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RESEARCH ARTICLE

On the non-collocated control of structures with optimal static output feedback: initial conditions dependence, sensors placement and sensitivity analysis

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Summary

It is well known that the linear quadratic regulator (LQR) exhibits significant frequency margins and reduced sensitivity properties. However, its application in real problems is restricted because it requires availability of all state variables of the controlled system to be measured. This problem is not satisfactorily overcome by state observers, since they are sensitive to spillover and their more complex structure can entail time delay. Another alternative to solve this problem is to use only linear combinations of measured signals for feedback, a technique known as optimal static output feedback (OSOF) or partial state feedback. In this paper, this method is studied considering additionally sensors locations as optimization variables. Necessary conditions of optimality are presented in order to highlight the dependence of the optimal solution on system initial conditions. A new approach to deal with this dependence, which is based on approaching the performances of OSOF and LQR for any initial condition, is developed and compared to the existing one. The method developed is tested on a simply supported plate modeled using the finite element method. The analyses of the results show that with a significantly reduced number of sensors, the OSOF controller has a performance equivalent to LQR. The developed methodology also provided the controlled system a relevant behavior to major problems of non-collocated control, such that it maintained stability and performance considering parameters variations and a large increase in frequency bandwidth.

KEYWORDS:
non-collocated control, sensor placement optimization, optimal static output feedback, active vibration control, sensitivity and spillover

1 | INTRODUCTION

Vibrations are present in virtually all engineering problems involving flexible structures subject to dynamic loads, and in many cases it represents a phenomenon that must be mitigated or suppressed[1,2]. Due to several factors, such as adaptation to different types of excitation, suppression of vibration over a wide range of frequencies or even availability of multi-purpose piezoelectric patches, the engineer can choose to apply an active control technique. When dealing with active control[3,4] and health monitoring[5,6] of flexible structures, or systems whose model is given by partial differential equations, sensors locations are usually variables of the design process, and can affect substantially the performance of the controlled system. Both the control law variables[7–10] and locations of sensors and actuators[11] are generally designed in order to achieve or optimize a performance criterion. Among these criteria, there is a functional quadratic in state and control variables, which is used to determine the optimal gain in the linear quadratic regulator (LQR) problem.

The LQR control has been applied in many researches of vibration control[12–16], which may be due to some advantageous properties. Besides minimizing a cost function that represents a trade-off between control effort and performance, the control law obtained by the solution of the LQR problem still provides infinite gain margin and phase margin of at least 60°, for each plant input channel, and reduces closed loop system sensitivity to controller and plant variations[17]. However, the practical implementation of the LQR is limited, since it requires that all state variables must be measured. A common approach to solve this problem is to use LQG control, which consists of using an observer to estimate unmeasured states. However, it still has drawbacks. An observer uses output, input and model information to reconstruct the actual state, producing a dynamic controller with complex hardware, causing time delay. Besides that, LQG control does not have guaranteed stability margins, and the loop transfer recovery procedures are limited to minimum phase plants and tend to produce high gains[18]. Finally, LQG control is sensitive to the spillover phenomenon[19]. Given these considerations, the technique presented in this paper is different from the ones that considered dynamic output feedback, e.g., Oliveira and Geromel[20] and Feng et al[21].

These problems motivated the study of a control technique that still consists on minimizing the quadratic cost function, but with the constraint of using only linear combinations of measured signals for feedback[22,23], which is known as optimal static output feedback (OSOF) or partial state feedback. Despite simplification of hardware, this technique posed some theoretical challenges: the optimization problem is non-convex, the optimal gain is dependent on system initial conditions and the existence of a stabilizing static output gain is an open problem in control theory[24]. This control technique was applied in vibration control problems, considering some approaches to the mentioned challenges. Lim et al[25] considered the application of OSOF control with collocated sensors and actuators to reduce vibration and noise of a clamped plate. Moon[26] and Gharib et al[27] obtained an acceptable performance using OSOF controller, but below that obtained for the LQR. Although this result was expected, since full state feedback controller is optimal for any initial condition, the performance and properties of both controllers tend to be equivalent as the expected values of both cost functions are closer[23]. As the performance index is also a function of sensors locations, it is expected that the performance may be even closer to the LQR considering also the locations of sensors as optimization variables. Abdullah[28] considered simultaneous determination of control gain and placement of collocated sensor/actuator pairs to control a beam subject to wind excitation, but without making comparisons with LQR. Cai and Lim[29] proposed a formulation to determine which states have significant effect on control performance, in order to use only these states for feedback. However, this formulation has limitations for structural models in modal coordinates, because the measurement of each state may involve the use of several sensors.

The OSOF control formulation was also used in other areas involving flexible structures, such as energy harvesting[30], design of PPF controllers[31] and semi-active control[32]. This technique also has a hardware structure similar to discrete modal filters[33–35] and other control techniques with static output feedback[36,37] (feedback of linear combinations of measured signals), showing that developments in this technique can benefit a large area of interest.

Most of the above mentioned articles considered the suggestion given by Levine and Athans[22] to handle the dependence of the optimal solution with respect to system initial conditions and none compared the use of different conditions. This comparison was performed by Morris and Yang[38], but for determination of actuators locations with full state feedback. Different strategies to solve the non-convex optimization were proposed, from a descent direction method[26] with different initial guesses to a genetic algorithm[27]. None of the considered articles indicated a problem for finding combinations of design variables that makes the closed loop system stable.

In this paper, all the problems mentioned above are addressed. Necessary conditions of optimality are presented considering also sensors locations as optimization variables, evidencing the dependence on system initial condition. A new approach to deal with this dependence, which is based on approaching the performances of OSOF and LQR for any initial condition, is proposed and compared to that given by Levine and Athans[22]. It is also suggested a possible set of initial guesses that can be used with a line search or trust region optimization algorithm[39]. The methodology developed is applied to control a simply supported plate, and the results are compared with LQR. The cases studied, in which the number of sensors is different from the number of actuators, may present common problems of non-collocated control. From this perspective, a final analysis considering performance and stability due to variations around the optimal gains and sensors' locations and the effects of the spillover phenomenon is performed.

