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Nilpotent spaces: some inequalities
on nilpotency degrees.

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NILPOTENT SPACES: SOME INEQUALITIES ON NILPOTENCY DEGREES.

by

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Introduction.

The idea of dealing with the subject considered in this paper arose in a talk between the author and Professor Peter Hilton.

In [H.M.R.] the authors proved that if X is a connected CW -complex and $\pi_1(X)$ is nilpotent, then $\pi_1(X)$ acts nilpotently on $\pi_n(X)$ ($n \geq 2$) if and only if $\pi_1(X)$ acts nilpotently on $H_n(\tilde{X})$, ($n \geq 2$), where \tilde{X} is the universal cover of X .

This fact gave rise to a natural question: compare the nilpotency degrees of the actions mentioned in the theorem above. It was already known by the authors that $\pi_1(X)$ acted on $\pi_2(X)$ and on $H_2(\tilde{X})$ with the same nilpotency degree, due to Hurewicz's Theorem that $\pi_2(X) \cong H_2(\tilde{X})$.

It remained, therefore, to investigate the case $n \geq 3$. This is exactly the problem which Professor Hilton suggested that we studied.

We have been successful in finding inequalities, when $n \leq 7$, between $nil_{\pi_1(X)}\pi_n(X)$ (the nilpotency degree of the action of $\pi_1(X)$ on $\pi_n(X)$) and $nil_{\pi_1(X)}H_n(\tilde{X})$, and this is the subject of this paper. The inequalities we obtained are the content of Theorem 12, which is our main result.

This work is part of the author's Ph.D. thesis done under the guidance of Professor Peter John Hilton.

The author is very much indebted to Professor Daciberg Lima Gonçalves, at whose suggestion this work was developed.

Let $\pi \xrightarrow{w} \text{Aut}(A)$ be an action of a group π on an abelian group A .

We recall that w yields, $\forall n \geq 0$, an action w_n of π on $H_n(K(A, m))$ (here m is a fixed integer, $m \geq 1$, and $K(A, m)$ is an Eilenberg-McLane space), defined by:

$$(\forall x \in \pi) \exists! [f_x] \in [K(A, m), *; K(A, m), *] \text{ such that } f_{x*} = w(x).$$

(For a more detailed description see [W], pg.100 and pg.225).

We define $w_n(x) = f_{x*} : H_n(K(A, m)) \rightarrow H_n(K(A, m))$.

The proof of the first proposition is known (see [H.M.R.] Lemma 2.17). We decided, nevertheless, to present it here owing to the technique used in its proof, which is going to be repeatedly used in the sequel.

Proposition 1. w nilpotent $\Rightarrow w_n$ nilpotent, $\forall n \geq 0$.

Proof. We argue by induction on $c = \text{nil} w = \text{nil}_\pi A$.

If $c = 1$ it follows from the definition that w_n is trivial and therefore nilpotent.

If $c > 1$ we take $\Gamma = \Gamma_w^c \neq (0)$ and the fibration

$$\begin{array}{ccc} K(\Gamma, m) & \hookrightarrow & K(A, m) \\ & & \downarrow \\ & & K(A/\Gamma, m) \end{array}$$

This fibration gives rise to a spectral sequence (Serre) in which $E_{r,s}^2 = H_r(K(A/\Gamma, m); H_s(K(\Gamma, m)))$. (Here, we should point out that the homology is taken with trivial coefficients, since if $m \geq 2$, then the base is simply-connected and if $m = 1$ we obtain $H_r(A/\Gamma; H_s(\Gamma))$ and A/Γ acts trivially on Γ , so that A/Γ acts trivially on $H_s(\Gamma)$).

We now invoke the universal coefficients Theorem to get the following exact sequence:

$$H_r(K(A/\Gamma, m) \otimes H_s(K(\Gamma, m))) \rightarrow E_{r,s}^2 \rightarrow \text{Tor}(H_{r-1}(K(A/\Gamma, m)), H_s(K(\Gamma, m))).$$

Taking into account the induction hypothesis, Lemma 1.1 from [H] and proposition I.4.3, pg.35, [H.M.R.], we can claim that π acts nilpotently on $E_{r,s}^2$.

Again, the reiterated use of the proposition I.4.3, pg.35, [H.M.R.] assures that π acts nilpotently on $E_{r,s}^\infty$, and therefore on $H_n(K(A, m))$. ■

Lemma 2. $nil_{\pi} H_n(K(A, m)) \leq \sum_{j=0}^n nil_{\pi} E_{n-j, j}^2; (m \geq 1)$. (According to the notation used in the proof of the proposition 1.)

Proof. It is known that the spectral sequence invoked in the previous proposition is formed by π -modules (the actions are induced by w) $E_{r, s}^k$ and the differentials $d_{r, s}^k$ are π -module homomorphisms. Also,

$$E_{r+2, s-1}^2 \xrightarrow{d_{r+2, s-1}^2} E_{r, s}^2 \xrightarrow{d_{r, s}^2} E_{r-2, s+1}^2 \quad \text{and}$$

$$E_{r, s}^3 = \frac{\ker(d_{r, s}^2)}{\text{im}(d_{r+2, s-1}^2)}.$$

Thus we get the exact sequence of π -modules,

$$0 \longrightarrow \text{im } d_{r+2, s-1}^2 \longrightarrow \ker d_{r, s}^2 \longrightarrow E_{r, s}^3 \longrightarrow 0.$$

It follows from proposition I.4.3, [H.M.R.] that

$$nil_{\pi} E_{r, s}^3 \leq nil_{\pi} \ker d_{r, s}^2 \leq nil_{\pi} E_{r, s}^2.$$

It is now readily seen, by induction, that

$$nil_{\pi} E_{r, s}^k \leq nil_{\pi} E_{r, s}^2; \quad \forall k \geq 2.$$

In particular, $nil_{\pi} E_{r, s}^{\infty} \leq nil_{\pi} E_{r, s}^2$. Finally, recalling that

$$\begin{array}{ccccccc} E_{s, n}^{\infty} & \subset & F_{1, n-1} & \subset & \cdots & \subset & F_{n, 0} = H_n(K(A, m)), \\ & & \downarrow & & & & \downarrow \\ & & E_{1, n-1}^{\infty} & & & & E_{n, 0}^{\infty} \end{array}$$

considering the exact sequence of π -modules

$$F_{i-1, n-i+1} \longrightarrow F_{i, n-i} \longrightarrow E_{i, n-i}^{\infty}$$

and using several times proposition I.4.3, [H.M.R.], we obtain

$$nil_{\pi} H_n(K(A, m)) \leq \sum_{j=0}^m nil_{\pi} E_{n-j, j}^{\infty} \leq \sum_{j=0}^m nil_{\pi} E_{n-j, j}^2$$

