbit in $\mathcal{N}_n$, which proves the first assertion of Theorem 1.5. The second assertion is a consequence of the following result.

Corollary 5.2. *The Jordan algebras μ_{Heis} and μ_{hyp} are solitons. Further, the maximum value of E_n is* 5, *and it is attained exactly at the G-orbit of μ_{Heis}.*

Proof. Suppose μ is a point of maximum of E_n. This implies that μ is a critical point of E_n on $G \cdot \mu$, and hence on V_n. By Kirwan–Ness theory (Theorem 2.1), μ is a point of minimum of E_n on $G \cdot \mu$, which implies that E_n is constant along $G \cdot \mu$. Now every point in $G \cdot \mu$ is a point of minimum of E_n on $G \cdot \mu$, so the G- and K-orbits through μ agree up to scaling, that is $G \cdot \mu = \mathbb{C}^\times \cdot K \cdot \mu$, again by Kirwan–Ness. It follows that the only possible degeneration of μ is to the trivial algebra. By Theorem 5.1, μ is isomorphic to one of μ_{Heis} or μ_{hyp}. A simple calculation yields

$$M_{\mu_{\mathrm{Heis}}} = -5I + \begin{pmatrix} 3 & & & & \\ & 6 & & & \\ & & 5 & & \\ & & & \ddots & \\ & & & & 5 \end{pmatrix}, \quad E_n(\mu_{\mathrm{Heis}}) = 5$$

and

$$M_{\mu_{\mathrm{hyp}}} = -\left(\frac{n+1}{2}\right) I + \begin{pmatrix} 0 & & & \\ & \frac{n+1}{2} & & \\ & & \ddots & \\ & & & \frac{n+1}{2} \end{pmatrix}, \quad E_n(\mu_{\mathrm{hyp}}) = 1.$$

This finishes the proof. □

6. Stratification

In this section we review the Kirwan–Ness stratification of the null cone in the setting of commutative algebras. We follow the formulation and notation of [10, 12] (for which we refer the reader, regarding the missing proofs below) and note that the results are exactly the same as for the case of skew-symmetric algebras.

Consider again the action of $G = GL(n,\mathbb{C})$ on $V_n = S^2(\mathbb{C}^{n*}) \otimes \mathbb{C}^n$. Since $\lim_{t\to 0} g_t^{-1} \cdot \mu = \lim_{t\to 0} t\mu = 0$ for $g_t = tI$, every commutative algebra degenerates to the trivial algebra, and therefore the null cone $N = V_n$. Let $\mathfrak{h}$ be the subalgebra of diagonal matrices of $i\mathfrak{k}$, choose a positive Weyl chamber $\mathfrak{h}^+$ in $\mathfrak{h}$ and denote its closure by $\overline{\mathfrak{h}^+}$. The critical set of E is K-invariant and it decomposes into a finite union of disjoint closed subsets $\{C_\beta\}_{\beta\in\mathcal{B}}$, where $\mathcal{B}$ is a finite subset of $\overline{\mathfrak{h}^+}$, such that C_β is mapped under m onto the adjoint orbit $K \cdot \beta$ in $i\mathfrak{k}$. Let S_β be the set of all points of $V_n \setminus \{0\}$ that flow into C_β under $-\nabla E_n$ (the stable manifold of C_β). Then S_β is G-invariant, Zariski-locally closed, irreducible and non-singular, and we have

$$V_n \setminus \{0\} = \dot{\bigcup}_{\beta\in\mathcal{B}} S_\beta \quad \text{(disjoint union)}, \tag{17}$$

where

$$\bar{S}_\beta \setminus S_\beta \subset \bigcup_{||\beta'||>||\beta||} S_{\beta'}. \tag{18}$$

Write $\mu = \sum_{ijk} \mu_{ij}^k e_i' e_j' \otimes e_k$, for an orthonormal basis $e_1, \dots, e_n$ of $\mathbb{C}^n$ and its dual basis $e_1', \dots, e_n'$. For $\mu \in V_n \setminus \{0\}$, define

$$\beta_\mu = \text{the convex combination of smallest norm of the elements } \alpha_{ij}^k \in \mathfrak{h} \text{ with } \mu_{ij}^k \neq 0.$$

