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Review

Optical forces and optical force categorizations exerted on quadrupoles in the framework of generalized Lorenz–Mie theory

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ABSTRACT

In the framework of a systematic study of the categorization of optical forces in generalized Lorenz–Mie theory, the present paper is devoted to the categorization of forces exerted on quadrupoles and, more importantly, to their explicit expressions in terms of beam shape coefficients (and of Mie coefficients) which allow one to calculate any kind of forces of the categorization in terms of the essential quantities involved in them. Three levels of categorization are presented. The first level, known since decades, distinguishes between mixing forces and recoil forces. The second level distinguishes between gradient and non-gradient forces. Although these non-gradient forces are usually denoted as scattering forces, we argue that a third level may be introduced in which the so-called scattering forces are actually decomposed in scattering forces and non-standard forces, this last distinction having been found necessary in previous studies devoted to electric dipoles, including Rayleigh scattering, and to magnetodielectric small particles.

1. Introduction

The present paper is embedded in the framework defined by the generalized Lorenz–Mie theory (GLMT) which is an analytical and rigorous theory describing the interaction between an electromagnetic arbitrary shaped beam and a homogeneous spherical particle characterized by its diameter and its complex refractive index, e.g. [1,2] and references therein dating back to 1982 [3]. Although the primary motivation of the theory concerned the field of optical particle sizing and characterization, e.g. [4,5], there has been an early effort concerning the evaluation of optical forces in the GLMT framework, with a review in Gouesbet [6], as soon as 1985 [7]. In GLMT, optical forces are expressed in terms of beam shape coefficients denoted $g_{n,TM}^m$ and $g_{n,TE}^m$ (TM standing for "Transverse Magnetic", TE for "Transverse Electric", with n ranging from 1 to infinity, and $-n \leq m \leq +n$) which encode the structure of the illuminating beam and of the Mie coefficients a_n and b_n which encode the structure of the illuminated scatterer. As we shall see, although the use of the subscripts TM and TE may seem superfluous (other notations have been introduced without using these complementary subscripts), they will be of great value to emphasize the structures of the different kinds of optical forces exhibited by the three-level categorization discussed in the present paper. Basic expressions published in 1988 [8] and valid for arbitrary shaped beams [9], will form the starting blocks of the present work, and already introduce the first level of categorization between mixing and recoil forces (although this last terminology was not used in the original papers). The issue of optical forces has been discussed as well by Barton et al. [10], relying on the previous efforts in GLMT, in a theory that the authors called Arbitrary Beam Theory (ABM), although GLMT is an ABM as well, both approaches being however strictly equivalent as discussed in Gouesbet and Gréhan [11], pp. 46–47.

The GLMT formalism (*stricto sensu*, i.e. in the case of homogeneous spherical particles) is valid as well to the cases of multilayered particles when the expressions of the BSCs are unchanged, requiring only to modify the expressions of the Mie coefficients [12,13], and to other kinds of particles leading to expressions which are formally identical to the ones of the GLMT *stricto sensu*, namely assemblies of spheres and aggregates [14–16] and spheres with an eccentrically located spherical inclusion [17–19], Wang et al. [20]. Examples of optical force studies may concern (to cite a few, recent ones) optical tweezers [21,22], stretching and deforming [23], transporting and sorting [24], binding [25], and pushing and pulling [26]. Many more examples may be found in Gouesbet [27,28].

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After Arthur Ashkin's work, it has been traditional to think of the optical forces in terms of a partition between gradient and scattering forces which is the second-level categorization. However, although it may look strange for a theory completed in 1987 and 1988, GLMT studies aiming to a categorization of optical forces in terms of gradient and scattering forces had to wait for a long time before a first occurrence. As far as we know, such a first occurrence would be due to Lock [29], in the course of a study devoted to the calculation of radiation trapping forces in optical tweezers using GLMT with illuminating Gaussian beams. In this study, Gaussian beams were discussed in the weak confinement limit and scattering forces were defined as being proportional to the Poynting vector. It was then found that, indeed, GLMT performed well to distinguish between gradient and scattering forces. However, the use of a weak confinement limit prevented the author to uncover that there exist as well non-gradient (non-conservative) forces which are not proportional to the Poynting vector and therefore do not deserve the name of scattering forces. Such forces which are both non-gradient (non-conservative) forces and non-scattering forces (in the sense that they are not proportional to the Poynting vector) have been explicitly observed in the GLMT framework within a study devoted to the interaction between Bessel beams and dipoles [30]. At this time, they have been called axicon forces and have been later renamed non-standard forces when it has been recognized that their existence was not restricted to the case of beams exhibiting axicon angles. The case of Gaussian beams outside of the limited case of a weak confinement limit has been discussed in Gouesbet and Ambrosio [31].The categorization of non-gradient (non-conservative) forces in terms of scattering and non-standard forces constitutes the third-level categorization to be discussed in the present paper where it is a central theme.

A systematic study of the categorization of the optical forces in the GLMT is very recent. It started in 2020 in the case of lossless dieletric particles in the Rayleigh regime [30] and has been pursued up to applications to non-dark axisymmetric on-axis beams of the second kind and to dark axisymmetric on-axis beams [31], in which we only retained the first electric Mie coefficient a_1, followed by a study of optical forces exerted on magnetodielectric particles [32], in which the first Mie coefficients, both electric for a_1 and magnetic for b_1, are retained. An accurate understanding of the situation required several papers which were briefly reviewed in Section 10 of [31]. One of the results of this effort has been the (unexpected) exhibition of the third-level categorization of non-gradient (non-conservative) optical forces in terms of scattering and non-standard forces, made more explicit by several comparisons between the GLMT expressions restricted to dipoles and the dipole theory of forces (possibly in the Rayleigh limit of the dipole theory of forces), both numerically in the case of Bessel beams [33,34], and formally in the case of arbitrary shaped beam illumination [35–37]. The present paper considers an extension of the GLMT formulation to the categorization of optical forces exerted on quadrupoles in which the second electric and magnetic coefficients a_2 and b_2 are retained as well, and therefore pursues the work devoted to the application of GLMT to optical forces. A subsequent paper will consider the cases of optical forces for arbitrary sized particles [38] and will close the series.

But it is important, for a refined understanding of the optical force categorization, to compare dipoles and quadrupoles which, although providing strong similarities, exhibit as well significant differences which may be missed if the case of arbitrary sized particles was considered straight away. Let us indeed introduce a definition according to which we define N -forces as subforces obtained from the general expressions of forces by extracting in the summations the terms involving the Mie coefficients a_N and b_N, and their products with Mie coefficients of order $(N+1)$. Then, the main difference is that, in the case of dipoles dealing with 1 -forces, the definition of non-standard forces relies on a physical property (being non-conservative forces which are not proportional to Poynting vector) while, in the case of 2-forces, the definition of non-standard forces relies on a structural property which is shared by 1-forces for the case of dipoles but never shared by scattering forces, as will be discussed in the bulk of the paper. Similarities and differences are also well exhibited in Section 7 in which, as a similarity, we observe that the classification of mixing forces is valid both for 1-forces and 2-forces (Section 7.1), in contrast with the fact that we have to distinguish between type-1 recoil forces for 1-forces (Section 7.2) and type-1 recoil forces for 2-forces (Section 7.3).

As a consequence of the previous discussion, the paper is organized as follows. Section 2 recalls a background on optical forces which will display the first-level categorization in terms of mixing and recoil forces as already displayed in the GLMT framework nearly four decades ago and which will serve to derive the expressions of 1-forces and 2-forces (in terms of BSCs and of Mie coefficients) required for the study of quadrupoles. A background concerning the Poynting vector in the GLMT framework will be as well recalled. Section 3 displays the expressions of 1-forces while Section 4 will provide their interpretation. Section 5 displays the expressions of 2-forces while Section 6 will provide their interpretation. Section 7 is a summary and a discussion of the results generated by the 3-level categorization and by a 2-level decomposition (to be introduced in the sequel). Section 8 is a conclusion. Two appendices will refer to gradient 2-forces exhibited in Eq. (24) of Zheng et al. [39].

2. Background

2.1. Optical forces

The present paper is carried out in the framework of GLMT with usual notations, e.g. [8,32]. In the longitudinal direction $O_P z$, the radiation pressure cross-section component $C_{pr,z}$ is then expressed by the relation (e.g. Eq. (3.146) in Gouesbet and Gréhan [2]):

$$C_{pr,z} = \int_0^{\pi}\int_0^{2\pi} \frac{1}{2} Re(E_{\varphi}^{i} H_{\theta}^{s*} + E_{\varphi}^{s} H_{\theta}^{i*} - E_{\theta}^{i} H_{\varphi}^{s*} - E_{\theta}^{s} H_{\varphi}^{i*}) r^2 \cos\theta \ \sin\theta \ d\theta d\varphi - \int_0^{\pi}\int_0^{2\pi} (I_{\theta}^{s} + I_{\varphi}^{s}) r^2 \ \cos\theta \ \sin\theta \ d\theta d\varphi \tag{1}$$

in which the integrations are performed on the surface of a sphere S, centered at O_P, and of radius $r_p \gg \lambda$ (with λ the wavelength), **E** and **H** denote the electric and magnetic fields respectively, the superscripts i and s denote incident and scattered fields respectively, and (r, θ, φ) are usual spherical coordinates attached to the Cartesian coordinates (x, y, z). Also I_{θ}^{s} and I_{φ}^{s} denote scattered intensities given by Eqs. (3.107)–(3.111) in Gouesbet and Gréhan [2]. They correspond to the usual scattered intensities of the Lorenz–Mie theory but generalized to the case of GLMT.

Cross-sections may be viewed as forces expressed in square meters and will be conveniently called forces in the sequel (for the relationship between forces in square meters and forces in newtons, see Eq. (226) in Appendix B). With this vocabulary, the first force in the r.h.s. represents the forward momentum removed from the beam, now called a mixing force, and the second force represents minus the forward momentum given by the scatterer to the scattered wave, now called the recoil force. Eq. (1) therefore expresses a first categorization of optical forces in terms of mixing and recoil forces, a partition already introduced in 1985 in Gouesbet et al. [7], although we here use a different language to express it. Such a categorization occurs as well for transverse forces, see

Section 3.12.3 of Gouesbet and Gréhan [2]. Eq. (1) may be rewritten as (Eq. (3.147) in Gouesbet and Gréhan [2]):

$$C_{pr,z} = \overline{\cos\theta}\ C_{ext} - \overline{\cos\theta}\ C_{sca} \tag{2}$$

in which $\overline{\cos\theta}$ indicates integrations weighted by cosθ, C_{ext} is the extinction cross-section (Eq. (3.142) in Gouesbet and Gréhan [2]), and C_{sca} is the scattering cross-section (Eq. (3.137) in Gouesbet and Gréhan [2]). The forces $\overline{\cos\theta}\ C_{ext}$ and $\overline{\cos\theta}\ C_{sca}$of Eq. (2) may be evaluated in terms of BSCs according to Eqs. (3.158) and (3.155) in Gouesbet and Gréhan [2], to which the reader is kindly requested to refer. Once they are inserted in Eq. (2), we obtain $C_{pr,z}$ in terms of BSCs and Mie coefficients according to (Eq. (3.159) in Gouesbet and Gréhan [2]):

$$\begin{aligned} C_{pr,z} = \frac{\lambda^2}{\pi}\sum_{n=1}^{\infty}\sum_{p=-n}^{+n}\Bigg\{ &\frac{1}{(n+1)^2}\frac{(n+1+|p|)!}{(n-|p|)!} \\ &\mathrm{Re}[(\mathrm{a_n}+\mathrm{a}^*_{\mathrm{n+1}}-2\mathrm{a_n}\mathrm{a}^*_{\mathrm{n+1}})\mathrm{g}^{\mathrm{p}}_{\mathrm{n,TM}}\mathrm{g}^{\mathrm{p}*}_{\mathrm{n+1,TM}} \\ &+(b_n+b^*_{n+1}-2b_nb^*_{n+1})g^p_{n,TE}g^{p*}_{n+1,TE}] \\ &+p\frac{2n+1}{n^2(n+1)^2}\frac{(n+|p|)!}{(n-|p|)!}\mathrm{Re}[\mathrm{i}(2\mathrm{a_n}\mathrm{b}^*_{\mathrm{n}}-\mathrm{a_n}-\mathrm{b}^*_{\mathrm{n}})\mathrm{g}^{\mathrm{p}}_{\mathrm{n,TM}}\mathrm{g}^{\mathrm{p}*}_{\mathrm{n,TE}}]\Bigg\} \end{aligned} \tag{3}$$