Theorem 3: If $\text{nil}_\pi A = \text{nil } w = c \geq 2$ and $m \geq 2$, then

- (i) $\text{nil}_\pi H_n(K(A, m)) \leq c$, if $0 \leq n < 2m$.
- (ii) $\text{nil}_\pi H_n(K(A, m)) \leq \frac{c(c+1)}{2}$, if either $n = 2m$ or $n = 2m + 1$.
- (iii) $\text{nil}_\pi H_{2m+2}(K(A, m)) \leq c^2$, if $m \geq 3$.
- (iv) $\text{nil}_\pi H_6(K(A, 2)) \leq \frac{c(c+1)(c+2)}{6}$.
- (v) $\text{nil}_\pi H_{2m+3}(K(A, m)) \leq 2c^2 - c$, if $m \geq 4$.
- (vi) $\text{nil}_\pi H_9(K(A, 3)) \leq \frac{c}{8}(c^2 - 9c - 4)$.
- (vii) $\text{nil}_\pi H_7(K(A, 2)) \leq \frac{c}{3}(c^2 + 3c - 1)$.

Proof. In the Serre spectral sequence we have

$$E_{n-j, j}^2 = H_{n-j}(K(A/\Gamma, m); H_j(K(\Gamma, m)));$$

So $E_{n, 0}^2 \cong H_n(K(A/\Gamma, m))$ and $E_{0, n}^2 \cong H_n(K(\Gamma, m))$ (isomorphisms of π -modules).

Thus,

- (i) The inequality is trivially true if $0 \leq n < m$, and if $n = m$ the result follows from the Hurewicz isomorphism, since $w_n = w$. In fact, $\text{nil}_\pi H_m(K(A, m)) = c \leq c$.

It remains to consider the case in which $m < n < 2m$.

Fix j , such that $0 < j < n$.

If $0 < j < m$, then $H_j(K(\Gamma, m)) = (0)$, so $E_{n-j, j}^2 = (0)$.

If $m \leq j < n$, then $0 < n - j < m$. Therefore $H_{n-j}(K(A/\Gamma, m)) = 0$, so $E_{n-j, j}^2 = 0$.

It follows from lemma 2 that

$$\text{nil}_\pi H_n(K(A, m)) \leq \text{nil}_\pi E_{0, n}^2 + \text{nil}_\pi E_{n, 0}^2 \leq 1 + \text{nil}_\pi H_n(K(A/\Gamma, m)),$$

since π acts trivially on Γ .

It turns out, then, by induction on c , that $\text{nil}_\pi H_n(K(A, m)) \leq c$.

(ii) Again, $0 < j < m \Rightarrow E_{n-j,j}^2 = 0$ and

$m < j < n = 2m \Rightarrow 0 < n - j < m$. Therefore $E_{n-j,j}^2 = 0$

and $E_{m,m}^2 \cong H_m(K(A/\Gamma, m)) \otimes H_m(K(\Gamma, m)) \cong A/\Gamma \otimes \Gamma$.

We now invoke the inequality (1.3) from [H.R.S.] in order to state that $\text{nil}_\pi E_{m,m}^2 \leq \text{nil}_\pi A/\Gamma = c - 1$. Thus, by using lemma 2, we obtain:

$$\text{nil}_\pi H_{2m}(K(A, m)) \leq 1 + (c - 1) + \text{nil}_\pi H_{2m}(K(A/\Gamma, m)).$$

It follows, therefore, by induction, that

$$\text{nil}_\pi H_{2m}(K(A, m)) \leq c + (c - 1) + \dots + 1 = \frac{c(c+1)}{2}.$$

For $n = 2m + 1$, we have again $E_{n-j,j}^2 = 0$, if either $0 < j < m$, or $m + 1 < j < 2m + 1$.

Also $E_{m,m+1}^2 = 0$, for $H_{m+1}(K(\Gamma, m)) = 0$ (Hurewicz) ($m \geq 2$) and

$$E_{m+1,m}^2 \cong \text{Tor}(H_m(K(A/\Gamma, m)); H_m(\Gamma, m)) \cong \text{Tor}(A/\Gamma, \Gamma).$$

It is easily checked, from the proof of lemma 1,1, [H], that

$$\text{nil}_\pi \text{Tor}(A, B) \leq (\text{nil}_\pi A)(\text{nil}_\pi B), \quad \text{whence}$$

$$\text{nil}_\pi E_{m+1,m}^2 \leq \text{nil}_\pi A/\Gamma = c - 1.$$

Hence, $\text{nil}_\pi H_{2m+1}(K(A, m)) \leq 1 + (c - 1) + \text{nil}_\pi H_{2m+1}(K(A/\Gamma, m))$, whence, by induction,

$$\text{nil}_\pi H_{2m+1}(K(A, m)) \leq c + (c - 1) + \dots + 1 = \frac{c(c+1)}{2}.$$

(iii) Now let's suppose that $m \geq 3$ and $n = 2m + 2$.

$E_{n-j,j}^2 = 0$, if either $0 < j < m$ or $m + 2 < j < 2m + 2$.

$E_{m+1,m+1}^2 = 0$ (for $H_{m+1}(K(\Gamma, m)) = 0$).

$E_{m+2,m}^2 \cong H_{m+2}(K(A/\Gamma, m)) \otimes H_m(K(\Gamma, m))$.

Therefore, $E_{m+2,m}^2 \cong H_{m+2}(K(A/\Gamma, m)) \otimes \Gamma$.

