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TWO CASES OF ASYMPTOTIC SMOOTHNESS FOR FUZZY DYNAMICAL SYSTEMS

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Asymptotic Smoothness of the Zadeh's Extensions

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Abstract

The concept of asymptotic smooth transformation was introduced by J. Hale [5] and it is a very important property for a transformation between complete metric spaces to have a global attractor. This property has also consequences on asymptotic stability of attractors. In our work we study the conditions under which the Zadeh's extension of a continuous map $f : \mathbf{R}^n \to \mathbf{R}^n$ is asymptotic smooth in the complete metric space $\mathcal{F}(\mathbf{R}^n)$ of normal fuzzy sets with the induced Hausdorff metric d_∞ (see Kloeden and Diamond [4]).

1 Introduction

The question of determine whether a discrete dynamical system has a compact attractor when the system evolves on a infinite dimensional vector space has been studied very intensively in the last ten years. Some new concepts that were irrelevant for the finite dimensional cases shows up to be of decisive importance in infinite dimensional cases. Among these concepts we will center our attention to the asymptotic smoothness of a function.

The concept of asymptotic smoothness was introduced by Jack Hale in [5]. Every continuous transformation between finite dimensional vector spaces are asymptotic smooth, this is not the case for infinite dimensional Banach spaces or metric spaces.. In particular, for such transformation every compact set which attracts locally points, also attracts locally compact sets. In his book Hale also gives some examples of asymptotic smooth transformations in infinite dimensional spaces. Our objective in this paper is to provide another class of such functions using the Zadeh's extensions of continuous transformation in $\mathbb{R}^n$. We think that these results will be also important to analyze the interaction of dynamical properties between a transformation in $\mathbb{R}^n$ and its Zadeh's Extension.

In the next section we give the main definitions and results in the literature. We include a small section discussing some properties of the compact sets in $\mathcal{F}(\mathbb{R}^n)$. In the last section we present the results we have up to now that are positive steps in the direction of the main conjecture: the Zadeh's extension of a continuous transformation is asymptotically smooth.

2 Preliminaries

If X is a metric space and $T : X \to X$ is a continuous transformation then we have a discrete dynamical system. For basic notation on dynamical systems we recommend [5]. We say that T is *asymptotically smooth* (see [5]) if, for each nonempty bounded and closed set $B \subset X$ for which $T(B) \subset B$, there is a compact set $J \subset B$ such that J attracts B. We recall that J attracts B if for each neighborhood of J there is a positive n_0 such that $T^n(B)$ is contained in that neighborhood for all $n \geq n_0$ (see Hale [5] page 9). If T is asymptotically smooth then a set attracts locally points if and only if attracts locally compact sets. Cooperman [3] and Brumley [2] have given examples where this is not true for general transformations on infinite dimensional Banach Spaces.

The concepts of limit sets of a dynamical systems are classical. Here we will need the following: the *ω-limit* of a subset B of X is given by:

$$\omega(B) = \bigcap_{n>0} \mathrm{cl}(\bigcup_{k\geq n} T^k(B))$$

If B is such that $T(B) \subset B$ then

$$\omega(B) = \bigcap_{n>0} \mathrm{cl}(T^n(B))$$

The next Lemma can be found in Hale [5] (page 11, Cor. 2.2.4)

Lemma 1. *If T is asymptotically smooth and B is a nonempty bounded set such that its positive orbit is bounded, then $\omega(B)$ is nonempty, compact, and invariant and $\omega(B)$ attracts B.*

The problem we addressed here is: are the Zadeh's extensions of continuous transformations on $\mathbb{R}^n$ asymptotically smooth? We recall some definitions on fuzzy metric spaces.

The family of all compact nonempty subsets of $\mathbb{R}^n$ will be denoted as $\mathcal{Q}(\mathbb{R}^n)$. We also set $\mathcal{F}(\mathbb{R}^n)$ for the family of fuzzy sets $u : \mathbb{R}^n \to [0,1]$ whose α-level:

$$[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\} \quad 0 < \alpha \leq 1 \text{ and } [u]^0 = \mathrm{cl}\{x \in \mathbb{R}^n : u(x) > 0\}$$

are in $\mathcal{Q}(\mathbb{R}^n)$.

It is known that the metric

$$d_\infty(u,v) = \sup_{0\leq\alpha\leq 1} h([u]^\alpha, [v]^\alpha)$$

where h is the Hausdorff metric in $\mathcal{Q}(\mathbb{R}^n)$, makes the spaces $(\mathcal{F}(\mathbb{R}^n), d_\infty)$ into complete metric spaces [6].

