Algebraic K-Theory and Rings with many Units

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The results on the algebraic K-theory of rings with many units, presented by the second author in the Séminaire Structures Algébriques Ordonnées of the Equipe de Logique, University of Paris VII in January 2006, will appear as section 4 of the forthcoming paper [DM7] and so here we shall present just a short account of the results, omitting proofs. The interested reader may consult the aforementioned paper.

In all that follows, the word **ring** stands for *unitary commutative ring.*

We shall employ the theories of special groups (SG) and reduced special groups (RSG), introduced in [DM2], that is our basic reference for notation and nomenclature. In particular, the acronym SG-morphism stands for a morphism of special groups.

Let $G = \langle G, \equiv_G, -1 \rangle$ be a special group and write D_G for the representation relation in G. Thus, if a, b, $c \in G$, $a \in D_G(b, c)$ mean that *a is represented by b and c* (see Chapter 1 of [DM2] for details).

The K-theory of G, introduced in [DM3], is the graded $\mathbb{F}_2$-algebra,

$$k_*G = \langle \mathbb{F}_2, k_1G, \ldots, k_nG, \ldots \rangle,$$

constructed as follows :

* k_1G is G written additively, that is, we fix an **isomorphism**

$$\lambda : G \longrightarrow k_1G, \text{ with } \lambda(ab) = \lambda(a) + \lambda(b).$$

In particular, $\lambda(1)$ is the zero of k_1G and k_1G has exponent 2, i.e., for $a \in G$, $\lambda(a) = -\lambda(a)$;

* k_*G is the quotient of the graded tensor algebra $\langle \mathbb{F}_2, k_1G, \ldots, \underbrace{k_1G \otimes \cdots \otimes k_1G}_{n\ times}, \ldots \rangle$ over $\mathbb{F}_2$, by the ideal generated by $\{\lambda(a)\lambda(ab) : a \in D_G(1, b)\}$. Thus, for each $n \geq 2$, k_nG is the quotient of the n-fold tensor product $k_1G \otimes \cdots \otimes k_1G$ over $\mathbb{F}_2$, by the subgroup consisting of finite sums of elements of the type $\lambda(a_1)\cdots\lambda(a_n)$, where for some $1 \leq i \leq n-1$ and $b \in G$, we have $a_{i+1} = a_ib$ and $a_i \in D_G(1, b)$. An element of the type $\lambda(x_1)\cdots\lambda(x_n)$ is a **generator** of k_nG.

* There is a graded ring morphism of degree 1, $\lambda(-1)\,(\cdot) : k_nG \longrightarrow k_{n+1}G$, taking $\eta \in k_nG$ to $\lambda(-1)\eta \in k_{n+1}G$. A special group is **[SMC]** if for all $n \geq 1$, multiplication by $\lambda(-1)$ is an injection. Any [SMC] special group is *reduced* ([DM3], Lemma 6.2, p. 173).

* A SG-morphism, $f : G \longrightarrow H$, induces a morphism of degree 0 of graded $\mathbb{F}_2$-algebras

$$f_* : k_*G \longrightarrow k_*H,$$

$f_* = \{f_n : n \geq 0\}$, where $f_0 = Id_{\mathbb{F}_2}$ and for $n \geq 1$, $f_n : k_nG \longrightarrow k_nH$ is the unique group morphism whose value on generators is given by $f_n(\lambda(a_1)\cdots\lambda(a_n)) = \lambda(f(a_1))\cdots\lambda(f(a_n))$.

Our main results are :

* A model-theoretic criterion for a subring to inherit the property of having many units.
* If A is a ring with many units, the ring-theoretic analog of Milnor's K-theory of fields, introduced in [Gu], when reduced mod 2, is canonically isomorphic to the K-theory of the special group naturally associated to A, presented in [DM5].

To begin with, we recall

Definition 1 *Let R be a ring.*

a) Write R^* for the group of units in R.

b) A polynomial $f \in R[X_1, \ldots, X_n]$ has **local unit values relative to maximal ideals** *if for all maximal ideals $\mathfrak{m}$ in R, there is $\overline{u} \in R^n$ such that $f(\overline{u}) \notin \mathfrak{m}$. Similarly, one defines the notion f having local unit values relative to prime ideals in R.*

c) R is a **ring with many units** *if for all $f \in R[X_1, \ldots, X_n]$, if f has local unit values relative to maximal ideals, then there is $\overline{y} \in R^n$ such that $f(\overline{y})$ is a unit in R.*

Remark 2 Since every maximal ideal is prime and all (proper) prime ideals are contained in a maximal ideal, a ring R has many units iff for all $f(X_1, \ldots, X_n) \in R[X_1, \ldots, X_n]$,

f has local unit values relative to all prime ideals in R $\Rightarrow$ $\exists\, \overline{z} = \langle z_1, \ldots, z_n \rangle \in R^n$ such that $f(\overline{z})$ is a unit in R.