Still, $E_{m,m+2}^2 \cong A/\Gamma \otimes H_{m+2}(K(\Gamma, m))$. So,

$$\text{nil}_\pi E_{m,m+2}^2 \leq \text{nil}_\pi A/\Gamma = c - 1.$$

Also, taking into account that $m \geq 3$ (therefore, $m + 2 < 2m$) it follows from (i) that:

$$\text{nil}_\pi E_{m+2,m}^2 \leq \text{nil}_\pi H_{m+2}(K(A/\Gamma, m)) \leq c - 1 \quad (\text{for } \text{nil}_\pi A/\Gamma = c - 1).$$

Thus, $nil_{\pi} H_{2m+2}(K(A, m)) \leq 1 + (c-1) + (c-1) + nil_{\pi} H_{2m+2}(K(A/\Gamma, m)) = (2c-1) + nil_{\pi} H_{2m+2}(K(A/\Gamma, m))$.

Then, it arises, by induction, that

$$nil_{\pi} H_{2m+2}(K(A, m)) \leq \sum_{j=1}^c (2j-1) = c^2.$$

(iv) $m = 2$ and $n = 2m + 2 = 6$.

The same kind of calculation that we have used in (iii) leads us to conclude that:

$$nil_{\pi} E_{4,2}^2 \leq nil_{\pi} H_4(K(A/\Gamma, 2)) \leq \frac{c(c-1)}{2} \quad (\text{due to (ii)})$$

$$\text{and } nil_{\pi} E_{2,4}^2 \leq c-1.$$

$$\text{So, } nil_{\pi} H_6(K(A, 2)) \leq 1 + (c-1) + \frac{(c-1)c}{2} + nil_{\pi} H_6(K(A/\Gamma, 2)) = \frac{c(c+1)}{2} + nil_{\pi} H_6(K(A/\Gamma, 2)).$$

$$\text{Thus, } nil_{\pi} H_6(K(A, 2)) \leq \sum_{j=1}^c \frac{j(j+1)}{2} = \frac{c(c+1)(c+2)}{6}.$$

(v) $m \geq 4$ and $n = 2m + 3$.

$$E_{2m+3-j, j}^2 = 0, \text{ if either } 0 < j < m \text{ or } m+3 < j < 2m+3 \text{ or } j = m+1.$$

$$E_{m, m+3}^2 \cong A/\Gamma \otimes H_{m+3}(K(\Gamma, m))$$

$$E_{m+1, m+2}^2 \cong Tor(A/\Gamma, H_{m+2}(K(\Gamma, m))) \quad \text{and the sequence is exact}$$

$$H_{m+3}(K(A/\Gamma, m)) \otimes \Gamma \longrightarrow E_{m+3, m}^2 \longrightarrow Tor(H_{m+2}(K(A/\Gamma, m)), \Gamma).$$

So, due to lemma 2,

$$nil_{\pi} H_{2m+3}(K(A, m)) \leq 1 + nil_{\pi} H_{2m+3}(K(A/\Gamma, m)) + nil_{\pi} A/\Gamma + nil_{\pi} A/\Gamma + nil_{\pi} H_{m+3}(K(A/\Gamma, m)) + nil_{\pi} H_{m+2}(K(A/\Gamma, m)) \dots (*).$$

So, due to (i) and remembering that $m \geq 4$,

$$nil_{\pi} H_{2m+3}(K(A, m)) \leq (4c-3) + nil_{\pi} H_{2m+3}(K(A/\Gamma, m)).$$

Then,

$$nil_{\pi} H_{2m+3}(K(A, m)) \leq \sum_{j=1}^c (4j-3) = c(2c-1).$$

(vi) $m = 3$ and $n = 2m + 3 = 9$.

Using (*) from the previous item we get:

$$\begin{aligned} \text{nil}_\pi H_9(K(A, 3)) &\leq \text{nil}_\pi H_9(K(A/\Gamma, 3)) + (2c - 1) + \text{nil}_\pi H_6(K(A/\Gamma, 3)) + \\ &+ \text{nil}_\pi H_5(K(A/\Gamma, 3)) \leq \text{nil}_\pi H_9(K(A/\Gamma, 3)) + (2c - 1) + \frac{c(c-1)}{2} \\ &+ c - 1, \text{ due to items (i) and (ii) as well.} \end{aligned}$$

Therefore, $\text{nil}_\pi H_9(K(A, 3)) \leq \frac{c^2+5c-4}{2} + \text{nil}_\pi H_9(K(A/\Gamma, 3))$,

whence $\text{nil}_\pi H_9(K(A, 3)) \leq \sum_{j=1}^c \frac{j^2+5j-4}{2} = \frac{c}{6}(c^2 - 9c - 4)$.

(vii) $m = 2$ and $n = 2m + 3 = 7$. Again, we invoke (*) from item (v), as well as item (ii) to get:

$$\begin{aligned} \text{nil}_\pi H_7(K(A, 2)) &\leq (2c - 1) + \text{nil}_\pi H_7(K(A/\Gamma, 2)) + \text{nil}_\pi H_5(K(A/\Gamma, 2)) + \\ &+ \text{nil}_\pi H_4(K(A/\Gamma, 2)) \leq (2c - 1) + \text{nil}_\pi H_7(K(A/\Gamma, 2)) + \\ &+ \frac{c(c-1)}{2} + \frac{c(c-1)}{2} = (c^2 + c - 1) + \text{nil}_\pi H_7(K(A/\Gamma, 2)), \end{aligned}$$

whence $\text{nil}_\pi H_7(K(A, 2)) \leq \sum_{j=1}^c (j^2 + j - 1) = \frac{c}{3}(c^2 + 3c - 1)$. ■

Example 4: Let $Z \xrightarrow{w} \text{Aut}(Z \otimes Z)$ be an action given by

$$w(1) \cdot (1, 0) = (1, 1) \quad \text{and} \quad w(1) \cdot (0, 1) = (0, 1), \quad \text{that is,}$$

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

is the matrix attached to the automorphism $w(1)$.

$(M - I_2)^2 = 0 \Rightarrow w$ is the nilpotent and $\text{nil } w = 2$.

Let $X = K(Z \oplus Z, 2) \cong K(Z, 2) \times K(Z, 2)$ and set $w_n : Z \rightarrow \text{Aut}(H_n(X))$ the action induced by w . In order to calculate $\text{nil } w_n$, we remember that $H_*(K(Z, 2)) \cong D[x_2] =$ polynomial algebra divided with one generator of degree 2. (x_2). (I.e., $x_{2i} \cdot x_{2j} = \binom{i+j}{i} x_{2(i+j)}$).

It follows from the definition that w_n is compatible with the multiplicative structure in $H_n(X)$.