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping then we define the Zadeh's extension as:

$$\hat{f}(u)(x) = \begin{cases} \sup_{\tau\in f^{-1}(x)} u(\tau) & \text{if} \quad f^{-1}(x) \neq \emptyset \\ 0 & \text{if} \quad f^{-1}(x) = \emptyset \end{cases}$$

for all fuzzy set u.

The proof of the following results can be found in [1].

Theorem 1. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous then $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is well defined and for all $\alpha \in [0,1]$ we have*

$$[\hat{f}(u)]^\alpha = f([u]^\alpha).$$

We will need also a recent result of Roman-Flores et al. [7]

Theorem 2. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous then $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is continuous.*

3 Some facts on compact sets on $\mathcal{F}(\mathbb{R}^n)$

Our strategy is to prove the conjecture using the definition. Then we should be able to determine when a subset of $\mathcal{F}(\mathbb{R}^n)$ is compact or not. In the book of Kloeden and Diamond [4] one find the characterization of compact sets in the space of fuzzy sets with convex levels which does not fits our purpose since our levels are only compact sets in $\mathbb{R}^n$. The better approach we have found in the literature is the article of Rojas et al. [8]. In that paper it is shown how difficult is to find a compact set in $\mathcal{F}(\mathbb{R}^n)$. A result that is of great importance here is the following: If K is a compact set in $\mathbb{R}^n$ then $J_K = \{u \in \mathcal{F}(\mathbb{R}^n) : [u]^0 \subset K\}$ is compact if and only if K has diameter zero! Our candidate to be an attractor will be of this type but we have also good properties for this candidate.

Lemma 2. *Let $K \subset \mathbb{R}^n$ be a compact set and $A = \{u \in \mathcal{F}(\mathbb{R}^n) : [u]^0 \subset K\}$. Then A is a bounded closed set of the metric space $(\mathcal{F}(\mathbb{R}^n), d_\infty)$*

Proof: To see that A is bounded note that the distance of A to a point $\hat{0}$ (the characteristic function at 0) is finite. Indeed, denoting H the Hausdorff metric between compact sets

$$d_\infty(\hat{0}, A) = \inf_{u \in A} d_\infty(\hat{0}, u) = \inf_{u \in A} \sup_{\alpha \in [0,1]} H(\{0\}, [u]^\alpha) \leq \sup_{x \in K} d(0, x)$$

this last number is a number $M < \infty$ because K is a compact set in $\mathbb{R}^n$.

Now A is closed. In fact, consider a convergent sequence u_n in A with limit u, the convergence being in the metric d_∞ then we have in particular that $H([u_n]^0, [u]^0) \to 0$ and since $[u_n]^0 \subset K$ then $[u]^0 \subset K$ proving that A contains all its cluster points and then is closed. *QED*

4 The asymptotic smoothness problem

The main results will follow from a sequence of lemmas. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous transformation and $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ its Zadeh's extension which is continuous according Theorem 2. To prove that $\hat{f}$ is asymptotically smooth we have to prove that for each closed bounded set $B \subset \mathcal{F}(\mathbb{R}^n)$ such that $\hat{f}(B) \subset B$ there is a compact set $J \subset B$ that attracts B. We take a closed bounded nonempty B. Note that for each $u \in \hat{f}(B)$ and $v \in B$ such that $\hat{f}(v) = u$ we have by definition $[u]^\alpha = [\hat{f}(v)]^\alpha = f([v]^\alpha)$.

Lemma 3. *Define $B_\alpha = cl(\bigcup_{u \in B}[u]^\alpha) \subset \mathbb{R}^n$. There exists a compact set $K \subset \mathbb{R}^n$ such that $[u]^0 \subset K$ for all $u \in B$. Hence B_α is bounded for each $\alpha \in [0,1]$*

Proof: Take a point x in $\mathbb{R}^n$ and denote as $\hat{x}$ its characteristic function. Since B is bounded there exists a $r > 0$ such that the ball with center in $\hat{x}$ and radius r contains the entire set B. In other words, for all $u \in B$ $d_\infty(\hat{x}, u) \leq r$. According to a result that can be found in [1] this metric can be written as

$$d_\infty(\hat{x}, u) = \sup_{0 \leq \alpha \leq 1} \inf\{a : [u]^\alpha \subset B_a(x) \text{ and } x \in B_a([u]^\alpha)\}$$

where $B_a(x)$ denote the Euclidean ball centered in x. Then it follows immediately that $[u]^0 \subset B_r(x)$ what proves the lemma. *QED*

Lemma 4. *Consider B_α as in Lemma 3. Then B_α are closed bounded and satisfies $f(B_\alpha) \subset B_\alpha$. Therefore there is a compact set $J_\alpha \subset B_\alpha$ that attracts B_α.*

Proof: In fact the only assertion that has to be proved is that $f(B_\alpha) \subset B_\alpha$. The rest follows immediately from definitions and the fact that every continuous transformation in $\mathbb{R}^n$ is asymptotically smooth.