Examples of rings with many units are semi-local rings, arbitrary products of rings with many units and more generally, the ring of global sections of a sheaf of rings over a partitionable space, whose stalks are rings with many units. In particular, the ring of global sections of a sheaf of rings over a Boolean space, whose stalks are rings with many units, is a ring with many units. The reader can find more information, as well as the proof of these results in [DM5], where it is also shown that, under mild assumptions, the RSGs associated to rings of this type faithfully represent the quadratic form theory over free modules (Theorems 3.15 and 3.16, [DM5]). $\diamondsuit$

We also recall the following

Definition 3 *Let L be a first-order language with equality.*

Let A, B be L-structures, let $A \xrightarrow{f} B$ be a map and let $\varphi(v_1, \ldots, v_n)$ be a formula of L in the free variables $\overline{v} = \langle v_1, \ldots, v_n \rangle$. For $\overline{a} = \langle a_1, \ldots, a_n \rangle \in A^n$, write $f(\overline{a})$ for $\langle f(a_1), \ldots, f(a_n) \rangle$ in B^n.

a) f **preserves** *φ if for all $\overline{a} \in A^n$, $A \models \varphi[\overline{a}] \;\Rightarrow\; B \models \varphi[f(\overline{a})]$; f* **reflects** *φ if the reverse implication holds.*

b) If f is a L-morphism, we say that **A is positively existentially closed in B along f** *if f reflects all positive existential L-formulas. Whenever A is a substructure of B, and f is the inclusion, we say that* **A is positively existentially closed in B**.

c) A formula in L is **positive primitive (pp-formula)** *if it is of the form $\exists \overline{v}\, \varphi(\overline{v}; \overline{t})$, where φ is a conjunction of atomic formulas.*

Proposition 4 *Let R be a ring with many units.*

a) If S is a positively existentially closed subring of R, then S is a ring with many units.

b) If e is an idempotent in R, then $Re = \{ae : a \in R\}$, a ring with identity e, also has many units.

c) Let T be a ring and let $T \xrightarrow{f} R$ be a map that preserves addition, multiplication and 0 [1]*. If T is positively existentially closed in R along f, then T has many units.* $\diamondsuit$

To establish our second main result, we adapt to our purposes a condition introduced in [Gu] (page 29) :

Definition 5 *Let A be a ring and let $m \geq 1$ be an integer. Recall that A^* is the group of units in A. We say that*

[1] So f is a morphism with respect to the language of rings *without* identity.

a) *A satisfies* [H1-m] ($A \models$ [H1-m]) *if for all* $n \geq 2$ *and all* $1 \leq k \leq m$, *if* $\{f_1, \ldots, f_k\}$ *is a family of surjective linear forms over the free A-module* A^n, *there is* $v \in A^n$ *such that* $f_j(v) \in A^*$, $1 \leq j \leq k$.

b) *A satisfies* [H1] *if* $A \models$ [H1-m] *for all* $m \geq 1$.

It is mentioned in the Examples given on page 33 of [Gu] that all semilocal rings whose residue fields are infinite verify [H1]. Generalizing this observation we have

Proposition 6 *Let* $m \geq 2$ *be an integer. If A is a ring with many units, whose residue fields all have cardinality* $\geq m$, *then* $A \models$ [H1-m]. ◇

We now wish to present a mod 2 K-theory of rings, patterned after the construction in section 3 of [Gu]. Let A be a ring. We set $K_0A = \mathbb{Z}$ and let K_1A be A^* written additively, that is, we fix an isomorphism

$$l : A^* \longrightarrow K_1A, \text{ such that } l(ab) \;=\; l(a) + l(b), \;\; \forall\, a, b \in A^*.$$