Similarly, the transverse forces components along x and y may be decomposed into mixing and recoil forces according to (Eqs. (3.162) and (3.163) in Gouesbet and Gréhan [2]):

$$C_{pr,x} = \overline{\sin\theta\ \cos\varphi}\ C_{ext} - \overline{\sin\theta\ \cos\varphi}\ C_{sca} \tag{4}$$

$$C_{pr,y} = \overline{\sin\theta\ \sin\varphi}C_{ext} - \overline{\sin\theta\ \sin\varphi}C_{sca} \tag{5}$$

For the subterms in Eqs. (4) and (5), the reader is kindly requested to refer to Gouesbet and Gréhan [2], Eq. (3.180) for $\overline{\sin\theta\ \cos\varphi}\ C_{ext}$ (with the subscript in the rightmost summation corrected from $m = p - 1$ to $m = p - 1 \neq 0$ as in Eq. (158) of Gouesbet et al. [8]), Eq. (3.174) for $\overline{\sin\theta\ \cos\varphi}\ C_{sca}$, to be complemented by Eqs. (3.177) and (3.178), with the rightmost subscript of Eq. (3.178) corrected from TE to TM, as in Eq. (156) of Gouesbet et al. [8], and (3.167), (3.168). As a whole, $C_{pr,x}$ is provided by Eqs. (3.181) and (3.182) in Gouesbet and Gréhan [2]. Establishing the expression for $C_{pr,y}$ is fully similar. It is found that $\overline{\sin\theta\ \sin\varphi}\ C_{sca}$ and $\overline{\sin\theta\ \sin\varphi}\ C_{ext}$ are deduced from $\overline{\sin\theta\ \cos\varphi}\ C_{sca}$ and $\overline{\sin\theta\ \cos\varphi}\ C_{ext}$ respectively, by changing Re to Im. The final expression for $C_{pr,y}$ is therefore identical to the one of $C_{pr,x}$ but with Re replaced by Im as well.

Optical forces expressed in square meters may be denoted **F**. Then, the first-level categorization expresses the forces **F** as a sum of mixing forces with a subscript "*ext*" and of recoil forces with a subscript "*sca*" according to:

$$\mathbf{F} = \mathbf{F}_{ext} + \mathbf{F}_{sca} \tag{6}$$

in which:

$$\mathbf{F}_{ext} = \begin{pmatrix} F_{ext,x} \\ F_{ext,y} \\ F_{ext,z} \end{pmatrix} = \begin{pmatrix} \overline{\sin\theta\cos\varphi}C_{ext} \\ \overline{\sin\theta\sin\varphi}C_{ext} \\ \overline{\cos\theta}C_{ext} \end{pmatrix} \tag{7}$$

$$\mathbf{F}_{sca} = \begin{pmatrix} F_{sca,x} \\ F_{sca,y} \\ F_{sca,z} \end{pmatrix} = -\begin{pmatrix} \overline{\sin\theta\cos\varphi}C_{sca} \\ \overline{\sin\theta\sin\varphi}C_{sca} \\ \overline{\cos\theta}C_{sca} \end{pmatrix} \tag{8}$$

The minus sign in the r.h.s. of Eq. (8) is the consequence of the minus signs in Eqs. (2), (4), (5). The word "force" will be conveniently used to discuss forces in the vectorial sense or, by metonymy, to discuss force components, without any risk of ambiguity due to the contexts. For the recoil forces, the components will be discussed modulo the minus sign in the r.h.s. of Eq. (8). Also, it must be noted that the forces in GLMT have been obtained by using a radiative balance of momentum rather than by using the more abstract Maxwell stress tensor, although both approaches are equivalent, as checked and stated for example in Gouesbet and Gréhan [11], p.66.

2.2. Poynting vector

The study of optical force categorizations (for 1-forces) relies on the Poynting vector which has been used to the introduction of scattering forces (as proportional to Poynting vector components) and, *a contrario*, to the introduction of non-standard forces (which are not proportional to Poynting vector components), in the case of small dipolar particles, e.g. [32]. For the expression of Poynting vector components versus BSCs in the GLMT framework, see [40,41]. The reader is kindly requested to refer to these references for expressions which are not repeated here. A particular attention is to be paid to the Poynting vector components evaluated at the origin O_P of the coordinate system. These components, with P used as a subscript, read as Gouesbet [30], Gouesbet et al. [32], Gouesbet and Ambrosio [42], Gouesbet [43]:

$$[S_x]_P = \mathrm{Re}\{\mathrm{i}[\mathrm{g}^0_{1,\mathrm{TM}}(\mathrm{g}^{-1*}_{1,\mathrm{TE}} - \mathrm{g}^{1*}_{1,\mathrm{TE}}) + \mathrm{g}^0_{1,\mathrm{TE}}(\mathrm{g}^{1*}_{1,\mathrm{TM}} - \mathrm{g}^{-1*}_{1,\mathrm{TM}})]\} \tag{9}$$

$$[S_y]_P = \mathrm{Re}\{[\mathrm{g}^0_{1,\mathrm{TE}}(\mathrm{g}^{1*}_{1,\mathrm{TM}} + \mathrm{g}^{-1*}_{1,\mathrm{TM}}) - \mathrm{g}^0_{1,\mathrm{TM}}(\mathrm{g}^{-1*}_{1,\mathrm{TE}} + \mathrm{g}^{1*}_{1,\mathrm{TE}})]\} \tag{10}$$

$$[S_z]_P = 2\mathrm{Re}[\mathrm{i}(\mathrm{g}^{-1}_{1,\mathrm{TM}}\mathrm{g}^{-1*}_{1,\mathrm{TE}} - \mathrm{g}^1_{1,\mathrm{TM}}\mathrm{g}^{1*}_{1,\mathrm{TE}})] \tag{11}$$

3. Expressions versus BSCs of the 1-forces

Among the 1-forces, those which have been required for dealing with dipoles only involve Mie coefficients with subscripts 1, i.e. a_1, b_1, and the product a_1b_1 [32,37]. Other 1-forces, which have not been used when dealing with dipolar forces but which are necessary for dealing with quadrupoles, are discussed as well in this section. They are derived from the general expressions discussed in Section 2.1.

3.1. Longitudinal 1-forces

The mixing 1-forces, denoted as $\overline{\cos\theta}C^1_{ext}$, are given by Eqs. (15)–(17) of Gouesbet et al. [32]. They are conveniently repeated below to train the reader to deal with a change of notations useful in the more general context considered in this paper. We then have:

$$\overline{\cos\theta}C^1_{ext} = \frac{3\lambda^2}{2\pi}\mathrm{Re}(\mathrm{a_1}\mathrm{Z}^1_{\mathrm{E}} + \mathrm{b_1}\mathrm{Z}^1_{\mathrm{H}}) \tag{12}$$

in which:

$$Z^1_E = g^{-1}_{1,TM}(g^{-1*}_{2,TM} + ig^{-1*}_{1,TE}) + g^1_{1,TM}(g^{1*}_{2,TM} - ig^{1*}_{1,TE}) + \frac{1}{3}g^0_{1,TM}g^{0*}_{2,TM} \tag{13}$$

$$Z^1_H = g^{-1}_{1,TE}(g^{-1*}_{2,TE} - ig^{-1*}_{1,TM}) + g^1_{1,TE}(g^{1*}_{2,TE} + ig^{1*}_{1,TM}) + \frac{1}{3}g^0_{1,TE}g^{0*}_{2,TE} \tag{14}$$

in which Z^1_E and Z^1_H were denoted G_E and G_H respectively in Gouesbet et al. [32]. The superscript 1 refers to 1 -forces (the superscript 2 will be later used for 2-forces). The subscripts E and H denote electric forces and magnetic forces respectively. Magnetoelectric forces will be denoted by using the subscript EH. Such magnetoelectric forces occur for the recoil 1-forces which are found to read as:

$$\overline{\cos\theta}C^1_{sca} = \frac{3\lambda^2}{\pi}\mathrm{Re}(\mathrm{i}\mathrm{a_1}\mathrm{b}^*_1\mathrm{Z}^{11}_{\mathrm{EH}} + \mathrm{a_1}\mathrm{a}^*_2\mathrm{Z}^{12}_{\mathrm{E}} + \mathrm{b_1}\mathrm{b}^*_2\mathrm{Z}^{12}_{\mathrm{H}}) \tag{15}$$

in which:

$$Z_{EH}^{11} = g_{1,TM}^{-1} g_{1,TE}^{-1*} - g_{1,TM}^{1} g_{1,TE}^{1*} \tag{16}$$

$$Z_{E}^{12} = g_{1,TM}^{-1} g_{2,TM}^{-1*} + g_{1,TM}^{1} g_{2,TM}^{1*} + \frac{1}{3} g_{1,TM}^{0} g_{2,TM}^{0*} \tag{17}$$

$$Z_{H}^{12} = g_{1,TE}^{-1} g_{2,TE}^{-1*} + g_{1,TE}^{1} g_{2,TE}^{1*} + \frac{1}{3} g_{1,TE}^{0} g_{2,TE}^{0*} \tag{18}$$

The forces associated with Z_E^{12} weighted by the product $a_1 a_2^*$ and Z_H^{12} weighted by the product $b_1 b_2^*$ are new, i.e. they did not occur in the paper related to the dipole theory of forces, see Eq. (22) in [32] to be compared with Eq. (15) above.

3.2. Transverse x-component of 1-forces

The x-component of the mixing 1-forces is given by Eqs. (24)–(26) of Gouesbet et al. [32] which is here to be understood with a change of notations in which H_1 is changed to X_E^1 and H_2 to X_H^1 (most of the time, changes of notations between [32] and the present paper are easy to deal with). The x -component of the recoil 1-forces is found to generalize Eq. (27) of Gouesbet et al. [32], and reads as:

$$\overline{\sin\theta\cos\varphi C_{sca}^1} = \frac{3\lambda^2}{2\pi} \mathrm{Re}(\mathrm{i}a_1 b_1^* X_{EH}^{11} + a_1 a_2^* X_E^{12} + b_1 b_2^* X_H^{12}) \tag{19}$$

in which:

$$X_{EH}^{11} = g_{1,TE}^{0*}(g_{1,TM}^{-1} - g_{1,TM}^{1}) + g_{1,TM}^{0}(g_{1,TE}^{-1*} - g_{1,TE}^{1*}) \tag{20}$$

$$X_E^{12} = \frac{1}{3} g_{2,TM}^{0*}(g_{1,TM}^{1} + g_{1,TM}^{-1}) - g_{1,TM}^{0}(g_{2,TM}^{1*} + g_{2,TM}^{-1*}) - 4(g_{1,TM}^{1} g_{2,TM}^{2*} + g_{2,TM}^{-2*} g_{1,TM}^{-1}) \tag{21}$$

$$X_H^{12} = \frac{1}{3} g_{2,TE}^{0*}(g_{1,TE}^{1} + g_{1,TE}^{-1}) - g_{1,TE}^{0}(g_{2,TE}^{1*} + g_{2,TE}^{-1*}) - 4(g_{1,TE}^{1} g_{2,TE}^{2*} + g_{2,TE}^{-2*} g_{1,TE}^{-1}) \tag{22}$$

The force associated with $ia_1 b_1^* X_{EH}^{11}$ is a dipolar force already introduced in Gouesbet et al. [32] while the $a_1 a_2^*$- and $b_1 b_2^*$-forces are new.

3.3. Transverse y-component of 1-forces

The y-component of the mixing 1-forces is given in Eqs. (38)–(40) of Gouesbet et al. [32], which is here to be understood with a change of notations in which H_1' is changed to Y_E^1 and H_2' to Y_H^1. The corresponding y-component of the recoil 1-forces is found to read as:

$$\overline{\sin\theta\sin\varphi C_{sca}^1} = \frac{3\lambda^2}{2\pi} \mathrm{Im}(\mathrm{i}a_1 b_1^* Y_{EH}^{11} + a_1 a_2^* Y_E^{12} + b_1 b_2^* Y_H^{12}) \tag{23}$$

in which:

$$Y_{EH}^{11} = g_{1,TE}^{0*}(g_{1,TM}^{-1} + g_{1,TM}^{1}) - g_{1,TM}^{0}(g_{1,TE}^{-1*} + g_{1,TE}^{1*}) \tag{24}$$

$$Y_E^{12} = \frac{1}{3} g_{2,TM}^{0*}(g_{1,TM}^{-1} - g_{1,TM}^{1}) - g_{1,TM}^{0}(g_{2,TM}^{1*} - g_{2,TM}^{-1*}) - 4(g_{1,TM}^{1} g_{2,TM}^{2*} - g_{2,TM}^{-2*} g_{1,TM}^{-1}) \tag{25}$$

$$Y_H^{12} = \frac{1}{3} g_{2,TE}^{0*}(g_{1,TE}^{-1} - g_{1,TE}^{1}) - g_{1,TE}^{0}(g_{2,TE}^{1*} - g_{2,TE}^{-1*}) - 4(g_{1,TE}^{1} g_{2,TE}^{2*} - g_{2,TE}^{-2*} g_{1,TE}^{-1}) \tag{26}$$

The force associated with $ia_1 b_1^* Y_{EH}^{11}$ has already been introduced in Gouesbet et al. [32] while forces associated with $a_1 a_2^* Y_E^{12}$ and $b_1 b_2^* Y_H^{12}$ are new.