Moreover, $w_2 = w$ (Hurewicz) and remembering that

$H_2(X) \cong [H_*(K(Z, 2)) \otimes H_2(K(Z, 2))] \oplus [H_2(K(Z, 2)) \otimes H_*(K(Z, 2))]$ (Künneth) we get:

$$w_2(1) \cdot (x_2 \otimes 1) = x_2 \otimes 1, \quad w_2(1)(1 \otimes x_2) = 1 \otimes x_2 + x_2 \otimes 1.$$

It is known that $\{1 \otimes x_4, x_2 \otimes x_2, x_4 \otimes 1\}$ is a base in $H_4(X)$ (where x_4 is a generator in $H_4(K(\mathbb{Z}, 2))$).

$$\begin{aligned} \text{Now, } 2w_4(1)(1 \otimes x_4) &= w_4(1)(1 \otimes 2x_4) = w_4(1)[(1 \otimes x_2) \cdot (1 \otimes x_2)] = \\ &= w_2(1)(1 \otimes x_2) \cdot w_2(1)(1 \otimes x_2) = [1 \otimes x_2 + x_2 \otimes 1] \cdot [1 \otimes x_2 + x_2 \otimes 1] = \\ &= 1 \otimes 2x_4 + 2x_2 \otimes x_2 + 2x_4 \otimes 1. \end{aligned}$$

$$\text{So, } w_2(1)(1 \otimes x_2) = 1 \otimes x_2 + x_2 \otimes x_2 + x_2 \otimes 1.$$

A similar calculation shows us that $w_4(1)(x_2 \otimes x_2) = x_2 \otimes x_2 + 2x_4 \otimes 1$ and $w_4(1)(x_4 \otimes 1) = x_4 \otimes 1$.

Therefore,

$$M_4 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

is the matrix attached to $w_4(1)$ relatively to the base previously indicated.

$$\text{Thus } (M_4 - I_3)^3 = 0, \text{ whence } \text{nil } w_4 = 3 = \frac{2(2+1)}{2}.$$

Likewise, fixing the base $\{1 \otimes x_6, x_2 \otimes x_4, x_4 \otimes x_2, x_6 \otimes 1\}$ in $H_6(X)$, we get

$$M_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

following the same steps of the previous argument.

$$\text{Then, } (M_6 - I_4)^4 = 0, \text{ whence } \text{nil } w_6 = 4 = \sum_{j=1}^2 \frac{j(j+1)}{2}.$$

Actually, this example can be generalized.

Example 5. Let $Z \xrightarrow{w} \text{Aut}(Z^c)$ be the action defined by the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

Thus $\text{nil } w = c$, since $(M - I_c)^c = 0$.

Let $X_c = K(Z^c, 2) \cong K(\mathbb{Z}, 2)^c$ and take $w_n : Z \rightarrow \text{Aut}(H_n(X_c))$ to be the action inducted by w .

Calculations similar to the ones used in the previous example show us that

$$\text{nil}_Z H_4(X_c) = \text{nil } w_4 = \frac{c(c+1)}{2} = \text{rank of } H_4(X_c) \text{ (by induction on } c), \text{ and}$$

$$\text{nil}_Z H_6(X_c) = \text{nil } w_6 = \sum_{j=1}^c \frac{j(j+1)}{2} = \text{rank of } H_6(X_c).$$

(Notice that by Kinneth Formula, we obtain $\text{rank } H_6(X_c) =$

$$\begin{aligned} \sum_{i=0}^3 \text{rank } H_{2i}(X_{c-1}) &= 1 + (c-1) + \frac{c(c-1)}{2} + \sum_{j=1}^{c-1} \frac{i(j-1)}{2} \text{ (induction)} \\ &= \sum_{j=1}^c \frac{j(j+1)}{2}. \end{aligned}$$

$M_4(c)$ and $M_6(c)$ turn out to be triangular, by induction on c as well.

Remark: This example shows us that the inequalities obtained in Theorem 3 (ii) ($n = 4$ and $m = 2$) and in 3(iv) are the best ones.

Proposition 6. If $\text{nil } w = \text{nil}_\pi A = c \geq 2$, then,

$$(i) \text{nil}_\pi H_2(A) \leq \frac{c(c+1)}{2}, \text{ and}$$

$$(ii) \text{nil}_\pi H_3(A) \leq \sum_{j=1}^c j^2 = \frac{c(c+1)(2c+1)}{6}.$$

Proof: We invoke the Lyndon-Hoschild-Serre spectral sequence attached to the exact sequence $\Gamma \rightarrow A \rightarrow A/\Gamma$ (where $\Gamma = \Gamma_w^c \neq 0$) to get $E_{r,s}^2 = H_r(A/\Gamma; H_s(\Gamma))$ with trivial coefficients.

$$(i) E_{2,0}^2 = H_2(A/\Gamma); E_{0,2}^2 \cong H_2(\Gamma); E_{1,1}^2 \cong A/\Gamma \otimes \Gamma.$$

So, $\text{nil}_\pi H_2(A) \leq 1 + (c-1) + \text{nil}_\pi H_2(A/\Gamma)$ due to lemma 2 and the inequality (1.3) [H.R.S.].

Thus, we get (by induction again):

$$\text{nil}_\pi H_2(A) \leq c + (c-1) + \dots + 2 + 1 = \frac{c(c+1)}{2}$$

$$(ii) E_{3,0}^2 \cong H_3(A/\Gamma); E_{0,3}^2 \cong H_3(\Gamma); E_{1,2}^2 \cong A/\Gamma \otimes H_2(\Gamma) \text{ and}$$

$$H_2(A/\Gamma) \otimes \Gamma \rightarrow E_{2,1}^2 \rightarrow \text{Tor}(A/\Gamma, \Gamma) \text{ is exact.}$$

$$\begin{aligned}
 \text{So, } \text{nil}_\pi H_3(A) &\leq 1 + \text{nil}_\pi H_3(A/\Gamma) + (c-1) + (c-1) + \text{nil}_\pi H_2(A/\Gamma) \leq \\
 &\leq 2c-1 + (c-1)^2 + \text{nil}_\pi H_3(A/\Gamma) \quad (\text{due to the previous case}) = \\
 &= c^2 + \text{nil}_\pi H_3(A/\Gamma)
 \end{aligned}$$

