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REPRESENTATIONS OF GALOIS ALGEBRAS

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Abstract

Galois algebras allow an effective study of their representation theory based on the invariant skew group structure. In particular, this leads to many remarkable results on Gelfand-Tsetlin representations of the general linear Lie algebra $\mathfrak{gl}_n$, quantum $\mathfrak{gl}_n$, Yangians of type A and finite W-algebras of type A.

1 Introduction

A classical problem of the representation theory of simple complex finite dimensional Lie algebras is the classification of simple modules. Today such a classification is known only for the Lie algebra $\mathfrak{sl}_2$ Block [1981]. Special attention is addressed to the study of so-called *weight* modules, i.e. those on which a certain Cartan subalgebra is diagonalizable. By the results of Fernando [1990] and Mathieu [2000], the classification of simple weight modules with finite dimensional weight subspaces is well known for any simple finite dimensional Lie algebra. On the other hand, a classification of simple modules remains open even in the category of weight modules with infinite dimensional weight subspaces. The largest subcategory of the category of weight module with some understanding of simple objects is the category of Gelfand-Tsetlin modules. The Gelfand-Tsetlin theory has attracted considerable interest in the last 40 years after the pioneering work of Gelfand and Cetlin [1950] and was developed in Drozd, Ovsienko, and Futorny [1991], Graev [2004], Graev [2007], Drozd, Ovsienko, and Futorny [1989], Mazorchuk [1998], Mazorchuk [2001], Molev [2006], Želobenko [1973], among the others. Gelfand-Tsetlin integrable systems were studied by Guillemin and Sternberg [1983], Kostant and Wallach [2006a], Kostant and Wallach [2006b], Colarusso and Evens [2010], Colarusso and Evens [2014].

Gelfand-Tsetlin theory can be viewed in a more general context of Harish-Chandra categories Drozd, Futorny, and Ovsienko [1994], Futorny and Ovsienko [2007] which play very important role in the representation theory. These are the categories of modules over

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a given algebra defined by the restriction onto a fixed subalgebra. General setting for the study of Harish-Chandra categories was established in Drozd, Futorny, and Ovsienko [1994]. Examples of Harish-Chandra categories include classical Harish-Chandra modules over a finite dimensional Lie algebra defined with respect to a reductive subalgebra (Dixmier [1974]), weight modules over semisimple finite dimensional Lie algebras with respect to a Cartan subalgebra, Gelfand-Tsetlin modules over $\mathfrak{gl}_n$ (Drozd, Ovsienko, and Futorny [1991]), certain representations of Yangians (Futorny, Molev, and Ovsienko [2005]) etc. In the case of generalized Weyl algebras of rank 1 this approach led to a complete classification of simple modules (Bavula [1992], Bavula and van Oystaeyen [2004]).

Developed techniques turned out to be very useful in the study of Gelfand-Tsetlin modules for the Lie algebra $\mathfrak{gl}_n$ (Drozd, Ovsienko, and Futorny [1991], Ovsienko [2002]). Gelfand-Tsetlin modules form the full subcategory of weight $\mathfrak{gl}_n$-modules which are sums of finite dimensional modules over the *Gelfand-Tsetlin subalgebra* Γ (certain maximal commutative subalgebra of the universal enveloping algebra of $\mathfrak{gl}_n$) Drozd, Ovsienko, and Futorny [1991], Futorny and Ovsienko [2010]. These modules are weight modules with respect to some Cartan subalgebra of $\mathfrak{gl}_n$ but they allow to have infinite dimensional weight spaces.

Gelfand-Tsetlin theory had a successful development in Ovsienko [2002], where it was shows that simple Gelfand-Tsetlin modules over $\mathfrak{gl}_n$ are parametrized up to some finiteness by the maximal ideals of Γ. Different explicit constructions of Gelfand-Tsetlin modules for $\mathfrak{gl}_n$ were recently obtained in Futorny, Grantcharov, and Ramirez [2014], Futorny, Grantcharov, and Ramirez [2015], Futorny, Grantcharov, and Ramirez [2016b], Futorny, Grantcharov, and Ramirez [2016a], Futorny, Ramirez, and Zhang [2016], Zadunaisky [2017], Vishnyakova [2018], Vishnyakova [2017], Ramírez and Zadunaisky [2017]. Nevertheless, the problem remains open.

As an attempt to unify the representation theories of the universal enveloping algebra of $\mathfrak{gl}_n$ and of the generalized Weyl algebras a new concept of Galois orders was introduced in Futorny and Ovsienko [2010]. These algebras have a hidden skew (semi)group structure. In particular, the universal enveloping algebra of $\mathfrak{gl}_n$ is an example of such algebra where invariant skew group structure comes from the Gelfand-Tsetlin formulas. Representation theory of Galois algebras was developed in Futorny and Ovsienko [2014]. It provides a new framework for the study of representation of various classes of algebras. Recent paper of Hartwig (Hartwig [2017a]) discovers new examples of Galois algebras for which the theory can be effectively applied.

2 Harish-Chandra modules

We recall basic facts about Harish-Chandra module categories following Drozd, Futorny, and Ovsienko [1994]. Let U be an associative algebra over , $\Gamma \subset U$ a subalgebra. The set of maximal ideals $\mathbf{m}$ of Γ such that $\dim \Gamma/\mathbf{m} < \infty$ will be called the *cofinite spectrum* $\operatorname{cfs}\Gamma$ of Γ. Then every $\mathbf{m} \in \operatorname{cfs}\Gamma$ defines a unique simple Γ-module of dimension $l(\mathbf{m})$ where $\Gamma/\mathbf{m} \simeq M_{l(\mathbf{m})}()$.

We say that M is a *Harish-Chandra module* (with respect to Γ) if M a finitely generated U-module such that

$$M = \bigoplus_{\mathbf{m}\in \operatorname{cfs}\Gamma} M(\mathbf{m}),$$

where

$$M(\mathbf{m}) = \{x \in M \mid \exists k,\ \mathbf{m}^k x = 0\}.$$

The support of a Harish-Chandra module M is a subset of $\operatorname{cfs}\Gamma$ consisting of those $\mathbf{m}$ for which $M(\mathbf{m}) \neq 0$.

We denote by $\mathbb{H}(U,\Gamma)$ the full subcategory consisting of all Harish-Chandra modules in $U-$ mod. It is closed under the operations of taking submodules, quotients and direct sums.

