Geometria. — *Surfaces with non-zero normal curvature tensor.*
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RIASSUNTO. — Studiamo la topologia differenziale e la geometria delle superfici compatte con curvatura normale non-nulla in spazio della curvatura costante.

An important object in the geometry of higher-codimensional submanifolds is the curvature tensor of the normal bundle ν. The normal directions annihilating it are geometrically simple in the sense that their second fundamental tensors (= shape operators) commute with all the others. It is therefore natural to split the normal space ν_p into this annihilator ν_p^0 and its orthogonal complement ν_p^*. We shall consider here surfaces, for which this yields a vector bundle splitting of the normal bundle $\nu = \nu^* \oplus \nu^0$ with a plane bundle ν^*. We shall prove two global results, a topological one and a geometrical one, which are based on information about the 'intrinsic' curvature of the bundle ν^* in relation to that of ν. The topological result generalizes, in particular, a characterization of the Veronese surface due to Little [7], Asperti [1], while at the same time we simplify considerably the earlier proofs of that characterization. The geometrical result deals with minimal surfaces of constant normal curvature, and improves similar results of Itoh [4] and Kenmotsu [5].

Let M be an oriented, compact, Riemannian surface with metric $\langle \cdot\cdot\cdot \rangle$ and complex structure J. Let Q be an n-dimensional Riemannian manifold of constant curvature, and let $f : M \to Q$ be an isometric immersion with normal bundle $\nu = \nu(f)$. Then ν is in the obvious way a Riemannian vector bundle with a distinguished covariant derivative $\nabla^{\perp}$ and a corresponding curvature tensor $R^{\perp}$ given by

$$R^{\perp}(X, Y)\, u = \nabla^{\perp}_X \nabla^{\perp}_Y u - \nabla^{\perp}_Y \nabla^{\perp}_X u - \nabla^{\perp}_{[X,Y]} u$$

for tangent fields X, Y and normal field u.

For $p \in M$ put

$$\nu_p^0 := \{ u \in \nu_p \mid R^{\perp}(\cdot\cdot\cdot)\, u = 0 \}$$

and

$$\nu_p^* := \text{orthogonal complement of } \nu_p^0 \text{ in } \nu_p .$$

(*) Nella seduta del 25 novembre 1982.

Remarks. (1) The assumed orientability is not a serious restriction, since, if M is not oriented we can use the twofold oriented covering.

(2) The curvature of the ambient manifold does not appear in our theorems. This is less strange than it appears at first sight, because it is implicitly taken care of by the non-vanishing assumption on $R^{\perp}$ and the information we assume for the normal curvatures.

LEMMA 1. *If* $R_p^{\perp} \neq 0$, *then* $\dim \nu_p^* = 2$.

Proof: We use the Ricci equation

$$\begin{aligned} R^{\perp}(X, Y)\, u &= \alpha(X, A_u Y) - \alpha(A_u X, Y) \\ &= \alpha(X, \phi(u) Y) - \alpha(\phi(u) X, Y), \end{aligned}$$

where $\phi(u) := A_u - 1/2\,(\mathrm{trace}\ A_u)\,\mathrm{Id}$. Since the space of tracefree, symmetric endomorphisms of the two-dimensional tangent space is of dimension 2, we see that $\dim \nu_p^* = \operatorname{codim} \nu_p^0 \leq 2$. The skew-symmetry of $R^{\perp}(X, Y)$ on ν_p implies then $\dim \nu_p^* \neq 1$, and the assertion follows.

From now on we assume

$$R_p^{\perp} \neq 0 \qquad \text{for all} \quad p \in M .$$

Then

$$\nu = \nu^* \oplus \nu^0$$

as an orthogonal direct sum of Riemannian vector bundles. In particular, the plane bundle ν^* inherits a canonical covariant derivative ∇^*, and we define an orientation as follows: (u, v) is positively oriented, if $\langle R^{\perp}(X, JX)\, v, u\rangle > 0$ for one, and hence for all $X \in T_p M \setminus \{0\}$. (Note: If the codimension is 2, then $\nu^* = \nu$, but the orientation may differ from the one induced by the orientation of the ambient space).

We shall now prove that the oriented plane bundle ν^* is isomorphic with an intrinsically defined bundle.