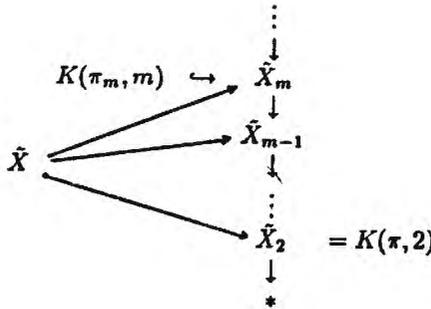
Therefore,

$$\text{nil}_\pi H_3(A) \leq \sum_{j=1}^c j^2 = \frac{c(c+1)(2c+1)}{6}$$

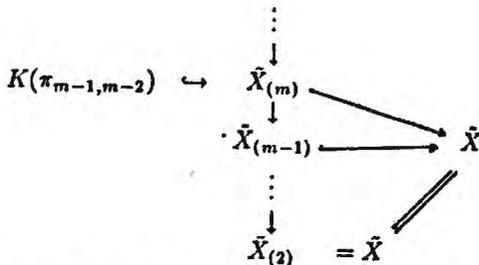
From now on X will denote a connected CW-complex, as well as \tilde{X} its universal covering; $\pi = \pi_1(X)$ and $\pi_n = \pi_n(X)$.

We are also going to use the decomposition of Postnikov of \tilde{X} and its dual called the Cartan-Serre-Whitehead decomposition. They will be indicated by:

(Postnikov)



(Dual)



Here we should recall that Lemma 1.2.18 and remark 2.19 from [H.M.R.] show us that π acts nilpotently on π_n , $\forall n$, $2 \leq n \leq k \leftrightarrow \pi$ acts nilpotently on

$$H_n(\tilde{X}), \quad \forall n, \quad 2 \leq n \leq k.$$

Our goal now is to obtain some results comparing such nilpotency classes.

Thus, we suppose from now on that π acts nilpotently on π_n , $\forall n \geq 2$. Concerning the previous decompositions, we are proving 2 lemmas to be used later.

$$\text{Lemma 7. (i) } \text{nil}_\pi H_{m+1}(\tilde{X}_m) \leq \text{nil}_\pi H_{m+1}(\tilde{X}_{m-1})$$

$$\text{(ii) } \text{nil}_\pi H_{m+2}(\tilde{X}_m) \leq \text{nil}_\pi H_{m+2}(\tilde{X}_{m-1}) + \text{nil}_\pi H_{m+2}(K(\pi_m(X), m)) \\ + \text{nil}_\pi[\pi_2(X) \otimes \pi_m(X)]$$

$$\text{(iii) } \text{nil}_\pi H_{m+3}(\tilde{X}_m) \leq \text{nil}_\pi H_{m+3}(\tilde{X}_{m-1}) + \text{nil}_\pi H_{m+3}(K(\pi_m(X), m)) \\ + \text{nil}_\pi[H_3(\tilde{X}_{m-1}) \otimes \pi_m(X)] + \text{nil}_\pi \text{Tor}(\pi_2(X), \pi_m(X))$$

$$\text{(iv) } \text{nil}_\pi H_{m+4}(\tilde{X}_m) \leq \text{nil}_\pi H_{m+4}(\tilde{X}_{m-1}) + \text{nil}_\pi H_{m+4}(K(\pi_m(X), m)) \\ + \text{nil}_\pi \pi_2(X) \otimes H_{m+2}(K(\pi_m(X), m)) + \text{nil}_\pi H_4(\tilde{X}_{m-1}) \otimes \pi_m(X) \\ + \text{nil}_\pi \text{Tor}(H_3(\tilde{X}_{m-1}), \pi_m(X)), \quad \forall m \geq 3.$$

Proof: We take the fibration ($m \geq 3$)

$$K(\pi_m(X), m) \hookrightarrow \tilde{X}_m \\ \downarrow \\ \tilde{X}_{m-1}$$

$$\text{to get (Serre) } E_{r,s}^2 = H_r(\tilde{X}_{m-1}, H_s(K(\pi_m(X), m)))$$

$$\text{Exactly as in Lemma 2 we have } \text{nil}_\pi H_n(\tilde{X}_m) \leq \sum_{j=0}^n \text{nil}_\pi E_{n-j,j}^2$$

$$\text{(i) } E_{0,m+1}^2 \cong H_{m+1}(K(\pi_m, m)) = 0 \quad \forall n \geq 0. \\ E_{m+1,0}^2 \cong H_{m+1}(\tilde{X}_{m-1}); \quad E_{1,m}^2 = 0 \quad \text{and} \quad E_{m+1-j,j}^2 = 0, \quad 0 < j < m. \\ \text{Therefore, } \text{nil}_\pi H_{m+1}(\tilde{X}_m) \leq \text{nil}_\pi H_{m+1}(\tilde{X}_{m-1}); \quad \forall m \geq 3.$$

$$\text{(ii) } E_{0,m+2}^2 \cong H_{m+2}(K(\pi_m, m)); \quad E_{m+2,0}^2 \cong H_{m+2}(\tilde{X}_{m-1}) \\ E_{1,m+1}^2 = 0 = E_{m+2-j,j}^2, \quad \text{if } 0 < j < m. \\ E_{2,m}^2 \cong H_2(\tilde{X}_{m-1}) \otimes \pi_m \cong \pi_2 \otimes \pi_m \quad (m-1 \geq 2) \\ \text{So, } \text{nil}_\pi H_{m+2}(\tilde{X}_m) \leq \text{nil}_\pi H_{m+2}(\tilde{X}_{m-1}) + \text{nil}_\pi H_{m+2}(K(\pi_m, m)) + \text{nil}_\pi \pi_2 \otimes \pi_m.$$

$$\text{(iii) } E_{0,m+3}^2 \cong H_{m+3}(K(\pi_m, m)); \quad E_{m+3,0}^2 \cong H_{m+3}(\tilde{X}_{m-1});$$

$$E_{1,m+2}^2 = 0 = E_{m+3-j,j}^2 \quad \text{if } 0 < j < m; \quad E_{2,m+1}^2 = 0,$$

and $H_3(\tilde{X}_m) \otimes \pi_m \rightarrow E_{3,m}^2 \rightarrow \text{Tor}(\pi_2(X), \pi_m)$ is an exact sequence of π -modules.