Recall that $\alpha_{ij}^k = -E_{ii} - E_{jj} + E_{kk}$, so $\mathrm{Tr}\beta_\mu = -1$. Now another description is

$$S_\beta = \{\mu \in V_n \setminus \{0\} \mid \beta \text{ is of maximal norm in } \{\beta_{g\cdot\mu} \mid g \in G\}\},$$

and $\mathcal{B} = \{\beta \in \bar{\mathfrak{h}}^+ \mid S_\beta \neq \varnothing\}$. Define also

$$W_\beta = \{\mu \in V_n \mid (\beta, \alpha_{ij}^k) \geq ||\beta||^2 \text{ for all } \mu_{ij}^k \neq 0\},$$

that is the sum of eigenspaces of $d\pi(\beta)$ with eigenvalues $\geq ||\beta||^2$, its subset

$$Y_\beta = \{\mu \in W_\beta \mid (\beta, \alpha_{ij}^k) = ||\beta||^2 \text{ for at least one } \mu_{ij}^k \neq 0\}$$

and

$$Y_\beta^{ss} = Y_\beta \cap S_\beta.$$

Then one proves

$$\begin{aligned} Y_\beta^{ss} &= S_\beta \cap W_\beta \\ &= \{\mu \in S_\beta \mid \beta = \beta_\mu\} \end{aligned} \tag{19}$$

and

$$S_\beta = K \cdot Y_\beta^{ss}. \tag{20}$$

Finally,

$$W_\beta \setminus \{0\} \subset S_\beta \cup \bigcup_{||\beta'||>||\beta||} S_{\beta'}. \tag{21}$$

Lemma 6.1. *If $M_\mu \in \mathfrak{h}$, then*

$$M_\mu = \sum_{ijk} |\mu_{ij}^k|^2 \, \alpha_{ij}^k.$$

Proof. For the canonical basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$ we have

$$\langle M_\mu e_k, e_k\rangle = -2\sum_{ij} |\langle L^\mu_{e_k} e_i, e_j\rangle|^2 + \sum_{ij} |\langle L^\mu_{e_i} e_j, e_k\rangle|^2,$$

so

$$\begin{aligned} M_\mu &= \sum_k \langle M_\mu e_k, e_k\rangle E_{kk} \\ &= -\sum_{ijk} |\langle L^\mu_{e_i} e_j, e_k\rangle|^2 E_{ii} - \sum_{ijk} |\langle L^\mu_{e_i} e_j, e_k\rangle|^2 E_{jj} + \sum_{ijk} |\langle L^\mu_{e_i} e_j, e_k\rangle|^2 E_{kk} \\ &= \sum_{ijk} |\mu_{ij}^k|^2 \, \alpha_{ij}^k, \end{aligned}$$

as desired. □

Corollary 6.2. *If $m(\mu) \in \mathfrak{h}$ then $m(\mu) \in \mathrm{Conv}(\{\alpha_{ij}^k : \mu_{ij}^k \neq 0\})$ (convex hull); in particular,*

$$E(\mu) = ||m(\mu)||^2 \geq ||\beta_\mu||^2,$$

and equality holds if and only if $m(\mu) = \beta_\mu$ if and only if μ is a soliton, and in this case $\mu \in S_\beta$ for $\beta \in \mathcal{B}$ the unique element of $\bar{\mathfrak{h}}^+$ Ad_K-conjugate to β_μ. In general, since E is K-invariant, from Eqs. (19) *and* (20) *we get that*

$$E(\mu) \geq ||\beta||^2 \quad \text{for all } \mu \in S_\beta, \tag{22}$$

and equality holds if and only if μ is a soliton, in which case $m(\mu)$ is Ad_K-conjugate to β and to β_μ.

Corollary 6.3. *If μ is a soliton and* 0 *is an eigenvalue of D_μ, then $||\beta_\mu|| \leq 1$.*

Proof. By replacing μ by a Ad_K-conjugate, we may assume $\beta_\mu = m(\mu)$, so $\beta_\mu + ||\beta_\mu||^2 I = \frac{1}{||M_\mu||^2} D_\mu$ has 0 as an eigenvalue, that is $(\beta_\mu, E_{ii}) = -||\beta_\mu||^2$ for some $i = 1, \ldots, n$. Finally, $||\beta_\mu|| \geq |(\beta_\mu, E_{ii})| = ||\beta_\mu||^2$. □

Since the stratification (17) is G-invariant, it naturally induces a stratification of any G-invariant subvariety of $V_n \setminus \{0\}$.

