Adaptive Kriging Strategy for Risk Optimization with Time-Dependent Reliability

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H.M. Kroetz and A.T. Beck
Structural Engineering Department, University of São Paulo, Av. Trabalhador São-Carlense, 400, 13566-590 São Carlos, SP, Brazil

M. Moustapha and B. Sudret
Chair of Risk, Safety and Uncertainty Quantification, ETH Zurich, Stefano-Franscini-Platz 5, 8093 Zurich, Switzerland

ABSTRACT

In structural design, increasing the safety of structural systems usually implies additional costs, and sometimes cost savings can result in jeopardized safety. The Risk Optimization (RO) approach allows the designer to account for these conflicting goals, but not much has been proposed for the solution of such problems, particularly when the problem involves time-dependent reliability analysis. This paper presents a novel strategy for solving time-dependent risk optimization for quasi-static problems. Two coupled Kriging surrogates are employed to aid in the computationally heavy solution inherent to this kind of problem. Different adaptive strategies are proposed, considering the particularities of each step of the problem solution.

1 INTRODUCTION

In a competitive environment of scarce resources, it is important that structures are designed optimally. Structural optimization is the tool that aids the engineers in this task. Increasing the safety of structural systems usually implies additional costs, and sometimes cost savings can result in jeopardized safety. The additional considerations regarded in such analyses may lead to excessive computational burden. Hence the search for efficient approaches to solve such problems is an extremely important topic.

Two major approaches are usually considered in this context: Reliability-Based Design Optimization (RBDO) (Frangopol, 1985), in which deterministic cost functions are subjected to reliability constraints, and Risk Optimization (RO) (Enevoldsen and Sorensen, 1994; Aktas et al., 2001), where the cost is a function that implicitly includes the probabilities of failure associated to the reliability problems involved in the analysis. It is understood that RO leads to more comprehensive results, since costs associated to the entire life-cycle of the structure can be considered. Moreover, RO results tend to be more general, since the level of safety arises from the problem solution itself, as opposed to RBDO where the designer must set *a priori* the level of safety (Beck and Gomes, 2012). In both optimization frameworks, the determination of probabilities of failure is an important part of the solving procedure. This computation in itself may already be associated with significant computational burden. When loads are described as stochastic processes, or when the structural configuration changes with time, time-dependent probabilities of failure must be accounted for, which adds complexity to the problem. Naturally, the solution of such problems involves computationally intense tasks, and have not yet been thoroughly explored. This paper aims to contribute to the study of Risk Optimization in the context of degrading structures by providing a framework based on coupled adaptive surrogate models.

2 RISK OPTIMIZATION PROBLEM STATEMENT

In a context where structural failures are undesirable, but unavoidably associated with a small probability of occurrence, a comprehensive design approach should take into account the possible consequences of failure. The Risk Optimization formulation considers different costs associated with the life-cycle of the structure, including the expected cost of failure C_{EF}, which reads:

$$C_{EF} = \sum_{h=1}^{N_{ls}} P_{f_h} C_{f_h}, \tag{1}$$

where $h = \{1,\dots,N_{ls}\}$ denotes the limit state associated with a failure that occurs with a probability Pf_h and whose cost is C_{f_h}. Design and reliability constraints can also be considered, so that the optimization problem can be cast as:

$$\begin{aligned} \boldsymbol{d}^* &= \arg\min_{\boldsymbol{d}\in\mathbb{D}} C_T(\boldsymbol{d}), \\ \text{subject to:} \quad & \boldsymbol{\beta}_h \geq \bar{\boldsymbol{\beta}}_h, \quad h = \{1,\dots,N_{ls}\}, \end{aligned} \tag{2}$$

where

$$C_T(\boldsymbol{d}) = C_I(\boldsymbol{d}) + C_{I\&M}(\boldsymbol{d}) + C_O(\boldsymbol{d}) + C_{EF}(\boldsymbol{d}) \tag{3}$$

is the total cost associated to the design $\boldsymbol{d}$. This cost can be broken down into **I**nitial design costs C_I, **I**nspection and **M**aintenance costs $C_{I\&M}$, **O**peration costs C_O and the **E**xpected cost of **F**ailure C_{EF} defined in Eq. (1). Since the probabilities of failure are directly considered in the objective functions, constraints are often not necessary in this type of problem. In general, the reliability constraints are added to enforce the consideration of standards. Bound constraints may also be included to limit the solution domain to possible structural configurations, herein through the definition of the design space $\mathbb{D}$.