In Drozd, Futorny, and Ovsienko [ibid.] a concept of a Harish-Chandra subalgebra was introduced. For any $\mathbf{m} \in \operatorname{cfs}\Gamma$ denote by $L_{\mathbf{m}}$ the unique simple $\Gamma/\mathbf{m}$-module. We say that Γ is quasi-commutative if $\operatorname{Ext}^1(L_{\mathbf{m}}, L_{\mathbf{n}}) = 0$ for all $\mathbf{m} \neq \mathbf{n}$. We also say that Γ is quasi-central if for every $u \in U$, the Γ-bimodule $\Gamma u \Gamma$ is finitely generated as a left and as a right Γ-module. Clearly, for a noetherian Γ it is sufficiently to check this condition only for the generators of Γ (cf. Drozd, Futorny, and Ovsienko [ibid.], Proposition 8). A subalgebra Γ is called *Harish-Chandra* if it is quasi-central and quasi-commutative.

Example 2.1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $\mathfrak{F}$ its reductive Lie subalgebra. Then $\Gamma = U(\mathfrak{F})$ is a Harish-Chandra subalgebra of $U = U(\mathfrak{g})$. Indeed, Γ is quasi-commutative since any $\mathbf{m} \in \operatorname{cfs}\Gamma$ is cofinite. Also Γ is quasi-central since $\mathfrak{g}$ is finite dimensional.*

The concept of a Harish-Chandra subalgebra is essential for understanding the categories of Harish-Chandra modules. We address to Drozd, Futorny, and Ovsienko [ibid.] for further properties of Harish-Chandra subalgebras in the general setting. The most studied is the case of commutative Γ which we consider next.

2.1 Commutative Γ. Let U be an associative algebra and $\Gamma \subset U$ a noetherian commutative subalgebra. A natural idea is to try to parametrize simple modules in the Harish-Chandra category $\mathbb{H}(U,\Gamma)$ by simple Γ-modules. Since any simple Γ-module is 1-dimensional it defines a homomorphism from Γ to which we call a *character* of Γ. The kernel of any such character is a maximal ideal of Γ, thus there is a one-to-one correspondence between the characters of Γ and elements of $\operatorname{Specm}\Gamma$. The following problem of extension of characters to simple modules in $\mathbb{H}(U,\Gamma)$ is of prime importance for description of possible supports of simple Harish-Chandra modules.

Problem 1. Given $\mathbf{m} \in \operatorname{Specm}\Gamma$ is there a (simple) module $M \in \mathbb{H}(U,\Gamma)$ such that $M(\mathbf{m}) \neq 0$.

Recall that in the classical setting when both U and Γ are commutative and extension $\Gamma \subset U$ is integral then we have a map between the sets of prime (maximal) ideals φ : $\operatorname{Spec} U \to \operatorname{Spec}\Gamma$ and the fiber $\varphi^{-1}(\mathfrak{p})$ is non-empty for every $\mathfrak{p} \in \operatorname{Spec} A$. In particular, every character of Γ can be extended to a character of U. Moreover, if U is finitely generated module over Γ then all fibers $\varphi^{-1}(\mathfrak{p})$ are finite and the number of different extensions of each character of Γ is finite. By the Hilbert-Noether theorem this is the case when $U = S(V)$ is the symmetric algebra of a finite dimensional vector space V and Γ is the subalgebra of G-invariants of U for some finite subgroup G of $GL(V)$.

If U is noncommutative then the restriction functor from the category $\mathbb{H}(U,\Gamma)$ to the category of torsion Γ-modules induces a map Φ from $\operatorname{Specm}\Gamma$ to the set of isomorphism classes $Irr(U)$ of simple U-modules in $\mathbb{H}(U,\Gamma)$. Given a maximal ideal $\mathbf{m} \in \operatorname{Specm}\Gamma$, $\Phi(\mathbf{m})$ consist of those simple $V \in \mathbb{H}(U,\Gamma)$ such that $V(\mathbf{m}) \neq 0$ (or left maximal ideals of U which contain $\mathbf{m}$).

Example 2.2. *(i) Let $\mathfrak{g}$ be a reductive Lie algebra with a Cartan subalgebra $\mathfrak{H}$, $U = U(\mathfrak{g})$ and $\Gamma = U(\mathfrak{H})$. Then for any weight $\lambda \in \mathfrak{H}^*$ the fiber $\Phi(\lambda)$ is infinite (even in the category $\mathcal{O}$ which is a subcategory of $\mathbb{H}(U,\Gamma)$).*

(ii) Let $U = U_n = \mathbb{C}[S_n]$, $U_1 \subset \ldots \subset U_n$, Z_k the center of $\mathbb{C}[U_k]$. Then $\Gamma =< Z_1, \ldots, Z_n >$ is maximal commutative. It is generated by the Jucys-Murphy elements $X_i = (1i)+\ldots+(i-1i)$, $i = 1, \ldots, n$. The elements of $\operatorname{Specm}\Gamma$ *parametrize the irreducible representations of the group S_n Okounkov and Vershik [1996].*

The freeness of U over Γ as a right module guarantees the lifting of characters of Γ (all examples above are of this kind). Finding sufficient conditions for the fiber $\Phi(\mathbf{m})$ to be non-empty for any point $\mathbf{m} \in \operatorname{Specm}\Gamma$ is a difficult problem in general. In particular, if U is a *special filtered* such conditions were obtained in Futorny and Ovsienko [2005] generalizing the Kostant's theorem (see further examples in Futorny and Ovsienko [ibid.] and Futorny, Molev, and Ovsienko [2005]).

3 Galois algebras

A class of Galois rings (orders) was introduced in Futorny and Ovsienko [2010] to deal with the problem of extension of characters of commutative subalgebras.

Let R be a ring, $\mathcal{M}$ a monoid acting on R by ring automorphisms. We will denote the action of $m \in \mathcal{M}$ on $r \in R$ by r^m. Consider the skew monoid ring $R * \mathcal{M}$. Any element of $R * \mathcal{M}$ can be written in the form $x = \sum_{m \in \mathcal{M}} x_m m$. Define $supp\, x$ as the set of those $m \in \mathcal{M}$ for which x_m is not zero.

Let G be a finite group acting on $\mathcal{M}$ by conjugation. Then we have an action of G on $R * \mathcal{M}$ by ring automorphisms: $g(rm) = g(r)g(m)$, $g \in G, r \in R, m \in \mathcal{M}$.

Assume now that Γ is an integral domain, K the field of fractions of Γ and L a finite Galois extension of K with the Galois group $G = \mathrm{Gal}(L/K)$. Consider the action of G by conjugation on $\mathrm{Aut}(L)$. Let $\mathcal{M}$ be any G-invariant submonoid of $\mathrm{Aut}(L)$. For our purposes we will always require $\mathcal{M}$ to be *K-separating*, that is $m_1|_K = m_2|_K \Rightarrow m_1 = m_2$ for $m_1, m_2 \in \mathcal{M}$. The action of G on L and on $\mathcal{M}$ (by conjugations) extends to the action of G on the skew monoid ring $L * \mathcal{M}$. Denote by $\mathcal{K} = (L * \mathcal{M})^G$ the subring of invariants.