PROPOSITION. *Let* $\sigma_0 = \sigma_0(M)$ *denote the bundle of symmetric endomorphisms of the tangent bundle which have zero trace. Then*

$$B \to JB = -BJ$$

is a complex structure inducing an orientation of σ_0. *The map* $\phi : \nu^* \to \sigma_0$ *defined by*

$$\phi(u) := A_u - 1/2\ (\mathrm{trace}\ A_u)\ \mathrm{Id}$$

is an orientation-preserving isomorphism.

Proof: By the proof of the lemma above, ϕ is an isomorphism. Given $B \in \sigma_0(M)_p$, $B \neq 0$, and $u, v \in \nu_p^*$ such that $\phi(u) = B$, $\phi(v) = JB$, we have

$$\begin{aligned}\langle R^{\perp}(X, JX) v, u\rangle &= -\langle [A_u, A_v] X, JX\rangle \\ &= -\langle [\phi(u), \phi(v)] X, JX\rangle \\ &= -\langle (BJB - JBB) X, JX\rangle \\ &= \langle BJX, JBX\rangle + \langle BX, BX\rangle \\ &> 0 .\end{aligned}$$

for $X \neq 0$.

For the oriented plane bundle ν^* over the compact oriented surface M, we have an integer-valued Euler characteristic $\chi(\nu^*)$. Our first main result is

THEOREM 1. *For a compact, oriented, Riemannian surface* M *isometrically immersed with nowhere vanishing normal curvature tensor into a Riemannian manifold of constant curvature, we have*

$$\chi(\nu^*) = 2\,\chi(M) .$$

Proof. By the above proposition $\chi(\nu^*) = \chi(\sigma_0(M))$. For any $X \in T_p M$ let

$$B(X) := 2\langle X, \cdot\rangle X - \langle X, X\rangle \, \mathrm{Id} \in \sigma_0(M)_p .$$

Then $B(\cos t\, X + \sin t\, JX) = \cos 2t\, B(X) + \sin 2t\, JB(X)$. Therefore the index formula for the Euler characteristic applied to a generic vector field X, and to $B(X)$ respectively, yields the theorem. The intrinsic curvature K^* of ν^* is defined as

$$K^*(p) := \langle R^*(X, JX) v, u\rangle ,$$

where R^* is the curvature tensor of ν^*, and (X, JX) and (u, v) are positively oriented orthonormal bases of $T_p M$, and ν_p^* respectively. Then we have the

COROLLARY. *Under the hypotheses of Theorem 1, if* $K^* = 0$, *or* $K^* > 0$, *or* $K^* < 0$, *then* $\chi(M) = 0$, *or* $\chi(M) > 0$, *or* $\chi(M) < 0$ *respectively*.

If at each point $p \in M$ there exists $X \in T_p M \setminus \{0\}$ such that ν^* is parallel in ν in the direction of X, i.e.

$$(2) \qquad (\nabla_X^{\perp} \nu^*)_p \subset \nu_p^* ,$$

then $\chi(M) > 0$.

Proof. The first assertion is trivial, because $\chi(\nu^*) = \int_M K^*$, see [7]. Moreover (2) implies $\langle R^*(X, JX) v, u\rangle_p = \langle R^{\perp}(X, JX) v, u\rangle$ as an immediate consequence of the definition of ∇^*. Then $K^*(p) > 0$ according to our choice of orientaion for ν^*.

Remarks. (1) When $n = 4$, then $\nu^* = \nu$, and the Corollary implies $\chi(M) > 0$. Hence M must be a sphere. Since (by the proposition) its normal bundle is uniquely determined, it follows from a well-known fact of differential topology, that the immersion is regular homotopic with the twofold covering of a Veronese surface, if the ambient space is a sphere. This is a theorem of Little [7], Asperti [1].

(2) In [2] Dajczer generalizes the Little–Asperti theorem to higher codimensions, using a parallelity condition stronger than condition (2) of the Corollary.

