Exploring Mathematical Methodology

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We present a record of our explorations on the development of some kind of methodology of mathematics – at least the approach to mathematics that's called "theory-building" [4]. We do so via the exploration of a case study, which is transporting a proof across mathematical theories. We start with the observation of a similarity between a well-established result – the Curry-Howard Correspondence (CHC) –, and a new contribution in the form of a particular formal system – [6]'s computational paths. Throughout the process, we introduce several concepts relating to formal systems such as those under the CHC, plus a CHC-like theorem explaining some features of computational paths, and the start of what Christopher Alexander calls a "pattern language", but for mathematics. Finally, we relate the methods that we found with one account of how Grothendieck approached doing mathematics [5].

In our exploration, we chose to engage in what could be called praxis: performing the kind of goal-oriented activity we want to schematize, and then, by recording our actions while doing so, we can later reflect about our choices (which will inform our future actions). Faced with such a description, we analyze it, synthesize our findings, and come up with a first contender for a method. Following this, we may critique, suggest improvements, and test it all again. So, to develop our methods, we must half-paradoxically put them to practice.

We started with the observation of a similarity between two situations. On the one hand, is the traditional, pedestrian Curry-Howard Correspondence between intuitionistic propositional logic and the simply typed λ-calculus (with products, sums, etc). On the other, there's what [6] call "computational paths", which are term-like objects that describe changes require to transform a term into another. Intuitively, they looked not only analogous but actual instances of the same general pattern.

To investigate this matter, keeping in mind the need of maintaining records of it all, we started by writing down the statement and proof in a semi-formalized way (just plain language, but capturing as many details as possible). By performing what we later understood to be a form of presupposition analysis, we came up with a graph representing the structure of the theory, together with dependency relations between its components, the statement and each step of the proof of the theorem.

This graph displayed a remarkable amount of redundancy. This means that concepts can be factored out, and this is in practice a form of compression of the theory. But this is a generalizable pattern: we use our intuition to identify in the current domain of discourse some latent structure; we named it, factored it out, and connect it to the original context via some kind of implementation of that structure. The next crucial observation is that there is a relatively low-effort almost-trivial generalization of the CHC that could be pulled along this loose thread: "given two logics (seen as a form of fibration of types), a morphism between them, and a function between their total spaces of proofs which commutes with the rest of the arrows, we get a form of CHC".

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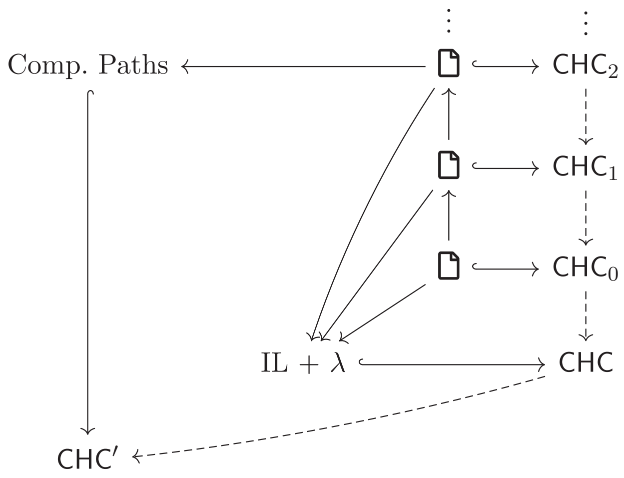
By repeating this process, each time seeking to explain the structures uncovered in the previous step, we engage in a process we call unraveling. And it gives us, besides a methodological guide, a particular form in which to record our results. That is, we go through this process, all the while writing down the generated artifacts in a structured way. We end up with a sequence of theories, each explaining the one below, but still being present in the original situation and even allowing for a new version of the original theorem.

All throughout the process of unraveling, we extract conclusions from each of the steps, such as that the relation between proofs and terms and their types is one of evaluation (maybe call it "splitting"); that weakening corresponds to variable renaming[1]; or that assumption introduction corresponds to variable declaration.

With the unraveling done, we turn to the original observation: the similarity of this situation with [6]'s computational paths. By this point the situation was clear enough. We just chose an appropriate level of abstraction whose structures were easily seen to be present in that other situation. By proving this, we effectively perform a sort of push-forward of the original theorem, thus creating a new theorem in this new context: that computational paths arise from intuitionistic type theory through a form of CHC-like transformation.

The resulting web of theories can be illustrated with the following graph:



We chose to present these methods by using Christopher Alexander's pattern languages [1]. Despite originating in architecture, this concept found ample adoption in the world of software development (cf. [3]), and can even be found in other areas (cf. [2]). In short, this means we can visualize the methods we found as a pattern, which conceivably could become part of a larger language of patterns. Besides, during our exploration we found several new concepts and constructions, not to mention a series of alternative readings of a well-established theorem relating to these structures. Finally, we also proved a theorem reconstructing [6]'s computational paths as an instance of a CHC-like construction.

References

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[1] Or de Brujn index shifting, depending on the chosen formalism.

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