Take a x in B_α. By definition x is the limit of a sequence x_n with $x_n \in [u_n]^\alpha$ and $u_n \in B$. Therefore we have

$$f(x_n) \in f([u_n]^\alpha) = [\hat{f}(u_n)]^\alpha \subset B_\alpha.$$

Since f is continuous $f(x_n)$ converges to $f(x) \in B_\alpha$ and this completes this proof. *QED*

Using the Lemma 1 we can define the special compact invariant sets $J_\alpha = \omega(B_\alpha)$. These are the attractors we will consider. Now we can prove

Lemma 5. *For $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ we have $J_{\alpha_2} \subset J_{\alpha_1}$*

Proof: We have that $B_{\alpha_2} \subset B_{\alpha_1}$. Observing that in this case the omega limits can be written as:

$$J_{\alpha_2} = \omega(B_{\alpha_2}) = \bigcap_{n \geq 0} f^n(B_{\alpha_2}) \subset \bigcap_{n \geq 0} f^n(B_{\alpha_1}) = J_{\alpha_1}$$

and then follows the result.

We define $J = \{u \in B : [u]^0 \subset J_0\}$.

Proposition 1. *If $\hat{f} : \mathcal{F}(\mathbf{R}^n) \to \mathcal{F}(\mathbf{R}^n)$ is asymptotically smooth then J is nonempty for all nonempty closed bounded B with $\hat{f}(B) \subset B$.*

Proof: We take a closed bounded B with $\hat{f}(B) \subset B$. Since we are assuming that $\hat{f}$ is asymptotic smooth, we can use Lemma 1 to construct the attractor $K = \omega(B)$ which is also nonempty, compact and invariant. Again take $K_0 = \operatorname{cl} \bigcup_{u \in K} [u]^0$. With respect to K_0 we can assert

a) K_0 is a compact set contained in B_0. And this is clear.

b) K_0 is an invariant set for f (i.e. $f(K_0) = K_0$). In fact, if $x \in K_0$ then we know that x is a limit of a sequence x_n such that $x_n \in [u_n]^0$ and $u_n \in K$ then follows: $f(x) = \lim_{n \to \infty} f(x_n)$. Now

$$f(x_n) \in f([u_n]^0) = [\hat{f}(u_n)]^0 \in K_0$$

this shows that $f(K_0) \subset K_0$.

To prove that $K_0 \subset f(K_0)$, we repeat the process taking $x \in K_0$ and x_n as above. Now since K is invariant for $\hat{f}$ we have that $u_n = \hat{f}(v_n)$ where $v_n \in K$. Then for each $n \geq 0$ we have $x_n = f(y_n)$ where $y_n \in [v_n]^0 \subset K_0$. Choosing a subsequence if necessary we take $y = \lim_{n \to \infty} y_n$. By continuity of f follows that $x = f(y)$. Hence K_0 is an invariant set of f.

Now J_0 attracts B_0 and also K_0. This means that for each $\epsilon \geq 0$ there is an n_0 such that for $n \geq n_0$, $f^n(K_0) \subset N(J_0, \epsilon)$ or $K_0 \subset N(J_0, \epsilon)$ using the invariance. Here $N(J_0, \epsilon)$ stands for an ϵ-neighborhood of J_0. This proves that in fact $K_0 \subset J_0$ because ϵ is arbitrary and then $J \neq \emptyset$. *QED.*

In particular if $J = \emptyset$ then $\hat{f}$ isn't asymptotically smooth.

Proposition 2. *Suppose that $\hat{f} : \mathcal{F}(\mathbf{R}^n) \to \mathcal{F}(\mathbf{R}^n)$ is asymptotically smooth, B a nonempty, bounded, closed subset of $\mathcal{F}(\mathbf{R}^n)$such that $\hat{f}(B) \subset B$, $K = \omega(B)$ and $J_\alpha = \omega(B_\alpha)$. Then $K_\alpha = J_\alpha$.*

Proof: The proof that $K_\alpha \subset J_\alpha$ follows as in the above Proposition changing the index 0 by α. It remains to prove that $J_\alpha \subset K_\alpha$. We know that K attracts B, therefore given $\epsilon > 0$ there exists n_0 such that for $n \geq n_0$, $\hat{f}^n(B) \subset N(K, \epsilon)$ this implies that for each $\alpha \in [0,1]$, $f^n(B_\alpha) \subset N^*(K_\alpha, \epsilon)$. Where

$$N^*(K_\alpha, \epsilon) = \{x \in \mathbb{R}^n : d(x, K_\alpha) \leq \epsilon\}$$