Definition 3.1. *A finitely generated Γ-subring U of $\mathcal{K}$ is called a* Galois ring over Γ *if* $UK = KU = \mathcal{K}$.

We have the following characterization of Galois rings:

We will always assume that all Galois rings are -algebras. In this case we say that a Galois ring is a Galois algebra over Γ.

Example 3.1. *Let $U = \Gamma(\sigma, a)$ be a* generalized Weyl algebra *of rank* 1 *(Bavula [1992]), where Γ is a unital integral domain, $a \in \Gamma$, σ an automorphism of Γ of infinite order. It is generated over Γ by X and Y such that $X\gamma = \sigma(\gamma)X$, $Y\gamma = \sigma^{-1}(\gamma)Y$, $YX = a$, $XY = \sigma(a)$. Let K be the field of fractions of Γ and $\mathcal{M} \simeq \mathbb{Z}$ is a subgroup of* $\mathrm{Aut}\,\Gamma$ *generated by σ. Then U can be embedded into the skew group algebra $K * \mathbb{Z}$ when $X \mapsto \sigma$ and $Y \mapsto a\sigma^{-1}$. Clearly, U is a Galois algebra over Γ. Note that $U \simeq \Gamma * \mathbb{Z}$ if a is invertible in Γ.*

3.1 Galois orders. Now we discuss a special class of Galois rings which are called *Galois orders*. Galois orders were introduced in Futorny and Ovsienko [2010] as a natural noncommutative generalization of a classical notion of order in skew group rings (cf. McConnell and Robson [1987]).

A Galois ring U over Γ. is *right (respectively left) Galois order*, if for any finite dimensional right (respectively left) K-subspace $W \subset U[S^{-1}]$ (respectively $W \subset [S^{-1}]U$), $W \cap U$ is a finitely generated right (respectively left) Γ-module. A Galois ring is *Galois order* if it is both right and left Galois order.

For a right Γ-submodule $M \subset U$ denote

$$\mathbb{D}_r(M) = \{u \in U \,|\, \exists \gamma \in \Gamma, \gamma \neq 0 \text{ such that } u \cdot \gamma \in M\}.$$

It follows immediately that $\mathbb{D}_r(M)$ is a Γ-module.

We have the following characterization of a Galois order.

Proposition 3.1 (Futorny and Ovsienko [2010], Corollary 5.1, 5.2)**.** *(i) A Galois ring U over a noetherian Γ is* right (left) Galois order *if and only if for every finitely generated right (left) Γ-module $M \subset U$, the right (left) Γ-module $\mathbb{D}_r(M)$ is finitely generated.*

(ii) If a Galois ring U over a noetherian domain Γ is projective as a right (left) Γ-module then U is a right (left) Galois order.

In the commutative case if K is the field of fractions of Γ, $U \subset K$ is finitely generated over Γ and the extension $\Gamma \subset U$ is integral then U is Galois order over Γ. Further examples of Galois orders include: generalized Weyl algebras over integral domains with infinite order automorphisms (e.g. the n-th Weyl algebra A_n, the quantum plane, the q-deformed Heisenberg algebra, quantized Weyl algebras, the Witten-Woronowicz algebra Bavula [1992]; the universal enveloping algebra of $\mathfrak{gl}_n$ over the Gelfand-Tsetlin subalgebra Drozd, Ovsienko, and Futorny [1991], Drozd, Futorny, and Ovsienko [1994], finite W-algebras Futorny, Molev, and Ovsienko [2005]).

There is a strong connection between Galois orders and maximality of Harish-Chandra subalgebras. Namely, we have

Theorem 3.1. *(i) Let Γ be a finitely generated domain over and U a Galois order over Γ. Then Γ is a Harish-Chandra subalgebra in U.*

(ii) Let U be a Galois ring over finitely generated -algebra Γ and $\mathcal{M}$ be a group. If Γ is a Harish-Chandra subalgebra in U then U is a Galois order if and only if U_e is an integral extension of Γ.

(iii) Let U be a Galois ring over a normal noetherian Harish-Chandra subalgebra Γ and $\mathcal{M}$ be a group. Then U is a Galois order over Γ if and only if Γ is maximal commutative in U.

Proof. First item follows from [Futorny and Ovsienko [2010], Corollary 5.4]. Second item follows from [Futorny and Ovsienko [ibid.], Theorem 5.2, (2)]. Let U be a Galois ring over a normal noetherian Harish-Chandra subalgebra Γ. If Γ is maximal then applying [Futorny and Ovsienko [ibid.], Corollary 5.6, (2)] and the fact that U has no torsion as a Γ-module we conclude that U is a Galois order over Γ. To prove the converse, it is sufficient to show that $U \cap K = \Gamma$ by [Futorny and Ovsienko [ibid.], Theorem 4.1]. Since $U \cap K$ is an integral extension of Γ by the second item, the statement follows from the normality of Γ.

□

The problem of lifting of characters for Galois orders was studied in Futorny and Ovsienko [2014]. In particular, sufficient conditions for the fiber $\Phi(\mathbf{m})$ to be nontrivial and finite were established. Let $U \subset (L * \mathcal{M})^G$ be a Galois ring over Γ. Consider the integral closure $\bar{\Gamma}$ of Γ in L. It is a standard fact that if Γ is finitely generated as a -algebra then any character of Γ has finitely many extensions to characters of $\bar{\Gamma}$.

Let $\bar{\mathbf{m}}$ be any lifting of $\mathbf{m}$ to the integral closure of Γ in L, and $\mathcal{M}_{\mathbf{m}}$ the stabilizer of $\bar{\mathbf{m}}$ in $\mathcal{M}$. Note that the group $\mathcal{M}_{\mathbf{m}}$ is defined uniquely up to G-conjugation. Thus the cardinality of $\mathcal{M}_{\mathbf{m}}$ does not depend on the choice of the lifting. We denote it by $|\mathbf{m}|$.