4. Interpretation of 1 forces

We have to distinguish between dipolar forces which have already been interpreted in Gouesbet et al. [32], whose interpretations will be recalled without any demonstration and the new forces which require new demonstrations (and which will be called non-dipolar forces).

4.1. Interpretation of dipolar 1-forces

4.1.1. z-component

Concerning $\overline{\cos\theta C_{ext}^1}$ of Eq. (12), it is the sum of electric forces and of magnetic forces, both of them being separable in two subforces, so that we have to face four kinds of forces, given by Eqs. (18)–(21) of [32], denoted respectively (with a new obvious slight change of notations) $\overline{\cos\theta C_{ext,E}^{1,I}}$, $\overline{\cos\theta C_{ext,E}^{1,R}}$, $\overline{\cos\theta C_{ext,H}^{1,I}}$ and $\overline{\cos\theta C_{ext,H}^{1,R}}$, in which the superscripts I and R are related to the imaginary and real parts respectively of the BSC-dependent terms (in the present case, the imaginary and real parts of the BSC-dependent terms correspond as well to the imaginary and real parts respectively of the Mie coefficients, but this will not be always the case). It has been demonstrated that $\overline{\cos\theta C_{ext,E}^{1,I}}$ and $\overline{\cos\theta C_{ext,H}^{1,I}}$ are mixing gradient electric and magnetic forces respectively, proportional respectively to $[\partial_z|\mathbf{E}|^2]_P$ and $[\partial_z|\mathbf{H}|^2]_P$, see Eqs. (56) and (58) of [32]. They may then conveniently be renamed $\overline{\cos\theta C_{ext,E}^{1,IG}}$ and $\overline{\cos\theta C_{ext,H}^{1,IG}}$, in which the superscript G stands for "gradient".

It has been demonstrated as well that $\overline{\cos\theta C_{ext,E}^{1,R}}$ is the sum of a mixing scattering electric force and of a mixing non-standard electric force, reading as respectively, see Eqs. (66) and (67) in Gouesbet et al. [32]:

$$\overline{\cos\theta C_{ext,E}^{1,RS}} = \frac{3\lambda^2}{2\pi} \mathrm{Re}(a_1)\mathrm{Re}[\mathrm{i}(g_{1,TM}^{-1} g_{1,TE}^{-1*} - g_{1,TM}^{1} g_{1,TE}^{1*})] \tag{27}$$

$$\overline{\cos\theta C_{ext,E}^{1,RNS}} = \frac{3\lambda^2}{2\pi} \mathrm{Re}(a_1)\mathrm{Re}[g_{1,TM}^{-1} g_{2,TM}^{-1*} + g_{1,TM}^{1} g_{2,TM}^{1*} + \frac{1}{3} g_{1,TM}^{0} g_{2,TM}^{0*}] \tag{28}$$

in which the superscripts S and NS stand for "scattering" and "non-standard" respectively. This introduces the definition of non-standard forces for 1-forces: the scattering force $\overline{\cos\theta C_{ext,E}^{1,RS}}$ is a non-gradient (non-conservative) force which is proportional to the corresponding component of the Poynting vector, see Eq. (11), while the non-standard force $\overline{\cos\theta C_{ext,E}^{1,RNS}}$ is a non-gradient (non-conservative) force which is *not proportional* to the corresponding component of the Poynting vector. Similarly, $\overline{\cos\theta C_{ext,H}^{1,R}}$ has been found to be the sum of a mixing scattering magnetic force $\overline{\cos\theta C_{ext,H}^{1,RS}}$ and of a mixing non-standard magnetic force $\overline{\cos\theta C_{ext,H}^{1,RNS}}$, see Eqs. (68) and (69) of [32]. For further use, it is useful to express $\overline{\cos\theta C_{ext,H}^{1,RS}}$ which reads as:

$$\overline{\cos\theta C_{ext,H}^{1,RS}} = \frac{3\lambda^2}{2\pi} \mathrm{Re}(b_1)\mathrm{Re}[\mathrm{i}(g_{1,TE}^{1} g_{1,TM}^{1*} - g_{1,TE}^{-1} g_{1,TM}^{-1*})] \tag{29}$$

4.1.2. x-component

Similarly, $\overline{\sin\theta\cos\varphi C_{ext}^1}$ in Eq. (4) may be decomposed into four terms $\overline{\sin\theta\cos\varphi C_{ext,E}^{1,I}}$, $\overline{\sin\theta\cos\varphi C_{ext,E}^{1,R}}$, $\overline{\sin\theta\cos\varphi C_{ext,H}^{1,I}}$ and $\overline{\sin\theta\cos\varphi C_{ext,H}^{1,R}}$, see Eqs. (32), (33), (35) and (36) of [32]. It has then been demonstrated that $\overline{\sin\theta\cos\varphi C_{ext,E}^{1,I}}$ and $\overline{\sin\theta\cos\varphi C_{ext,H}^{1,I}}$ (renamed $\overline{\sin\theta\cos\varphi C_{ext,E}^{1,IG}}$ and $\overline{\sin\theta\cos\varphi C_{ext,H}^{1,IG}}$) are mixing gradient electric forces and mixing gradient magnetic forces respectively, proportional to $[\partial_x|\mathbf{E}|^2]_P$ and $[\partial_x|\mathbf{H}|^2]_P$, respectively, see Eqs. (59) and (60) of [32]. Also, it has been demonstrated that

$\overline{\sin\theta\cos\varphi C^{1,R}_{ext,E}}$ is the sum of a mixing scattering electric force and of a mixing non-standard electric force, reading as (Eqs. (70) and (71) of [32]):

$$\overline{\sin\theta\cos\varphi C^{1,RS}_{ext,E}} = \frac{3\lambda^2}{4\pi}\mathrm{Re}(a_1)\mathrm{Re}\{i[g^0_{1,TM}(g^{-1*}_{1,TE} - g^{1*}_{1,TE}) + g^{0*}_{1,TE}(g^{-1}_{1,TM} - g^{1}_{1,TM})]\} \quad (30)$$

$$\begin{aligned}\overline{\sin\theta\cos\varphi C^{1,RNS}_{ext,E}} &= \frac{\lambda^2}{4\pi}\mathrm{Re}(a_1)\mathrm{Re}[g^{0*}_{2,TM}(g^{1}_{1,TM} + g^{-1}_{1,TM}) \\ &\quad -3g^{0}_{1,TM}(g^{1*}_{2,TM} + g^{-1*}_{2,TM}) - 12(g^{1}_{1,TM}g^{2*}_{2,TM} \\ &\quad +g^{-1}_{1,TM}g^{-2*}_{2,TM})] \\ &= \frac{3\lambda^2}{4\pi}\mathrm{Re}(a_1)\mathrm{Re}(X^{12}_{E})\end{aligned} \quad (31)$$

in which, for the interpretation of $\overline{\sin\theta\cos\varphi C^{1,RS}_{ext,E}}$ as a scattering force, we may compare Eqs. (30) and (9). Similarly, $\overline{\sin\theta\cos\varphi C^{1,R}_{ext,H}}$ has been found to be the sum of a mixing scattering magnetic force and of a mixing non-standard magnetic force, see Eqs. (72) and (73) of [32].

4.1.3. y-component

Similarly, $\overline{\sin\theta\sin\varphi C^{1}_{ext}}$, see Eq. (38) of [32], may be decomposed into four terms denoted $\overline{\sin\theta\sin\varphi C^{1,I}_{ext,E}}$, $\overline{\sin\theta\sin\varphi C^{1,R}_{ext,E}}$, $\overline{\sin\theta\sin\varphi C^{1,I}_{ext,H}}$ and $\overline{\sin\theta\sin\varphi C^{1,R}_{ext,H}}$, see Eqs. (47), (46) and (50), (49), respectively, of [32]. It has then been demonstrated that $\overline{\sin\theta\sin\varphi C^{1,R}_{ext,E}}$ and $\overline{\sin\theta\sin\varphi C^{1,R}_{ext,H}}$ (renamed $\overline{\sin\theta\sin\varphi C^{1,RG}_{ext,E}}$ and $\overline{\sin\theta\sin\varphi C^{1,RG}_{ext,H}}$) are mixing gradient electric forces and mixing gradient magnetic forces respectively, proportional to $[\partial_y|\mathbf{E}|^2]_P$ and $[\partial_y|\mathbf{H}|^2]_P$, see Eqs. (61) and (62) of [32]. Meanwhile, $\overline{\sin\theta\sin\varphi C^{1,I}_{ext,E}}$ has been found to be the sum of a mixing scattering electric force and of a mixing non-standard electric force, reading as respectively (Eqs. (74) and (75) of [32]):

$$\overline{\sin\theta\sin\varphi C^{1,IS}_{ext,E}} = \frac{3\lambda^2}{4\pi}\mathrm{Re}(a_1)\mathrm{Re}[g^{0*}_{1TE}(g^{1}_{1,TM} + g^{-1}_{1,TM}) - g^{0}_{1,TM}(g^{1*}_{1,TE} + g^{-1*}_{1,TE})] \quad (32)$$

$$\begin{aligned}\overline{\sin\theta\sin\varphi C^{1,INS}_{ext,E}} &= \frac{\lambda^2}{4\pi}\mathrm{Re}(a_1)\mathrm{Im}[g^{0*}_{2,TM}(g^{-1}_{1,TM} - g^{1}_{1,TM}) \\ &\quad -3g^{0}_{1,TM}(g^{1*}_{2,TM} - g^{-1*}_{2,TM}) - 12(g^{1}_{1,TM}g^{2*}_{2,TM} \\ &\quad -g^{-1}_{1,TM}g^{-2*}_{2,TM})] \\ &= \frac{3\lambda^2}{4\pi}\mathrm{Re}(a_1)\mathrm{Im}(Y^{12}_{E})\end{aligned} \quad (33)$$

in which, for the interpretation of the scattering force $\overline{\sin\theta\sin\varphi C^{1,IS}_{ext,E}}$, we may compare Eqs. (32) and (10). Furthermore, $\overline{\sin\theta\sin\varphi C^{1,I}_{ext,H}}$ has similarly been found to be the sum of a mixing scattering magnetic force and of a mixing non-standard magnetic force, see Eqs. (76) and (77) of [32].

4.1.4. Magnetoelectric dipolar 1-forces

The dipolar 1-forces considered up to now were mixing purely electric or purely magnetic forces. We now have to consider the first force of Eq. (15) which is again a dipolar force (i.e. a force involved in the GLMT limit of the dipole theory of forces as discussed in Ambrosio et al. [37]). It is the z-component of a recoil magnetoelectric force given by Eq. (22) of [32], which may be decomposed into two subforces denoted as $\overline{\cos\theta C^{1,R}_{sca,EH}}$ and $\overline{\cos\theta C^{1,I}_{sca,EH}}$ which are given in Eqs. (79) and (80) of subsection (4.4) of [32] in which they are interpreted. In particular, we have:

$$\begin{aligned}\overline{\cos\theta C^{1,R}_{sca,EH}} &= \frac{-3\lambda^2}{\pi}\mathrm{Re}(a_1 b_1^*)\mathrm{Re}[i(g^{1}_{1,TM}g^{1*}_{1,TE} - g^{-1}_{1,TM}g^{-1*}_{1,TE})] \\ &= \frac{3\lambda^2}{\pi}\mathrm{Re}(a_1 b_1^*)\mathrm{Re}(iZ^{11}_{EH})\end{aligned} \quad (34)$$

It was concluded by a direct argument that $\overline{\cos\theta C^{1,R}_{sca,EH}}$ (renamed $\overline{\cos\theta C^{1,RS}_{sca,EH}}$) is a scattering force. Furthermore, from Eqs. (34), (27) and (29), we obtain:

$$\overline{\cos\theta C^{1,RS}_{sca,EH}} = 2\frac{\mathrm{Re}(a_1 b_1^*)}{\mathrm{Re}(a_1)}\overline{\cos\theta C^{1,RS}_{ext,E}} = 2\frac{\mathrm{Re}(a_1 b_1^*)}{\mathrm{Re}(b_1)}\overline{\cos\theta C^{1,RS}_{ext,H}} \quad (35)$$

which expresses $\overline{\cos\theta C^{1,RS}_{sca,EH}}$ in terms of mixing scattering pure electric and magnetic forces, and confirms that $\overline{\cos\theta C^{1,RS}_{sca,EH}}$ is indeed a scattering force. In contrast, it was argued that $\overline{\cos\theta C^{1,I}_{sca,EH}}$ (which is not a scattering force and which is not a gradient force) is a recoil non-standard force, to be renamed $\overline{\cos\theta C^{1,INS}_{sca,EH}}$ (see complementary discussion, between Eqs. (80) and (81) of [32]). From Eq. (80) in Gouesbet et al. [32], it is observed that this recoil non-standard force only contains BSCs associated with $n=1$, without any coupling between $(n=1)$- and $(n=2)$- partial waves (i.e. involving only BSCs of the form $g^m_{1,X}$) and without any coupling between $(n=1)$- and $(n=2)$-Mie coefficients (i.e. involving only Mie coefficients a_1 and b_1). This force is both non-standard and magnetoelectric, with therefore is a noticeable property. Similar comments would apply as well to the x- and y-components of recoil non-standard magnetoelectric forces discussed below.