In general, to solve this problem, two simplifying assumptions are made. First, the inspection and maintenance costs are often neglected. In fact, even though many studies have already been conducted focusing on those cost terms, it seems to be a common practice to neglect them in Risk Optimization, hence considering only initial and expected failure costs (Gomes and Beck, 2013) Aissani et al. (2014) explains that the two inspection and maintenance costs can usually be regarded as deterministic whereas the failure cost is particularly important, because it highly affects the optimal solution in an uncertain context. The second aspect is the proper scaling of the cost with respect to time for structures whose life cycle is expected to span over decades. In fact, the costs of failure cannot be treated directly over time, since economic changes will affect the present value of money. One way to deal with this problem is to discretize the structure lifetime, bringing all costs to present value considering discount rates over each period (e.g. year discount rates), and considering the cumulative failure probabilities associated with each given period (Saad et al., 2016):

$$C_{EF}^{PV} = \sum_{h=1}^{N_{ls}} \sum_{l=1}^{\mathscr{T}} \frac{P_{fc_{hn}} C_{f_{hn}}}{(1+\eta)^l} \tag{4}$$

C_{EF}^{PV} is the expected cost of failure in present value, η is the discount rate, herein set to 1% per year, and $P_{fc_{kn}}$ and $C_{f_{hn}}$ are respectively the cumulative probability and cost of failure of the h-th limit-state in year n. In the remainder of this paper, instead of Eq. (1), Eq. (4) will be used to compute the expected cost of failure in Eq. (3).

3 TIME-VARIANT RELIABILITY

Let $\boldsymbol{X}(t,\omega)$ be a set of $M = p+q$ elements that represents the randomness of a mechanical problem. Material properties and geometric characteristics are typically described as random variables,

represented by $X_j(\omega)$, $j = \{1,\dots,p\}$. Loads can be modeled as random processes of time $X_k(t,\omega)$, $k = \{p+1,\dots,p+q\}$. In this notation, ω stands for the outcome in the space of outcomes Ω. Furthermore, for optimization problems, let $\boldsymbol{d}$ be a vector that gathers together all the system's design parameter. This vector may include parameters that describe moments of random variables, should tolerances on design dimensions be included in the analysis (Moustapha, 2016). Assume a structural limit state function that denotes safe states if it is greater than zero and failure if it is smaller than zero. Thus the boundary between desirable and undesirable structure responses is given by $g(\boldsymbol{d},t,\boldsymbol{X}(t,\omega))$, such that:

$$\begin{aligned} D_f(\boldsymbol{d},t) &= \{\boldsymbol{d},\boldsymbol{X}(t,\omega) : g(\boldsymbol{d},t,\boldsymbol{X}(t,\omega)) \leq 0\} \qquad \text{is the failure domain,} \\ D_s(\boldsymbol{d},t) &= \{\boldsymbol{d},\boldsymbol{X}(t,\omega) : g(\boldsymbol{d},t,\boldsymbol{X}(t,\omega)) > 0\} \qquad \text{is the safe domain.} \end{aligned} \tag{5}$$

For a given limit state, the instantaneous probability of failure P_{f_i} at a time $t = \tau$ is given by:

$$P_{f_i}(\boldsymbol{d};\tau) = \mathbb{P}\left(g(\boldsymbol{d},\tau,\boldsymbol{X}(\tau,\omega)) \leq 0\right) = \int_{D_f(\boldsymbol{d},\tau)} f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \tag{6}$$

where $\mathbb{P}(\cdot)$ denotes the probability of the event and $f_{\boldsymbol{X}}$ is the joint probability density function of all the random variables for a configuration $\boldsymbol{d}$ at a time τ. In this work, we are rather interested in the so-called *cumulative probability of failure* $P_{fc}(t_1,t_2)$ which is defined for a given $\boldsymbol{d}$ as the probability of occurrence of a structural failure within the time interval $[t_1,t_2]$:

$$P_{f_c}(\boldsymbol{d};t_1,t_2) = \mathbb{P}\left(\exists\tau \in [t_1,t_2] : g(\boldsymbol{d},\tau,\boldsymbol{X}(t,\omega)) \leq 0\right) \tag{7}$$