It follows from the definition of J_α that

$$J_\alpha \subset \bigcap_{n \geq n_0} f^n(B_\alpha) \subset N^*(K_\alpha, \epsilon).$$

This is true for every $\epsilon > 0$ then follows the result. *QED.*

As a conclusion let us say that we are at this stage: To prove that some subset J as above is empty ensure that $\hat{f}$ isn't asymptotically smooth. If all J are nonempty, it would be a candidate for an attractor even not being compact in most cases, but our guess is: if J is not empty, there is a compact set K containing J that attracts B.

5 Examples

We present two examples.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous transformation, such that $\{0\}$ is the unique global attractor of f. Then $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is asymptotically smooth.

It is clear that $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is continuous. We take a bounded closed set $B \in \mathcal{F}(\mathbb{R}^n)$ such that $\hat{f}(B) \subset B$. Now we shall prove that:

(A) $\chi_{\{0\}}$ attracts B, and

(B) $\chi_{\{0\}} \subset B$.

Since the set $\{\chi_{\{0\}}\}$ is compact in $\mathcal{F}(\mathbb{R}^n)$ it follows our result.

To prove (A) note that for each $\alpha \in [0,1]$ the set $B_\alpha = \text{cl}(\bigcup_{u \in B} [u]^\alpha) \subset \mathbb{R}^n$ is compact, and then $\{0\}$ attracts B_α. This means that for each $\varepsilon > 0$ there is an $n_\alpha \in \mathbb{N}$ such that $f^n(B_\alpha) \subset N(\{0\}, \varepsilon)$ for all $n > n_\alpha$. Here $N(\{0\}, \varepsilon)$ denote the ε-neighborhood of $\{0\}$. But since $B_\alpha \subset B_0$ we have $f^n(B_\alpha) \subset N(\{0\}, \varepsilon)$ for all $n > n_0$.

For each $u \in B$, it follows that $f^n([u]^\alpha) \subset N(\{0\}, \varepsilon)$ and since f is continuous $[\hat{f}^n(u)]^\alpha \subset N(\{0\}, \varepsilon)$. From this it follows the assertion (A).

For the item (B) we take n_0 such that $d_\infty(\hat{f}^n(B), \chi_{\{0\}}) \leq \varepsilon$ for $n > n_0$. This implies that $\chi_{\{0\}}$ is in a ε-neighborhood of $\hat{f}^n(B)$ and then in a ε-neighborhood of B, because $\hat{f}^n(B) \subset B$. As this last assertion is true for any $\varepsilon > 0$ we must have $\chi_{\{0\}} \in B$. The proof of the result is complete.

As a particular case, we can take $f(x) = Ax$ where A is a linear operator. We can restrict the analysis to the eigenspace associated to the eigenvalues whose absolute values are less than one, and the above result applies.

This next example shows that it is not always true that the Zadeh's extension is asymptotic smooth.

If $f : \mathbb{R}^n \to \mathbb{R}^n$ have a compact set K with infinite points as attractor, then one can easily see that the set

$$B = \{u \in \mathcal{F}(\mathbb{R}^n) : [u]^0 \subset K\}$$

is a bounded closed set for which $\hat{f}(B) = B$. Since B is not compact according to Rojas et al. [8], then $\hat{f}$ is not asymptotically smooth. The meaning of this last example is that if $\hat{f}$ have a global attractor it will be hard to detect it since we can not use the theory for asymptotic smooth transformation.

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DISSIPATIVENESS OF FUZZY DYNAMICAL SYSTEMS

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ABSTRACT. We study the preservation of dissipativeness of a fuzzified dynamical system. We show that Zadeh's extensions of continuous discrete dynamical systems are not always asymptotic smooth in the fuzzy space. We give some examples and prove some results related to asymptotic smoothness.

1. INTRODUCTION

A way to define dissipativeness of a dynamical system is to look for global attractors. These are compact invariant sets that attracts the system in some specific way. To define these " attraction" we take a class $\mathcal{D}$ of subsets of the state space, and check if for each element of $\mathcal{D}$ the orbit approaches the attractor. (We will give the precise definitions in the next section).

It is well known that for finite dimensional state space all the concepts of attraction does not depend on $\mathcal{D}$ and one could simply work with the concept of point attractor. For infinite dimensional spaces this is not true. The concept connected to the equivalence between attraction of points and attraction of compact sets is the asymptotic smoothness of the dynamical systems. In fact, one can prove that if a system is asymptotic smooth then both concepts of attraction are the same. In this paper we will study this equivalence for the Zadeh's extension of a continuous map f in $\mathbb{R}^n$

2. PRELIMINARIES

The common definitions and examples from the theory of dynamical systems can be found in the book of Jack Hale [6]. Here we will recall some known results and definitions for fix the notations.