$$\text{Thus, } \text{nil}_\pi H_{m+3}(\tilde{X}_m) \leq \text{nil}_\pi H_{m+3}(\tilde{X}_{m-1}) + \text{nil}_\pi H_{m+3}(K(\pi_m, m)) + \\ + \text{nil}_\pi(H_3(\tilde{X}_m) \otimes \pi_m) + \text{nil}_\pi \text{Tor}(\pi_2, \pi_m).$$

$$(iv) \quad E_{0,m+4}^2 \cong H_{m+4}(K(\pi_m, m)); \quad E_{m+4,0}^2 \cong H_{m+4}(\tilde{X}_{m-1});$$

$$E_{1,m+3}^2 = 0 = E_{m+4-j,j}^2, \quad \text{if } 0 < j < m; \quad E_{3,m-1}^2 = 0,$$

$$E_{2,m+2}^2 \cong \pi_2 \otimes H_{m+2}(K(\pi_m, m)) \quad \text{and}$$

$$H_4(\tilde{X}_{m-1}) \otimes \pi_m \rightarrow E_{4,m}^2 \rightarrow \text{Tor}(H_3(\tilde{X}_{m-1}), \pi_m) \quad \text{is exact.}$$

$$\text{Then, } \text{nil}_\pi H_{m+4}(\tilde{X}_m) \leq \text{nil}_\pi H_{m+4}(\tilde{X}_{m-1}) + \text{nil}_\pi H_{m+4}(K(\pi_m, m)) + \\ + \text{nil}_\pi \pi_2 \otimes H_{m+2}(K(\pi_m, m)) + \text{nil}_\pi H_4(\tilde{X}_{m-1}) \otimes \pi_m + \\ + \text{nil}_\pi \text{Tor}(H_3(\tilde{X}_{m-1}), \pi_m) \quad \blacksquare$$

Corollary 8: (i) $\text{nil}_\pi H_5(\tilde{X}_3) \leq \text{nil}_\pi H_5(K(\pi_2, 2)) + \text{nil}_\pi H_5(K(\pi_3, 3)) + \text{nil}_\pi \pi_2 \otimes \pi_3.$

$$(ii) \quad \text{nil}_\pi H_6(\tilde{X}_3) \leq \text{nil}_\pi H_6(K(\pi_2, 2)) + \text{nil}_\pi H_6(K(\pi_3, 3)) + \text{nil}_\pi \text{Tor}(\pi_2, \pi_3).$$

$$(iii) \quad \text{nil}_\pi H_7(\tilde{X}_3) \leq \text{nil}_\pi H_7(K(\pi_2, 2)) + \text{nil}_\pi H_7(K(\pi_3, 3)) + \\ + \text{nil}_\pi \pi_2 \otimes H_5(K(\pi_3, 3)) + \text{nil}_\pi H_4(K(\pi_2, 2)) \otimes \pi_3.$$

$$(iv) \quad \text{nil}_\pi H_6(\tilde{X}_4) \leq \text{nil}_\pi H_6(K(\pi_4, 4)) + \text{nil}_\pi H_6(K(\pi_3, 3)) + \\ + \text{nil}_\pi H_6(K(\pi_2, 2)) + \text{nil}_\pi \pi_2 \otimes \pi_4 + \text{nil}_\pi \text{Tor}(\pi_2, \pi_3)$$

$$(v) \quad \text{nil}_\pi H_7(\tilde{X}_4) \leq \text{nil}_\pi H_7(K(\pi_2, 2)) + \text{nil}_\pi H_7(K(\pi_3, 3)) + \\ + \text{nil}_\pi H_7(K(\pi_4, 4)) + \text{nil}_\pi H_3(\tilde{X}) \otimes \pi_4 + \\ + \text{nil}_\pi \text{Tor}(\pi_2, \pi_4) + \text{nil}_\pi \pi_2 \otimes H_5(K(\pi_3, 3)) + \\ + \text{nil}_\pi H_4(K(\pi_2, 2)) \otimes \pi_3.$$

$$(vi) \quad \text{nil}_\pi H_7(\tilde{X}_5) \leq \text{nil}_\pi H_7(K(\pi_2, 2)) + \text{nil}_\pi H_7(K(\pi_3, 3)) + \\ + \text{nil}_\pi H_7(K(\pi_4, 4)) + \text{nil}_\pi H_7(K(\pi_5, 5)) + \text{nil}_\pi \pi_2 \otimes \pi_5 \\ + \text{nil}_\pi \pi_4 \otimes H_3(\tilde{X}) + \text{nil}_\pi \text{Tor}(\pi_2, \pi_4) + \text{nil}_\pi \pi_2 \otimes H_5(K(\pi_3, 3)) \\ + \text{nil}_\pi \pi_3 \otimes H_4(K(\pi_2, 2)).$$

Proof: (i) It is enough to remark that $\tilde{X}_2 = K(\pi_2, 2)$ and use (i) from lemma 7.

(ii) Notice that $H_3(K(\pi_2, 2)) = 0$ and use (iii) from lemma 7.

(iii) It is readily seen from (iv) lemma 7 together with $H_3(K(\pi_2, 2)) = 0$.

(iv) It is enough to use lemma 7 (ii) and corollary 8 (ii).

(v) It is a consequence of lemma 7 (iii) together with corollary 8 (iii).

(vi) It follows from lemma 7 (ii) and corollary 8 (v).

Lemma 9. (i) $nil_\pi H_{m+1}(\tilde{X}_{(m)}) \leq nil_\pi H_{m+1}(\tilde{X}_{(m-1)}) +$

$$+ nil_\pi H_{m+1}(K(\pi_{m-1}, m-2)) + nil_\pi H_3(\tilde{X}_{(m-1)}) \otimes \pi_{m-1} +$$

$$+ nil_\pi Tor(H_2(\tilde{X}_{(m-1)}), \pi_{m-1}); \quad \forall m \geq 3.$$

(ii) $nil_\pi H_{m+2}(\tilde{X}_{(m)}) \leq nil_\pi H_{m+2}(\tilde{X}_{(m-1)}) + nil_\pi H_{m+2}(K(\pi_{m-1}, m-2))$

$$+ nil_\pi H_2(\tilde{X}_{(m-1)}) \otimes H_m(K(\pi_{m-1}, m-2)) + nil_\pi H_4(\tilde{X}_{(m-1)}) \otimes \pi_{m-1}$$

$$+ nil_\pi Tor(H_3(\tilde{X}_{(m-1)}), \pi_{m-1}); \quad \forall m \geq 3.$$

Proof. Let's take the fibration

$$K(\pi_{m-1}(X), m-2) \hookrightarrow \begin{array}{c} \tilde{X}_{(m)} \\ \downarrow \\ \tilde{X}_{(m-1)} \end{array}$$