For $\mathbf{m}, \mathbf{n} \in \operatorname{Specm} \Gamma$ set

$$S(\mathbf{m}, \mathbf{n}) = \{m \in \mathcal{M} \mid \bar{\mathbf{n}} \in GmG \cdot \bar{\mathbf{m}}\},$$

which is a G-invariant subset in $\mathcal{M}$. If $\mathcal{M}$ is a group then we have

$$|S(\mathbf{m}, \mathbf{n})/G| \leq |\{x \in \mathcal{M} \mid x\bar{\mathbf{m}} = \bar{\mathbf{n}}\}|.$$

Denote by $r(\mathbf{m}, \mathbf{n})$ the minimal number of generators of $U(S(\mathbf{m}, \mathbf{n}))$ as a right Γ-module.

Theorem 3.2. *[Futorny and Ovsienko [ibid.], Theorem A, , Theorem 8] Let Γ be a commutative domain which is finitely generated as a -algebra, $U \subset (L * \mathcal{M})^G$ a right Galois order over Γ, $\mathbf{m} \in \operatorname{Specm} \Gamma$. Suppose $|\mathbf{m}|$ is finite.*

(i) The fiber $\Phi(\mathbf{m})$ is non-empty.

(ii) If U is a Galois order over Γ, then the fiber $\Phi(\mathbf{m})$ is finite.

(iii) Let U be a Galois order over normal noetherian Γ, $M \in \mathbb{H}(U, \Gamma)$ a simple U-module and $\mathbf{m} \in \operatorname{Specm} \Gamma$. If U is free as a right Γ-module then for any $\mathbf{n}$

$$\dim M(\mathbf{n}) \leq |S(\mathbf{m}, \mathbf{n})/G|.$$

3.2 Principal Galois orders. Further examples of Galois orders were recently constructed by Hartwig [2017a]. As before denote $\mathcal{K} = (L * \mathcal{M})^G$. Clearly, $\mathcal{K}$ is its own Galois subring over Γ. But, $\mathcal{K}$ is a Galois order if and only if $\Lambda = L$, where Λ is the integral closure of Γ in L [Hartwig [ibid.], Corollary 2.15].

Let $x = \sum_{\phi \in \mathcal{M}} x_\phi \phi \in L * \mathcal{M}$ and $a \in L$. Define the evaluation $x(a) := \sum_{\phi \in \mathcal{M}} x_\phi \phi(a)$ [Hartwig [ibid.], Definition 2.18]. Then we have

Theorem 3.3 (Hartwig [2017a], Theorem 2.21)**.**

$$\mathcal{K}_\Gamma = \{x \in \mathcal{K} \mid x(\gamma) \in \Gamma \text{ for all } \gamma \in \Gamma\}$$

is a Galois order over Γ *in* $\mathcal{K}$.

One immediately sees that any Galois subring of $\mathcal{K}_\Gamma$ over Γ is a Galois order. Such orders are called *principal* Hartwig [ibid.].

A new class of principal Galois orders, *rational Galois orders*, was introduced in Hartwig [ibid.]. These structures are attached to an arbitrary finite reflection group and a set of difference operators with rational function coefficients. In particular, the parabolic subalgebras of finite W-algebras of type A are rational Galois orders [Hartwig [ibid.], Theorem 1.2]. This extends the result of Futorny, Molev, and Ovsienko [2005] for W-algebras of type A. Other examples of principal Galois orders include *orthogonal Gelfand-Tsetlin algebras* (Hartwig [2017a], Theorem 4.6) introduced in Mazorchuk [1999] and *quantum orthgonal Gelfand-Tsetlin algebras* (Hartwig [2017a], Theorem 5.6) introduced in Hartwig [2017b]. The family of quantum orthogonal Gelfand-Tsetlin algebras includes in particular quantized universal enveloping algebra $U_q(\mathfrak{gl}_n)$ and, as a consequence, implies the maximality of the Gelfand-Tsetlin subalgebra of $U_q(\mathfrak{gl}_n)$ when q is not a root of unity (this was conjectured by Mazorchuk and Turowska [2000]).

4 Gelfand-Tsetlin modules

Now we address the Lie algebra $\mathfrak{gl}_n$ consisting of all $n \times n$ complex matrices with the standard basis of elementary matrices $\{e_{i,j} \mid 1 \leq i, j \leq n\}$. For each $k \leqslant n$ denote by $\mathfrak{gl}_k$ the Lie subalgebra of $\mathfrak{gl}_n$ spanned by $\{e_{ij} \mid i, j = 1, \dots, k\}$. We have the following embeddings of Lie subalgebras

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \ldots \subset \mathfrak{gl}_n.$$

We have corresponding embeddings $U_1 \subset U_2 \subset \ldots \subset U_n$ of the universal enveloping algebras $U_k = U(\mathfrak{gl}_k)$, $1 \leq k \leq n$. Set $U = U_n$.

Let Z_k be the center of U_k. This is the polynomial algebra generated by the following elements

$$c_{ks} = \sum_{(i_1,\dots,i_s)\in\{1,\dots,k\}^s} e_{i_1 i_2} e_{i_2 i_3} \dots e_{i_s i_1}, \tag{1}$$

$s = 1, \dots, k$.

Let Γ be the subalgebra of $U(\mathfrak{gl}_n)$ generated by the centers Z_k, $k = 1, \dots, n$, the *Gelfand-Tsetlin subalgebra* Drozd, Ovsienko, and Futorny [1991]. The generators c_{ks}, $k = 1, \dots, n$, $s = 1, \dots, k$ are algebraically independent Želobenko [1973].

Let Λ be the polynomial algebra in the variables $\{\lambda_{ij} \mid 1 \leqslant j \leqslant i \leqslant n\}$. Consider the embedding $\pi : \Gamma \longrightarrow \Lambda$ such that

$$c_{ks} \mapsto \sum_{i=1}^{k} (\lambda_{ki} + k - 1)^s \prod_{j \neq i} \left(1 - \frac{1}{\lambda_{ki} - \lambda_{kj}}\right).$$

One can easily check that $\pi(c_{ks})$ is a symmetric polynomial in Λ of degree s in variables $\lambda_{k1}, \dots, \lambda_{kk}$. Let $G = \prod_{i=1}^{n} S_i$ be the product of symmetric groups. Then G acts naturally on Λ where S_k permutes the variables $\lambda_{k1}, \dots, \lambda_{kk}, k = 1, \dots, n$. The image of Γ, $\pi(\Gamma)$, coincides with the subalgebra of G-invariant polynomials in Λ which we identify with Γ.