Let (X, d) a metric space and $T : X \to X$ a continuous map. We refer just as the dynamical system generated by T the discrete dynamical system $S : \mathbb{N} \times X \to X$ given by $S(n, x) = T^n(x)$. We assume at this point familiarity with the first chapter of [6]; in particular the concepts of positive and negative orbits, invariant sets and limit sets.

The distance between two subsets of X is defined as

$$\mathrm{dist}(A,B) = \sup_{a\in A}\inf_{b\in B} d(a,b)$$

This, sometimes, is called the semi distance of Hausdorff. The Hausdorff distance between these sets is

$$h(A,B) = \max\{\mathrm{dist}(A,B)\ ,\mathrm{dist}(B,A)\ \}.$$

To define the concept of attraction it is convenient to consider a class $\mathcal{D}$ of subsets of X such that if $D \in \mathcal{D}$ then $T(D) \in \mathcal{D}$. We will say that such a class is invariant for T. Recall that a set A attracts a set B if

$$\lim_{n\to\infty} \mathrm{dist}(T^n(B), A) = 0.$$

Then a compact invariant set A attracts $\mathcal{D}$ if it attracts each element D of that class. Then a compact invariant set A will be called a $\mathcal{D}$-attractor if it attracts $\mathcal{D}$. A is maximal with these properties.

We observe that if a $\mathcal{D}$-attractor exists then it is unique, but it is not necessary that A belongs to $\mathcal{D}$.

The map T is *asymptotically smooth* if, for every nonempty bounded set $B \subset X$, such that $T(B) \subset B$, there is a compact set $J \subset B$ which attracts B.

Let $V \subset X$ be an open set. Denote by $\mathcal{D}_p(V)$ the class of subsets consisting of exactly one point in V. If J is a compact invariant set for which there exists a neighborhood $J \subset V$ such that J is the $\mathcal{D}_p(V)$-attractor, we say that J attracts points locally.

Let $W \subset X$ be a bounded set. Denote by $\mathcal{D}_c(W)$ the class of compact sets contained in W. A compact invariant set $J \subset W$ attracts compact sets in W if it attracts $\mathcal{D}_c(W)$. The following Theorem is from Hale [6, p.12]

Theorem 1. *If T is asymptotically smooth and J is a compact invariant set that attracts point locally, then the statements are equivalent:*

(i) There is a bounded neighborhood W such that J attracts $\mathcal{D}_c(W)$.

(ii) J is stable.

(iii) J is uniformly asymptotically stable.

We will also need the following concepts on fuzzy spaces.

The family of all compact nonempty subsets of $\mathbb{R}^n$ will be denoted as $\mathcal{Q}(\mathbb{R}^n)$. We also set $\mathcal{F}(\mathbb{R}^n)$ for the family of fuzzy sets $u : \mathbb{R}^n \to [0,1]$ whose α-level:

$$[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\} \quad 0 < \alpha \leq 1 \text{ and } [u]^0 = \mathrm{cl}\{x \in \mathbb{R}^n : u(x) > 0\}$$

are in $\mathcal{Q}(\mathbb{R}^n)$.

It is known that the metric

$$d_\infty(u,v) = \sup_{0\le\alpha\le1} h([u]^\alpha,[v]^\alpha)$$

where h is the Hausdorff metric in $\mathcal{Q}(\mathbb{R}^n)$, makes the spaces $(\mathcal{F}(\mathbb{R}^n), d_\infty)$ into complete metric spaces [7].

Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be a mapping then we define the Zadeh's extension as:

$$\hat{f}(u)(x) = \begin{cases} \sup_{\tau\in f^{-1}(x)} u(\tau) & \text{if} \quad f^{-1}(x) \neq \emptyset \\ 0 & \text{if} \quad f^{-1}(x) = \emptyset \end{cases}$$

for all fuzzy set u.

The proof of the following results can be found in [1].

Theorem 2. *If $f:\mathbb{R}^n \to \mathbb{R}^n$ is continuous then $\hat{f}:\mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is well defined and for all $\alpha \in [0,1]$ we have*

$$[\hat{f}(u)]^\alpha = f([u]^\alpha).$$

We will need also a recent result of Roman-Flores et al. [8]

Theorem 3. *If $f:\mathbb{R}^n \to \mathbb{R}^n$ is continuous then $\hat{f}:\mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is continuous.*

Two others spaces of fuzzy sets are also found in the literature and they will of interest here, because they are invariant sets in $\mathcal{F}(\mathbb{R}^n)$. We denote by $\mathcal{F}^*(\mathbb{R}^n)$the subset of $\mathcal{F}(\mathbb{R}^n)$consisting of the elements u which don't have proper maximal local points (see [10] for a reference). And let $\mathcal{F}_C(\mathbb{R}^n)$to be the set of $u \in \mathcal{F}(\mathbb{R}^n)$ such that the mapping $\alpha \mapsto [u]^\alpha$ is continuous.