Similarly, from the x-component of the recoil force of Eq. (19), we may extract a dipolar recoil magnetoelectric force, again without any coupling between $n=1$ and $n=2$ partial waves, denoted $\overline{\sin\theta\cos\varphi C^{1}_{sca,EH}}$, and given by Eq. (27) of [32]. This force may be decomposed into two subforces denoted as $\overline{\sin\theta\cos\varphi C^{1,R}_{sca,EH}}$ and $\overline{\sin\theta\cos\varphi C^{1,I}_{sca,EH}}$ which are respectively given by Eqs. (82) and (83) of [32], and interpreted in subsection (4.4) of [32], in which it is found that $\overline{\sin\theta\cos\varphi C^{1,R}_{sca,EH}}$, to be renamed $\overline{\sin\theta\cos\varphi C^{1,RS}_{sca,EH}}$, is a scattering force while, arguing as for $\overline{\cos\theta C^{1,I}_{sca,EH}}$, we have that $\overline{\sin\theta\cos\varphi C^{1,I}_{sca,EH}}$, renamed $\overline{\sin\theta\cos\varphi C^{1,INS}_{sca,EH}}$ is a recoil non-standard (magnetoelectric) force, again without any coupling involving $n=1$ and $n=2$ partial waves. Furthermore, similarly to Eq. (35), we demonstrate that:

$$\begin{aligned}\overline{\sin\theta\cos\varphi C^{1,RS}_{sca,EH}} &= 2\frac{\mathrm{Re}(a_1 b_1^*)}{\mathrm{Re}(a_1)}\overline{\sin\theta\cos\varphi C^{1,RS}_{ext,E}} \\ &= 2\frac{\mathrm{Re}(a_1 b_1^*)}{\mathrm{Re}(b_1)}\overline{\sin\theta\cos\varphi C^{1,RS}_{ext,H}}\end{aligned} \quad (36)$$

which confirms that $\overline{\sin\theta\cos\varphi C^{1,RS}_{sca,EH}}$ is a scattering force which is expressed in terms of mixing scattering pure electric and magnetic forces.

Similarly, from the y-component of the recoil force of Eq. (23), we extract a non-coupling recoil magnetoelectric force given by Eq. (41) of [32] which may be decomposed into two subforces denoted $\overline{\sin\theta\sin\varphi C^{1,R}_{sca,EH}}$ proportional to $\mathrm{Re}(iY^{11}_{EH})$ and $\overline{\sin\theta\sin\varphi C^{1,I}_{sca,EH}}$ proportional to $\mathrm{Im}(iY^{11}_{EH})$. We here have to take into account for a change of notation which, using $\mathrm{Re}(iz) = -\mathrm{Im}(z)$ and $\mathrm{Im}(iz) = \mathrm{Re}(z)$, shows that $\overline{\sin\theta\sin\varphi C^{1,R}_{sca,EH}}$ and $\overline{\sin\theta\sin\varphi C^{1,I}_{sca,EH}}$, due to a change of location of the imaginary unit "i", actually correspond to Eqs. (86) and (85) – i.e. not with (85) and (86) – of [32]. In subsection 4.4 of [32], it is then concluded that $\overline{\sin\theta\sin\varphi C^{1,I}_{sca,EH}}$ is a scattering force (renamed $\overline{\sin\theta\sin\varphi C^{1,IS}_{sca,EH}}$), while arguing as for the z- and x-components, $\overline{\sin\theta\sin\varphi C^{1,R}_{sca,EH}}$ (renamed $\overline{\sin\theta\sin\varphi C^{1,RNS}_{sca,EH}}$) denotes a non-coupling recoil non-standard force. Note that the interpretations of these x- and y-components require to interchange the superscripts R and I which is the consequence of the change from Re to Im in the general expressions for the x- and y-forces as mentioned in Section 2.1. Similarly as for Eqs. (35) and (36), we then demonstrate that:

$$\overline{\sin\theta\sin\varphi C^{1,IS}_{sca,EH}} = 2\frac{\mathrm{Re}(a_1 b_1^*)}{\mathrm{Re}(a_1)}\overline{\sin\theta\sin\varphi C^{1,IS}_{ext,E}} = 2\frac{\mathrm{Re}(a_1 b_1^*)}{\mathrm{Re}(b_1)}\overline{\sin\theta\sin\varphi C^{1,IS}_{ext,H}} \quad (37)$$

confirming that $\overline{\sin\theta\sin\varphi C^{1,IS}_{sca,EH}}$ is a scattering force, which furthermore can be expressed in terms of mixing scattering pure electric and magnetic forces.

4.2. Interpretation of non-dipolar 1-forces

4.2.1. z-component

We now have to discuss non-dipolar 1-forces, that is to say forces which are not required to match the dipole theory of forces, once again see [32] where they have not been implemented. These new forces, not yet interpreted, are all recoil forces with a coupling between $(n = 1)$ and $(n = 2)$ partial waves, these couplings being associated with products $a_1 a_2^*$ and $b_1 b_2^*$ of Mie coefficients. We again begin with a z-component. From the second and third forces of Eq. (15), we have:

$$\overline{\cos\theta C^{1}_{sca,E}} = \frac{3\lambda^2}{\pi}\mathrm{Re}(a_1 a_2^* Z_E^{12}) \quad (38)$$

$$\overline{\cos\theta C^{1}_{sca,H}} = \frac{3\lambda^2}{\pi}\mathrm{Re}(b_1 b_2^* Z_H^{12}) \quad (39)$$

in which we recall that Z_E^{12} and Z_H^{12} are given in Eqs. (17) and (18).

The electric recoil force $\overline{\cos\theta C^{1}_{sca,E}}$ may be decomposed into two subforces denoted $\overline{\cos\theta C^{1,R}_{sca,E}}$ and $\overline{\cos\theta C^{1,I}_{sca,E}}$, which, using Eq. (17) read as:

$$\overline{\cos\theta C^{1,R}_{sca,E}} = \frac{3\lambda^2}{\pi}\mathrm{Re}(a_1 a_2^*)\mathrm{Re}(Z_E^{12}) = \frac{3\lambda^2}{\pi}\mathrm{Re}(a_1 a_2^*)\mathrm{Re}(g_{1,TM}^{-1} g_{2,TM}^{-1*} + g_{1,TM}^{1} g_{2,TM}^{1*} + \frac{1}{3} g_{1,TM}^{0} g_{2,TM}^{0*}) \quad (40)$$

$$\overline{\cos\theta C^{1,I}_{sca,E}} = \frac{-3\lambda^2}{\pi}\mathrm{Im}(a_1 a_2^*)\mathrm{Im}(Z_E^{12}) = \frac{-3\lambda^2}{\pi}\mathrm{Im}(a_1 a_2^*)\mathrm{Im}(g_{1,TM}^{-1} g_{2,TM}^{-1*} + g_{1,TM}^{1} g_{2,TM}^{1*} + \frac{1}{3} g_{1,TM}^{0} g_{2,TM}^{0*}) \quad (41)$$

From Eq. (28), we see that $\overline{\cos\theta C^{1,R}_{sca,E}}$ is a non-standard force according to:

$$\overline{\cos\theta C^{1,R}_{sca,E}} = \overline{\cos\theta C^{1,RNS}_{sca,E}} = \frac{2\mathrm{Re}(a_1 a_2^*)}{\mathrm{Re}(a_1)}\overline{\cos\theta C^{1,RNS}_{ext,E}} \quad (42)$$

showing that we have a recoil non-standard electric force which may be expressed in terms of a mixing non-standard electric force.

For $\overline{\cos\theta C^{1,I}_{sca,E}}$, we first establish, from Eqs. (13) and (17) that:

$$Z_E^{12} = Z_E^1 - i(g_{1,TM}^{-1} g_{1,TE}^{-1*} - g_{1,TM}^{1} g_{1,TE}^{1*}) \quad (43)$$

We may then decompose $\overline{\cos\theta C^{1,I}_{sca,E}}$ into two subforces denoted $\overline{\cos\theta C^{1,I\alpha}_{sca,E}}$ and $\overline{\cos\theta C^{1,I\beta}_{sca,E}}$, reading as:

$$\overline{\cos\theta C^{1,I\alpha}_{sca,E}} = \frac{-3\lambda^2}{\pi}\mathrm{Im}(a_1 a_2^*)\mathrm{Im}(Z_E^1) \quad (44)$$

$$\overline{\cos\theta C^{1,I\beta}_{sca,E}} = \frac{3\lambda^2}{\pi}\mathrm{Im}(a_1 a_2^*)\mathrm{Im}[i(g_{1,TM}^{-1} g_{1,TE}^{-1*} - g_{1,TM}^{1} g_{1,TE}^{1*})] \quad (45)$$

Let us consider Eq. (44). $\mathrm{Im}(Z_E^1)$ occurs in Eqs. (18) and (56) of [32] for $\overline{\cos\theta C^{1,I}_{ext,E}}$ and $[\partial_z|\mathbf{E}|^2]_P$ respectively (notwithstanding an easy change of notations). We then find that $\overline{\cos\theta C^{1,I\alpha}_{sca,E}}$ is a recoil gradient electric force, which may be expressed versus a mixing gradient electric force, according to:

$$\overline{\cos\theta C^{1,I\alpha}_{sca,E}} = \overline{\cos\theta C^{1,IG}_{sca,E}} = \frac{2\mathrm{Im}(a_1 a_2^*)}{\mathrm{Im}(a_1)}\overline{\cos\theta C^{1,I}_{ext,E}} = \frac{3\lambda^2}{4k\eta\pi}\mathrm{Im}(a_1 a_2^*)[\partial_z|\mathbf{E}|^2]_P \quad (46)$$

We now consider $\overline{\cos\theta C^{1,I\beta}_{sca,E}}$. The Im[] in Eq. (45) may be expressed in terms of $\overline{\cos\theta C^{1,I}_{sca,EH}}$ by Eq. (80) of [32], showing that it is a recoil non-standard electric force which may be expressed in terms of a recoil non-standard magneto-electric force according to:

$$\overline{\cos\theta C^{1,I\beta}_{sca,E}} = \overline{\cos\theta C^{1,INS}_{sca,E}} = \frac{-\mathrm{Im}(a_1 a_2^*)}{\mathrm{Im}(a_1 b_1^*)}\overline{\cos\theta C^{1,INS}_{sca,EH}} \quad (47)$$

Similarly, $\overline{\cos\theta C^{1}_{sca,H}}$ may be decomposed into two subforces denoted $\overline{\cos\theta C^{1,R}_{sca,H}}$ and $\overline{\cos\theta C^{1,I}_{sca,H}}$, reading as:

$$\overline{\cos\theta C^{1,R}_{sca,H}} = \frac{3\lambda^2}{\pi}\mathrm{Re}(b_1 b_2^*)\mathrm{Re}(Z_H^{12}) = \frac{3\lambda^2}{\pi}\mathrm{Re}(b_1 b_2^*)\mathrm{Re}(g_{1,TE}^{-1} g_{2,TE}^{-1*} + g_{1,TE}^{1} g_{2,TE}^{1*} + \frac{1}{3} g_{1,TE}^{0} g_{2,TE}^{0*}) \quad (48)$$

$$\overline{\cos\theta C^{1,I}_{sca,H}} = \frac{-3\lambda^2}{\pi}\mathrm{Im}(b_1 b_2^*)\mathrm{Im}(Z_H^{12}) = \frac{-3\lambda^2}{\pi}\mathrm{Im}(b_1 b_2^*)\mathrm{Im}(g_{1,TE}^{-1} g_{2,TE}^{-1*} + g_{1,TE}^{1} g_{2,TE}^{1*} + \frac{1}{3} g_{1,TE}^{0} g_{2,TE}^{0*}) \quad (49)$$