2 | NECESSARY CONDITIONS FOR OPTIMALITY

To determine the necessary conditions of optimality, it is considered that the dynamic behavior of the flexible structure can be described by the standard equations of linear time invariant systems:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{1}$$

$$\mathbf{y} = \mathbf{C}(\boldsymbol{\xi})\mathbf{x} \tag{2}$$

$$\mathbf{u} = -\mathbf{K}\mathbf{y} \tag{3}$$

in which $\mathbf{x}(t) \in \mathbb{R}^{2n}$ is the state vector, $\mathbf{u}(t) \in \mathbb{R}^{a}$ is the control vector, $\mathbf{y}(t) \in \mathbb{R}^{s}$ is the output vector, t is the time variable and the dot represents time derivative, $\mathbf{A} \in \mathbb{R}^{2n\times 2n}$ is the state matrix, $\mathbf{B} \in \mathbb{R}^{2n\times a}$ is the input matrix, $\mathbf{C} \in \mathbb{R}^{s\times 2n}$ is the output matrix, $\mathbf{K} \in \mathbb{R}^{a\times s}$ is the control gain and $\boldsymbol{\xi} \in \Omega$ represents sensors locations, in which $\Omega \subset \mathbb{R}^{s\times m}$ is the structure domain, m is the structure dimension, n is the number of degrees of freedom considered in the structure's model and a and s are the number of available actuators and sensors, respectively. Equation (3) characterizes the output feedback constraint. The control gain and sensors locations are variables to be determined in order to minimize the following cost function:

$$J = \int_0^\infty \mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} + \mathbf{u}^\mathsf{T}\mathbf{R}\mathbf{u}\, dt \tag{4}$$

that represents the traditional LQR cost. Considering that the system is stabilizable with output feedback and using a procedure like the one given by Lewis et al[40], this optimization problem can be rewritten as

$$\underset{(\mathbf{K},\boldsymbol{\xi})}{\text{minimize}} \quad \operatorname{tr}\{\mathbf{P}\mathbf{x}_0\mathbf{x}_0^\mathsf{T}\} \tag{5}$$

$$\text{subject to} \quad \mathbf{A}_\mathbf{c}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A}_\mathbf{c} + \mathbf{Q} + (\mathbf{K}\mathbf{C})^\mathsf{T}\mathbf{R}\mathbf{K}\mathbf{C} = 0 \tag{6}$$

in which $\mathbf{x}_0$ is the vector of system initial conditions, tr is the trace operator and $\mathbf{A}_\mathbf{c}$ is the closed loop state matrix ($\mathbf{A}_\mathbf{c} = \mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{C}$). The constraint given in Equation (6) is a Lyapunov equation and has a unique solution $\mathbf{P}$ for every $\mathbf{A}_c$ that is Hurwitz. Therefore, for every pair $(\mathbf{K}, \boldsymbol{\xi})$ that makes the closed loop system stable, Equation (6) can be solved to determine a matrix $\mathbf{P}$ that will be used to evaluate the cost function (5) for a given initial condition. As the purpose of this section is only to demonstrate the dependence on the initial condition, extrema values of the cost function corresponding to the boundary of the structure were not considered, in order to simplify the analysis and use only the equality constraint given in Equation (6). Using this assumption, the Lagrange function is defined

$$L = \operatorname{tr}\{\mathbf{P}\mathbf{x}_0\mathbf{x}_0^\mathsf{T}\} + \operatorname{tr}\{\mathbf{S}(\mathbf{A}_\mathbf{c}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A}_\mathbf{c} + \mathbf{Q} + (\mathbf{K}\mathbf{C})^\mathsf{T}\mathbf{R}\mathbf{K}\mathbf{C})\} \tag{7}$$

in which $\mathbf{S}$ is a symmetric matrix of Lagrange multipliers. The first order necessary conditions for optimality can be determined from the partial derivatives of L with respect to the independent variables:

$$\frac{\partial L}{\partial \mathbf{S}} = \mathbf{A}_\mathbf{c}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A}_\mathbf{c} + \mathbf{Q} + \mathbf{C}^\mathsf{T}\mathbf{K}^\mathsf{T}\mathbf{R}\mathbf{K}\mathbf{C} = 0 \tag{8}$$

$$\frac{\partial L}{\partial \mathbf{P}} = \mathbf{A}_\mathbf{c}\mathbf{S} + \mathbf{S}\mathbf{A}_\mathbf{c}^\mathsf{T} + \mathbf{x}_0\mathbf{x}_0^\mathsf{T} = 0 \tag{9}$$

$$\frac{1}{2}\frac{\partial L}{\partial \mathbf{K}} = \mathbf{R}\mathbf{K}\mathbf{C}\mathbf{S}\mathbf{C}^\mathsf{T} - \mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{S}\mathbf{C}^\mathsf{T} = 0 \tag{10}$$

$$\frac{1}{2}\frac{\partial L}{\partial \xi_{ij}} = \operatorname{tr}\left\{\left(\mathbf{S}\mathbf{C}^\mathsf{T}\mathbf{K}^\mathsf{T}\mathbf{R}\mathbf{K} - \mathbf{S}\mathbf{P}\mathbf{B}\mathbf{K}\right)\frac{\partial \mathbf{C}}{\partial \xi_{ij}}\right\} = 0 \qquad i = 1,\dots,s \qquad j = 1,\dots,m \tag{11}$$

In the case of full state feedback, the matrix $\mathbf{C}$ can be replaced by the identity matrix and $\mathbf{K}$ can be determined as a function of $\mathbf{P}$, such that

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^{\mathsf{T}}\mathbf{P} \tag{12}$$

Substituting this value of $\mathbf{K}$ in Equation (8) gives the algebraic Riccati equation:

$$\mathbf{A}^{\mathsf{T}}\mathbf{P} + \mathbf{PA} + \mathbf{Q} - \mathbf{PBR}^{-1}\mathbf{B}^{\mathsf{T}}\mathbf{P} = 0 \tag{13}$$

In this case, Equation (9), which is the only equation that has a dependence on system initial condition, is not necessary to determine the optimal gain, showing that it is optimal for any initial condition. When there is limited state information, matrix $\mathbf{C}$ is not square and the same procedure to decouple equations (8-11) cannot be applied. Since Equation (9) is dependent on system initial condition, so do the optimal gain and locations of sensors.

3 | DEALING WITH THE DEPENDENCE ON SYSTEM INITIAL CONDITIONS

A common way to deal with the dependency on system initial conditions is to adopt the suggestion given by Levine and Athans[22], which consists on optimizing the expected value of the cost function (5) assuming that the initial condition is a random variable uniformly distributed on the surface of a unit hyper-sphere. This proposition implies replacing the matrix $\mathbf{x}_0\mathbf{x}_0^{\mathsf{T}}$ by the identity matrix, which is the covariance matrix of $\mathbf{x}_0$ for this distribution, in equations (5) and (9). Despite simplifying the evaluation of the cost function, this approach involves an assumption on the distribution of the system initial conditions, which are usually unknown, and also gives the same importance for each state. Since any output controller has a performance criterion below that of the full state feedback controller, it would be desirable that the values of the cost functions for both controllers would be as closer as possible for any initial condition. This idea is the basis of the approach proposed in this paper, which consists in minimizing the ratio between the cost functions of the OSOF and LQR for the initial condition that maximizes this ratio, such that

$$\min_{(\mathbf{K},\xi)} \max_{\mathbf{x}_0} \frac{\mathbf{x}_0^{\mathsf{T}}\mathbf{P}_o(\mathbf{K},\xi)\mathbf{x}_0}{\mathbf{x}_0^{\mathsf{T}}\mathbf{P}_l\mathbf{x}_0} \tag{14}$$

in which $\mathbf{P}_o$ and $\mathbf{P}_l$ are the matrices associated with the OSOF and LQR cost functions, respectively. Given the weighting matrices $(\mathbf{Q}, \mathbf{R})$, the matrix $\mathbf{P}_l$ is a constant, and can be calculated by solving Equation (13). Since the cost function of the LQR is minimal for any initial condition, this ratio is always greater than or equal to one, and it also gives a measure of the maximum difference between LQR and OSOF considering performance and control effort. This ratio can also be interpreted as a controller performance metric, which has the benefit of being independent of measurement and excitation points of the structure.