3. Asymptotic Smoothness of extensions

First of all we are led to ask whether the Zadeh's extension of continuous map $f:\mathbb{R}^n \to \mathbb{R}^n$ are asymptotically smooth. We know that every continuous map in $\mathbb{R}^n$ are asymptotically smooth. But the space $\mathcal{F}(\mathbb{R}^n)$has an important defect in this case: there are closed bounded sets in $\mathcal{F}(\mathbb{R}^n)$which are not compact. The following class of sets is a fundamental example in this case. Take K a compact set in $\mathbb{R}^n$ and consider $J_K = \{u \in \mathcal{F}(\mathbb{R}^n) : [u]^0 \subset K\}$. This set is closed and bounded but it is compact if and only if K has diameter zero. In view of this fact, one has that the identity map in $\mathcal{F}(\mathbb{R}^n)$ isn't asymptotically smooth. We will consider more examples in the next section.

In a previous work [2] we have studied which are the consequences of asymptotic smoothness of the Zadeh's extension, in particular we've obtained a procedure to construct the attractors when they exist. We'll give the main results of that work.

Suppose that $B \subset \mathcal{F}(\mathbb{R}^n)$ such that $\hat{f}(B) \subset B$. Put $B_\alpha = \mathrm{cl}(\bigcup_{u\in B}[u]^\alpha) \subset \mathbb{R}^n$ and $J_\alpha = \omega(B_\alpha) \subset B_\alpha$, now define $J = \{u \in B : [u]^0 \subset J_0\}$. We have

Proposition 1. *If $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is asymptotically smooth then J is nonempty for all nonempty closed bounded B with $\hat{f}(B) \subset B$.*

and

Proposition 2. *Suppose that $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is asymptotically smooth, B a nonempty, bounded, closed subset of $\mathcal{F}(\mathbb{R}^n)$such that $\hat{f}(B) \subset B$, $K = \omega(B)$ and $J_\alpha = \omega(B_\alpha)$. Then $K_\alpha = J_\alpha$.*

The details and proofs are in [2].

4. Examples

First we'll prove that if f has one global asymptotically stable fixed point, then we have a global attractor in $\mathcal{F}(\mathbb{R}^n)$.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous transformation, such that $\{0\}$ is the unique global attractor of f. Then $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is asymptotically smooth.

It is clear that $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is continuous. We take a bounded closed set $B \in \mathcal{F}(\mathbb{R}^n)$ such that $\hat{f}(B) \subset B$. Now we shall prove that:

(A) $\chi_{\{0\}}$ attracts B, and

(B) $\chi_{\{0\}} \subset B$.

Since the set $\{\chi_{\{0\}}\}$ is compact in $\mathcal{F}(\mathbb{R}^n)$ it follows our result.

To prove (A) note that for each $\alpha \in [0,1]$ the set $B_\alpha = \mathrm{cl}(\bigcup_{u\in B}[u]^\alpha) \subset \mathbb{R}^n$ is compact, and then $\{0\}$ attracts B_α. This means that for each $\varepsilon > 0$ there is an $n_\alpha \in \mathbb{N}$ such that $f^n(B_\alpha) \subset N(\{0\}, \varepsilon)$ for all $n > n_\alpha$. Here $N(\{0\}, \varepsilon)$ denote the ε-neighborhood of $\{0\}$. But since $B_\alpha \subset B_0$ we have $f^n(B_\alpha) \subset N(\{0\}, \varepsilon)$ for all $n > n_0$.

For each $u \in B$, it follows that $f^n([u]^\alpha) \subset N(\{0\}, \varepsilon)$ and since f is continuous $[\hat{f}^n(u)]^\alpha \subset N(\{0\}, \varepsilon)$. From this it follows the assertion (A).

For the item (B) we take n_0 such that $d_\infty(\hat{f}^n(B), \chi_{\{0\}}) \leq \varepsilon$ for $n > n_0$. This implies that $\chi_{\{0\}}$ is in a ε-neighborhood of $\hat{f}^n(B)$ and then in a ε-neighborhood of B, because $\hat{f}^n(B) \subset B$. As this last assertion is true for any $\varepsilon > 0$ we must have $\chi_{\{0\}} \in B$. The proof of the result is complete.