Consider the Harish-Chandra category $H(U, \Gamma)$. We will call the modules of $H(U, \Gamma)$ *Gelfand-Tsetlin modules*. If $M \in H(U, \Gamma)$ then

$$M = \bigoplus_{\mathsf{m} \in \operatorname{Specm} \Gamma} M(\mathbf{m}),$$

where $M(\mathsf{m}) = \{v \in M | \mathbf{m}^k v = 0 \text{ for some } k \geq 0\}$. Clearly, any simple Gelfand-Tsetlin module over $\mathfrak{gl}(n)$ is a weight module with respect to the Cartan subalgebra spanned by $e_{ii}, i = 1, \dots, n$. Moreover, for $n = 2$ any simple weight module is a Gelfand-Tsetlin module. For $n > 2$ this is not true in general, but holds for modules with finite weight multiplicities. For a Gelfand-Tsetlin module $M(\mathbf{m}) \in H(U, \Gamma)$ and $\mathbf{m} \in \operatorname{Specm} \Gamma$ we call the dimension of $M(\mathbf{m})$ *the Gelfand-Tsetlin multiplicity* of $\mathbf{m}$.

4.1 Finite dimensional modules over $\mathfrak{gl}_n$. We recall a classical result of Gelfand and Cetlin [1950] which gives an explicit basis for all simple finite dimensional $\mathfrak{gl}_n$-modules.

For convenience we consider the elements of the space $\mathbb{C}^k$ as k-tuples whose entries are labeled as follows $(v_{k1,\dots,v_{kk}})$. We also identify $\mathbb{C}^{\frac{n(n+1)}{2}}$ with $T_n(\mathbb{C}) = \mathbb{C}^n \times \mathbb{C}^{n-1} \times \dots \times \mathbb{C}$. Then every vector v in $\mathbb{C}^{\frac{n(n+1)}{2}}$ can be written in the following form:

$$v = (v_{n1}, \dots, v_{nn} | v_{n-1,1}, \dots, v_{n-1,n-1} | \cdots | v_{21}, v_{22} | v_{11})$$

to which we associate the following array $T(v)$

$$\begin{array}{ccccccc}
\boxed{v_{n1}} & \boxed{v_{n2}} & & \cdots & & \boxed{v_{n,n-1}} & \boxed{v_{nn}} \\
& \boxed{v_{n-1,1}} & & \cdots & & \boxed{v_{n-1,n-1}} & \\
& & \cdots & \cdots & \cdots & & \\
& & & \boxed{v_{21}} \; \boxed{v_{22}} & & &
\end{array}$$

$$\boxed{v_{11}}$$

Such an array will be called a *Gelfand-Tsetlin tableau* of height n.

For a fixed element $v = (v_{ij})_{j\le i=1}^{n} \in T_n(\mathbb{C})$ consider the set

$$v + T_{n-1}(\mathbb{Z}) = \{v + w \mid w = (w_{ij})_{j\le i=1}^{n}, w_{ij} \in \mathbb{Z}, w_{nk} = 0\,, k = 1, \dots, n\}.$$

Denote by $V(T(v))$ the complex vector space spanned by the set $v + T_{n-1}(\mathbb{Z})$ as a basis. Clearly, the spaces $V(T(v))$ and $T_n(\mathbb{C})$ are not isomorphic as $T(v+w) \neq T(v) + T(w)$ in $V(T(v))$.

A Gelfand-Tsetlin tableau $T(v)$ of height n is called *standard* if $v_{ki} - v_{k-1,i} \in \mathbb{Z}_{\ge 0}$ and $v_{k-1,i} - v_{k,i+1} \in \mathbb{Z}_{>0}$ for all $1 \le i \le k \le n-1$.

Theorem 4.1 (Gelfand and Cetlin [1950]). *Let $L(\lambda)$ be the simple finite dimensional $\mathfrak{gl}_n$-module of highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$. Then the set of all standard tableaux $T(v)$ with fixed top row $v_{ni} = \lambda_i - i + 1$, $i = 1, \dots, n$ forms a basis of $L(\lambda)$. Moreover, the action of the generators of $\mathfrak{gl}(n)$ on $L(\lambda)$ is given by the* Gelfand-Tsetlin formulas*:*

$$e_{k,k+1}(T(v)) = -\sum_{i=1}^{k} \left(\frac{\prod_{j=1}^{k+1}(v_{ki} - v_{k+1,j})}{\prod_{j\neq i}^{k}(v_{ki} - v_{kj})} \right) T(v + \delta^{ki}),$$

$$e_{k+1,k}(T(v)) = \sum_{i=1}^{k} \left(\frac{\prod_{j=1}^{k-1}(v_{ki} - v_{k-1,j})}{\prod_{j\neq i}^{k}(v_{ki} - v_{kj})} \right) T(v - \delta^{ki}),$$

$$e_{kk}(T(v)) = \left(k - 1 + \sum_{i=1}^{k} v_{ki} - \sum_{i=1}^{k-1} v_{k-1,i} \right) T(v).$$

If $e_{k,k+1}(T(v))$ or $e_{k+1,k}(T(v))$ contains a summand with a non-standard $T(v \pm \delta^{ki})$, then the summand is assumed to be zero.

These formulas define a Gelfand-Tsetlin modules where the action of the generators of Γ is given by:

$$c_{mk}(T(v)) = \gamma_{mk}(v) T(v),$$

where

$$\gamma_{mk}(v) := \sum_{i=1}^{m} (v_{mi} + m - 1)^k \prod_{j\neq i} \left(1 - \frac{1}{v_{mi} - v_{mj}} \right). \tag{2}$$

To every $w \in v + T_{n-1}(\mathbb{Z})$ we associate the maximal ideal of Λ generated by $\lambda_{ij} - w_{ij}$ and the maximal ideal $\mathbf{m}_w$ of Γ generated by $c_{ij} - \gamma_{ij}(w)$, where $\gamma_{ij}(w)$ are symmetric polynomials defined in (2). Again, the correspondence $w \mapsto \mathbf{m}_w$ is not one-to-one, a given $\mathbf{m} \in \operatorname{Specm}\Gamma$ defines the finite fiber of maximal ideals of Λ corresponding to the set $\widehat{\mathbf{m}}$ of $w \in v + T_{n-1}(\mathbb{Z})$ with $\mathbf{m}_w = \mathbf{m}$.

The basis of tableaux defined in Theorem 4.1 is called the *Gelfand-Tsetlin basis*. Discovery of Gelfand-Tsetlin bases are among the most remarkable results of the representation theory of classical Lie algebras. It provides a convenient realization of every simple finite dimensional representation of the Lie algebra $\mathfrak{gl}_n$. For other types of simple finite dimensional Lie algebras we refer to Molev [2006].