The problem of determining the initial conditions that are stationary points of the ratio given in Equation (14) is similar to that of determining modes and natural frequencies using the Rayleigh's quotient[41]. Both quotients involves the ratio of two quadratic forms, and in the present case the matrices $\mathbf{P}_o$ and $\mathbf{P}_l$ play the roles of the stiffness and mass matrices, respectively. A minimization of the Rayleigh's quotient leads to the generalized eigenvalue problem involving the mass and stiffness matrix, such that the lowest eigenvalue represents the minimum value of Rayleigh's quotient[41]. In a similar fashion, it can be shown that the initial condition that maximizes the ratio given in Equation (14) is given by the eigenvector associated with the largest eigenvalue of the following generalized eigenvalue problem:

$$\mathbf{P}_o\mathbf{v} = \lambda\mathbf{P}_l\mathbf{v} \tag{15}$$

Substituting the eigenvector ($\boldsymbol{\phi}_m$) associated with the largest eigenvalue (λ_m) in the Equation (15), left multiplying by $\boldsymbol{\phi}_m^{\mathsf{T}}$ and isolating λ_m, it is shown that the largest value of the ratio of cost functions is equal to λ_m:

$$\lambda_m = \frac{\boldsymbol{\phi}_m^{\mathsf{T}}\mathbf{P}_o\boldsymbol{\phi}_m}{\boldsymbol{\phi}_m^{\mathsf{T}}\mathbf{P}_l\boldsymbol{\phi}_m} \tag{16}$$

Thus, the optimization proposed in Equation (14) can be rewritten as:

$$\min_{(\mathbf{K},\xi)} \lambda_m(\mathbf{P}_o, \mathbf{P}_l) \tag{17}$$

4 | DESCRIPTION OF OPTIMIZATION PROCEDURE

The proposal presented in the previous section to deal with the dependence on the initial condition involves the calculation of variables associated with the LQR cost function. Therefore, the first step in the design process is to choose weighting matrices ($\mathbf{Q}, \mathbf{R}$) and actuators locations that provide a full state feedback controller with desirable performance. These variables are used to calculate the matrix related to the LQR cost function ($\mathbf{P}_l$) and the full state feedback optimal gain ($\mathbf{G}$). With the LQR controller designed, it is possible to perform the optimization given in Equation (17), in order to obtain an OSOF controller that attempts to reproduce the performance of the full state feedback controller. This process can be applied to different models for flexible structures, whether continuous or discrete and including distributed or discrete sensors.

In this paper, the optimization considers a continuous model with discrete sensors, whose stiffness and inertia properties do not affect the structure dynamics. Different strategies can be used to solve the non-convex optimization given in Equation (17); we choose to apply the numerical method SQP with random initial guesses ($\mathbf{K}_0, \xi_0$). The random values ($\mathbf{K}_0, \xi_0$) were generated until the matrix $\mathbf{A}_c$ was Hurwitz, so that it was possible to solve Equation (6) for $\mathbf{P}_o$ and then Equation (15) to evaluate the cost function λ_m. The values ($\mathbf{K}_0, \xi_0$) that meet this requirement were used as an initial guess for the SQP algorithm to determine local optima. Throughout the optimization process, it was observed that the best local optima were given by products $\mathbf{KC}$ whose terms had the same order of magnitude as the terms of $\mathbf{G}$. This fact was used to improve the choice of the set of possible initial guesses for the output feedback gain. Lastly, no problems were found to determine combinations of the variables ($\mathbf{K}_0, \xi_0$) that stabilized the closed loop system. A schematic representation of the optimization procedure using a generic optimization algorithm is depicted in Figure 1.

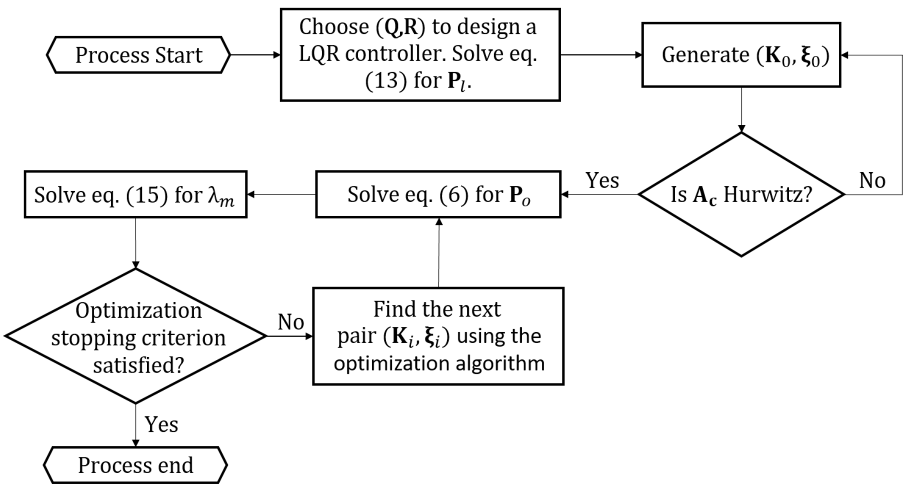

FIGURE 1 Schematic representation of the optimization procedure.

5 | NUMERICAL EXAMPLE

The methods outlined in this paper were tested on a common bi-dimensional structural element. The main challenges associated with non-collocated control, such as sensitivity and spillover, were taken into account in the analysis of the controller designed.

5.1 | Model and LQR design

A simply supported plate, whose properties are given in Table 1, was chosen to test the proposed methodology. This system was modelled using the finite element method (FEM) considering Kirchhoff-Love hypotheses. The Hermite cubics for nonconforming elements given by Reddy[42] were used as interpolation functions to construct a regular 20×20 mesh. A frequency bandwidth of 1 kHz was considered, which entailed a reduced-order model with 18 modes. Given the hypothesis that the sensors and actuators do not change the mass and stifness of the system, a continuous model was constructed using the approximate eigenfunctions in order to simplify the implementation of the SQP algorithm. These functions were calculated as in the Rayleigh-Ritz method since, except for convergence and embedding properties of mass and stiffness matrices, the FEM can be treated as a Rayleigh-Ritz method[41]. It was considered only one actuator that acts as a concentrated transversal force at position $\boldsymbol{\alpha} = [211.6 \;\; 162.9]\, mm$ (Figure 2) and sensors whose signals are processed in order to give the velocity of measured points. The actuator location was chosen so that it was close to the plate center and did not coincide with any mode shape nodes. Finally, it was adopted a damping factor of 0.5% for every mode, obtaining the state space matrices given in Equation (18),

TABLE 1 Plate properties.