Proposition 6.4. *The stratum $S_\beta \cap \mathcal{J}_n$ for $\beta = -\frac{1}{n}I$ precisely consists of the n-dimensional semisimple Jordan algebras.*

Proof. It follows from Proposition 4.5 that if $\mu \in \mathcal{J}_n$ is semisimple then there is $g \in G$ with $M_{g\cdot\mu}$ a scalar matrix. Therefore $\mu \in S_\beta \cap \mathcal{J}_n$. Conversely, assume that $\mu \in S_\beta \cap \mathcal{J}_n$. Then the integral curve $\{\mu(t)\}$ of the $(-\nabla E_n)$-flow with $\mu(0) = \mu$ is contained in $G \cdot \mu$ and converges to a soliton in S_β. In particular all the eigenvalues of $M_{\mu(t)}$ are negative for sufficiently large t. In the second half of the proof of Proposition 4.5, the argument only needs this information to imply that $\mu(t)$ must be semisimple for sufficiently large t (cf. inequality for $\langle M_\mu x, x\rangle$ on p. 18). Hence μ is semisimple, too. □

6.1. *Proofs of Proposition* 1.3 *and Theorem* 1.4

If $\mu \to \nu$ then $G \cdot \nu \subset \overline{G \cdot \mu}$. Say $\mu \in S_\beta$ for some $\beta \in \mathcal{B}$. Then (18) implies that $\nu \in S_\beta \cup \bigcup_{||\beta'||>||\beta|} S_{\beta'}$. Owing to Corollary 6.2, we have

$$E(\nu) \geq ||\beta||^2 = E(\mu),$$

proving Proposition 1.3.

We move to the proof of the theorem. Recall that $S_\beta \cap \mathcal{J}_n$ for $\beta = -\frac{1}{n}I$ precisely consists of the n-dimensional semisimple Jordan algebras (Proposition 6.4), so it is open in $\mathcal{J}_n$, say thanks to Albert's criterion for semisimplicity in terms of the trace form (Sec. 2.1). Suppose μ is semisimple and $\nu \to \mu$. Then ν is semisimple. Moreover $\mu \in \overline{G \cdot \nu}$ implies that $\dim G \cdot \mu \leq \dim G \cdot \nu$ and therefore $\dim \mathrm{Der}(\mu) \geq \dim \mathrm{Der}(\nu)$. We will prove the reverse inequality by using a result of [6] asserting that every derivation D of a semisimple Jordan algebra $\mathfrak{A}$ (over a field of characteristic zero) is inner, in the sense that it is given as $D = \sum_i [L_{a_i}, L_{b_i}]$ for some a_i, $b_i \in \mathfrak{A}$. It implies that

$$\dim \mathrm{Der}(\mu) = \dim \mathrm{span}\{[L^\mu_{x_i}, L^\mu_{x_j}]\}_{ij} \leq \dim \mathrm{span}\{[L^\nu_{x_i}, L^\nu_{x_j}]\}_{ij} = \dim \mathrm{Der}(\nu) \quad (23)$$

for a fixed basis $x_1, \ldots, x_n$ of $\mathbb{C}^n$, by lower semicontinuity of the dimension of the span in terms of the multiplication in V_n.

Now $G \cdot \mu$ and $G \cdot \nu$ have the same dimension, which implies that $G \cdot \mu = G \cdot \nu$, that is μ and ν are isomorphic. This means that $G \cdot \mu$ is closed in $S_\beta \cap \mathcal{J}_n$ for $\beta = -\frac{1}{n}I$. Since there are only finitely many G-orbits in $S_\beta \cap \mathcal{J}_n$ (Proposition 6.4), they are all open in $S_\beta \cap \mathcal{J}_n$, and hence in $\mathcal{J}_n$. In particular $G \cdot \mu$ is open in $\mathcal{J}_n$ and hence μ is rigid. This finishes the proof of Theorem 1.4.

7. Low-Dimensional Jordan Algebras and Other Examples

The proof of Theorem 1.6 regarding Jordan algebras of dimension at most 4 is given in Tables 1–9. In this section we explain how to read them and explain certain cases in more detail. We start with some remarks of a general nature.

7.1. *Decomposable algebras*

A Jordan algebra is called *indecomposable* if it is not isomorphic to the direct product of two Jordan algebras, and *decomposable* otherwise. The following lemma, whose proof is easy, shows that we can restrict our search for solitons to the indecomposable Jordan algebras.