Then, Milnor's K-theory of A is the graded ring (Definition 3.2, p. 47, [Gu])

$$K_*A \;=\; \langle \mathbb{Z}, K_1A, \ldots, K_nA, \ldots \rangle,$$

obtained as the quotient of the graded tensor algebra over $\mathbb{Z}$,

$$\langle \mathbb{Z}, K_1A, \ldots, \underbrace{K_1A \otimes \ldots \otimes K_1A}_{n \text{ times}}, \ldots \rangle$$

by the ideal generated by $\{l(a) \otimes l(b) : a, b \in A^* \text{ and } a + b = 1 \text{ or } 0\}$. Hence, for each $n \geq 2$, K_nA is the quotient of the n-fold tensor product over $\mathbb{Z}$, $K_1A \otimes \ldots \otimes K_1A$, by the subgroup consisting of sums of generators $l(a_1) \otimes \ldots \otimes l(a_n)$, such that $a_i + a_{i+1} = 1$ or 0, for some $1 \leq i \leq n-1$. As usual, we shall write the generators in K_nA as $l(a_1) \cdots l(a_n)$, omitting the tensor operation. As a consequence of (the proof of) Proposition 3.2.3 in [Gu] (p. 48) and Proposition 6 we have

Lemma 7 *Let A be a ring with many units whose residue fields all have at least 7 elements. Then,* K_*A *is the graded ring obtained as the quotient of the graded tensor algebra over* $\mathbb{Z}$,

$$\langle \mathbb{Z}, K_1A, \ldots, \underbrace{K_1A \otimes \ldots \otimes K_1A}_{n \text{ times}}, \ldots \rangle$$

by the graded ideal generated by $\{l(a)l(b) : a, b \in A^* \text{ and } a + b = 1\}$. ◇

Definition 8 *If A is a ring, we define the* **mod 2 K-theory of A**, *as the graded ring*

$$k_*A \;=\; \langle k_0A, k_1A, \ldots, k_nA, \ldots \rangle \;=_{def}\; K_*A/2K_*A,$$

that is, for each $n \geq 0$, k_nA *is the quotient of* K_nA *by the subgroup* $\{2\eta \in K_nA : \eta \in K_nA\}$.

We have $k_0A = \mathbb{F}_2$ and $k_1A \approx A^*/A^{*2}$, via an isomorphism still denoted by l. A generator in k_nA will be written $l(a_1) \cdots l(a_n)$. Clearly, k_nA is a group of exponent 2, i.e., $\eta + \eta = 0$, for all $\eta \in k_nA$.

Lemma 9 *If A is a ring verifying* [H1-6], *then for all* $b, a, a_1, \ldots, a_n \in A^*$ *and all permutations* σ *of* $\{1, \ldots, n\}$

a) *In* k_2A, $\;l(a)l(-a) \;=\; 0$.

b) *In* k_2A, $\;l(a)l(-1) \;=\; l(a)^2$.

c) *In* k_2A, $\;l(a)l(b) \;=\; l(b)l(a)$.

d) *In* k_nA, $\;l(a_1) \cdots l(a_n) \;=\; l(a_{\sigma(1)} \cdots l(a_{\sigma(n)})$.

e) *If* $t_1, \ldots, t_n \in A^*$, *then in* k_nA, $\;l(t_1^2a_1) \cdots l(t_n^2a_n) \;=\; l(a_1) \cdots l(a_n)$. ◇

Our next order of business is to connect the mod 2 K-theory of a ring with many units satisfying certain conditions with the K-theory of the special group naturally associated to it in [DM5].

Lemma 10 *Let A be a ring with many units, whose residue fields all have at least 7 elements. Let $a, b, a_1, \ldots, a_n \in A^*$, with $a \in D_A(1, b)$. If $a_i = a$ and $a_j = ab$ for some $1 \leq i \neq j \leq n$, then $l(a_1) \cdots l(a_n) = 0$ in $k_n A$.*

Theorem 11 *Let A be a ring with many units such that $2 \in A^*$ and whose residue fields all have at least 7 elements. Then, $G(A) = \langle G(A), \equiv, -1 \rangle$ (defined in [DM5]) is a special group. Moreover, the rules $\alpha_0 = Id_{\mathbb{F}_2}$ and $\alpha_n : k_n A \longrightarrow k_n G(A)$, defined on generators by $\alpha_n(l(a_1) \cdots l(a_n)) = \lambda(\overline{a}_1) \cdots \lambda(\overline{a}_n)$, for $n \geq 1$, determine a graded ring isomorphism between the mod 2 K-theory of A and the K-theory of the special group $G(A)$.* $\diamondsuit$

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