Density (kg/m^3)	2700
Elastic Modulus (GPa)	69
Poisson Ratio	0.33
Length, x-direction (mm)	545
Width, y-direction (mm)	400
Thickness, z-direction (mm)	3

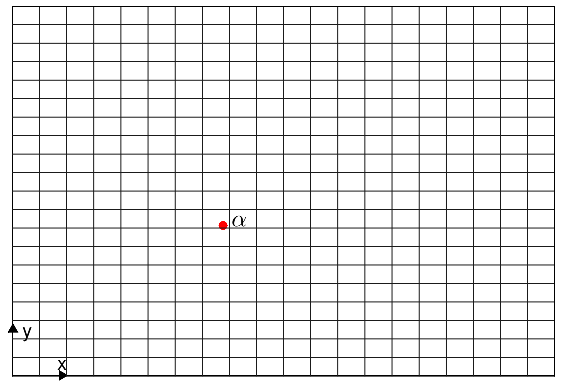

FIGURE 2 Plate mesh with actuator location.

$$\mathbf{x} = \begin{bmatrix} \boldsymbol{\eta} \\ \dot{\boldsymbol{\eta}} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\boldsymbol{\Lambda} & -\mathbf{D} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\phi}(\boldsymbol{\alpha}) \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{0} & \dots & \mathbf{0} \\ \boldsymbol{\phi}(\xi_1) & \dots & \boldsymbol{\phi}(\xi_s) \end{bmatrix}^{\mathsf{T}} \tag{18}$$

in which $\boldsymbol{\eta} \in \mathbb{R}^n$ is the modal coordinate vector, $\mathbf{I} \in \mathbb{R}^{n \times n}$ is the identity matrix, $\boldsymbol{\Lambda} \in \mathbb{R}^{n \times n}$ is a diagonal matrix of system eigenvalues or natural frequencies squared, $\mathbf{D} \in \mathbb{R}^{n \times n}$ is a diagonal matrix of damping and $\boldsymbol{\phi} \in \mathbb{R}^n$ is the vector of approximated eigenfunctions.

The weighting matrix **Q** was adopted as suggested by Meirovitch[43]:

$$\mathbf{Q} = \begin{bmatrix} \boldsymbol{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{19}$$

which leads to a minimization of system energy in modal coordinates. The value of the matrix **R** was chosen in order to obtain a LQR control with good compromise between control effort and performance. After some simulations, it was found that the value 0.001 satisfied this requirement.

5.2 | OSOF design and analysis

To determine the optimal locations and output gain, the optimization process described in section 4 was applied considering different numbers of sensors. When only one sensor was considered, its location was fixed as that of the actuator, producing a convex optimization for the determination of the output gain. Although this procedure was not necessary, it was carried out in order to observe the improvement in the cost function of the multi-sensor non-collocated case compared to that of the single-sensor collocated case. For the other cases, many iterations using different initial guesses were performed and some relevant local optima were determined, which are identified in Table 2. The largest difference between the cost functions of the OSOF and LQR controllers is expressed in percentage by $\lambda_m^* = (\lambda_m - 1) \times 100$. It is also shown the expected value of the cost function (J^*) assuming a spherical distribution for the initial condition, in order to make later comparisons with another way of dealing with the dependence on initial conditions.

As expected, the performance of the OSOF controller improved as the number of sensors increased – i.e., more information of the system configuration was available for feedback. The results given in Table 2 also show that with a significantly reduced

TABLE 2 Optimal gain and locations of sensors from the optimization of λ_m.

Nº Sensors	Gain (Ns/m)	Location (mm)	λ_m^*	J^*
1	$k = 24.87$	$\xi = [\,211.6 \quad 162.9\,]$	7.31%	5.232×10^6
2	$k_1 = 21.35$ $k_2 = 8.46$	$\xi_1 = [\,224.2 \quad 170.2\,]$ $\xi_2 = [\,166.1 \quad 118.2\,]$	4.05%	5.216×10^6
3	$k_1 = 10.87$ $k_2 = 13.00$ $k_3 = 13.68$	$\xi_1 = [\,261.9 \quad 123.7\,]$ $\xi_2 = [\,160.0 \quad 137.0\,]$ $\xi_3 = [\,223.8 \quad 197.0\,]$	1.91%	5.176×10^6
4	$k_1 = -4.49$ $k_2 = 6.89$ $k_3 = 24.41$ $k_4 = 5.81$	$\xi_1 = [\,434.9 \quad 164.3\,]$ $\xi_2 = [\,227.9 \quad 54.8\,]$ $\xi_3 = [\,201.4 \quad 170.0\,]$ $\xi_4 = [\,310.4 \quad 164.3\,]$	1.69%	5.159×10^6

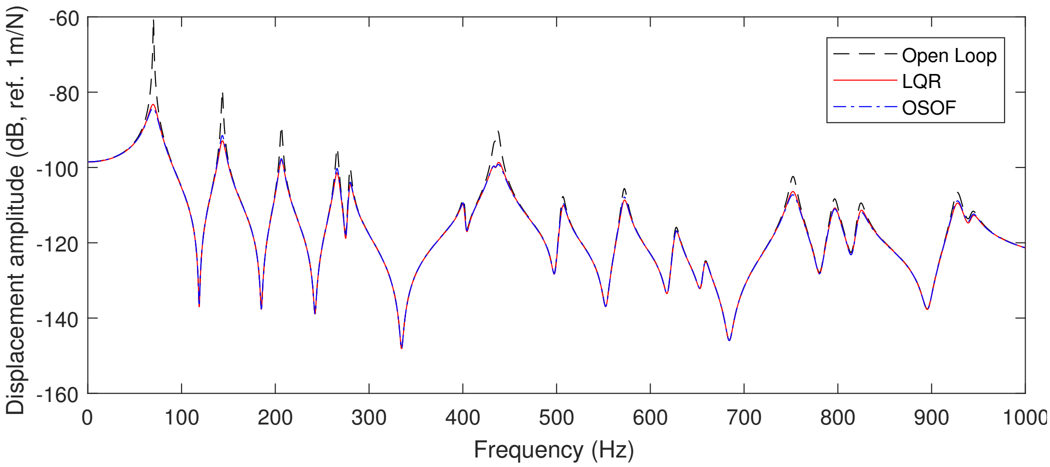

FIGURE 3 Frequency response of the transversal displacement of the plate in position $\boldsymbol{\alpha}$, in open- and closed-loop.