For a surface with nowhere vanishing $R^{\perp}$ we define the normal curvature

$$K^{\nu}(p) := \langle R^{\perp}(X, JX)\, v, u \rangle ,$$

where (X, JX) and (u, v) are positively oriented orthonormal bases of $T_p M$ and ν_p^* respectively. (Hence K^{ν} is, by definition, positive). When we consider surfaces with constant mean curvature and constant K^{ν}, then a sign condition for K^* yields a rigidity theorem:

THEOREM 2. *Let* M *be a compact, oriented, Riemannian surface isometrically immersed with nowhere vanishing normal curvature tensor into a Riemannian manifold of constant curvature. Assume that the mean curvature vector of the immersion is covariantly constant in the normal bundle, and the normal curvature* K^{ν} *is constant.*

If $K^* \geq 0$, then K^* and the Gaussian curvature K of M are constant, and moreover

$$K = 1/2\, K^* .$$

Remarks. (1) If M is minimal in a sphere, and $K^* > 0$, then we obtain a minimal S^2 of constant curvature in S^n. These minimal immersions were classified by Do Carmo and Wallach [3]; see also Otsuki [8]. Itoh [4] has a similar theorem under the assumption $K^{\nu} > 4\, k_2^2$, where k_2 is the second curvature. This condition implies $K^* > 0$, but is in general stronger. Our proof of Theorem 2 uses ideas of [4].

(2) For a minimal M with $K^* = 0$ we obtain a flat minimal torus. These were studied by Kenmotsu [6].

Before we turn to the proof of Theorem 2, we note the following: Let $\sigma_0(M)$ be endowed with the metric $\langle A, B \rangle := \frac{1}{2}$ trace (AB). If for some $p \in M$ the isomorphism $\phi_p : \nu_p^* \to \sigma_0(M)$ is not homothetic, then there exist a neighborhood U of p, and smooth, positively oriented orthonormal fields u, v in $\nu^* \mid U$, such that $\phi(u), \phi(v)$ are orthogonal. (This follows from the fact that $\sphericalangle(\phi(u), \phi(v)) + \sphericalangle(\phi(v), \phi(-(u)) = \pi)$. Let λ and μ denote the positive eigenvalue functions of $\phi(u)$ and $\phi(v)$. We may assume that on U there exists a unit tangent field X such that $\phi(u)\, X = \lambda X$. Then, with B as in the proof of Theorem 1,

$$\phi(u) = \lambda B(X) \tag{3}$$
$$\phi(v) = \mu J B(X) .$$

On the other hand, if ϕ is homothetic on a neighborhood of p, we can start with *any* positively oriented orthonormal fields u, v, and obtain the same result. In this case $\lambda = \mu$. If $\lambda(p) \neq \mu(p)$, we may in addition assume that $\lambda > \mu$ on U: switch to $(v, -u)$ instead of (u, v), if necessary.

From the Ricci identity (1) and (3) we obtain

$$K^{\nu} = 2\lambda\mu . \tag{4}$$

Let H denote the mean curvature normal vector characterized by

$$\langle H, u\rangle = 1/2 \text{ trace } A_u \tag{5}$$

for all u.

Proof of Theorem 2:

Case 1. There exists $p \in M$ such that ϕ_p is not homothetic. Then we choose U, u, v, etc. as above. We put $C := B(X)$. Since $\langle C, C\rangle = 0$, we have

$$\nabla C = \omega \otimes JC$$

for some 1-form ω, which is easily identified as

$$\omega = 2\langle \nabla X, JX\rangle .$$

We set

$$\theta := \langle \nabla u, v\rangle .$$

Recall that $A_u = \phi(u) + \frac{1}{2}(\text{trace } A_u)\, \text{Id} = \lambda G + \langle H, u\rangle\, \text{Id}$, and $A_{\nabla_z u} = A_{\theta(z)v} = \theta(z)\,\mu JC + \theta(z)\langle H, v\rangle\, \text{Id}$.