Different approaches have been suggested to compute P_{f_c}. The so-called *out-crossing approach* has been widely used. Examples of well-known methods include the *PHI2 approach* (Andrieu-Renaud et al., 2004) and the *asymptotic PHI2 method* (Sudret, 2008). The accuracy of such methods are however impeded by the introduction of approximation methods such as first-order reliability method (FORM). FORM is indeed known to lead to spurious results in presence of highly non-linear limit states and multiple design points. Henceforth, an approach based on direct simulation is instead considered in this paper. As will be shown in the sequel, this approach is coupled to surrogate modeling to lower the computational burden.

3.1 Monte Carlo-based estimation of the cumulative failure probability

The basic idea is to sample trajectories of the limit-state function over a given time interval and then count the number of such trajectories for which failure occurs. To achieve this, the random process of interest is first discretized, *i.e.* represented by a finite set of random variables (Sudret and Der Kiureghian, 2000). The discretization method employed in this work is the *expansion optimal linear estimation* (EOLE), as presented by Li and Der Kiureghian (1993).

Let us consider a scalar Gaussian random process $X(t,\omega)$, with mean $m(t)$, standard deviation $\sigma(t)$ and autocorrelation coefficient function $\rho_X(t_1,t_2)$. P time points are selected in the interval $[0,\mathscr{T}]$, so that $t_1 = 0$ and $t_P = \mathscr{T}$. The EOLE expansion is then given by:

$$X(t,\omega) \approx m(t) + \sigma(t)\sum_{i=1}^{r} \frac{\xi_i(\omega)}{\sqrt{\lambda_i}}\phi_i^T C_{t,t_i}(t), \tag{8}$$

where $\{\xi_i(\omega), i = 1,\dots,P\}$ are independent standard normal variables, $\{\phi_i,\lambda_i, i = 1,\dots,r\}$ are the eigenvectors and eigenvalues of the correlation matrix $\boldsymbol{C}$ sorted in decreasing order, with $C_{ij} = \rho_X(t_i,t_j), i,j = \{1,\dots,P\}$. Note that here the expansion is truncated to $r \leq P$ terms, the value of which defines the so-called *order of expansion*.

Once a proper discretization is obtained one is able to draw trajectories of the limit state function $g(\boldsymbol{d},t,\boldsymbol{X}(t,\omega))$ in the time interval $[0,\mathscr{T}]$, for a given $\boldsymbol{d}$. This is done by considering both the EOLE

expansions of the random processes $X_k(t,\omega), k = \{p+1,\dots,p+q\}$ and realizations of the time independent random variables $X_j(\omega)$, $j = \{1,\dots,p\}$. In practice, the simulated values are stored in an array G of dimension $1 \times N$, where N is the number of time instants in which the limit state equation is discretized. Each position i of this array corresponds to a time $t_i = (i-1)\cdot\Delta t$, where $\Delta t = \frac{\mathscr{T}}{N-1}$ is the sampling step, assuming a uniform discretization. For each t_i, consider a counter k_j that is increased every time g presents the first outcrossing in the interval $[t_i, t_{i+1}]$. A brute Monte Carlo estimation for the cumulative probability of failure is then given by:

$$P_{fc_{MC}}(0,t_i) = \frac{1}{N_{MC}} \sum_{j=1}^{N} k_j, \tag{9}$$

4 KRIGING A.K.A. GAUSSIAN PROCESS MODELING

When a complex computational model $\mathscr{M}$ has to be evaluated a large number of times, the idea of replacing it by a surrogate model, which mimics its behavior, but is far less expensive to evaluate arises. The Kriging technique consists in approximating such expensive models through the realization of a Gaussian process. In order to build it, a set of inputs and outputs of the model to be surrogated is considered, so that it is viewed as a *black box*. This set is know as the *design of experiments* (DOE):