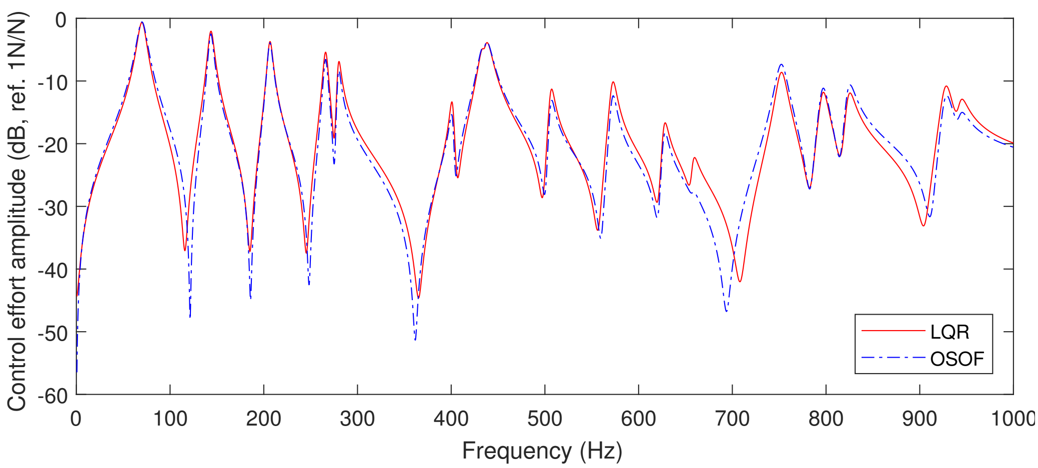

FIGURE 4 Frequency response of the control effort for LQR and OSOF.

number of sensors, the OSOF controller had a performance equivalent to LQR in terms of magnitude of the cost function. Since the cost function is given by a combination of factors, the proximity between the values does not guarantee equal performance or control effort. However, it was observed that both frequency and time responses had an almost negligible difference. The frequency response functions for the open- and closed-loop systems, using LQR and OSOF with 3 sensors, positioned according to Table 2, are shown in Figure 3. The vibration amplitude reductions for the first three modes, as shown in Figure 3, are 22.7, 13.5 and 9.1 dB for LQR and 23.7, 12.1 and 8.8 dB for OSOF. It is also worthwhile noticing that the control effort required for such performance, show in Figure 4, is also very similar for both controllers, LQR and OSOF.

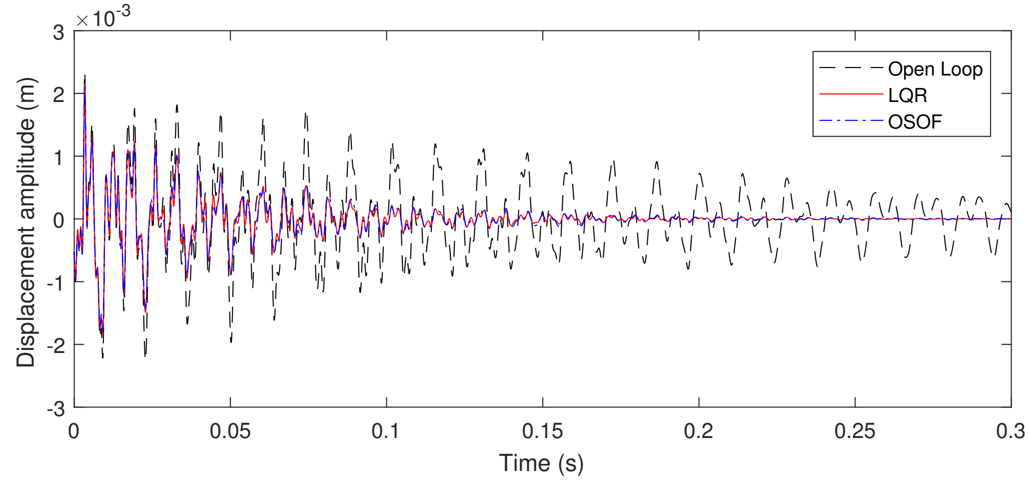

FIGURE 5 Time response of the transversal displacement of the plate in position $\boldsymbol{\alpha}$, in open- and closed-loop.

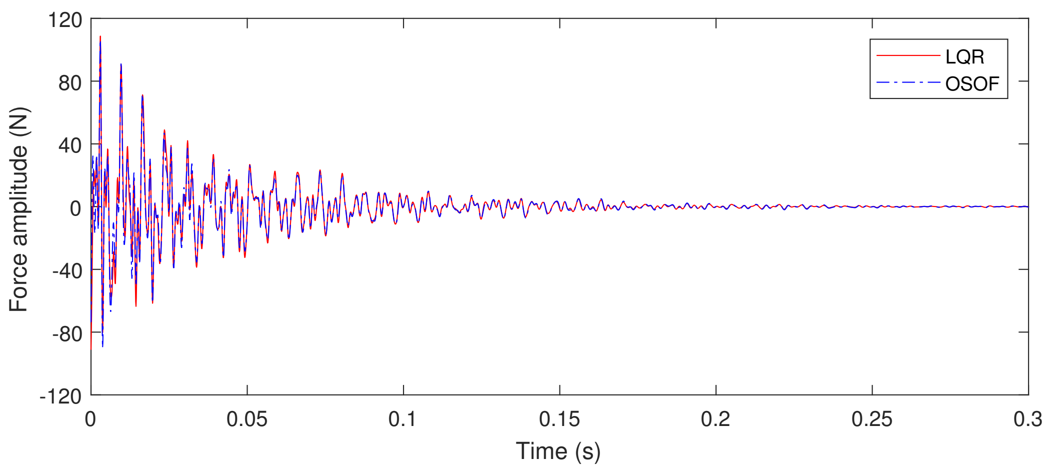

FIGURE 6 Time response of the control effort for LQR and OSOF.

The open- and closed-loop time responses of the transversal displacement of the plate at position $\boldsymbol{\alpha}$ are shown in Figure 5. The time response of the required control effort is shown in Figure 6. These responses were obtained considering the initial condition for which the largest difference between OSOF and LQR was observed ($\lambda_m^* = 1.91\%$). Notice that this initial condition is such that the displacement is within reasonable values, that is, smaller than the plate thickness in the present case.

To analyze how the optimal solution can be affected by the way of dealing with the dependence on the initial condition, it was also considered the approach presented by Levine and Athans[22], which consists in optimizing the expected value of the cost function ($\boldsymbol{J}^*$) assuming that the initial condition is a random variable uniformly distributed on the surface of a unit hyper-sphere. This optimization was performed varying the number of sensors and the results obtained are indicated in Table 3. An interesting result in this scenario is that, despite having a lower expected value of the cost function, the case with two sensors

TABLE 3 Optimal gain and locations of sensors from the optimization of $\boldsymbol{J}^*$.