Its attached spectral sequence yields

$$E_{r,s}^2 = H_r(\tilde{X}_{(m-1)}), \quad H_s(K(\pi_{m-1}, m-2)) \quad \text{and}$$

$$nil_\pi H_n(\tilde{X}_{(m)}) \leq \sum_{j=0}^n nil_\pi E_{n-j,j}^2 \cdots (*).$$

(i) $E_{0,m}^2 \cong H_{m+1}(K(\pi_{m-1}, m-2)); \quad E_{m+1,0}^2 \cong H_{m+1}(\tilde{X}_{(m-1)});$

$$E_{m+1-j,j}^2 = 0, \quad \text{if } 0 < j < m-2; \quad E_{2,m-1}^2 = 0 = E_{1,m}^2 \quad \text{and the sequence}$$

$$H_3(\tilde{X}_{(m-1)}) \otimes \pi_{m-1} \twoheadrightarrow E_{3,m-2}^2 \twoheadrightarrow Tor(H_2(\tilde{X}_{(m-1)}), \pi_{m-1}) \quad \text{is exact.}$$

Now the result follows from (*).

(ii) $E_{0,m+2}^2 \cong H_{m+2}(K(\pi_{m-1}, m-2)); \quad E_{m+2,0}^2 \cong H_{m+2}(\tilde{X}_{(m-1)}).$

$$E_{1,m+1}^2 = 0 = E_{m+2-j,j}^2 \quad \text{if } 0 < j < m-2; \quad E_{3,m-1}^2 = 0; \quad \text{and}$$

$$H_4(\tilde{X}_{(m-1)}) \otimes \pi_{m-1} \twoheadrightarrow E_{4,m-2}^2 \twoheadrightarrow Tor(H_3(\tilde{X}_{(m-1)}), \pi_{m-1}) \quad \text{is exact.}$$

Finally, by remembering that $E_{2,m}^2 \cong H_2(\tilde{X}_{(m-1)}) \otimes H_m(K(\pi_{m-1}, m-2))$ and by using (*), we obtain the result.

Corollary 10. (i) $nil_{\pi} H_4(\tilde{X}_{(3)}) \leq nil_{\pi} H_4(\tilde{X}) + nil_{\pi} H_4(K(\pi_2, 1)) +$
 $+ nil_{\pi} H_3(\tilde{X}) \otimes \pi_2 + nil_{\pi} Tor(H_2(\tilde{X}); \pi_2).$

(ii) $nil_{\pi} H_5(\tilde{X}_{(3)}) \leq nil_{\pi} H_5(\tilde{X}) + nil_{\pi} H_5(K(\pi_2, 1)) + nil_{\pi} H_2(\tilde{X}) \otimes H_3(\pi_2, 1)$
 $+ nil_{\pi} H_4(\tilde{X}) \otimes \pi_2 + nil_{\pi} Tor(H_3(\tilde{X}), \pi_2).$

(iii) $nil_{\pi} H_5(\tilde{X}_{(4)}) \leq nil_{\pi} H_5(\tilde{X}_{(3)}) + nil_{\pi} H_5(K(\pi_3, 2)) + nil_{\pi} \pi_3 \otimes \pi_3$

(iv) $nil_{\pi} H_{m+1}(\tilde{X}_{(m)}) \leq nil_{\pi} H_{m+1}(\tilde{X}_{(m-1)}) + nil_{\pi} H_{m+1}(K(\pi_{m-1}, m-2))$, provided that
 $m \geq 5.$

(v) $nil_{\pi} H_6(\tilde{X}_{(4)}) \leq nil_{\pi} H_6(\tilde{X}_{(3)}) + nil_{\pi} H_6(K(\pi_3, 2)) +$
 $+ nil_{\pi} H_4(\tilde{X}_{(3)}) \otimes \pi_3 + nil_{\pi} Tor(\pi_3, \pi_3).$

(vi) $nil_{\pi} H_7(\tilde{X}_{(5)}) \leq nil_{\pi} H_7(\tilde{X}_{(4)}) + nil_{\pi} H_7(K(\pi_4, 3)) + nil_{\pi} \pi_4 \otimes \pi_4.$

(vii) $nil_{\pi} H_{m+2}(\tilde{X}_{(m)}) \leq nil_{\pi} H_{m+2}(\tilde{X}_{(m-1)}) + nil_{\pi} H_{m+2}(K(\pi_{m-1}, m-2)).$

Proof. In order to prove (i) and (ii) it is enough to remember that $\tilde{X}_{(2)} = \tilde{X}.$

(iii) follows from the facts that $H_2(\tilde{X}_{(3)}) = (0)$ and $H_3(\tilde{X}_{(3)}) \cong \pi_3(X).$

For (iv), $H_3(\tilde{X}_{(m-1)}) = 0 = H_2(\tilde{X}_{(m-1)})$ if $m \geq 5.$

For (v), $H_2(\tilde{X}_{(m-1)}) = 0$ and $H_3(\tilde{X}_{(3)}) \cong \pi_3.$

Similarly, we obtain (vi) and (vii). ■

Lemma 11. (i) $nil_{\pi} H_m(\tilde{X}) \leq nil_{\pi} \pi_m(X) + nil_{\pi} H_m(\tilde{X}_{m-1}); m \geq 3.$

(ii) $nil_{\pi} \pi_m \leq nil_{\pi} H_m(\tilde{X}_{(m-1)}) + nil_{\pi} H_m(K(\pi_{m-1}, m-2)) +$
 $+ nil_{\pi} H_2(\tilde{X}_{(m-1)}) \otimes \pi_{m-1}; \forall m \geq 3.$

Proof: Let

$$K(\pi_m, m) \hookrightarrow \begin{array}{c} \tilde{X}_m \\ \downarrow \\ \tilde{X}_{m-1} \end{array}$$

the fibration obtained by means of Postnikov's decomposition of \tilde{X} .