$$\mathscr{D} = \left\{(\boldsymbol{x_i}, y_i) : \boldsymbol{x_i} \in \mathbb{R}^M, y_i = \mathscr{M}(\boldsymbol{x}_i), i = 1...n\right\} \tag{10}$$

where n is the number of observations available in the DOE. The Kriging model is then written as (Santner et al., 2003):

$$\mathscr{M}(\boldsymbol{x}) \approx \tilde{\mathscr{M}}(\boldsymbol{x}) = \sum_{j=1}^{p} \beta_j f_j(\boldsymbol{x}) + Z(\boldsymbol{x}) \tag{11}$$

where $\boldsymbol{\beta}$ are coefficients to be determined, $\boldsymbol{f}(\boldsymbol{x})$ is a set of basis functions. $Z(\boldsymbol{x})$ is a zero-mean, stationary Gaussian stochastic process, which will interpolate the known output information of the DOE. It is defined by an auto-covariance function $\mathrm{Cov}\left[Z(\boldsymbol{x}), Z(\boldsymbol{x}')\right] = \sigma^2 R(\boldsymbol{x}, \boldsymbol{x}'; \boldsymbol{\theta})$, where R stands for auto-correlation function with hyper-parameters $\boldsymbol{\theta}$, and σ^2 denotes its variance. In this work, ordinary Kriging is employed, *i.e.* $p = 1$ and $f_1(\boldsymbol{x}) = 1$. The auto-correlation function is chosen to belong to the Matérn $5/2$ family. The mean Kriging predictor reads:

$$\mu_{\tilde{\mathscr{M}}}(\boldsymbol{x}) = f^T(\boldsymbol{x})\boldsymbol{\beta} + r^T(\boldsymbol{x})\boldsymbol{R}^{-1}\left(\boldsymbol{y} - \boldsymbol{F}^T\boldsymbol{\beta}\right). \tag{12}$$

Here $\beta = \left(\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{F}\right)^{-1}\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{y}$ are weight coefficients obtained by *least squares regression*, and $\boldsymbol{F}$ is a matrix that gathers the basis functions applied to the DOE points, such that $F_{ij} = f_j(\boldsymbol{x}^{(i)})$, which reduces to a column vector of ones in the case of ordinary Kriging. Since the predictor is a Gaussian process, it is also possible to obtain its variance:

$$\sigma^2_{\tilde{\mathscr{M}}}(\boldsymbol{x}) = \sigma^2\left(1 - \boldsymbol{r}^T(\boldsymbol{x})\boldsymbol{R}^{-1}\boldsymbol{r}(x) + \boldsymbol{u}^T(\boldsymbol{x})\left(\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{F}\right)^{-1}\boldsymbol{u}(\boldsymbol{x})\right) \tag{13}$$

where $\boldsymbol{u} = \left(\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{F}\right)^{-1}\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{y}$ and $\boldsymbol{r}(\boldsymbol{x}) = [\boldsymbol{R}(\boldsymbol{x},\boldsymbol{x}_1)...\boldsymbol{R}(\boldsymbol{x},\boldsymbol{x}_n)]$. With this information, it is possible to evaluate the regions of the process where the prediction (Eq.12) is associated with lower or higher uncertainty.

4.1 Efficient Global Optimization (EGO)

The Kriging model is built here in order to approximate the cost function in the optimization process. The selection of an adequate DOE is crucial for this surrogate to provide sound results. In a context where each evaluation of the total cost is expensive, or only a limited budget of observations is available, the selection of an optimal DOE is necessary. Considering that the Kriging predictor provides not only an

estimation for the actual value of the surrogated model, but also a measure of the uncertainty associated to the prediction in that point, Jones et al. (1998) proposes an iterative way to concomitantly build an optimal DOE and find the minimizer of the underlying function. A first Kriging model $\tilde{\mathscr{M}}^0$ is built from an initial dataset $\mathscr{D} = \{(\boldsymbol{x}_i, y_i), i = 1, ..., m\}$. An improvement function $I(\boldsymbol{x})$ is defined as the difference between $\tilde{\mathscr{M}}^0(\boldsymbol{x})$ and $y_{\min} = \min(y_i), i = 1, ..., m \in \mathscr{D}$, if positive, and 0 otherwise. The expected value of $I(\boldsymbol{x})$ leads to the so-called *expected improvement function* (Bichon, 2010):