Proceeding similarly as for the electric force $\overline{\cos\theta C^{1}_{sca,E}}$ (or, more rapidly, invoking a duality between electric and magnetic forces), it is then found that $\overline{\cos\theta C^{1,R}_{sca,H}}$ is a recoil non-standard magnetic force which may be expressed in terms of a mixing non-standard magnetic force according to:

$$\overline{\cos\theta C^{1,R}_{sca,H}} = \overline{\cos\theta C^{1,RNS}_{sca,H}} = \frac{2\mathrm{Re}(b_1 b_2^*)}{\mathrm{Re}(b_1)}\overline{\cos\theta C^{1,RNS}_{ext,H}} \quad (50)$$

while $\overline{\cos\theta C^{1,I}_{sca,H}}$ may be decomposed in two subforces denoted $\overline{\cos\theta C^{1,I\alpha}_{sca,H}}$ and $\overline{\cos\theta C^{1,I\beta}_{sca,H}}$, which, using the magnetic counterpart of Eq. (43), read as:

$$\overline{\cos\theta C^{1,I\alpha}_{sca,H}} = \frac{-3\lambda^2}{\pi}\mathrm{Im}(b_1 b_2^*)\mathrm{Im}(Z_H^1) \quad (51)$$

$$\overline{\cos\theta C^{1,I\beta}_{sca,H}} = \frac{-3\lambda^2}{\pi}\mathrm{Im}(b_1 b_2^*)\mathrm{Im}[i(g_{1,TE}^{-1} g_{1,TM}^{-1*} - g_{1,TE}^{1} g_{1,TM}^{1*})] \quad (52)$$

Similarly as for the electric case, it is then found that $\overline{\cos\theta C^{1,I\alpha}_{sca,H}}$ is a recoil gradient magnetic force, which may be expressed versus a mixing gradient magnetic force, according to:

$$\overline{\cos\theta C^{1,I\alpha}_{sca,H}} = \overline{\cos\theta C^{1,IG}_{sca,H}} = \frac{2\mathrm{Im}(b_1 b_2^*)}{\mathrm{Im}(b_1)}\overline{\cos\theta C^{1,IG}_{ext,H}} = \frac{3\lambda^2\eta}{4k\pi}\mathrm{Im}(b_1 b_2^*)[\partial_z|\mathbf{H}|^2]_P \quad (53)$$

while $\overline{\cos\theta C^{1,I\beta}_{sca,H}}$ is a recoil non-standard magnetic force, which may be expressed versus a recoil non-standard magnetoelectric force, reading as:

$$\overline{\cos\theta C^{1,I\beta}_{sca,H}} = \overline{\cos\theta C^{1,INS}_{sca,H}} = \frac{\mathrm{Im}(b_1 b_2^*)}{\mathrm{Im}(a_1 b_1^*)}\overline{\cos\theta C^{1,INS}_{sca,EH}} \tag{54}$$

4.2.2. x-component

From the second and third forces of Eq. (19), we have to deal with:

$$\overline{\sin\theta\cos\varphi C^{1}_{sca,E}} = \frac{3\lambda^2}{2\pi}\mathrm{Re}(a_1 a_2^* X_E^{12}) \tag{55}$$

$$\overline{\sin\theta\cos\varphi C^{1}_{sca,H}} = \frac{3\lambda^2}{2\pi}\mathrm{Re}(b_1 b_2^* X_H^{12}) \tag{56}$$

We begin with $\overline{\sin\theta\cos\varphi C^{1}_{sca,E}}$ which may decomposed into two subforces denoted $\overline{\sin\theta\cos\varphi C^{1,R}_{sca,E}}$ and $\overline{\sin\theta\cos\varphi C^{1,I}_{sca,E}}$, reading as:

$$\overline{\sin\theta\cos\varphi C^{1,R}_{sca,E}} = \frac{3\lambda^2}{2\pi}\mathrm{Re}(a_1 a_2^*)\mathrm{Re}(X_E^{12}) \tag{57}$$

$$\overline{\sin\theta\cos\varphi C^{1,I}_{sca,E}} = \frac{-3\lambda^2}{2\pi}\mathrm{Im}(a_1 a_2^*)\mathrm{Im}(X_E^{12}) \tag{58}$$

Using Eq. (31), we obtain:

$$\overline{\sin\theta\cos\varphi C^{1,R}_{sca,E}} = \overline{\sin\theta\cos\varphi C^{1,RNS}_{sca,E}} = 2\frac{\mathrm{Re}(a_1 a_2^*)}{\mathrm{Re}(a_1)}\overline{\sin\theta\cos\varphi C^{1,RNS}_{ext,E}} \tag{59}$$

showing that $\overline{\sin\theta\cos\varphi C^{1,R}_{sca,E}}$ is a recoil non-standard electric force which may be expressed in terms of a mixing non-standard electric force.

For $\overline{\sin\theta\cos\varphi C^{1,I}_{sca,E}}$, we observe that X_E^{12} of Eq. (21) may be expressed in terms of X_E^1 of Eq. (25) of [32] (in which it was denoted as H_1), according to:

$$X_E^{12} = \frac{1}{3}X_E^1 - i[g^0_{1,TM}(g^{-1*}_{1,TE} - g^{1*}_{1,TE}) + g^{0*}_{1,TE}(g^{-1}_{1,TM} - g^{1}_{1,TM})] \tag{60}$$

Therefore, $\overline{\sin\theta\cos\varphi C^{1,I}_{sca,E}}$ may be decomposed in two subforces denoted $\overline{\sin\theta\cos\varphi C^{1,I\alpha}_{sca,E}}$ and $\overline{\sin\theta\cos\varphi C^{1,I\beta}_{sca,E}}$, reading as:

$$\overline{\sin\theta\cos\varphi C^{1,I\alpha}_{sca,E}} = \frac{-\lambda^2}{2\pi}\mathrm{Im}(a_1 a_2^*)\mathrm{Im}(X_E^1) \tag{61}$$

$$\overline{\sin\theta\cos\varphi C^{1,I\beta}_{sca,E}} = \frac{3\lambda^2}{2\pi}\mathrm{Im}(a_1 a_2^*)\mathrm{Im}\{i[g^0_{1,TM}(g^{-1*}_{1,TE} - g^{1*}_{1,TE}) + g^{0*}_{1,TE}(g^{-1}_{1,TM} - g^{1}_{1,TM})]\} \tag{62}$$

The quantity $\mathrm{Im}(X_E^1)$ is available as well from Eqs. (32), (59) of [32], leading to the fact that $\overline{\sin\theta\cos\varphi C^{1,I\alpha}_{sca,E}}$ is a recoil gradient electric force which may be expressed versus a mixing gradient electric force according to:

$$\begin{aligned}\overline{\sin\theta\cos\varphi C^{1,I\alpha}_{sca,E}} &= \overline{\sin\theta\cos\varphi C^{1,IG}_{sca,E}} \\ &= 2\frac{\mathrm{Im}(a_1 a_2^*)}{\mathrm{Im}(a_1)}\overline{\sin\theta\cos\varphi C^{1,IG}_{ext,E}} \\ &= \frac{3\lambda^2}{4k\eta\pi}\mathrm{Im}(a_1 a_2^*)[\partial_x|\mathbf{E}|^2]_P\end{aligned} \tag{63}$$

Next, from Eqs. (20), (62) can be rewritten as:

$$\overline{\sin\theta\cos\varphi C^{1,I\beta}_{sca,E}} = \frac{3\lambda^2}{2\pi}\mathrm{Im}(a_1 a_2^*)\mathrm{Im}(iX_{EH}^{11}) \tag{64}$$

which, using Eq. (83) of [32], leads to:

$$\overline{\sin\theta\cos\varphi C^{1,I\beta}_{sca,E}} = \overline{\sin\theta\cos\varphi C^{1,INS}_{sca,E}} = \frac{-\mathrm{Im}(a_1 a_2^*)}{\mathrm{Im}(a_1 b_1^*)}\overline{\sin\theta\cos\varphi C^{1,INS}_{sca,EH}} \tag{65}$$

showing that it is a recoil non-standard electric force which maybe expressed in terms of a recoil non-standard magnetoelectric force.

Proceeding similarly as for the corresponding magnetic force $\overline{\sin\theta\cos\varphi C^{1}_{sca,H}}$ of Eq. (56), we use a decomposition into two subforces denoted as $\overline{\sin\theta\cos\varphi C^{1,R}_{sca,H}}$ and $\overline{\sin\theta\cos\varphi C^{1,I}_{sca,H}}$ in the usual way. Then, using Eq. (73) of [32], we establish that $\overline{\sin\theta\cos\varphi C^{1,R}_{sca,H}}$, renamed $\overline{\sin\theta\cos\varphi C^{1,RNS}_{sca,H}}$, is a recoil non-standard magnetic force which may be expressed versus a mixing non-standard magnetic force according to:

$$\overline{\sin\theta\cos\varphi C^{1,R}_{sca,H}} = \overline{\sin\theta\cos\varphi C^{1,RNS}_{sca,H}} = 2\frac{\mathrm{Re}(b_1 b_2^*)}{\mathrm{Re}(b_1)}\overline{\sin\theta\cos\varphi C^{1,RNS}_{ext,H}} \tag{66}$$

Expressing X_H^{12} of Eq. (22) versus X_H^1 of Eq. (26) of [32] (where it is denoted H_2), it is found that $\overline{\sin\theta\cos\varphi C^{1,I}_{sca,H}}$ may be decomposed into two subforces, denoted $\overline{\sin\theta\cos\varphi C^{1,I\alpha}_{sca,H}}$ and $\overline{\sin\theta\cos\varphi C^{1,I\beta}_{sca,H}}$, reading as:

$$\overline{\sin\theta\cos\varphi C^{1,I\alpha}_{sca,H}} = \frac{-\lambda^2}{2\pi}\mathrm{Im}(b_1 b_2^*)\mathrm{Im}(X_H^1) \tag{67}$$

$$\overline{\sin\theta\cos\varphi C^{1,I\beta}_{sca,H}} = \frac{3\lambda^2}{2\pi}\mathrm{Im}(b_1 b_2^*)\mathrm{Im}\{i[g^0_{1,TE}(g^{1*}_{1,TM} - g^{-1*}_{1,TM}) + g^{0*}_{1,TM}(g^{1}_{1,TE} - g^{-1}_{1,TE})]\} \tag{68}$$

From Eqs. (35), (60) of [32], we conclude that $\overline{\sin\theta\cos\varphi C^{1,I\alpha}_{sca,H}}$ is a recoil gradient magnetic force which may be expressed versus a mixing gradient magnetic force according to:

$$\begin{aligned}\overline{\sin\theta\cos\varphi C^{1,I\alpha}_{sca,H}} &= \overline{\sin\theta\cos\varphi C^{1,IG}_{sca,H}} \\ &= 2\frac{\mathrm{Im}(b_1 b_2^*)}{\mathrm{Im}(b_1)}\overline{\sin\theta\cos\varphi C^{1,IG}_{ext,H}} \\ &= \frac{3\lambda^2\eta}{4k\pi}\mathrm{Im}(b_1 b_2^*)[\partial_x|\mathbf{H}|^2]_P\end{aligned} \tag{69}$$

while, using Eq. (83) of [32] and Eqs. (20), (68) shows that $\overline{\sin\theta\cos\varphi C^{1,I\beta}_{sca,H}}$ is a recoil non-standard magnetic force which may be expressed versus a recoil non-standard magnetoelectric force according to:

$$\overline{\sin\theta\cos\varphi C^{1,I\beta}_{sca,H}} = \overline{\sin\theta\cos\varphi C^{1,INS}_{sca,H}} = \frac{\mathrm{Im}(b_1 b_2^*)}{\mathrm{Im}(a_1 b_1^*)}\overline{\sin\theta\cos\varphi C^{1,INS}_{sca,EH}} \tag{70}$$