Lemma 7.1. *if $\mu \in V_n$ and $\nu \in V_m$ are solitons and $M_\mu = c_\mu I + D_\mu$, $M_\nu = c_\nu I + D_\nu$, then $\mu \times c\nu \in V_{n+m}$ is a soliton, where $c = \sqrt{\frac{c_\mu}{c_\nu}}$.*

7.2. *Unitalization of Jordan algebras*

The following result is an easy check using, say, Lemma 6.1.

Lemma 7.2. *If a Jordan algebra $\mathfrak{A} = (\mathbb{C}^n, \mu)$ does not carry a unit element and we adjoin a unit element to $\mathfrak{A}$ to obtain $\hat{\mathfrak{A}} = (\mathbb{C}^{n+1}, \hat{\mu})$, then the moment matrix of $\hat{\mu}$ is*

$$M_{\hat{\mu}} = \begin{pmatrix} M_\mu & \\ & -(2n+1) \end{pmatrix}.$$

In particular, if $\mu \in \mathcal{J}_n$ is a soliton and $M_\mu = c_\mu I + D_\mu$, then $\widehat{\sqrt{c}\mu} \in \mathcal{J}_{n+1}$ is a soliton, where $c = \frac{2n+1}{-c_\mu}$, and

$$M_{\widehat{\sqrt{c}\mu}} = -(2n+1)I + \begin{pmatrix} cD_\mu & \\ & 0 \end{pmatrix};$$

in this case $E_{n+1}(\widehat{\sqrt{c}\mu}) = \frac{E_n(\mu)}{E_n(\mu)+1}$.

Basic examples are $\widehat{\mathfrak{A}_{2,2}} = \mathfrak{A}_{3,4}$, $\widehat{\mathfrak{A}_{2,3}} = \mathfrak{A}_{3,7}$, $\widehat{\mathfrak{A}_{2,5}} = \mathfrak{A}_{3,3}$, $\widehat{\mathfrak{A}_{3,13}} = \mathfrak{A}_{4,25}$, $\widehat{\mathfrak{A}_{3,17}} = \mathfrak{A}_{4,39}$ (see Tables 1–6).

7.3. *The regular representation of a semisimple Jordan algebra*

Let $\mathfrak{S} = (\mathbb{C}^n, \mu)$ be a semisimple Jordan algebra, and let $\mathfrak{N}$ be the underlying vector space of $\mathfrak{S}$, which we consider as a $\mathfrak{S}$-module under the regular representation. We put $\mathfrak{N}^2 = 0$ so that $\mathfrak{A} = \mathfrak{S} + \mathfrak{N} = (\mathbb{C}^{2n}, \tilde{\mu})$ is a Jordan algebra. Choose the Hermitian product such that $\mathfrak{S} \perp \mathfrak{N}$, on $\mathfrak{S}$ it is the Hermitian product that makes μ a soliton with $M_\mu = -\frac{1}{n}I$, and on $\mathfrak{N}$ it is isometric to $\mathfrak{S}$. Then a simple calculation shows that $M_{\tilde{\mu}} = -\frac{1}{n}I = -\frac{2}{2n}I$. Basic examples are $\widetilde{\mathfrak{A}_{1,1}} = \mathfrak{A}_{2,1}$ and $\widetilde{\mathfrak{A}_{2,4}} = \mathfrak{A}_{4,22}$ (see Tables 1–6).

7.4. *Jordan algebras in dimensions 1, 2 and 3*

Excluding the trivial algebras, there is one isomorphism class of complex one-dimensional Jordan algebras, five isomorphism classes of complex two-dimensional

Jordan algebras, and 19 isomorphism classes of complex three-dimensional Jordan algebras [7]; those are listed in Tables 1, 3 and 5 in soliton form. One computes the moment matrices using Proposition 3.2, and then uses Proposition 1.1 to find a soliton in each isomorphism class. It is essentially enough to work with non-semisimple and indecomposable algebras. The stratification and invariants are collected in Tables 2, 4 and 6. To give one example in dimension 3, consider $\mathfrak{A}_{3,4}$. One replaces the standard basis e_1, e_2, n_1 given in [7] (see Table 2 therein, where the algebra is listed as $\mathcal{T}_{10}$) by f_1, f_2, n_1, where $f_1 = e_1 + e_2$, $f_2 = e_1 - e_2$, in order to diagonalize the moment matrix, and then replaces f_2 by $f_2' = kf_2$, where $k^4 = 5/3$, in order to find a soliton.