Then, $E_{0,m}^2 \cong \pi_m$; $E_{m,0}^2 \cong H_m(\tilde{X}_{m-1})$ and

$$E_{m-j,j}^2 = 0 \quad \text{if } 0 < j < m.$$

So, remembering that $H_m(\tilde{X}_m) \cong H_m(X)$ we get (i).

(iii) Let's consider the dual fibration

$$K(\pi_{m-1}, m-2) \hookrightarrow \begin{array}{c} \tilde{X}(m) \\ \downarrow \\ \tilde{X}_{m-1} \end{array}$$

So, $E_{0,m}^2 \cong H_m(K(\pi_{m-1}, m-2))$

$E_{m,0}^2 \cong H_m(\tilde{X}_{(m-1)})$; $E_{1,m-1}^2 = 0 = E_{m-j,j}^2$, if $0 < j < m-2$

and $E_{2,m-2}^2 \cong H_2(\tilde{X}_{(m-1)}) \otimes \pi_{m-1}$.

(Remark that $E_{2,m-2}^2 = 0$, if $m \geq 4$ and $E_{2,1}^2 \cong \pi_2 \otimes \pi_2$, $m = 3$).

Taking into account that $\pi_m(X) \cong H_m(\tilde{X}_{(m)})$ (Hurewicz), it follows that

$$\begin{aligned} \text{nil}_\pi \pi_m(X) &= \text{nil}_\pi H_m(\tilde{X}_{(m)}) \leq \text{nil}_\pi H_m(\tilde{X}_{(m-1)}) + \text{nil}_\pi H_m(K(\pi_{m-1}, m-2)) \\ &\quad + \text{nil}_\pi H_2(\tilde{X}_{(m-1)}) \otimes \pi_{m-1} \end{aligned} \quad \blacksquare$$

Theorem 12. Under the conditions that π acts nilpotently on π_n , $2 \leq n \leq 7$ we have:

- (i) $\text{nil}_\pi H_3(\tilde{X}) \leq \text{nil}_\pi \pi_3(X) \leq \text{nil}_\pi H_3(\tilde{X}) + \text{nil}_\pi H_2(\tilde{X}) \otimes H_2(\tilde{X})$
 $\quad + \text{nil}_\pi H_3(H_2(\tilde{X}))$.
- (ii) $\text{nil}_\pi \pi_4 \leq \text{nil}_\pi H_4(\tilde{X}) + \text{nil}_\pi H_4(K(\pi_3, 2)) + \text{nil}_\pi H_4(K(\pi_2, 1)) +$
 $\quad + \text{nil}_\pi H_3(\tilde{X}) \otimes H_2(\tilde{X}) + \text{nil}_\pi \text{Tor}(H_2(\tilde{X}), H_2(\tilde{X}))$
- (iii) $\text{nil}_\pi H_4(\tilde{X}) \leq \text{nil}_\pi \pi_4(X) + \text{nil}_\pi H_4(K(\pi_2, 2))$
- (iv) $\text{nil}_\pi H_5(\tilde{X}) \leq \text{nil}_\pi \pi_5(X) + \text{nil}_\pi \pi_2 \otimes \pi_3 + \text{nil}_\pi H_5(K(\pi_2, 2)) +$
 $\quad + \text{nil}_\pi H_5(K(\pi_3, 3))$

$$(v) \quad \text{nil}_\pi H_6(\tilde{X}) \leq \text{nil}_\pi \pi_6(X) + \text{nil}_\pi \pi_2 \otimes \pi_4 + \text{nil}_\pi H_6(K(\pi_2, 2)) + \\ + \text{nil}_\pi H_6(K(\pi_3, 3)) + \text{nil}_\pi H_6(K(\pi_4, 4))$$

$$(vi) \quad \text{nil}_\pi H_7(\tilde{X}) \leq \text{nil}_\pi \pi_7(X) + \text{nil}_\pi \pi_2 \otimes \pi_5 + \text{nil}_\pi \pi_4 \otimes H_3(\tilde{X}) + \\ + \text{nil}_\pi \text{Tor}(\pi_2, \pi_4) + \text{nil}_\pi \pi_3 \otimes H_4(K(\pi_2, 2)) + \\ + \text{nil}_\pi \pi_2 \otimes H_5(K(\pi_3, 3)) + \text{nil}_\pi H_7(K(\pi_2, 2)) + \\ + \text{nil}_\pi H_7(K(\pi_3, 3)) + \text{nil}_\pi H_7(K(\pi_4, 4)) + \\ + \text{nil}_\pi H_7(K(\pi_5, 5)).$$

Proof. To get (i) we recall that $\tilde{X}_{(2)} = \tilde{X}$ and that $H_3(\tilde{X}_2) = H_3(K(\pi_2, 2)) = 0$. Then, it is enough to use lemma 11 (i) and (ii).

(ii) results from the use of lemma 11 (ii) and corollary 10 (i).

(iii) follows from lemma 11 (i) and lemma 7 (i).

(iv) follows from lemma 11 (i) together with lemma 7 (i) and corollary 8 (i).

Finally, (v) is a consequence of lemma 11 (i), lemma 7 (i) and corollary 8 (iv). ■

Remark. The inequalities obtained for $\text{nil}_\pi \pi_n$ and $\text{nil}_\pi H_m(\tilde{X})$ turn out to be very complicated, if $n > 4$ and $m > 7$. As a matter of fact, if $n = 4$ and $m = 7$, they are not so simple, as the items (ii) and (vi) from the previous theorem show us.

Example 13. Let X be a connected CW-complex such that

$\pi_1(X) \cong \mathbb{Z}$, $\pi_2(X) \cong \mathbb{Z}^c$, $\pi_i(X) = 0$; $i > 2$, and the action of $\pi_1(X)$ or $\pi_2(X)$ is given by M as in the example 5.

Thus, $\tilde{X} = K(\mathbb{Z}^c, 2)$ and according to example 5 we have $\text{nil}_{\pi_1(X)} \pi_2(X) = c$ and $\text{nil}_{\pi_1(X)} H_4(\tilde{X}) = \frac{c(c+1)}{2}$.

Owing to the fact that $\pi_4(X) = 0$, we may state that the inequality (iii) obtained in theorem 12 is in this case an equality.

Likewise, this example yields a situation in which inequality (v) is an equality!

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