4.2.3. y-component

The y-component is treated quite similarly as the x-component so that we shall omit the details of the demonstrations. They can be "copied" by the reader from the case of the x-component (taking into account, once again, the interchange of the superscripts R and I arising between x- and y-components). From Eq. (23), we have to deal with two new subforces denoted as $\overline{\sin\theta\sin\varphi C^{1}_{sca,E}}$ and $\overline{\sin\theta\sin\varphi C^{1}_{sca,H}}$. The recoil electric force $\overline{\sin\theta\sin\varphi C^{1}_{sca,E}}$ may decomposed again into two subforces denoted as $\overline{\sin\theta\sin\varphi C^{1,R}_{sca,E}}$ and $\overline{\sin\theta\sin\varphi C^{1,I}_{sca,E}}$. It is then found, from Eq. (33), that $\overline{\sin\theta\sin\varphi C^{1,I}_{sca,E}}$, to be renamed $\overline{\sin\theta\sin\varphi C^{1,INS}_{sca,E}}$, is a recoil non-standard electric force which may be expressed in terms of a mixing non-standard electric force according to:

$$\overline{\sin\theta\sin\varphi C^{1,I}_{sca,E}} = \overline{\sin\theta\sin\varphi C^{1,INS}_{sca,E}} = 2\frac{\mathrm{Re}(a_1a_2^*)}{\mathrm{Re}(a_1)}\overline{\sin\theta\sin\varphi C^{1,INS}_{ext,E}} \tag{71}$$

Next, $\overline{\sin\theta\sin\varphi C^{1,R}_{sca,E}}$ may be decomposed into two subforces denoted as $\overline{\sin\theta\sin\varphi C^{1,R\alpha}_{sca,E}}$ and $\overline{\sin\theta\sin\varphi C^{1,R\beta}_{sca,E}}$. We afterward show that $\overline{\sin\theta\sin\varphi C^{1,R\alpha}_{sca,E}}$ is a recoil gradient electric force which may be expressed versus a mixing gradient electric force, according to:

$$\begin{aligned}\overline{\sin\theta\sin\varphi C^{1,R\alpha}_{sca,E}} &= \overline{\sin\theta\sin\varphi C^{1,RG}_{sca,E}} \\ &= 2\frac{\mathrm{Im}(a_1a_2^*)}{\mathrm{Im}(a_1)}\overline{\sin\theta\sin\varphi C^{1,R}_{ext,E}} \\ &= \frac{3\lambda^2}{4k\eta\pi}\mathrm{Im}(a_1a_2^*)[\partial_y|\mathbf{E}|^2]_P\end{aligned} \tag{72}$$

while $\overline{\sin\theta\sin\varphi C^{1,R\beta}_{sca,E}}$ is a recoil non-standard electric force which may be expressed versus a recoil non-standard magnetoelectric force, according to:

$$\overline{\sin\theta\sin\varphi C^{1,R\beta}_{sca,E}} = \overline{\sin\theta\sin\varphi C^{1,RNS}_{sca,E}} = \frac{-\mathrm{Im}(a_1a_2^*)}{\mathrm{Im}(a_1b_1^*)}\overline{\sin\theta\sin\varphi C^{1,RNS}_{sca,EH}} \tag{73}$$

Similarly, $\overline{\sin\theta\sin\varphi C^{1}_{sca,H}}$ may be decomposed into two subforces denoted as $\overline{\sin\theta\sin\varphi C^{1,R}_{sca,H}}$ and $\overline{\sin\theta\sin\varphi C^{1,I}_{sca,H}}$. We then establish, using Eq. (77) of [32], that $\overline{\sin\theta\sin\varphi C^{1,I}_{sca,H}}$, to be renamed $\overline{\sin\theta\sin\varphi C^{1,INS}_{sca,H}}$, is a recoil non-standard magnetic force which may be expressed versus a mixing non-standard magnetic force according to:

$$\overline{\sin\theta\sin\varphi C^{1,I}_{sca,H}} = \overline{\sin\theta\sin\varphi C^{1,INS}_{sca,H}} = 2\frac{\mathrm{Re}(b_1b_2^*)}{\mathrm{Re}(b_1)}\overline{\sin\theta\sin\varphi C^{1,INS}_{ext,H}} \tag{74}$$

and we establish, using Eq. (40) of [32] and Eq. (26), that $\overline{\sin\theta\sin\varphi C^{1,R}_{sca,H}}$ may be decomposed into two subforces denoted as $\overline{\sin\theta\sin\varphi C^{1,R\alpha}_{sca,H}}$ and $\overline{\sin\theta\sin\varphi C^{1,R\beta}_{sca,H}}$. From Eqs. (49) and (62) of [32], we then have that $\overline{\sin\theta\sin\varphi C^{1,R\alpha}_{sca,H}}$ is a recoil gradient magnetic force which may be expressed versus a mixing gradient magnetic force according to:

$$\begin{aligned}\overline{\sin\theta\sin\varphi C^{1,R\alpha}_{sca,H}} &= \overline{\sin\theta\sin\varphi C^{1,RG}_{sca,H}} = 2\frac{\mathrm{Im}(b_1b_2^*)}{\mathrm{Im}(b_1)}\overline{\sin\theta\sin\varphi C^{1,RG}_{ext,H}} \\ &= \frac{3\lambda^2\eta}{4k\pi}\mathrm{Im}(b_1b_2^*)[\partial_y|\mathbf{H}|^2]_P\end{aligned} \tag{75}$$

while from Eq. (86) of [32] and Eq. (24), the recoil magnetic force $\overline{\sin\theta\sin\varphi C^{1,R\beta}_{sca,H}}$ is a non-standard force which may be expressed versus a recoil non-standard magnetoelectric force according to:

$$\overline{\sin\theta\sin\varphi C^{1,R\beta}_{sca,H}} = \overline{\sin\theta\sin\varphi C^{1,RNS}_{sca,H}} = \frac{\mathrm{Im}(b_1b_2^*)}{\mathrm{Im}(a_1b_1^*)}\overline{\sin\theta\sin\varphi C^{1,RNS}_{sca,EH}} \tag{76}$$

5. Expressions versus BSCs of the 2-forces

All expressions for the 2-forces are derived from the general expressions discussed in Section 2.1, similarly as for 1-forces.

5.1. Longitudinal 2-forces

From Eq. (3.158) of [2], the longitudinal mixing 2-force is found to read as:

$$\overline{\cos\theta C^{2}_{ext}} = \frac{\lambda^2}{\pi}\mathrm{Re}(a_2Z^2_E + b_2Z^2_H) \tag{77}$$

in which:

$$\begin{aligned}Z^2_E = &\frac{1}{2}g^{0*}_{1,TM}g^{0}_{2,TM} + \frac{1}{3}g^{0}_{2,TM}g^{0*}_{3,TM} + \frac{3}{2}(g^{-1*}_{1,TM}g^{-1}_{2,TM} + g^{1*}_{1,TM}g^{1}_{2,TM}) \\ &+\frac{8}{3}(g^{-1}_{2,TM}g^{-1*}_{3,TM} + g^{1}_{2,TM}g^{1*}_{3,TM}) + \frac{40}{3}(g^{-2}_{2,TM}g^{-2*}_{3,TM} + g^{2}_{2,TM}g^{2*}_{3,TM}) \\ &+i[\frac{5}{6}(g^{-1}_{2,TM}g^{-1*}_{2,TE} - g^{1}_{2,TM}g^{1*}_{2,TE}) + \frac{20}{3}(g^{-2}_{2,TM}g^{-2*}_{2,TE} - g^{2}_{2,TM}g^{2*}_{2,TE})]\end{aligned} \tag{78}$$

$$\begin{aligned}Z^2_H = &\frac{1}{2}g^{0*}_{1,TE}g^{0}_{2,TE} + \frac{1}{3}g^{0}_{2,TE}g^{0*}_{3,TE} + \frac{3}{2}(g^{-1*}_{1,TE}g^{-1}_{2,TE} + g^{1*}_{1,TE}g^{1}_{2,TE}) \\ &+\frac{8}{3}(g^{-1}_{2,TE}g^{-1*}_{3,TE} + g^{1}_{2,TE}g^{1*}_{3,TE}) + \frac{40}{3}(g^{-2}_{2,TE}g^{-2*}_{3,TE} + g^{2}_{2,TE}g^{2*}_{3,TE}) \\ &+i[\frac{5}{6}(g^{1*}_{2,TM}g^{1}_{2,TE} - g^{-1*}_{2,TM}g^{-1}_{2,TE}) + \frac{20}{3}(g^{2*}_{2,TM}g^{2}_{2,TE} - g^{-2*}_{2,TM}g^{-2}_{2,TE})]\end{aligned} \tag{79}$$

Eq. (77) may be compared with Eq. (12), the difference by a factor 3/2 being due to a difference in prefactors involved in the definitions of the Z^M_N-terms. Let us note that Z^2_H and Z^2_E are related by an interchange of the subscripts TM and TE, and by another one concerning the sign of the "i-term". Such subscript symmetries between electric and magnetic forces may be recurrently observed, although we shall not mention it any more. Let us note as well that the expression for Z^2_E (and for Z^2_H, and for many other expressions) can be written in a compact form as Gouesbet et al. [38]:

$$\begin{aligned}[Z^N_E]_{N=2} = \Bigg[&\frac{1}{N^2}\sum_{p=-N+1}^{N-1}\frac{(N+|p|)!}{(N-1-|p|)!}g^{p*}_{N-1,TM}g^{p}_{N,TM} \\ &+\frac{1}{(N+1)^2}\sum_{p=-N}^{+N}\frac{(N+1+|p|)!}{(N-|p|)!}g^{p}_{N,TM}g^{p*}_{N+1,TM} \\ &-\frac{(2N+1)i}{N^2(N+1)^2}\sum_{p=-N}^{+N}p\frac{(N+|p|)!}{(N-|p|)!}g^{p}_{N,TM}g^{p*}_{N,TE}\Bigg]_{N=2}\end{aligned} \tag{80}$$

When dealing with arbitrary sized particles, compact forms such as in Eq. (80) are compulsory. In the present paper, expanded forms such as in Eq. (78) are preferred because they better exhibit the internal structure of the equations. It must furthermore be remarked that the general form of Eq. (80) is not valid for $N = 1$ because it would generate BSCs $g^0_{0,TM}$ which do not exist. This is a supplementary reason to distinguish 1-forces and N-forces ($N > 1$) as done in the present paper with the consequence that the forthcoming paper for arbitrary sized particles will focus only on $N > 1$. Let us note that indeed we could force Eq. (80) to remain valid whatever N by multiplying it by a prefactor $(1 - \delta_{N-1,0})$ but such a procedure would nevertheless insist on the peculiar character of the case $N = 1$.

From Eq.(3.155) of [2], the longitudinal recoil 2-force is found to read as:

$$\overline{\cos\theta C^{2}_{sca}} = \frac{-2\lambda^2}{\pi}\mathrm{Re}(ia_2b_2^*Z^{22}_{EH} + a_2a_3^*Z^{23}_E + b_2b_3^*Z^{23}_H) \tag{81}$$

in which:

$$\begin{aligned}Z^{22}_{EH} = &\frac{5}{6}(g^{1}_{2,TM}g^{1*}_{2,TE} - g^{-1}_{2,TM}g^{-1*}_{2,TE}) \\ &+\frac{20}{3}(g^{2}_{2,TM}g^{2*}_{2,TE} - g^{-2}_{2,TM}g^{-2*}_{2,TE})\end{aligned} \tag{82}$$

$$Z^{23}_E = -\frac{1}{3}g^{0}_{2,TM}g^{0*}_{3,TM} \tag{83}$$

$$-\frac{8}{3}(g^{1}_{2,TM}g^{1*}_{3,TM}+g^{-1}_{2,TM}g^{-1*}_{3,TM})$$
$$-\frac{40}{3}(g^{2}_{2,TM}g^{2*}_{3,TM}+g^{-2}_{2,TM}g^{-2*}_{3,TM})$$

$$Z^{23}_{H}=-\frac{1}{3}g^{0}_{2,TE}g^{0*}_{3,TE}-\frac{8}{3}(g^{1}_{2,TE}g^{1*}_{3,TE}+g^{-1}_{2,TE}g^{-1*}_{3,TE})-\frac{40}{3}(g^{2}_{2,TE}g^{2*}_{3,TE}+g^{-2}_{2,TE}g^{-2*}_{3,TE}) \quad (84)$$