Nº Sensors	Gain (Ns/m)	Location (mm)	λ_m^*	$\boldsymbol{J}^*$
1	$k = 19.03$	$\xi = [\,211.6 \quad 162.9\,]$	17.29%	5.189×10^6
2	$k_1 = 19.92$ $k_2 = -5.22$	$\xi_1 = [\,211.1 \quad 153.1\,]$ $\xi_2 = [\,224.2 \quad 365.5\,]$	18.78%	5.170×10^6
3	$k_1 = 11.75$ $k_2 = 18.93$ $k_3 = 15.31$	$\xi_1 = [\,258.2 \quad 169.1\,]$ $\xi_2 = [\,188.0 \quad 170.2\,]$ $\xi_3 = [\,216.4 \quad 17.9\,]$	3.60%	5.148×10^6
4	$k_1 = 18.56$ $k_2 = 10.10$ $k_3 = 6.52$ $k_4 = -2.18$	$\xi_1 = [\,189.6 \quad 172.6\,]$ $\xi_2 = [\,258.3 \quad 171.3\,]$ $\xi_3 = [\,220.6 \quad 60.8\,]$ $\xi_4 = [\,441.3 \quad 178.6\,]$	2.33%	5.143×10^6

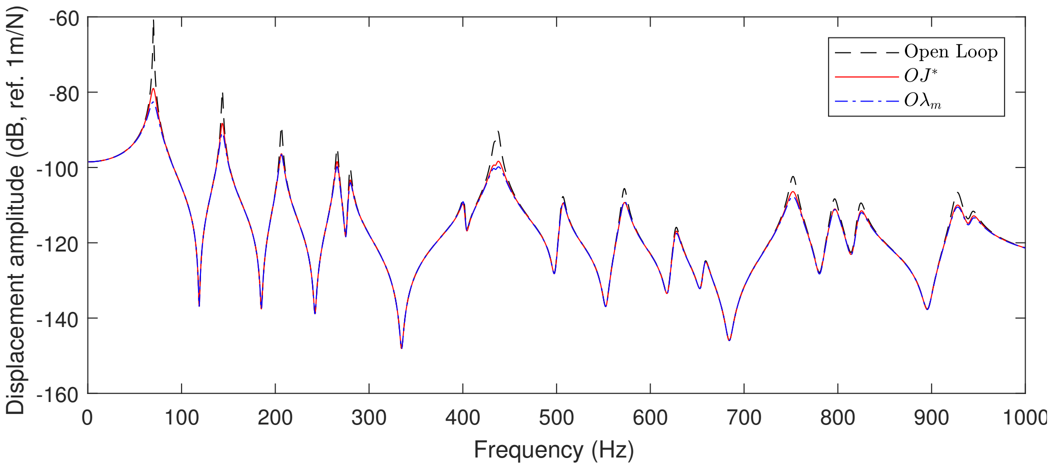

FIGURE 7 Frequency response of the transversal displacement of the plate in position $\boldsymbol{\alpha}$, in open- and closed-loop.

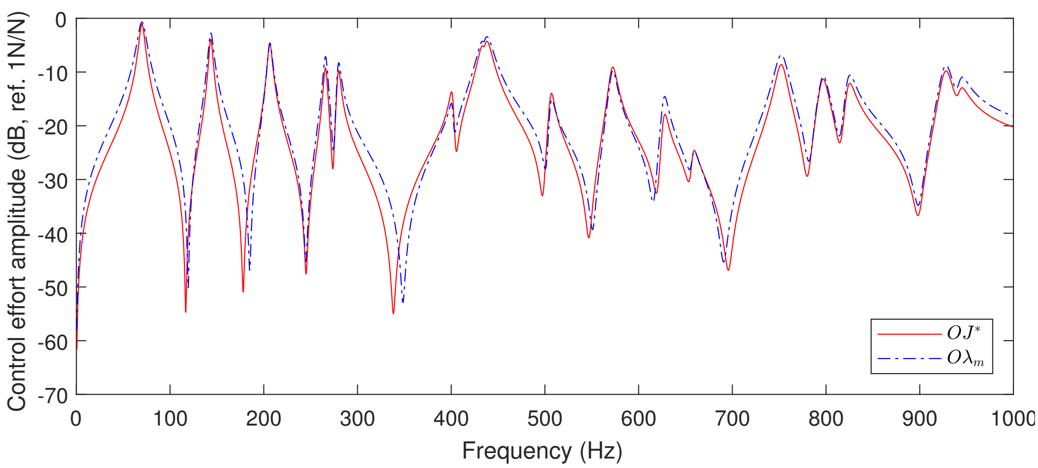

FIGURE 8 Frequency response of the control effort for the controllers obtained from the optimization of λ_m and J^*.

had a larger difference for the LQR than the case with one sensor when considering the initial condition that maximizes the ratio between the cost functions.

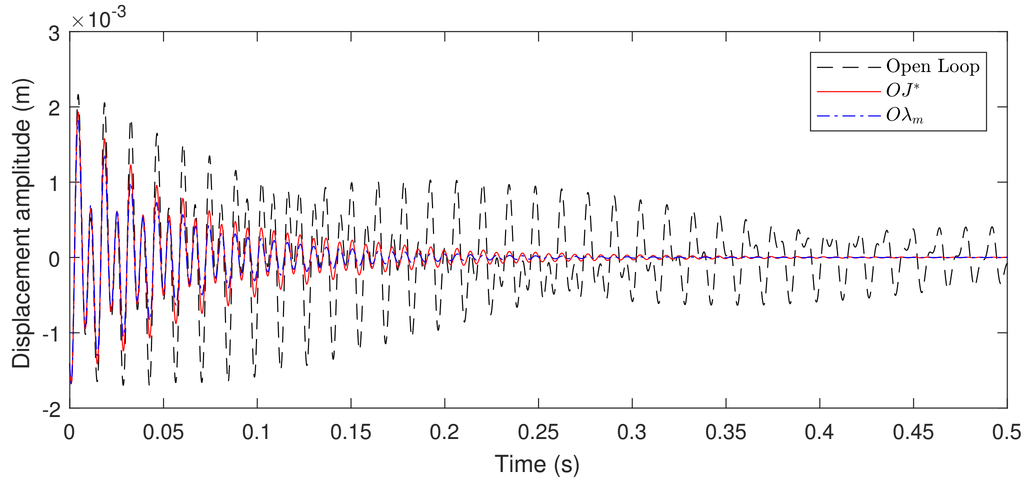

FIGURE 9 Time response of the transversal displacement of the plate in position $\boldsymbol{\alpha}$, in open- and closed-loop.