4.2 $U(\mathfrak{gl}_n)$ is a Galois order over Γ. We identify $T_{n-1}(\mathbb{Z})$ with the free abelian group $\mathcal{M} \simeq \mathbb{Z}^{\frac{n(n-1)}{2}}$ generated by δ^{ij}, $1 \leqslant j \leqslant i \leqslant n-1$, where $(\delta^{ij})_{ij} = 1$ and all other $(\delta^{ij})_{k\ell}$ are zero, $1 \leq j \leq i \leq n-1$. The group $\mathcal{M}$ acts naturally on $T_n(\mathbb{C})$ by translations. We also have the action of $G = S_n \times S_{n-1} \times \cdots \times S_1$ on $T_n(\mathbb{C})$ as follows:

$$\sigma(v) := (v_{n,\sigma^{-1}[n](1)}, \ldots, v_{n,\sigma^{-1}[n](n)}| \ldots |v_{1,\sigma^{-1}[1](1)}).$$

where $v \in T_n(\mathbb{C}), \sigma \in G$ and $\sigma[k] \in S_k$. This leads to the action of the semidirect product $G \ltimes T_{n-1}(\mathbb{Z})$ on $T_n(\mathbb{C})$.

Denote by K the field of fractions of Γ and by L the field of fractions of Λ. We have $L^G = K, \Lambda^G = \Gamma$ and $G = G(L/K)$ is the Galois group of the field extension $K \subset L$. The group $\mathcal{M}$ acts naturally on L and U is a subalgebra of $(L * \mathcal{M})^G$. Following Futorny and Ovsienko [2014], define a linear map $\tau : U \longmapsto (L * \mathcal{M})^G$ where

$$\tau(e_{mm}) = e_{mm} * e, \ \tau(e_{m\,m+1}) = \sum_{i=1}^{m} a_{mi}^{+} \delta^{mi}, \ \tau(e_{m+1\,m}) = \sum_{i=1}^{m} a_{mi}^{-} (\delta^{mi})^{-1},$$

where

$$a_{mi}^{\pm} = \mp \frac{\prod_j (\lambda_{m\pm1,j} - \lambda_{mi})}{\prod_{j \neq i} (\lambda_{mj} - \lambda_{mi})},$$

and e is the identity element of the group $\mathcal{M}$.

In fact, the map τ is algebra homomorphism since the defining relations of $\mathfrak{gl}_n$ are given by some rational functions which agree on finite dimensional modules, thus relations are satisfied.

Moreover, we have

Theorem 4.2 (Futorny and Ovsienko [2010], Proposition 7.2)**.** *τ is an embedding and U is a Galois order over Γ.*

Applying Theorem 3.2 we obtain that the number of isomorphism classes of simple Gelfand-Tsetlin U-modules with a given maximal ideal of Γ in the support is bounded by $\prod_{i=1}^{n-1} i!$. Another consequence of Theorem 4.2 is the following. If M is a Gelfand-Tsetlin U-module and $\mathbf{m} = \mathbf{m}_v \in \operatorname{Specm}\Gamma$ for some $v \in T_n(\mathbb{C})$ then

$$e_{k,k\pm1} M(\mathbf{m}) \subset \sum_{i=1}^{k} M(\mathbf{m}_{v\pm\delta^{ki}}).$$

From here and Theorem 3.1 one easily obtains

Corollary 4.1. *(i)* Γ *is a Harish-Chandra subalgebra of* U.

(ii) Γ *is maximal commutative in* U.

4.3 Tableaux modules. The explicit nature of the Gelfand-Tsetlin formulas in Theorem 4.1 and the fact that the coefficients in the formulas are rational functions on the entries of the tableaux, naturally raises the question whether this construction can be extended for more general tableaux.

If V is a Gelfand-Tsetlin modules which has a basis parametrized by the tableaux and the action of Γ is determined by the entries of tableaux as in (2) then such V will be called *tableau module*. The problem of constructing of tableaux modules was studied by Gelfand and Graev in Gelfand and Graev [1965] and by Lemire and Patera (for $n = 3$) in Lemire and Patera [1979], Lemire and Patera [1985]. Tableux relalization for Generalzied Verma modules was considered in Mazorchuk [1998].

If the action of the generators of $\mathfrak{gl}_n$ on a tableau Gelfand-Tsetlin module V is given by the classical Gelfand-Tsetlin formulas as in Theorem 4.1 then V will be called *standard tableau module*. Modules considered in Gelfand and Graev [1965], Lemire and Patera [1979], Lemire and Patera [1985] are standard tableau modules.

We call $T(v)$ a *generic tableau* and v a *generic vector* if $v_{rs} - v_{rt} \notin \mathbb{Z}$ for any $r < n$ and all possible $s \neq t$. For a generic tableau all denominators in the Gelfand-Tsetlin formulas are nonintegers and one can use the same formulas to define *generic* standard tableau Gelfand-Tsetlin module $V(T(v))$ (Drozd, Futorny, and Ovsienko [1994], Section 2.3). All Gelfand-Tsetlin multiplicities of maximal ideals of $V(T(v))$ are 1.

Definition 4.1. *For each generic vector w and any $1 \leq r, s \leq n$ define*

$$d_{rs}(w) := \begin{cases} -\frac{\prod_{j=1}^{s}(w_{s-1,1}-w_{s,j})}{\prod_{j=2}^{s-1}(w_{s-1,1}-w_{s-1,j})} \prod_{j=r}^{s-2} \left(\frac{\prod_{t=2}^{j+1}(w_{j1}-w_{j+1,t})}{\prod_{t=2}^{j}(w_{j1}-w_{jt})} \right), & \textit{if} \quad r < s, \\ \frac{\prod_{j=1}^{s-1}(w_{s1}-w_{s-1,j})}{\prod_{j=2}^{s}(w_{s1}-w_{sj})} \prod_{j=s+2}^{r} \left(\frac{\prod_{t=2}^{j-2}(w_{j-1,1}-w_{j-2,t})}{\prod_{t=2}^{j-1}(w_{j-1,1}-w_{j-1,t})} \right), & \textit{if} \quad r > s, \\ r-1+\sum_{i=1}^{r} w_{ri} - \sum_{i=1}^{r-1} w_{r-1,i}, & \textit{if} \quad r = s, \end{cases}$$

Let $1 \leq r < s \leq n-1$. Set $\varepsilon_{rs} := \delta^{r,1} + \delta^{r+1,1} + \ldots + \delta^{s-1,1} \in T_n(\mathbb{Z})$, $\varepsilon_{rr} = 0$ and $\varepsilon_{sr} = -\varepsilon_{rs}$.