$$EI(\boldsymbol{x}) = (y_{\min} - \mu_{\tilde{\mathscr{M}}}(\boldsymbol{x}))\Phi\left(\frac{y_{\min} - \mu_{\tilde{\mathscr{M}}}(\boldsymbol{x})}{\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x})}\right) + \sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x})\varphi\left(\frac{y_{\min} - \mu_{\tilde{\mathscr{M}}}(\boldsymbol{x})}{\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x})}\right) \tag{14}$$

where φ and Φ are the standard normal PDF and CDF. This equation indicates how much the cost function evaluated in a given point $\boldsymbol{x}$ is expected to be smaller than the current predicted minimum. The next point to be added to the DOE, $\boldsymbol{x}_{m+1}$, is the one that maximizes $EI(\boldsymbol{x})$. Thus, a trade-off between moving towards a minimum and exploring the regions associated with high variance is carried out. As more points are iteratively added, $\boldsymbol{x}_{m+1}$ is expected to converge to the global optimum of a surrogate which is precise in the region of more relevance to the optimization problem. Since $\mu_{\tilde{\mathscr{M}}}(\boldsymbol{x})$ should be much faster to evaluate than $\mathscr{M}(\boldsymbol{x})$, one simple way to proceed is to define a large sample set (e.g. 10^5 or 10^6 samples) on the space of the design variables $[\boldsymbol{d}_l, \boldsymbol{d}_u]$, and evaluate the surrogate mean and variance in every point. EI is then calculated for every point. The next point to enrich the DOE is the one with the greatest EI.

4.2 Efficient global reliability analysis (EGRA)

Even with the employment of EGO, a time-variant risk optimization problem may still be computationally intractable. In a more limited scope, where time-variant problems can be pointwise represented by time-independent models (e.g. quasi-static problems), the limit state equations can also be easily surrogated, further improving the efficiency of the solution. In reliability problems one is mostly concerned with the sign of a limit state function. Any configuration of a given problem which is associated to a point that belongs to the failure domain D_f is considered infeasible, and the points in the vicinity of the limit state equation are the only ones where a small error could lead to a misinterpretation of the failure behavior. The different nature between a generic global optimization and a reliability analysis problem has led to adaptations of the EGO approach, which is not very suitable for the latter problem. Seeking for an optimum way of building a Kriging predictor that surrogates a limit state function, Bichon et al. (2008) proposes the *Expected Feasibility Function*:

$$\begin{aligned} EF(\boldsymbol{x}) = \mu_{\tilde{\mathscr{M}}}(\boldsymbol{x})\left[2\Phi\left(\frac{\mu_{\tilde{\mathscr{M}}}(\boldsymbol{x})}{\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x})}\right) - \Phi\left(\frac{-2\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x}) - \mu_{\tilde{\mathscr{M}}}(\boldsymbol{x})}{\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x})}\right) - \Phi\left(\frac{2\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x}) - \mu_{\tilde{\mathscr{M}}}(\boldsymbol{x})}{\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x})}\right)\right] \\ -\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x})\left[2\varphi\left(\frac{\mu_{\tilde{\mathscr{M}}}(\boldsymbol{x})}{\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x})}\right) - \varphi\left(\frac{-2\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x}) - \mu_{\tilde{\mathscr{M}}}(\boldsymbol{x})}{\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x})}\right) - \varphi\left(\frac{2\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x}) - \mu_{\tilde{\mathscr{M}}}(\boldsymbol{x})}{\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x})}\right)\right] \\ +2\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x})\left[\Phi\left(\frac{2\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x}) - \mu_{\tilde{\mathscr{M}}}(\boldsymbol{x})}{\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x})}\right) - \Phi\left(\frac{-2\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x}) - \mu_{\tilde{\mathscr{M}}}(\boldsymbol{x})}{\sigma_{\tilde{\mathscr{M}}}(\boldsymbol{x})}\right)\right]. \end{aligned} \tag{15}$$

Echard et al. (2011) makes use of a similar function to propose an iterative method, suitable for the adaptive construction of a limit state function, the so-called *Active Kriging - Monte Carlo Simulation* (AK-MCS). An initial Kriging model is built from a dozen points DOE, randomly selected from a larger set of size N_{mc}. The predictor is evaluated in all these points, and EF is computed for all of them. The next point to be added to the DOE is the one with the maximum expected feasibility. The probability of failure is thus calculated performing a Monte Carlo simulation on the predictor.