5.2. Transverse x-component of 2-forces

From Eq.(3.180) of [2], with the subscript of the rightmost summation changed from $m=p-1$ to $m=p-1\neq 0$, the transverse x-component of the mixing 2-force is found to read as:

$$\overline{\sin\theta\cos\varphi C^{2}_{ext}}=\frac{\lambda^2}{2\pi}\mathrm{Re}(a_2X^2_E+b_2X^2_H) \quad (85)$$

in which it is convenient to express X^2_E and X^2_H as a summation of three terms according to:

$$X^2_E=X^{\alpha 2}_E+X^{\beta 2}_E+iX^{\gamma 2}_E \quad (86)$$

$$X^2_H=X^{\alpha 2}_H+X^{\beta 2}_H+iX^{\gamma 2}_H \quad (87)$$

with:

$$X^{\alpha 2}_E=\frac{1}{2}g^{0}_{2,TM}(g^{1*}_{1,TM}+g^{-1*}_{1,TM})+\frac{2}{3}g^{0*}_{3,TM}(g^{1}_{2,TM}+g^{-1}_{2,TM})+\frac{8}{3}(g^{1*}_{3,TM}g^{2}_{2,TM}+g^{-1*}_{3,TM}g^{-2}_{2,TM}) \quad (88)$$

$$X^{\alpha 2}_H=\frac{1}{2}g^{0}_{2,TE}(g^{1*}_{1,TE}+g^{-1*}_{1,TE})+\frac{2}{3}g^{0*}_{3,TE}(g^{1}_{2,TE}+g^{-1}_{2,TE})+\frac{8}{3}(g^{1*}_{3,TE}g^{2}_{2,TE}+g^{-1*}_{3,TE}g^{-2}_{2,TE}) \quad (89)$$

$$X^{\beta 2}_E=\frac{-3}{2}g^{0*}_{1,TM}(g^{-1}_{2,TM}+g^{1}_{2,TM})-\frac{4}{3}g^{0}_{2,TM}(g^{-1*}_{3,TM}+g^{1*}_{3,TM})-6(g^{1*}_{1,TM}g^{2}_{2,TM}+g^{-1*}_{1,TM}g^{-2}_{2,TM})-\frac{40}{3}(g^{1}_{2,TM}g^{2*}_{3,TM}+g^{-1}_{2,TM}g^{-2*}_{3,TM})-80(g^{2}_{2,TM}g^{3*}_{3,TM}+g^{-3*}_{3,TM}g^{-2}_{2,TM}) \quad (90)$$

$$X^{\beta 2}_H=\frac{-3}{2}g^{0*}_{1,TE}(g^{-1}_{2,TE}+g^{1}_{2,TE})-\frac{4}{3}g^{0}_{2,TE}(g^{-1*}_{3,TE}+g^{1*}_{3,TE})-6(g^{1*}_{1,TE}g^{2}_{2,TE}+g^{-1*}_{1,TE}g^{-2}_{2,TE})-\frac{40}{3}(g^{1}_{2,TE}g^{2*}_{3,TE}+g^{-1}_{2,TE}g^{-2*}_{3,TE})-80(g^{2}_{2,TE}g^{3*}_{3,TE}+g^{-3*}_{3,TE}g^{-2}_{2,TE}) \quad (91)$$

$$X^{\gamma 2}_E=\frac{5}{6}g^{0}_{2,TM}(g^{-1*}_{2,TE}-g^{1*}_{2,TE})+\frac{5}{6}g^{0*}_{2,TE}(g^{-1}_{2,TM}-g^{1}_{2,TM})+\frac{10}{3}(g^{-2}_{2,TM}g^{-1*}_{2,TE}+g^{-2*}_{2,TE}g^{-1}_{2,TM}-g^{1}_{2,TM}g^{2*}_{2,TE}-g^{1*}_{2,TE}g^{2}_{2,TM}) \quad (92)$$

$$X^{\gamma 2}_H=\frac{5}{6}g^{0*}_{2,TM}(g^{1}_{2,TE}-g^{-1}_{2,TE})+\frac{5}{6}g^{0}_{2,TE}(g^{1*}_{2,TM}-g^{-1*}_{2,TM})+\frac{10}{3}(g^{1*}_{2,TM}g^{2}_{2,TE}+g^{1}_{2,TE}g^{2*}_{2,TM}-g^{-2*}_{2,TM}g^{-1}_{2,TE}-g^{-2}_{2,TE}g^{-1*}_{2,TM}) \quad (93)$$

while, from Eq.(3.174) of [2], the transverse x -component of the recoil 2-force is found to read as:

$$\overline{\sin\theta\cos\varphi C^{2}_{sca}}=\frac{\lambda^2}{\pi}\mathrm{Re}(ia_2b^*_2X^{22}_{EH}+a_2a^*_3X^{23}_E+b_2b^*_3X^{23}_H) \quad (94)$$

in which:

$$X^{22}_{EH}=\frac{5}{6}[g^{0*}_{2,TE}(g^{-1}_{2,TM}-g^{1}_{2,TM})+g^{0}_{2,TM}(g^{-1*}_{2,TE}-g^{1*}_{2,TE})]+\frac{10}{3}(g^{-2*}_{2,TE}g^{-1}_{2,TM}-g^{2*}_{2,TE}g^{1}_{2,TM}+g^{-2}_{2,TM}g^{-1*}_{2,TE}-g^{2}_{2,TM}g^{1*}_{2,TE}) \quad (95)$$

$$X^{23}_E=\frac{2}{3}g^{0*}_{3,TM}(g^{1}_{2,TM}+g^{-1}_{2,TM})-\frac{4}{3}g^{0}_{2,TM}(g^{1*}_{3,TM}+g^{-1*}_{3,TM})+\frac{8}{3}(g^{2}_{2,TM}g^{1*}_{3,TM}+g^{-2}_{2,TM}g^{-1*}_{3,TM})-\frac{40}{3}(g^{1}_{2,TM}g^{2*}_{3,TM}+g^{-1}_{2,TM}g^{-2*}_{3,TM})-80(g^{2}_{2,TM}g^{3*}_{3,TM}+g^{-2}_{2,TM}g^{-3*}_{3,TM}) \quad (96)$$

$$X^{23}_H=\frac{2}{3}g^{0*}_{3,TE}(g^{1}_{2,TE}+g^{-1}_{2,TE})-\frac{4}{3}g^{0}_{2,TE}(g^{1*}_{3,TE}+g^{-1*}_{3,TE})+\frac{8}{3}(g^{2}_{2,TE}g^{1*}_{3,TE}+g^{-2}_{2,TE}g^{-1*}_{3,TE})-\frac{40}{3}(g^{1}_{2,TE}g^{2*}_{3,TE}+g^{-1}_{2,TE}g^{-2*}_{3,TE})-80(g^{2}_{2,TE}g^{3*}_{3,TE}+g^{-2}_{2,TE}g^{-3*}_{3,TE}) \quad (97)$$

5.3. Transverse y-component of 2-forces

Similarly, the transverse y-component of the mixing 2-force is found to read as:

$$\overline{\sin\theta\sin\varphi C^{2}_{ext}}=\frac{\lambda^2}{2\pi}\mathrm{Im}(a_2Y^2_E+b_2Y^2_H) \quad (98)$$

$$Y^2_E=Y^{\alpha 2}_E+Y^{\beta 2}_E+iY^{\gamma 2}_E \quad (99)$$

$$Y^2_H=Y^{\alpha 2}_H+Y^{\beta 2}_H+iY^{\gamma 2}_H \quad (100)$$

with:

$$Y^{\alpha 2}_E=\frac{1}{2}g^{0}_{2,TM}(g^{1*}_{1,TM}-g^{-1*}_{1,TM})+\frac{2}{3}g^{0*}_{3,TM}(g^{-1}_{2,TM}-g^{1}_{2,TM})+\frac{8}{3}(g^{-1*}_{3,TM}g^{-2}_{2,TM}-g^{1*}_{3,TM}g^{2}_{2,TM}) \quad (101)$$

$$Y^{\alpha 2}_H=\frac{1}{2}g^{0}_{2,TE}(g^{1*}_{1,TE}-g^{-1*}_{1,TE})+\frac{2}{3}g^{0*}_{3,TE}(g^{-1}_{2,TE}-g^{1}_{2,TE})+\frac{8}{3}(g^{-2}_{2,TE}g^{-1*}_{3,TE}-g^{1*}_{3,TE}g^{2}_{2,TE}) \quad (102)$$

$$Y^{\beta 2}_E=\frac{-3}{2}g^{0*}_{1,TM}(g^{-1}_{2,TM}-g^{1}_{2,TM})-\frac{4}{3}g^{0}_{2,TM}(g^{1*}_{3,TM}-g^{-1*}_{3,TM})-6(g^{-1*}_{1,TM}g^{-2}_{2,TM}-g^{1*}_{1,TM}g^{2}_{2,TM})-\frac{40}{3}(g^{1}_{2,TM}g^{2*}_{3,TM}-g^{-1}_{2,TM}g^{-2*}_{3,TM})-80(g^{2}_{2,TM}g^{3*}_{3,TM}-g^{-3*}_{3,TM}g^{-2}_{2,TM}) \quad (103)$$

$$Y^{\beta 2}_H=\frac{-3}{2}g^{0*}_{1,TE}(g^{-1}_{2,TE}-g^{1}_{2,TE})-\frac{4}{3}g^{0}_{2,TE}(g^{1*}_{3,TE}-g^{-1*}_{3,TE})-6(g^{-1*}_{1,TE}g^{-2}_{2,TE}-g^{1*}_{1,TE}g^{2}_{2,TE})-\frac{40}{3}(g^{1}_{2,TE}g^{2*}_{3,TE}-g^{-1}_{2,TE}g^{-2*}_{3,TE})-80(g^{2}_{2,TE}g^{3*}_{3,TE}-g^{-3*}_{3,TE}g^{-2}_{2,TE}) \quad (104)$$

$$Y^{\gamma 2}_E=\frac{5}{6}[g^{0*}_{2,TE}(g^{-1}_{2,TM}+g^{1}_{2,TM})-g^{0}_{2,TM}(g^{-1*}_{2,TE}+g^{1*}_{2,TE})]+\frac{10}{3}(g^{1*}_{2,TE}g^{2}_{2,TM}+g^{-2}_{2,TM}g^{-1*}_{2,TE}-g^{-2*}_{2,TE}g^{-1}_{2,TM}-g^{2*}_{2,TE}g^{1}_{2,TM}) \quad (105)$$

$$Y^{\gamma 2}_H=\frac{5}{6}[g^{0}_{2,TE}(g^{-1*}_{2,TM}+g^{1*}_{2,TM})-g^{0*}_{2,TM}(g^{-1}_{2,TE}+g^{1}_{2,TE})] \quad (106)$$

$$-\frac{10}{3}(g_{2,TM}^{1*}g_{2,TE}^{2}+g_{2,TE}^{-2}g_{2,TM}^{-1*}-g_{2,TM}^{-2*}g_{2,TE}^{-1}-g_{2,TM}^{2*}g_{2,TE}^{1})$$

while the transverse y-component of the recoil 2-force is found to read as:

$$\overline{\sin\theta\sin\varphi C_{sca}^{2}}=\frac{\lambda^2}{\pi}\mathrm{Im}(\mathrm{i}a_2b_2^*Y_{EH}^{22}+a_2a_3^*Y_E^{23}+b_2b_3^*Y_H^{23}) \quad (107)$$

in which:

$$Y_{EH}^{22}=\frac{5}{6}[g_{2,TE}^{0*}(g_{2,TM}^{-1}+g_{2,TM}^{1})-g_{2,TM}^{0}(g_{2,TE}^{-1*}+g_{2,TE}^{1*})] \quad (108)$$
$$+\frac{10}{3}(g_{2,TM}^{-2}g_{2,TE}^{-1*}+g_{2,TM}^{2}g_{2,TE}^{1*}-g_{2,TE}^{-2*}g_{2,TM}^{-1}-g_{2,TE}^{2*}g_{2,TM}^{1})$$

$$Y_E^{23}=\frac{2}{3}g_{3,TM}^{0*}(g_{2,TM}^{-1}-g_{2,TM}^{1})+\frac{4}{3}g_{2,TM}^{0}(g_{3,TM}^{-1*}-g_{3,TM}^{1*}) \quad (109)$$
$$+\frac{8}{3}(g_{2,TM}^{-2}g_{3,TM}^{-1*}-g_{2,TM}^{2}g_{3,TM}^{1*})+\frac{40}{3}(g_{2,TM}^{-1}g_{3,TM}^{-2*}-g_{2,TM}^{1}g_{3,TM}^{2*})$$
$$+80(g_{2,TM}^{-2}g_{3,TM}^{-3*}-g_{2,TM}^{2}g_{3,TM}^{3*})$$

$$Y_H^{23}=\frac{2}{3}g_{3,TE}^{0*}(g_{2,TE}^{-1}-g_{2,TE}^{1})+\frac{4}{3}g_{2,TE}^{0}(g_{3,TE}^{-1*}-g_{3,TE}^{1*}) \quad (110)$$
$$+\frac{8}{3}(g_{2,TE}^{-2}g_{3,TE}^{-1*}-g_{2,TE}^{2}g_{3,TE}^{1*})+\frac{40}{3}(g_{2,TE}^{-1}g_{3,TE}^{-2*}-g_{2,TE}^{1}g_{3,TE}^{2*})$$
$$+80(g_{2,TE}^{-2}g_{3,TE}^{-3*}-g_{2,TE}^{2}g_{3,TE}^{3*})$$

6. Interpretation of 2 forces

In the case of 1-forces, we distinguished between dipolar and non-dipolar forces, motivated by the fact that GLMT restricted to 1-forces exhibits more kinds of forces than the ones strictly required for its agreement with the dipole theory of forces, e.g. [35,36]. For the sake of coherency, we shall therefore distinguish between quadrupolar forces and non-quadrupolar forces. Then, similarly as for dipolar and non-dipolar forces, by definition, quadrupolar 2-forces are then associated with Mie coefficients a_2, b_2 and with products $a_2b_2^*$ of Mie coefficients, while the non-quadrupolar forces are associated with products $a_2a_3^*$ and $b_2b_3^*$.