7.5. *Jordan algebras in dimension* 4

Excluding the trivial algebra, there are 72 isomorphism classes of complex four-dimensional Jordan algebras [7]. Those are listed in Tables 7 and 8 in soliton form, except for $\mathfrak{A}_{4,63}$. The stratification and invariants are collected in Table 9. We omit the tedious calculations and only give a few typical examples.

7.5.1. *The orbit of* $\mathfrak{A}_{4,66}$ *is distinguished*

We search for solitons among isomorphic algebras of the form $n_1^2 = \alpha n_2$, $n_4^2 = \beta n_3$, $n_1 n_2 = \gamma n_3$. Using Propositions 3.2 and 1.1 we find one with $\alpha^2 = \beta^2 = 4$ and $\gamma^2 = 3$.

7.5.2. *The orbits of* $\mathfrak{A}_{4,16}$*,* $\mathfrak{A}_{4,17}$ *and* $\mathfrak{A}_{4,25}$ *are distinguished*

We compute that the energy of $\mathfrak{A}_{4,16}$ in the basis given in [7, Table 3] is 27/49. Consider a point of minimum ν of E_4 in $\overline{G \cdot \mu}$. According to [13], the first level degenerations of μ are $\mathfrak{A}_{4,28}$, $\mathfrak{A}_{4,31}$ and $\mathfrak{A}_{4,50}$, whose orbits contain solitons with energy levels, respectively 1, 1 and 3/4. Since $E(\nu) < 27/49$, ν cannot lie in those orbits. Still according to [13], the only other possible degenerations of μ are to $\mathfrak{A}_{4,i}$, where $i = 48, 64, 66, 67, 68, 70$. However $\mathfrak{A}_{4,48}$ has a soliton with energy level 5/6 and the other ones have solitons with energy level well above 1, so again ν cannot lie in those orbits. The only remaining possibility is that $\nu \in G \cdot \mu$. Hence $\mathfrak{A}_{4,16}$ is distinguished. It follows from Corollary 6.2 that $E(\nu) = ||\beta_\nu||^2 = ||\beta_\mu||^2 = 1/2$.

We compute that the energy of $\mathfrak{A}_{4,17}$ and $\mathfrak{A}_{4,25}$ are 3/5 and 5/9, respectively, and proceed similarly in those cases.

In Table 7 we have written the approximate values of the structural constants for these solitons, which were obtained by using computer software.

7.5.3. *The orbit of* $\mathfrak{A}_{4,63}$ *is not distinguished*

Denote $(\mathbb{C}^4, \mu) = \mathfrak{A}_{4,63}$ and $(\mathbb{C}^4, \nu) = \mathfrak{A}_{4,64}$. It is known that $\mu \to \nu$. Indeed, let $g_t = \begin{pmatrix} 1 & & & \\ & t & & \\ & & t & \\ & & & t \end{pmatrix} \in G$ and put $\mu_t = g_t^{-1} \cdot \mu$. Then $\lim_{t \to 0} \mu_t = \nu$. Note that ν is a soliton and put $\beta = m(\nu) = \begin{pmatrix} -1 & & & \\ & -\frac{1}{2} & & \\ & & 0 & \\ & & & \frac{1}{2} \end{pmatrix} \in \bar{\mathfrak{h}}^+$. It is immediate to see that $\mu \in W_\beta$. Since $\inf E_4(G \cdot \mu) = ||\beta||^2 = \frac{3}{2}$, we deduce from (18), (21) and (22) that $\mu \in S_\beta$.

Now suppose, to the contrary, that $\lambda \in G \cdot \mu$ is a soliton. Then $m(\lambda)$ is Ad_K-conjugate to β (Corollary 6.2), and by replacing λ by an element in its K-orbit we may assume $m(\lambda) = \beta$. Since $\frac{1}{||\lambda||^2} D_\lambda = \beta + ||\beta||^2 I = \frac{1}{2} \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix}$, this implies that this matrix is a derivation of λ. This matrix has pairwise different eigenvalues, so λ must be given by

$$\lambda(n_1, n_1) = an_2, \quad \lambda(n_1, n_2) = bn_3, \quad \lambda(n_1, n_3) = cn_4, \quad \lambda(n_2, n_2) = dn_4, \quad (24)$$

and the other products zero, for some complex constants a, b, c, d.