Then, since H is parallel,

$$\begin{aligned}(\nabla_Z A)_u &= \nabla_Z A_u - A_{\nabla_Z} u \\ &= d\lambda(Z)\, C + \lambda\nabla_Z C + \langle \nabla_Z H, u\rangle + \langle H, \theta(Z)\, v\rangle\, \text{Id} - \theta(Z)\,\mu JC - \\ &\qquad - \theta(z)\langle H, v\rangle\, \text{Id} \\ &= d\lambda(Z)\, C + (\lambda\omega(Z) - \mu\theta(Z))\, JC .\end{aligned}$$

Using Codazzi's equation on Z and JZ this gives

$$d\lambda \circ J = \lambda\omega - \mu\theta . \tag{6}$$

Similarly, starting with v instead of u, we have

$$d\mu \circ J = \mu\omega - \lambda\theta . \tag{7}$$

Since $K^{\nu} = 2\lambda\mu =$ constant, the equations (6) and (7) imply

$$\begin{aligned}0 &= 2\lambda\omega - (\lambda^2 + \mu^2)\,\theta \\ &= K^{\nu}\,\omega - (\lambda^2 + \mu^2)\,\theta\end{aligned} \tag{8}$$

and

$$1/2\, d\,(\lambda^2+\mu^2)\circ J = (\lambda^2+\mu^2)\,\omega - 2\,\lambda\mu\theta$$

or, using (8),

$$(9) \qquad 1/2\, d\,(\lambda^2+\mu^2)\circ J = (\lambda^2+\mu^2)\,\omega - K^\nu\,\theta = \frac{(\lambda^2+\mu^2)^2-(K^\nu)^2}{\lambda^2+\mu^2}\,\omega\,,$$

and

$$(10) \qquad 1/2\, d\,(\lambda^2-\mu^2)\circ J = (\lambda^2-\mu^2)\,\omega\,.$$

Differentiating (8) we have, by the definitions of ω and θ,

$$2\,K^\nu\langle R\,(Y,Z)\,X\,,JX\rangle = (\lambda^2+\mu^2)\,\langle R^*(Y,Z)\,u\,,v\rangle + (d\,(\lambda^2+\mu^2)\wedge\theta)\,(Y,Z)\,,$$

whence

$$\begin{aligned}2\,K^\nu\,K &= (\lambda^2+\mu^2)\,K^* + (d\,(\lambda^2+\mu^2)\wedge\theta)\,(JX\,,X)\\ &= (\lambda^2+\mu^2)\,K^* - \frac{(\lambda^2+\mu^2)^2-(K^\nu)^2}{(\lambda^2+\mu^2)}\,(\omega\circ J)\wedge\frac{K^\nu}{\lambda^2+\mu^2}\,\omega\,(JX\,,X)\\ &= (\lambda^2+\mu^2)\,K^* + K^\nu\,\frac{(\lambda^2+\mu^2)^2-(K^\nu)^2}{(\lambda^2+\mu^2)^2}\,\|\,\omega\,\|^2\,,\end{aligned}$$

or

$$(11) \qquad 2\,K = (\lambda^2+\mu^2)\,\frac{K^*}{K^\nu} + \frac{(\lambda^2+\mu^2)^2-(K^\nu)^2}{(\lambda^2+\mu^2)^2}\,\|\,\omega\,\|^2 \geq 0\,.$$

On the other hand, since $(\lambda^2+\mu^2)^2-(K^\nu)^2 = (\lambda^2-\mu^2)^2$, (10) gives

$$1/2\, d\,(\log\,(\lambda^2-\mu^2))\circ J = \omega$$

or

$$1/4\, d\,(\log\,(\lambda^2+\mu^2)^2-(K^\nu)^2))\circ J = \omega\,.$$

Differentiation yields

$$(12) \qquad 1/4\,\Delta\,\log\,((\lambda^2+\mu^2)^2-(K^\nu)^2) = 2\,K \geq 0\,.$$

From (11) and (12) we see that $\log\,((\lambda^2+\mu^2)^2-(K^\nu)^2)$ is subharmonic on U. Hence $(\lambda^2+\mu^2)^2-(K^\nu)^2$ is subharmonic on its support, and therefore on all of M, whence it is a constant. Relation (12) implies $K=0$, and (11) implies $K^*=0$.

Case 2. ϕ_p is homothetic at every $p\in M$. Then $(\lambda^2+\mu^2)^2-(K^\nu)^2=0$. As in the first case we obtain equation (11). Hence $2\,K = K^*$. Finally, $(\lambda^2+\mu^2) = K^\nu$ implies the constancy of λ and μ. Then K is constant by the equation of Gauss.

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