Let $\widetilde{S}_k$ be the subset of S_n consisting of the transpositions $(1, i)$, $i = 1, \ldots, k$. For $s < \ell$, set $\Phi_{s\ell} = \widetilde{S}_{\ell-1} \times \cdots \times \widetilde{S}_s$. For $s > \ell$ we set $\Phi_{s\ell} = \Phi_{\ell s}$. Finally we let $\Phi_{\ell\ell} = \{\mathrm{Id}\}$. Every σ in $\Phi_{s\ell}$ will be written as a $|s-\ell|$-tuple of transpositions $\sigma[k]$ (where $\sigma[k]$ is the k-th component of σ).

Proposition 4.1 (Futorny, Grantcharov, and Ramirez [2015])**.** *Let $v \in T_n(\mathbb{C})$ be generic. Then the $\mathfrak{gl}_n$-module structure on $V(T(v))$ is defined by the formulas:*

$$e_{k\ell}(T(v+z)) = \sum_{\sigma \in \Phi_{k\ell}} d_{k\ell}(\sigma(v+z)) T(v+z+\sigma(\varepsilon_{k\ell})), \tag{3}$$

for $z \in T_{n-1}(\mathbb{Z})$ and $1 \leq k, \ell \leq n$. Moreover, $V(T(v))$ is a Gelfand-Tsetlin module with action of Γ given by the formulas (2).

Note that if $v - v' \in T_{n-1}(\mathbb{Z})$ then $V(T(v))$ and $V(T(v'))$ are isomorphic as vectors spaces but not necessarily as the $\mathfrak{gl}_n$-modules. Simple generic Gelfand-Tsetlin modules were described in Futorny, Grantcharov, and Ramirez [ibid.].

The main difficulty in the defining of a tableau Gelfand-Tsetlin module structure on $V(T(v))$ is the existence of entries in one row of $T(v)$ that have integer difference. Let $v \in \mathbb{C}^{\frac{n(n+1)}{2}}$. A pair of entries (v_{ki_j}, v_{ki_s}) such that $k > 1$ and $v_{ki_j} - v_{ki_s} \in \mathbb{Z}$ is called a *singular pair*. We say that v (and $T(v)$) is *singuar* if v has singular pairs. First examples of infinite dimensional tableau Gelfand-Tsetlin modules with singular tableaux were considered in Gelfand and Graev [1965], Lemire and Patera [1979], Lemire and Patera [1985]. A new effective method of constructing standard tableau simple Gelfand - Tsetlin modules was proposed in Futorny, Ramirez, and Zhang [2016]. It allowed to obtain a large family of simple modules that have a basis consisting of Gelfand-Tsetlin tableaux and the action of the generators of $\mathfrak{gl}_n$ is given by the classical Gelfand-Tsetlin formulas. All examples obtained in Gelfand and Graev [1965], Lemire and Patera [1979], Lemire and Patera [1985] are particular cases of this construction. But the class of modules defined in

Futorny, Ramirez, and Zhang [2016] is more general. They build out of a tableau satisfying certain *FRZ-condition*.

A tableau $T(v)$ is *critical* if it has equal entries in one or more rows different from the top row. Otherwise, tableau is *noncritical*.

Theorem 4.3 (Futorny, Ramirez, and Zhang [ibid.], Theorem II)**.** *Let $T(w)$ be a tableau satisfying the FRZ-condition. There exists a unique simple Gelfand-Tsetlin $\mathfrak{gl}_n$-module V_w having the following properties:*

(i) $V_w(\mathbf{m}_w) \neq 0$;

(ii) V_m has a basis consisting of noncritical tableaux and the action of the generators of $\mathfrak{gl}_n$ is given by the classical Gelfand-Tsetlin formulas (4.1).

(iii) All Gelfand-Tsetlin multiplicities of maximal ideals of Γ in the support of V_w equal 1.

Theorem 4.3 provides a combinatorial way to explicitly construct a large class of infinite dimensional simple Gelfand-Tsetlin modules.

Conjecture 1. If V is a simple Gelfand-Tsetlin $\mathfrak{gl}_n$-module which has a basis consisting of noncritical tableaux with classical action of the generators of $\mathfrak{gl}_n$ then $V \simeq V_w$ for some w satisfying the FRZ-condition.

The conjecture holds for $n \leq 4$.

A systematic study of singular modules was initiated in Futorny, Grantcharov, and Ramirez [2015] where the case of a singular tableau $T(v)$ with a unique singular pair was considered (*1-singular* case). A significant difference from all previous cases is the existence of *derivative* tableaux in the basis of $V(T(v))$ which reflects the fact that the exact bound for the Gelfand-Tsetlin multiplicities of $V(T(v))$ is 2. Alternative interpretation of a tableau Gelfand-Tsetlin module structure on $V(T(v))$ in 1-singular case was given independently in Vishnyakova [2018] and Zadunaisky [2017]. Simple subquotients of $V(T(v))$ were described in Gomes and Ramirez [2016].

We say that v is *singular of index $m \geq 2$* if:

(i) there exists a row k, $1 < k < n$, and m entries $v_{ki_1}, \ldots, v_{ki_m}$ on this row such that $v_{ki_j} - v_{ki_s} \in \mathbb{Z}$ for all $j, s \in \{1, \ldots, m\}$;

(ii) m is maximal with the property (i).

The case of arbitrary singularity of index $m = 2$ was solved in Futorny, Grantcharov, and Ramirez [2016a]. In this case any number of singular pairs (but not singular triples) and multiple singular pairs in the same row were allowed. Finally, the general case of an arbitrary singularity was solved in Vishnyakova [2017] (for p-singularity) and Ramírez and Zadunaisky [2017] (for arbitrary singularity). We provide the construction from Ramírez and Zadunaisky [ibid.] whose spirit is closer to our original approach.

Recall that L is the field of rational functions in λ_{ij}, $i = 1, \dots, n$, $j = 1, \dots, i$, and $a^{\pm}_{mi} \in L$ for all $1 \leq i \leq k < n$. Consider a set of all tableau with integral entries whose top row consists of zeros. We set $V_{\mathbb{C}}$ to be the $\mathbb{C}$-vector space with this basis, and $V_L = L \otimes_{\mathbb{C}} V_{\mathbb{C}}$. Since τ (4.2) is a homomorphism, V_L is a U-module, with the action of $\mathfrak{gl}_n$ given by the Gelfand-Tsetlin formulas.

The group G acts acts on V_L by the diagonal action, while $\mathcal{M}$ acts on Λ and L by translations: $\delta^{k,i} \cdot \lambda_{l,j} = \lambda_{l,j} + \delta_{k,l}\delta_{i,j}$.