As a particular case of the above example we can take a linear map $f(x) = Ax$ where all eigenvalues have absolute value less than one. It is clear that in this case f has a point as attractor. If, however, one of eigenvalues has module 1, the extension isn't asymptotically smooth; in particular the identity isn't asymptotically smooth. In fact, we take K as the compact unitary disk contained in the general eigenspace $E(\lambda)$ where $|\lambda| > 1$. Clear that $f(K) = K$ and $\hat{f}(J_K) = J_K$, but J_K isn't compact.

Another example to consider is when the $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map with the extension$\hat{f}$ restricted to the $\mathcal{F}^*(\mathbb{R}^n)$ space. We'll show that in this case, $\mathcal{F}^*(\mathbb{R}^n)$ is asymptotically smooth, if all the sets $B \subset \mathcal{F}(\mathbb{R}^n)$ nonempty, bounded and closed have their $w-limits$ nonempty. First of all, we need the following result , which prove is in [1].

Proposition 3. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map, then:*
(i) $\hat{f}(\mathcal{F}^*(\mathbb{R}^n)) \subset \mathcal{F}^*(\mathbb{R}^n)$;
(ii) $\hat{f}(\mathcal{F}^*(\mathbb{R}^n))$ is continuous in the d_∞ metric.

In this case, we prove the result using the below proposition due to Róman-Rojas of equivalence of convergences:

Theorem 4. *Let u, $u_p \in \mathcal{F}(\mathbb{R}^n)$. If u doesn't have proper maxim local points, then are equivalents:*
(1) $u \longrightarrow u_p$ in d_∞ metric;
(2) $u \longrightarrow u_p$ in h metric and $[u]^1 = \liminf_{p\to\infty}[u_p]^1$;
(3) $u \longrightarrow u_p$ in L and $\bigcup\limits_{p=1}^{\infty}$ is bounded and $\limsup_{p\to\infty}[u_p]^0 \in [u]^0$,
where L metric is the metric in each level.

Still a result about sequences is necessary:

Proposition 4. *Let $\{A_n\}$ e $\{B_n\}$ two sequences of compact sets. Let $A_n \subset B_n$ for all $n \in \mathbb{N}$, $B_n \longrightarrow B$ and $A_{n_j} \longrightarrow A$ in Hausdorff metric. Then $A \subset B$.*

We know that $\lim_{p\to\infty} h(A_p, A) = 0$, if and only if A_p converges to A in a Kuratowski sense, where $\limsup_{p\to\infty} A_p = \bigcap_{p=1}^{\infty}(\overline{\bigcup_{j=p}^{\infty} A_j})$. And then, $A = \bigcap_{n\geq 0} \bigcup_{k_j \geq n} A_{k_j}$ and $B = \bigcap_{n\geq 0} \bigcup_{k\geq n} B_k$. Let $x \in A$, so $x \in \bigcap_{k_j \geq n} B_{k_j}$, for all $n \geq 0$, concluding that $x \in B$.

Theorem 5. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a continuous map and $\hat{f}$ the Zadeh extension restricted to the $\mathcal{F}^*(\mathbb{R}^n)$ space. If for all $B \subset \mathcal{F}^*(\mathbb{R}^n)$ nonempty, bounded and closed, such that $\hat{f}(B) \subset B$ we have $J \neq \emptyset$, so that $J = \omega(B)$, then $\hat{f}$ is asymptotically smooth.*

Is enough to show that J is compact.

As $\mathcal{F}^*(\mathbb{R}^n)$ is a metric space, we'll prove that is sequentially compact.

Let (u_j) be a sequence in J. For all $\alpha \in [0,1]$, $[u_j]^\alpha \subset J_\alpha$ is a sequence of nonempty compact sets of $\mathbb{R}^n$. Looking $\wp(J_\alpha) \subset K^n$ with the Hausdorff metric we have $\wp(J_\alpha)$ is a compact subset of a complete metric space. So, for all $\alpha \in [0,1]$, there is at least one accumulation point $A_\alpha \subset J_\alpha$ such that exists a subsequence of $[u_j]^\alpha \longrightarrow A_\alpha$ in Hausdorff metric, ie, $h([u_{j_{k_\alpha}}]^\alpha, A_\alpha) \longrightarrow 0$ when $j_{k_\alpha} \longrightarrow \infty$.