Suppose now λ is given by (24). We finish by checking that: (i) λ can only be isomorphic to μ if $a = 0$ and b, c, $d \neq 0$; (ii) if $a = 0$, λ can be a soliton only if $b = \pm c$ and $d = 0$. This will prove that there are no solitons in $G \cdot \mu$.

Write $\mathfrak{A} = \mathfrak{A}_{4,63} = (\mathbb{C}^4, \mu)$ and $\mathfrak{B} = (\mathbb{C}^4, \lambda)$. Note that $\mathfrak{A}^2 = \mathrm{span}(n_3, n_4)$ and $\mathfrak{A}^3 = \mathrm{span}(n_4)$, so $\mathfrak{A}/\mathfrak{A}^3 = \mathfrak{A}_{3,18}$. On the other hand, $\mathfrak{B}^2 = \mathrm{span}(an_2, bn_3, cn_4, dn_4)$ and $\mathfrak{B}^3 = \mathrm{span}(abn_3, bcn_4, adn_4)$. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. If $a \neq 0$ then, owing to $\dim \mathfrak{B}^2 = 2$, we have $b = 0$ or $c = d = 0$. In both cases we get $\mathfrak{B}/\mathfrak{B}^3 = \mathfrak{A}_{3,19}$. This shows that $a = 0$. Now $\dim \mathfrak{B}^2 = 2$ and $\dim \mathfrak{B}^3 = 1$ imply that $b \neq 0$ and $c \neq 0$. We also have $d \neq 0$, for otherwise $\mathfrak{B}$ would be isomorphic to $\mathfrak{A}_{4,64}$. This proves (i).

We turn to (ii). Suppose $a = 0$ and λ is a soliton. We compute that

$$M_\lambda = \begin{pmatrix} -2b^2 - 2c^2 & & & \\ & -2b^2 - 2d^2 & & \\ & & -2c^2 + 2b^2 & \\ & & & 2c^2 + d^2 \end{pmatrix}. \quad (25)$$

Since M_λ is a multiple of β, this immediately gives that $b^2 = c^2$ and $d = 0$, which proves (ii).

8. Closed $SL(n, \mathbb{C})$-Orbits in $\mathcal{J}_n$

Since there are no closed $GL(n, \mathbb{C})$-orbits in $\mathcal{J}_n$ (not even in $V_n \backslash \{0\}$), precisely because of multiples of the identity, it arises the natural question of knowing whether the subgroup $SL(n, \mathbb{C})$ admits closed orbits in $\mathcal{J}_n$.

Proposition 8.1. *Let $\mu \in \mathcal{J}_n$, $\mu \neq 0$. Then*

(a) *The orbit $SL(n, \mathbb{C}) \cdot \mu$ is closed if and only if μ is a semisimple Jordan algebra.*
(b) *If μ is not semisimple then 0 lies in the closure of $SL(n, \mathbb{C}) \cdot \mu$.*

Proof. The orbit $SL(n,\mathbb{C})\cdot\mu$ is closed if and only the moment map m^{SL_n} of the $SL(n,\mathbb{C})$-action on V_n vanishes at some point $\nu\in SL(n,\mathbb{C})\cdot\mu$. Since m^{SL_n} is obtained from m^{GL_n} by post-composing with the projection $i\mathfrak{u}(n)=\mathbb{R}\oplus i\mathfrak{su}(n)\to i\mathfrak{su}(n)$, the latter is equivalent to $m^{GL_n}(\nu)=-\frac{1}{n}I$, which means that ν and μ are semisimple, owing to Proposition 4.5.

Suppose now μ is not semisimple. Then $SL(n,\mathbb{C})\cdot\mu$ is not closed and $\overline{SL(n,\mathbb{C})\cdot\mu}$ contains a closed orbit, say, $SL(n,\mathbb{C})\cdot\nu$. If $\nu\neq 0$ then ν is semisimple by part (a). Since μ is a deformation of ν, owing to the rigidity of semisimple Jordan algebras, the $SL(n,\mathbb{C})$-orbits of μ and ν coincide, a contradiction. □

9. Partial Results, Open Problems and Conjectures

It is interesting to note that the application of GIT to the study of (commutative) Jordan algebras has many similarities with the case of (anti-commutative) Lie algebras. However the Jordan identity (in each of its disguises) seems to be more difficult to use than the Jacobi identity. In particular, for Jordan algebras in general the left multiplications are not derivations of the algebra, in flagrant contrast with Lie algebras. So proofs of results for Lie algebras which depend on this property cannot be simply carried over to the context of Jordan algebras, and throughout this work we have tried to find alternative lines of arguments, with some success. The partial results that we collect in this section are somehow related to this situation.