5 Proposed Framework

To sum up, a framework based on two coupled surrogate models is proposed in this work. In fact, the optimization problem in Eq. (2) is solved using EGO, *i.e.* a Kriging model $\widetilde{\mathscr{M}_{C_T}}$ that approximates

the relationship $\boldsymbol{d} \mapsto C_T(\boldsymbol{d})$ is built adaptively with the ultimate aim of finding the optimal cost $\boldsymbol{d}^*$. This is achieved here by Monte Carlo simulation as explained in Section 3.1. This task is actually cumbersome and becomes untractable when the model underlying the limit-state function is expensive-to-evaluate. In this paper, we restrict the application scope to quasi-static problems, hence allowing for the metamodeling of the limit-state function independently of the stochastic loading. In practice, EGRA is used to adaptively build a *unique global* metamodel $\widetilde{\mathcal{M}_G}$ in the so-called *augmented space* following Moustapha et al. (2016). The augmented space combines both the design and random variables space, so that one single metamodel can be used to compute the failure probability regardless of the design choice. Once that model is deemed accurate enough, herein when EF (Eq. (15)) is below a threshold set to 10^{-4} for all the candidates to enrichment, it is used to compute the failure probability, henceforth the expected cost of failure, a component of the total cost. Thus, the optimization is carried out by EGO, using the surrogate models built by EGRA when a limit state function must be evaluated. The last point added in the EGO enrichment procedure is the optimum sought.

6 EXAMPLES

6.1 Steel beam subject to corrosion

Consider a steel bending beam with length $L = 5$ m and rectangular cross-section $\{b_0, h_0\}^T$, which is submitted to dead loads $\rho_{st} b_0 h_0$ (Nm^{-1}), where $\rho_{st} = 78.5$ kNm is the steel mass density, as well as a pinpoint load F applied midspan.

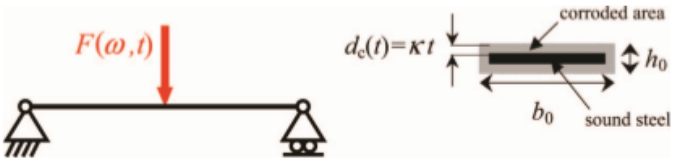

Figure 1: Corroded beam under a midspan load, after (Sudret, 2008)

The yield stress is denoted by f_y. The beam is also subjected to corrosion, in such a way that the corrosion depth d_c all around the section increases linearly with time, *i.e.* $d_c = \kappa t$. Furthermore, it is assumed that the corroded areas have lost all mechanical stiffness. The limit state function associated with the failure related to a plastic hinge at midspan reads:

$$g(\boldsymbol{d}, t, \mathbf{X}) = \frac{(b_0 - 2\kappa t)(h_0 - 2\kappa t)^2 f_y}{4} - \left(\frac{FL}{4} + \frac{\rho_{st} b_0 h_0 L^2}{8}\right), \tag{16}$$

where the yield stress is denoted by f_y. The time interval under consideration is [0, 10] years. The corrosion kinetics is controlled by $\kappa = 0.05$ mm year^{-1} . The load is modeled as a Gaussian random process with mean $6,000\,N$, coefficient of variation 0.3 and with a Gaussian autocorrelation function with correlation length $\lambda = 1$ month. The random parameters are gathered in Table 1. The risk optimization problem is defined by Eq. (17)

Table 1: Corroded bending beam – random variables and parameters

Parameter	Distribution	Mean	COV
Steel yield stress (MPa)	Lognormal	240	10%
Beam breadth (m)	Lognormal	b_0	3%
Beam Height (m)	Lognormal	h_0	3%