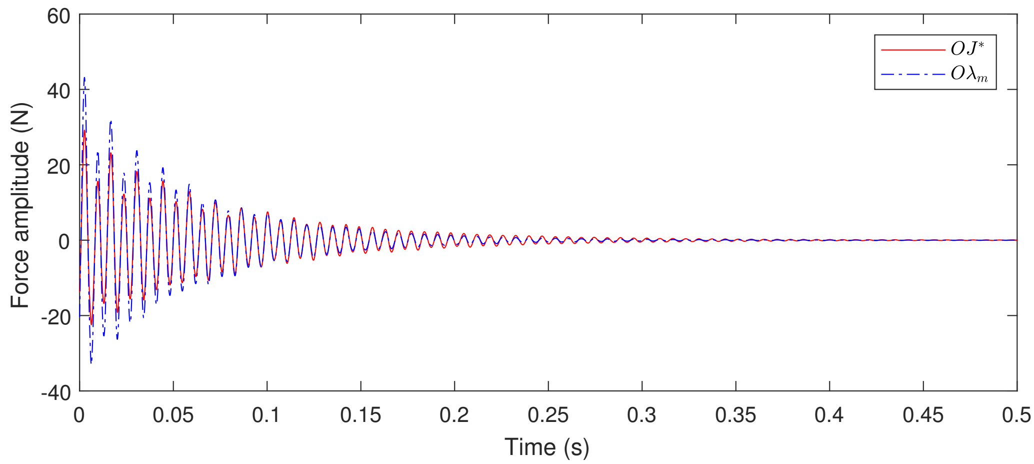

FIGURE 10 Time response of the control effort for the controllers obtained from the optimization of λ_m and $\boldsymbol{J}^*$.

In Figures 7 and 8, there is a comparison between frequency responses for the controller obtained from the optimization of λ_m ($O\lambda_m$) and the one obtained from the optimization of $\boldsymbol{J}^*$ ($\boldsymbol{OJ}^*$), both with two sensors. Although the response curves were slightly different, the case involving the optimization of λ_m stood out. This behavior was also observed for the other cases considering different numbers of sensors. Combining this result with the fact that the time necessary for the optimization of both cases is similar also contributes for the preference of the optimization of λ_m in the OSOF controller design. Figures 9 and 10 also compare the time response and control effort for both controllers with two sensors considering the initial condition associated with $\lambda_m^* = 18.78\%$.

5.3 | Effect of sensors locations, control gains variations and frequency range increase on the control performance

Major concerns of the application of non-collocated sensors and actuators for structural control are instability and spillover. To evaluate performance and stability when the design variables of the OSOF controller (gain and sensors locations) are varied, it was considered the case with two sensors obtained from the optimization of λ_m. Although this case has been chosen in order to obtain a graphical visualization of the results, similar conclusions can also be obtained for the cases with more sensors. First, the optimal gains given in Table 2 were fixed, the locations of both sensors were varied in both directions and the corresponding values of λ_m were calculated. The effects of both sensors locations on the relative performance of the controlled system in relation to LQR are indicated in Figure 11. It can be noticed in this figure that the variation of the position of the first sensor had greater impact on the cost function λ_m, which may be related to the fact that its gain is larger. To obtain a cost function of $\lambda_m = 1.15$, such that the controlled system still would perform better than the one with two sensors obtained from the optimization of $\boldsymbol{J}^*$ ($\lambda_m = 1.19$ from Table 3), it was necessary to change radially the location of first sensor in at least 18.4 mm, while for the second sensor this distance reached 36.7 mm.

A similar analysis was done for sensors gains, in which the sensors locations were fixed and the output gains were varied arbitrarily. The results for this case are indicated in Figure 12. To get a cost function of $\lambda_m = 1.15$, it was necessary to vary individually the first sensor gain in at least 26, 7% and the second sensor gain in at least 79.3%. It is noticed in both Figures 11 and 12 that for small variations around the optimal point, the performance of the control system is practically unaffected, which may be relevant in the case of lack of accuracy in the application for a real system. Other important result that can be drawn from the sensitivity analyses for both sensors and gains variations is that the closed system remained stable (finite λ_m) even for large variations around the optimal values.

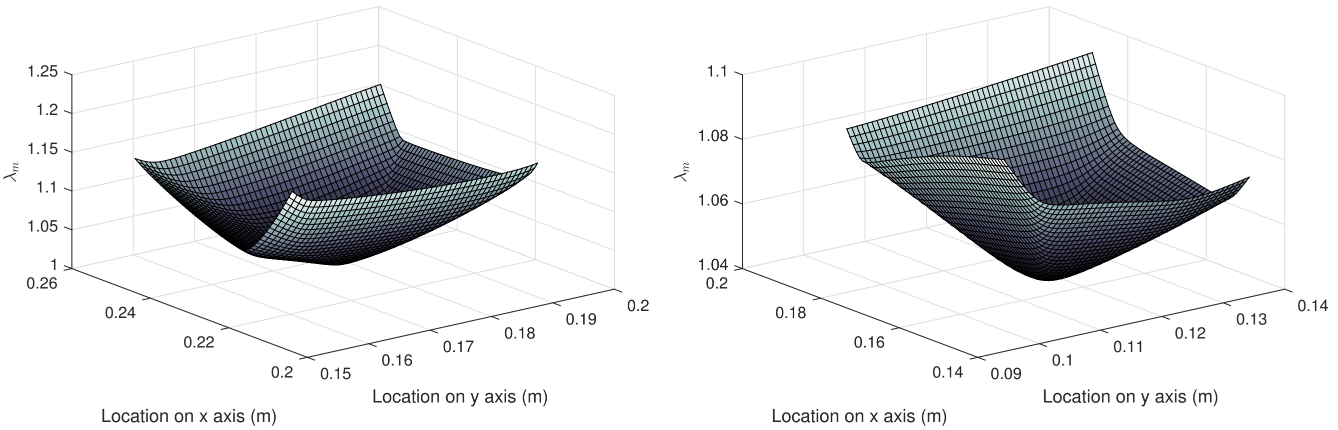

FIGURE 11 Effect of sensor 1 (left) and sensor 2 (right) displacements from the optimal locations on the cost function λ_m.

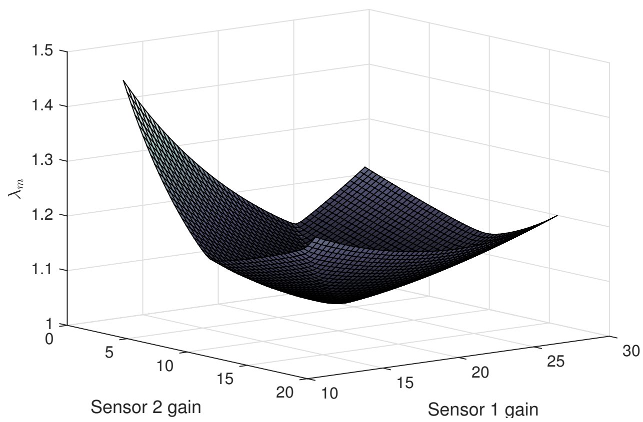

FIGURE 12 Effect of sensors gains variations from the optimal values on the cost function λ_m.