Unless explicitly stated, throughout this section we let $\mu\in\mathcal{J}_n$ be a Jordan soliton, $\mathfrak{A}=(\mathbb{C}^n,\mu)$. We write the moment matrix $M_\mu=c_\mu I+D_\mu$, where D_μ is a Hermitian derivation, according to Proposition 1.1. Let also $\mathfrak{N}$ denote the radical of $\mathfrak{A}$.

9.1. *The annihilator*

The annihilator of $\mathfrak{A}$ is $\mathrm{Ann}(\mathfrak{A})=\{x\in\mathfrak{A}:L^\mu_x=0\}$. It is clear from $L^\mu_{D_\mu x}=[D_\mu,L^\mu_x]$ that D_μ preserves $\mathrm{Ann}(\mathfrak{A})$, and it follows from Proposition 3.2 that the eigenvalues of D_μ on $\mathrm{Ann}(\mathfrak{A})$ are positive.

9.2. *Basic calculation*

Let x be an eigenvector of D_μ with eigenvalue d. Then

$$\begin{aligned}(M_\mu,[L^\mu_x,L^{\mu*}_x]) &= \mathrm{Tr}(M_\mu[L^\mu_x,L^{\mu*}_x])\\ &= \mathrm{Tr}(D_\mu[L^\mu_x,L^{\mu*}_x])\\ &= \mathrm{Tr}([D_\mu,L^\mu_x]L^{\mu*}_x)\\ &= \mathrm{Tr}(L^\mu_{D_\mu x}L^{\mu*}_x)=d||L^\mu_x||^2. \end{aligned}\tag{26}$$

On the other hand, due to (3) the left-hand side of (26) also equals $\langle[L^\mu_x,L^{\mu*}_x]\cdot\mu,\mu\rangle$, so we deduce

$$d||L^\mu_x||^2=||L^{\mu*}_x\cdot\mu||^2-||L^\mu_x\cdot\mu||^2.\tag{27}$$

We use this formula and some variations below.

Lemma 9.1. *Let $\mu \in \mathcal{J}_n$ be a soliton, and let x, $y \in \mathbb{C}^n$ be eigenvectors of D_μ with corresponding* (*real*) *eigenvalues d_x, d_y. If $d_x \neq d_y$ then $(L^\mu_x, L^\mu_y) = 0$.*

Proof. The basic calculation yields

$$d_x(L^\mu_x, L^\mu_y) = (M_\mu, [L^\mu_x, L^{\mu*}_y]) = \langle L^{\mu*}_y \cdot \mu, L^{\mu*}_x \cdot \mu\rangle - \langle L^\mu_x \cdot \mu, L^\mu_y \cdot \mu\rangle.$$

We interchange x and y in these equations to obtain that

$$d_x(L^\mu_x, L^\mu_y) = \overline{d_y(L^\mu_y, L^\mu_x)} = d_y(L^\mu_x, L^\mu_y),$$

which proves the desired result. □

9.3. *The kernel of D_μ*

Proposition 1.1(b) shows that the kernel of D_μ contains a maximal semisimple subalgebra of $\mathfrak{A} = (\mathbb{C}^n, \mu)$.

Question 9.2. For a soliton $\mu \in \mathcal{J}_n$ is it true that $\ker D_\mu$ is a maximal semisimple subalgebra? In other words, are the eigenvalues of D_μ restricted to $\mathfrak{N}$ different from zero?

9.4. *Positivity of D_μ on $\mathfrak{N}$*

A positive answer to Question 9.3 implies a positive answer to Question 9.2.

Question 9.3. For a soliton $\mu \in \mathcal{J}_n$, are the eigenvalues of D_μ on $\mathfrak{N}$ positive?

For soliton Jordan algebras $\mathfrak{A}$ satisfying $\mathfrak{A}^3 = 0$, trivially all left-multiplications are derivations of the algebra, and we can use a standard argument to answer yes to Question 9.3.

Proposition 9.4. *If the soliton $\mathfrak{A} = (\mathbb{C}^n, \mu)$ satisfies $\mathfrak{A}^3 = 0$, then all eigenvalues of D_μ are positive.*

Proof. The assumption $\mathfrak{A}^3 = 0$ implies that $\mathfrak{A}$ is a nilpotent Jordan algebra and hence L^μ_x is a nilpotent operator for all $x \in \mathfrak{A}$.