The initial costs are proportional to the cross section of the beam $C_I = \frac{bh}{0.008}$, and the failure costs are considered to be $1,000$ times higher, *i.e.* $C_f = 1,000 C_I$. The problem is to find $\boldsymbol{d} = \{b_0, h_0\}^T$ that

optimizes the total cost $C_T(\boldsymbol{d})$:

$$\begin{aligned} C_T &= C_I + \sum_{i=1}^{10} \frac{C_f P_{fc_i}}{(1+\eta)^i} \\ \text{s.t. } & 0.1 \leq b_0 \leq 0.5 \\ & 0.01 \leq h_0 \leq 0.06 \end{aligned} \tag{17}$$

Since the cost function has very steep regions, and the costs are always positive, the analysis was carried out considering the natural logarithm of the total costs. Figure 2 shows the contour plot of the cost function and the evolution of the EGO enrichment. The initial DOE points are marked with red squares. The subsequent points are shown by the yellow circles, and the optimum corresponds to the green diamond. The optimum obtained with the presented approach is $\boldsymbol{d}^*_{Krig} = \{0.1475, 0.0569\}$. A Particle Swarm Optimization (PSO) with 20 generations of 30 particles was also performed on the original problem, without the aid of surrogate models, to compare the results. The optimum point obtained with this approach is $\boldsymbol{d}^*_{PSO} = \{0.1339, 0.06\}$. The corresponding respective total costs are $C_T(\boldsymbol{d}^*_{Krig}) = 1.08$ and $C_T(\boldsymbol{d}^*_{PSO}) = 1.02$, which are relatively close thus validating the results of the proposed method.

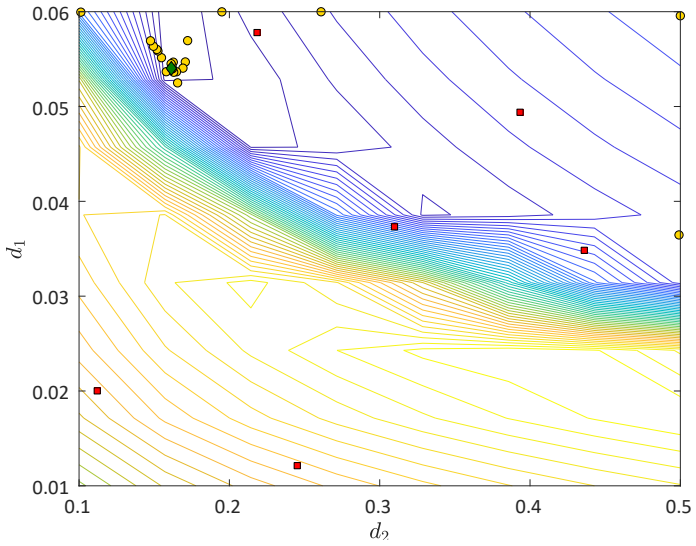

Figure 2: Convergence of $\log(C_T)$

6.2 23-bar Plane Truss

Consider the truss composed by 23 bars and 13 nodes represented in Figure 3, subjected to time varying loads applied on the upper nodes. The six vertical loads are modeled by a single stationary Gaussian process with mean value 50 kN, standard deviation 7.5 kN and Gaussian autocorrelation coefficient function with a correlation length of $\lambda = 1$ year. There are two types of bars, with different cross-sectional areas and materials, as indicated in Figure 3.

The bars are also subjected to uniform corrosion, with each dimension decreasing linearly in time, *i.e.* $d_c = \kappa t$. The cross-section is considered to be square, so that it can be totally defined by $A(t) = (l_0 - d_c)^2$, where l_0 is the square root of the area in the beginning of the analysis, and $\kappa = 0.1$ mm year^{-1}. The random variables of the problem are described in Table 2. The limit state equation is defined implicitly by a finite element model, and is written in terms of the vertical displacement of the mid-span node, herein denoted by V_1. The serviceability of the truss is given by a 0.1 m allowed displacement of the mid-span node :

$$g(\boldsymbol{d}, t, \mathbf{X}) = V_1(\boldsymbol{d}, t, \mathbf{X}) - 0.1. \tag{18}$$