Let's choose $A_1 = \{x_0\}$, where x_0 is an accumulation point of $[u]^1$. Let $[u_{n_k}]^1 \longrightarrow \{x_0\}$ be a sequence that converges to $\{x_0\}$ and for $\alpha < 1$, let the set be $\mathcal{A}_\alpha$=$\{B \subset \mathbb{R}^n$ compact; B is an accumulation point of $[u_{n_k}]^\alpha$ and $\{x_0\} \subset B\}$. By the previous proposition, $\mathcal{A}_\alpha \neq \emptyset$. For each $\alpha < 1$, let's define $A_\alpha = \bigcap_{B \in \mathcal{A}_\alpha} B$

1. $\{x_0\} \subset A_\alpha$, for all $\alpha \in [0,1]$;
2. A_α is compact for all $\alpha \in [0,1]$;
3. For all $\beta < \alpha$, $A_\alpha \subset A_\beta$, by the previous proposition.

Let be v such that $[v]^\alpha = A_\alpha$ for all $\alpha \in [0,1]$.

So: $v \in \Im^*(\mathbf{R}^n)$.

In fact, as we've seen,

1. $[v]^0 = \{x_0\}$ is compact ;
2. $[v]^1 = v_1$ is nonempty and only has one element;
3. Let be a sequence $\alpha_k \nearrow \alpha$, $\alpha > 0$. Then $A_\alpha = \bigcap A_{\alpha_k}$. In fact,

•$A_\alpha \subset \bigcap A_{\alpha_k}$: As $\alpha_k \leq \alpha$, for all $k \geq 0$, so $A_\alpha \subset A_{\alpha_k}$, for all $k \geq 0$,ie, $A_\alpha \subset \bigcap A_{\alpha_k}$.

•$A_\alpha \supset \bigcap A_{\alpha_k}$: Observe that:

1. $\bigcap_{k \geq 0} [u_n]^{\alpha_k} = [u_n]^\alpha$, for a fixed n ;
2. $[u_n]^{\alpha_k} \longrightarrow B \in \mathcal{A}_{\alpha_k}$

Then, $\bigcap A_{\alpha_k} = \bigcap_{k \geq 0} \bigcap_{\mathcal{A}_{\alpha_k}} \bigcap_{j \geq 0} \bigcup_{l \geq j} [u_{n_l}]^{\alpha_k}$
$\subset \bigcap_{k \geq 0} \bigcap_{j \geq 0} \bigcup_{l \geq j} [u_{n_l}]^{\alpha_k}$
$= \bigcap_{j \geq 0} \bigcap_{k \geq 0} \bigcup_{l \geq j} [u_{n_l}]^{\alpha_k}$
$= \bigcap_{j \geq 0} \bigcup_{l \geq j} \bigcap_{k \geq 0} [u_{n_l}]^{\alpha_k}$
$= \bigcap_{j \geq 0} \bigcup_{l \geq j} [u_{n_l}]^{\alpha} \subset A_\alpha$

On the other hand,

$\bigcap A_{\alpha_k} = \bigcap_{k \geq 0} \bigcap_{\mathcal{A}_{\alpha_k}} \bigcup_{j \geq 0} \bigcap_{l \geq j} [u_{n_l}]^{\alpha_k}$
$\subset \bigcap_{k \geq 0} \bigcup_{j \geq 0} \bigcap_{l \geq j} [u_{n_l}]^{\alpha_k}$
$= \bigcup_{j \geq 0} \bigcap_{k \geq 0} \bigcap_{l \geq j} [u_{n_l}]^{\alpha_k}$
$= \bigcup_{j \geq 0} \bigcap_{l \geq j} \bigcap_{k \geq 0} [u_{n_l}]^{\alpha_k}$
$= \bigcup_{j \geq 0} \bigcap_{l \geq j} [u_{n_l}]^{\alpha} \subset A_\alpha$

Then $A_\alpha = \bigcap A_{\alpha_k}$.

Observe that $\bigcup_{p=1}^{\infty}[u_p]^0 \subset J_0$, hence is bounded. Applying the previous theorem $u_p \longrightarrow u$ in d_∞ metric, hence J is compact.

Concluding that $\hat{f}$ is asymptotically smooth.

5. Conclusions

We couldn't determine whether an extension in $\mathcal{F}^*(\mathbb{R}^n)$ is asymptotically smooth or not, but $\mathcal{F}^*(\mathbb{R}^n)$ is an invariant subset of $\mathcal{F}(\mathbb{R}^n)$ and, under some conditions, we can classify a class of extensions as asymptotically smooth. If we could prove that all the $w - limits$ are nonempty in this subspace, then all closed, bounded sets would be a compact set. We also have the subspace $\mathcal{F}_C(\mathbb{R}^n) \supset \mathcal{F}^*(\mathbb{R}^n)$. Our point is: Could we classify the extensions in this subspace? Is that true that all bounded and closed sets in this space is a compact set?

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