Let $x \in \mathbb{C}^n$ be an eigenvector of D_μ with corresponding eigenvalue $d \in \mathbb{R}$. Since $L^\mu_x \cdot \mu = 0$, the basic calculation (27) immediately gives $d \geq 0$. If in addition $d = 0$, then $L^{\mu*}_x$ is also a derivation of μ, thus $[L^{\mu*}_x, L^\mu_x] = L^\mu_{L^{\mu*}_x x}$ by the defining condition of a derivation.

Since D_μ is Hermitian, the orthogonal decomposition $\mathfrak{A} = \mathfrak{A}^2 \oplus \mathfrak{A}^{2\perp}$ is D_μ-invariant, so we may assume $x \in \mathfrak{A}^2$ (respectively, $x \in \mathfrak{A}^{2\perp}$). In any case $\langle L^{\mu*}_x x, y\rangle = \langle x, xy\rangle = 0$ for all $y \in \mathfrak{A}$, as $xy \in \mathfrak{A}^3 = 0$ (respectively, $xy \in \mathfrak{A}^2$). This shows $L^{\mu*}_x x = 0$.

Now L_x^μ is a normal and nilpotent operator, hence $x \in \mathrm{Ann}(\mathfrak{A})$. The result in Sec. 9.1 contradicts our assumption that $d = 0$. Hence $d > 0$. □

For non-associative Jordan algebras, we have the following partial result.

Proposition 9.5. *Let x, $y \in \mathbb{C}^n$ be eigenvectors of D_μ with corresponding eigenvalues d_x, d_y. If $[L_x^\mu, L_y^\mu] \neq 0$ then $d_x + d_y \geq 0$.*

Proof. We consider $D := [L_x^\mu, L_y^\mu]$ and compute

$$\begin{aligned}[D_\mu, D] &= [[D_\mu, L_x^\mu], L_y^\mu] + [L_x^\mu, [D_\mu, L_y^\mu]] \\ &= [L^\mu_{D_\mu x}, L_y^\mu] + [L_x^\mu, L^\mu_{D_\mu} y] \\ &= (d_x + d_y)D.\end{aligned}$$

We proceed as in the basic calculation to get

$$(d_x + d_y)||D||^2 = \mathrm{Tr}([D_\mu, D]D^*) = \mathrm{Tr}(D_\mu[D, D^*]) = \mathrm{Tr}(M_\mu[D, D^*]) = ||D^* \cdot \mu||^2,$$

since D is a (inner) derivation, and the result follows. □

If $\mathfrak{N} = (\mathbb{C}^n, \mu)$ is a nilpotent Jordan soliton generated by a single element n, then n and its powers must be eigenvectors of D_μ. Owing to $\mathrm{Tr}(D_\mu) = \frac{\mathrm{Tr} D_\mu^2}{-c_\mu} > 0$, we also get $D_\mu > 0$ in this case.

9.5. *Orthogonality of $\mathfrak{S}$ and $\mathfrak{N}$*

Since D_μ is Hermitian, the orthogonal complement of $\mathfrak{N}$ with respect to the Hermitian product is D_μ-invariant.

Question 9.6. For a soliton $\mu \in \mathcal{J}_n$, $\mathfrak{A} = (\mathbb{C}^n, \mu)$, is $\mathfrak{N}^\perp$ a semisimple subalgebra of $\mathfrak{A}$?

If the answer to Question 9.6 is yes, then $\mathfrak{N}^\perp$ will be maximal semisimple. A positive answer to Question 9.2 implies a positive answer to Question 9.6.

9.6. *Reduction to nilpotent Jordan algebras*

Question 9.7. If $\mathfrak{A} = (\mathbb{C}^n, \mu)$ is a Jordan soliton and $\mathfrak{N}$ is the radical of $\mathfrak{A}$, is it true that $\mu|_{\mathfrak{N} \times \mathfrak{N}}$ is a soliton? Conversely, given a nilpotent Jordan soliton $\mathfrak{N}$ and a semisimple Jordan algebra $\mathfrak{S}$ such that $\mathfrak{A} = \mathfrak{S} + \mathfrak{N}$ is a Jordan algebra, can we extend the metric from $\mathfrak{N}$ to $\mathfrak{A}$ so that $\mathfrak{A}$ becomes a soliton?

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