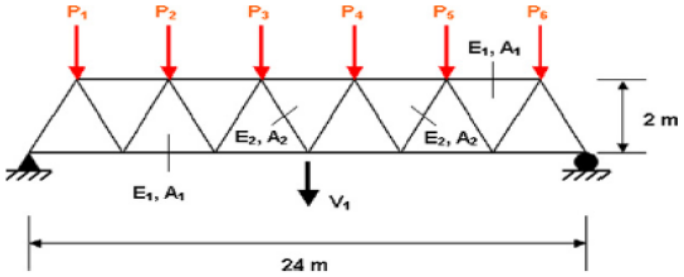

Figure 3: Corroded beam under a midspan load, after (Blatman and Sudret, 2010)

Table 2: Corroded bending beam – random variables and parameters

Parameter	Distribution	Mean	COV
$E_1(MPa)$	Lognormal	210000	10%
$E_2(MPa)$	Lognormal	210000	10%
$A_1(cm^2)$	Lognormal	d_1	10%
$A_2(cm^2)$	Lognormal	d_2	10%

The time interval under consideration is [0, 30] years, thus leading to the following formulation of the risk optimization problem:

$$\begin{aligned} C_T &= C_I + \sum_{i=1}^{30} \frac{C_f P_{fc_i}}{(1+\eta)^i}, \\ \text{s.t. } & 10\,\text{cm}^2 \leq d_1 \leq 30\,\text{cm}^2 \\ & 10\,\text{cm}^2 \leq d_2 \leq 30\,\text{cm}^2 \end{aligned} \tag{19}$$

The design costs are proportional to the volume of the structure, *i.e.* $C_I = 10^3(d_1 + d_2)$, and the cost of failure is obtained as $C_f = 1{,}000 C_I$. The results of optimum cost for 10 analyses are summarized in Figures 4 and 5. Except for one outlier, the methodology seems to provide consistent solutions. On average, a total of 15 model evaluations was needed to reach a solution.

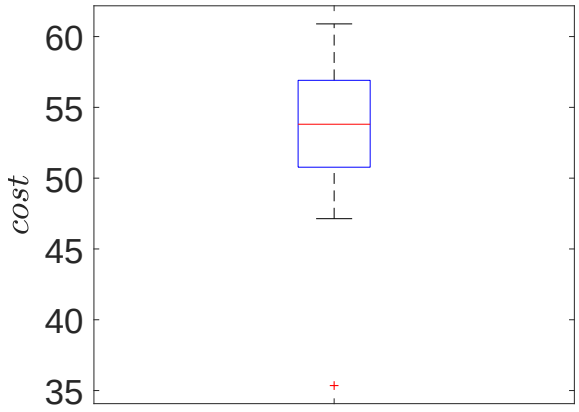

Figure 4: Optimum Costs

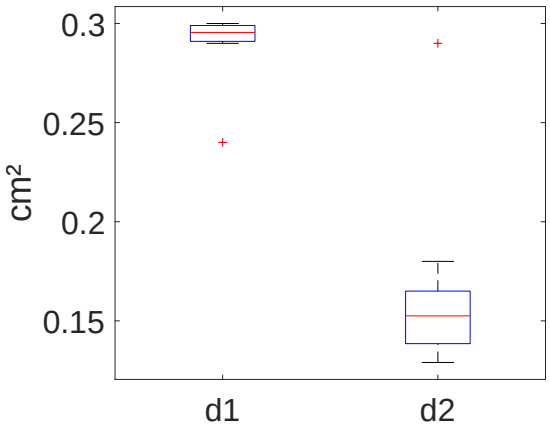

Figure 5: Optimum Value for Design Variables

7 Conclusion

A new strategy for the solution of time-dependent risk optimization problems was proposed. Two levels of adaptive Kriging models were applied. One analytical and one numerical example were studied, with satisfactory accuracy and convergence. On the other hand, number of evaluations of the inner surrogate model was found to be excessively large for this strategy to be applied in problems that combine extremely low failure probabilities together with time series that requires a large number of discretization points. Further studies are necessary in order to adequate the method to this kind of problems, and to increase the scope of the solution to involve dynamic problems that cannot be represented by pointwise surrogates of the limit state equations.

8 Acknowledgment

The first author thanks the Coordination for the Improvement of Higher Education Personnel (CAPES) for the financial support, program: Programa de Doutorado Sanduíche no Exterior, Process number: 88881.133186/2016-01.

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