Another source of instability is given by the presence of higher frequency modes not considered in the reduced model. To deal with this problem, one can increase the range of frequencies considered, in order to cover the entire range of possible excitation of the structure, or represent the effects of the neglected modes through modifications in the original model, such as static correction[19]. In order to evaluate how the OSOF controller would respond to the spillover effect, it was considered an extension of the frequency bandwidth to 2 kHz, which implied the use of a reduced model with 40 modes. The optimal gains and locations obtained for the system with 18 modes (Table 2) were applied to the extended system and, for all sensor numbers indicated in Table 2, it was found that the closed loop system remained stable. The response of the OSOF controller was also compared with a new LQR designed for the augmented system, using the same weighting matrices. Comparing with this LQR, the values of λ_m^*, considering the cases of 1 to 4 sensors, increased to 23.78%, 30.30%, 31.69% and 45.30%, respectively. These results show that the systems with a larger number of sensors are more sensitive to insertion of residual modes, and suggest that they may also be more sensitive to spillover.

The case with three sensors was taken as an example to illustrate the frequency response, which was also compared with LQR in Figures 13 and 14. The behavior of both controllers in the frequency bandwidth of 1 kHz remained practically unchanged when compared to Figure 3, presenting a larger difference in control effort out of the frequency range for which the OSOF was designed. Despite of that, even after the insertion of 22 residual modes, the OSOF controller with 3 sensors did not destabilize the system and still had a performance similar to LQR, which in this case would require at least 80 sensors. Similar results were observed when considering different numbers of sensors.

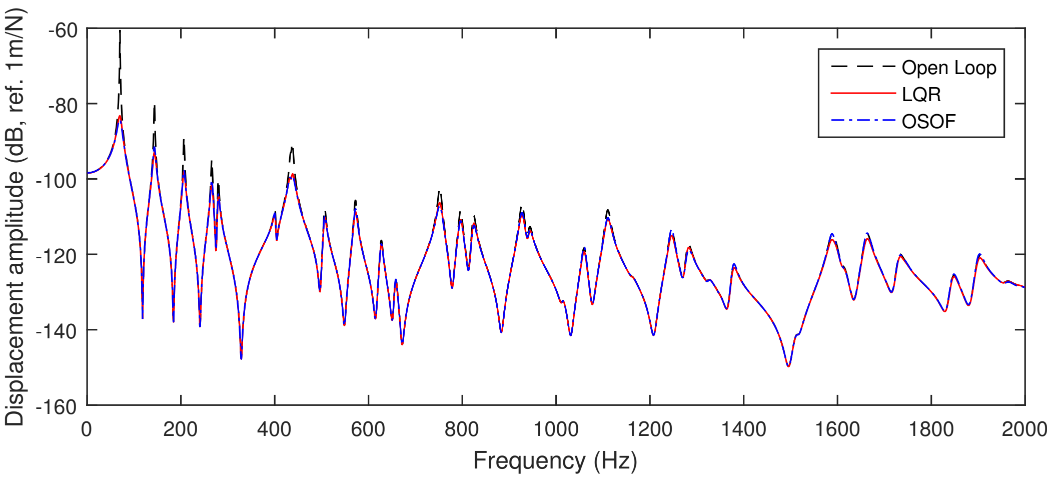

FIGURE 13 Frequency response of the transversal displacement of the plate in position $\boldsymbol{\alpha}$, in open- and closed-loop, considering an extended frequency range including vibration modes not considered in the OSOF control design.

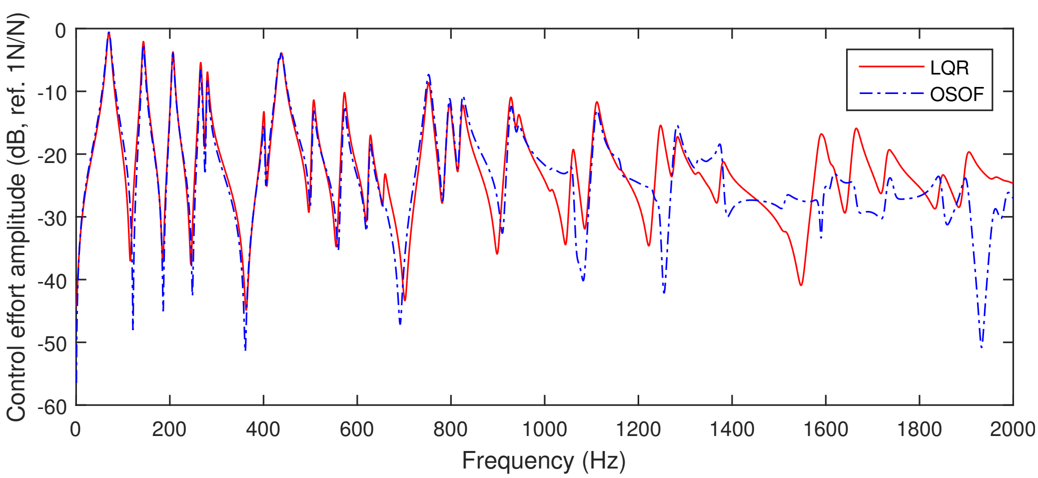

FIGURE 14 Frequency response of the control effort for the controllers LQR and OSOF, considering an extended frequency range including vibration modes not considered in the OSOF control design.

6 | CONCLUSIONS

In this paper, the problem of active vibration control of flexible structures using optimal static output feedback was studied. Necessary conditions for optimality, considering additionally sensors locations as optimization variables, were presented in order to highlight the dependence on system initial conditions. A new approach to deal with this dependence, which consists on approaching the performances of OSOF and LQR for any initial condition, was developed. The proposed methodology was applied to control a simply supported plate, and the results obtained were compared with LQR and an OSOF technique that uses another approach to deal with the dependence on system initial conditions. The numerical example studied considered a different number of sensors and actuators, giving rise to the non-collocated control. From this perspective, an additional analysis was done to evaluate how the designed controller would perform considering the major problems of non-collocated control, such as sensitivity and spillover. The main contributions that can be drawn from the results are:

- The methodology developed demonstrated that it is a powerful technique to find control gains and sensors locations, since it reproduces the response of the full state feedback controller with a significantly reduced number of sensors. In terms of the cost function, the difference between OSOF and LQR was less than 2% for a ratio of number of sensors by number of states of approximately 0.08;
- A comparison between the way of dealing with the initial conditions dependence proposed in the article and the one commonly used showed that the former reproduced better the behavior of the full state feedback controller when using few sensors;
- The controlled system presented low sensitivity to variations of control gains and sensors locations. After large variations around the optimal control gains and positions of sensors, the closed loop system maintained a difference for the LQR cost function of less than 15%;
- The analysis of the effects of the spillover showed that even after doubling the frequency bandwidth, the closed loop system remained stable and well-performing. The results also indicated that the cases with more sensors may be more sensitive to sensitive to the insertion of residual modes.

Further investigation of the proposed methodology will be directed to applications with piezoelectric sensors and actuators and a detailed analysis of robustness properties considering plant uncertainties.

7 | ACKNOWLEDGEMENTS

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