Denote by A the algebra of regular functions over generic tableaux, that is those elements in L which can be evaluated in any generic tableau, and let V_A to be the A-submodule of V_L generated by all integral tableaux. Given a generic tableau $T(v)$, we can recover the corresponding generic module $V(T(v))$ by specializing V_A at v. If $T(v)$ is a singular tableau then we replace A with an algebra $B \subset L$ such that there exists a B-submodule $V_B \subset V_L$ which is also a U-submodule and any element of V_B can be evaluated at v. Specialization at v finally defines $V(T(v))$.

Each point $v \in \mathbb{C}^{\frac{n(n+1)}{2}}$ defines the following refinement $\eta(v)$ of v, which measures of how far is v from being generic Ramírez and Zadunaisky [ibid.]. Fix $1 \leq k \leq n-1$. Construct a graph with vertices $i = 1, \dots, k$, put an edge between i and j if and only if $v_{k,i} - v_{k,j}$ is integer. The graph is the disjoint union of connected components, we set $\eta^{(k)}$ to be a sequence of their cardinalities arranged in descending order. The entries $v_{k,i}$ that form one connected component are called an *η-block* of v. If v is generic then $\eta^{(k)} = (1^k)$, a sequence consisting of n ones. We set $\eta = (\eta^{(1)}, \dots, \eta^{(n-1)}, 1^n)$ to be the *η-type* of v and $\eta(v)$ to be the element in $\mathbb{C}^{\frac{n(n+1)}{2}}$ obtained from v by rearranging of it components to match the η-blocks.

Let B be the localization of Λ by the multiplicative subset of Λ generated by the elements

$$\lambda_{k,i} - \lambda_{k,j} - z,\ 1 \leq i < j \leq k < n, z \in \mathbb{Z} \setminus \{0\}.$$

Following Ramírez and Zadunaisky [ibid.], we say that v is in an *η-normal form* if $v_{k,i} - v_{k,j} \in \mathbb{Z}_{\geq 0}$ implies that $v_{k,i}$ and $v_{k,j}$ belong to the same η-block of v and $i < j$. Clearly, the orbit $G \cdot v$ has at least one (but not necessarily unique) element in normal form. We also say that v is an *η-critical* if it is in η-normal form and $v_{k,i} - v_{k,j} \in \mathbb{Z}$ implies $v_{k,i} = v_{k,j}$.

Consider a subgroup $G_\eta \subset G$ consisting of those elements of G which preserve the block structure of $\eta(v)$.

Fix η and an η-critical v. Set B_η to be the localization of B by the multiplicative set generated by all $\lambda_{k,i} - \lambda_{k,j}$ such that $v_{k,i} \neq v_{k,j}$.

Now we consider divided difference operators that play a key role in the construction of $V(T(v))$. Denote by $s_i^{(k)}$ the simple transposition in G_η which interchanges $\lambda_{k,i}$ and $\lambda_{k,i+1}$ and fixes all other elements. The *divided difference* associated to $s_i^{(k)}$ is

$$\partial_i^{(k)} = \frac{1}{\lambda_{k,i} - \lambda_{k,i+1}}(id - s_i^{(k)}).$$

It can be viewed as an element of the smash product $L\#G_\eta$, where $(f \otimes \sigma) \cdot (g \otimes \sigma') = f\sigma(g) \otimes \sigma\sigma'$ for $f, g \in L$ and all $\sigma, \sigma' \in G_\eta$.

Let $G_{\eta,k} \subset G_\eta$ be the corresponding component of S_k, $k = 1, \ldots, n$. If $\sigma = s_{i_1}^{(k)} s_{i_2}^{(k)} \cdots s_{i_l}^{(k)}$ is a reduced decomposition for $\sigma \in G_{\eta,k}$ then set $\partial_\sigma = \partial_{i_1}^{(k)} \cdot \partial_{i_2}^{(k)} \cdots \partial_{i_l}^{(k)}$ which does not depend of the chosen reduced decomposition. This naturally extends to the whole group G_η.

For each $\sigma \in G_\eta$ we define the *symmetrized divided difference operator*

$$D_\sigma^\eta = \mathrm{sym}_\eta \cdot \partial_\sigma,$$

where $\mathrm{sym}_\eta = \frac{1}{|G_\eta|}\sum_{\sigma \in G_\eta} \sigma$.

Since B_η is closed under the action of G_η, we have $D_\sigma^\eta(f) \in B_\eta$ for all $f \in B_\eta$. Denote by $V_\eta \subset V_L$ the B_η-span of $\{D_\sigma^\eta T(z) \mid \sigma \in G_\eta, z \in T_{n-1}(\mathbb{Z})\}$. Then V_η is a U-submodule of V_L. Denote $\mathfrak{N}_\eta = \{z \in T_{n-1}(\mathbb{Z}) \mid v + z \text{ is in normal form}\}$. If $z \in \mathfrak{N}_\eta$ then the stabilizer subgroup of z in G_η is $G_{\epsilon(z)}$ where $\epsilon(z)$ is some refinement of η. Fix $z \in \mathfrak{N}$. We say that $\sigma \in G_\eta$ is a $\epsilon(z)$-shuffle if it is increasing in each $\epsilon(z)$-block. We denote the set of all $\epsilon(z)$-shuffles in G_η by $\mathrm{Shuffle}_{\epsilon(z)}^\eta$. Write $\bar{D}_\sigma^\eta(v + z) = 1 \otimes D_\sigma^\eta(z)$ for $z \in \mathfrak{N}$ and $\sigma \in \mathrm{Shuffle}_{\epsilon(z)}^\eta$.

Combining Theorems 5.3 and 5.6 of Ramírez and Zadunaisky [2017] we obtain

Theorem 4.4. *Let $V(T(v)) = \mathbb{C} \otimes_{B_\eta} V_\eta$, where $\mathbb{C}$ is a right B_η-module such that $1\dot{f} = f(v)$. Then $V(T(v))$ is a Gelfand-Tsetlin module with a basis $\{\bar{D}_\sigma^\eta(v+z) \mid z \in \mathfrak{N}_\eta, \sigma \in \mathrm{Shuffle}_{\epsilon(z)}^\eta\}$ and $\mathbf{m}_v$ belongs to the support of $V(T(v))$.*

Conjecture 2. Any simple Gelfand-Tsetlin module V with $V(\mathbf{m}_v) \neq 0$ is isomorphic to a subquotient of $V(T(v))$ for any singular v.

The conjecture was stated for any singular v of index 2 in Futorny, Grantcharov, and Ramirez [2016a]. It is known to be true for $n = 2$ and $n = 3$, and for the 1-singular v Futorny, Grantcharov, and Ramirez [2017]. In particular, it gives a complete classification of all simple Gelfand-Tsetlin $\mathfrak{gl}(3)$-modules, Futorny, Grantcharov, and Ramirez [2014].

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