6.1. Interpretation of quadrupolar 2-forces

6.1.1. z-component

Eq. (77) is the sum of electric and magnetic forces, which are separable into two subforces, so that we have to face four kinds of forces, reading as:

$$\overline{\cos\theta C_{ext,E}^{2,I}}=\frac{-\lambda^2}{\pi}\mathrm{Im}(a_2)\mathrm{Im}(Z_E^2) \quad (111)$$

$$\overline{\cos\theta C_{ext,E}^{2,R}}=\frac{\lambda^2}{\pi}\mathrm{Re}(a_2)\mathrm{Re}(Z_E^2) \quad (112)$$

to be complemented respectively by magnetic forces $\overline{\cos\theta C_{ext,H}^{2,I}}$ and $\overline{\cos\theta C_{ext,H}^{2,R}}$, with a_2 changed to b_2, and Z_E^2 to Z_H^2.

In the same way that $\overline{\cos\theta C_{ext,E}^{1,I}}$ and $\overline{\cos\theta C_{ext,H}^{1,I}}$ are gradient forces, we shall show in Section 6.1.4 that $\overline{\cos\theta C_{ext,E}^{2,I}}$ and $\overline{\cos\theta C_{ext,H}^{2,I}}$ are gradient forces as well. Let us next consider $\overline{\cos\theta C_{ext,E}^{2,R}}$ and recall that $\overline{\cos\theta C_{ext,E}^{1,R}}$ is the sum of a scattering force and of a non-standard force according to Eqs. (27) and (28). The term Z_E^1 of Eq. (13) may be decomposed as:

$$Z_E^1=Z_E^{1,S}+Z_E^{1,NS} \quad (113)$$

in which:

$$Z_E^{1,S}=i(g_{1,TM}^{-1}g_{1,TE}^{-1*}-g_{1,TM}^{1}g_{1,TE}^{1*}) \quad (114)$$

$$Z_E^{1,NS}=g_{1,TM}^{-1}g_{2,TM}^{-1*}+g_{1,TM}^{1}g_{2,TM}^{1*}+\frac{1}{3}g_{1,TM}^{0}g_{2,TM}^{0*} \quad (115)$$

which served as arguments in Eqs. (27) and (28) for $\overline{\cos\theta C_{ext,E}^{1,RS}}$ and $\overline{\cos\theta C_{ext,E}^{1,RNS}}$ respectively.

We then observe structural differences between $Z_E^{1,S}$ and $Z_E^{1,NS}$ as follows: $Z_E^{1,S}$ (i) exhibits a sum of products of BSCs multiplied by "i" (ii) exhibits superscripts -1, $+1$ for 1 -forces with (iii) all subscripts equal to $+1$ and (iv) couplings between TM- and TE-waves exhibited in the products of BSCs. By contrast, $Z_E^{1,NS}$ (i) does not exhibit explicitly any multiplication by "i", (ii) exhibits three superscripts -1, 0, $+1$ for 1-forces, (iii) with subscripts displaying a coupling between partial waves of orders 1 and 2 and (iv) without any coupling between partial waves of different modes.

Similarly, Z_E^2 may be decomposed into:

$$Z_E^2=Z_E^{2,S}+Z_E^{2,NS} \quad (116)$$

in which:

$$Z_E^{2,S}=i[\frac{5}{6}(g_{2,TM}^{-1}g_{2,TE}^{-1*}-g_{2,TM}^{1}g_{2,TE}^{1*})+\frac{20}{3}(g_{2,TM}^{-2}g_{2,TE}^{-2*}-g_{2,TM}^{2}g_{2,TE}^{2*})] \quad (117)$$

$$Z_E^{2,NS}=\frac{1}{2}g_{1,TM}^{0*}g_{2,TM}^{0}+\frac{1}{3}g_{2,TM}^{0}g_{3,TM}^{0*}+\frac{3}{2}(g_{1,TM}^{-1*}g_{2,TM}^{-1}+g_{1,TM}^{1*}g_{2,TM}^{1})$$
$$+\frac{8}{3}(g_{2,TM}^{-1}g_{3,TM}^{-1*}+g_{2,TM}^{1}g_{3,TM}^{1*})+\frac{40}{3}(g_{2,TM}^{-2}g_{3,TM}^{-2*}+g_{2,TM}^{2}g_{3,TM}^{2*}) \quad (118)$$

and the structural distinction between $Z_E^{1,S}$ and $Z_E^{1,NS}$ propagates to a similar structural distinction between $Z_E^{2,S}$ and $Z_E^{2,NS}$. Indeed, $Z_E^{2,S}$ (i) exhibits a sum of products of BSCs multiplied by "i" (ii) exhibits superscripts $-2, -1, +1, +2$ for 2 -forces with (iii) all subscripts equal to $+2$ and (iv) couplings between TM- and TE-waves exhibited in the products of BSCs. By contrast, $Z_E^{2,NS}$ (i) does not exhibit explicitly any multiplication by "i", (ii) exhibits five superscripts $-2, -1$, 0, $+1, +2$ for 2-forces, (iii) with subscripts displaying a coupling between partial waves of orders 1 and 2, and 2 and 3 and (iv) without any coupling between partial waves of different modes.

We then decompose $\overline{\cos\theta C_{ext,E}^{2,R}}$ into a sum of scattering and non-standard forces according to:

$$\overline{\cos\theta C_{ext,E}^{2,RS}}=\frac{\lambda^2}{\pi}\mathrm{Re}(a_2)\mathrm{Re}(Z_E^{2,S}) \quad (119)$$

$$\overline{\cos\theta C_{ext,E}^{2,RNS}}=\frac{\lambda^2}{\pi}\mathrm{Re}(a_2)\mathrm{Re}(Z_E^{2,NS}) \quad (120)$$

We recall that, for 1-forces, the distinction between the scattering force $\overline{\cos\theta C_{ext,E}^{1,RS}}$ and the non-standard force $\overline{\cos\theta C_{ext,E}^{1,RNS}}$, which are both non-gradient (non-conservative) forces, relies on the fact that the scattering force is proportional to the corresponding Poynting vector component while the non-standard force is a non-gradient (non-conservative) force which is not a scattering force. More specifically, for 1-force dipoles (which are located at the origin of coordinates), the Poynting vector component was to be taken as well at the location of the particle, i.e. of the dipole, a fact which is underlined by the use of the subscript P in Eqs. (9)–(11). A related fact is that both the involved component of the Poynting vector and the associated scattering force rely only on partial wave of order $n = 1$, e.g. Eq. (27).

Up to now however, it has not been possible for 2-forces to discriminate between scattering and non-standard forces using arguments related to the physical meaning of them, as it happened for

1-forces in which scattering forces were directly related by proportionality to the Poynting vector, in contrast with non-standard forces. Therefore, the discrimination between scattering forces and non-standard forces for the 2 -forces relies on (i) the fact that it already exists in the case of 1 -forces, where it is the consequence of precise and accurate definitions of scattering and non-standard forces, and (ii) on structural differences between scattering and non-standard forces in 2-forces which were already observed in the case of 1-forces and propagate to the case of 2-forces. The validity of the distinction between scattering and non-standard forces will be reinforced in Section 7 where we shall observe that this distinction indeed shows a deep coherency in the optical force partition. Another issue is that the non-standard 1-forces provide a contribution to spin-curl forces, e.g. [44–48] in the context of dipole theory of forces and [33–36], Gouesbet and Ambrosio [31], Gouesbet et al. [32], Ambrosio et al. [37], Gouesbet and Ambrosio [49] in a GLMT context. It is likely that the non-standard forces of N-forces ($N > 1$) might receive a more or less analogous physical interpretation, still to be revealed, an issue which is postponed to future works.

Similarly, we decompose $\overline{\cos\theta C^{2,R}_{ext,H}}$, see comment after Eq. (112), into a summation of scattering and non-standard forces according to:

$$\overline{\cos\theta C^{2,RS}_{ext,H}} = \frac{\lambda^2}{\pi}\mathrm{Re}(\mathrm{b}_2)\mathrm{Re}(\mathrm{Z}^{2,\mathrm{S}}_{\mathrm{H}}) \tag{121}$$

$$\overline{\cos\theta C^{2,RNS}_{ext,H}} = \frac{\lambda^2}{\pi}\mathrm{Re}(\mathrm{b}_2)\mathrm{Re}(\mathrm{Z}^{2,\mathrm{NS}}_{\mathrm{H}}) \tag{122}$$

according to:

$$Z^2_H = Z^{2,S}_H + Z^{2,NS}_H \tag{123}$$

in which:

$$Z^{2,S}_H = i[\frac{5}{6}(g^{1*}_{2,TM}g^1_{2,TE} - g^{-1*}_{2,TM}g^{-1}_{2,TE}) + \frac{20}{3}(g^{2*}_{2,TM}g^2_{2,TE} - g^{-2*}_{2,TM}g^{-2}_{2,TE})] \tag{124}$$

$$\begin{aligned} Z^{2,NS}_H = &\frac{1}{2}g^{0*}_{1,TE}g^0_{2,TE} + \frac{1}{3}g^0_{2,TE}g^{0*}_{3,TE} + \frac{3}{2}(g^{-1*}_{1,TE}g^{-1}_{2,TE} + g^{1*}_{1,TE}g^1_{2,TE}) \\ &+ \frac{8}{3}(g^{-1}_{2,TE}g^{-1*}_{3,TE} + g^1_{2,TE}g^{1*}_{3,TE}) + \frac{40}{3}(g^{-2}_{2,TE}g^{-2*}_{3,TE} + g^2_{2,TE}g^{2*}_{3,TE}) \end{aligned} \tag{125}$$

6.1.2. x-component

Comments done for the z-component could be transferred to the cases of x- and y-components, but will be omitted for the sake of conciseness. From Eq. (85), we then have to deal with four kinds of forces reading as:

$$\overline{\sin\theta\cos\varphi C^{2,I}_{ext,E}} = \frac{-\lambda^2}{2\pi}\mathrm{Im}(\mathrm{a}_2)\mathrm{Im}(\mathrm{X}^2_{\mathrm{E}}) \tag{126}$$

$$\overline{\sin\theta\cos\varphi C^{2,R}_{ext,E}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(\mathrm{a}_2)\mathrm{Re}(\mathrm{X}^2_{\mathrm{E}}) \tag{127}$$

and similarly for magnetic forces $\overline{\sin\theta\cos\varphi C^{2,I}_{ext,H}}$ and $\overline{\sin\theta\cos\varphi C^{2,R}_{ext,H}}$, with a_2 changed to b_2, and X^2_E to X^2_H. Similarly as for the z-component, and for similar reasons, $\overline{\sin\theta\cos\varphi C^{2,I}_{ext,E}}$ and $\overline{\sin\theta\cos\varphi C^{2,I}_{ext,H}}$ will be found to be gradient forces (see Section 6.1.4). Conversely, similarly as for the z-component, and for similar reasons, $\overline{\sin\theta\cos\varphi C^{2,R}_{ext,E}}$ and $\overline{\sin\theta\cos\varphi C^{2,R}_{ext,H}}$ may be decomposed into a sum of scattering and non-standard forces according to:

$$\overline{\sin\theta\cos\varphi C^{2,R}_{ext,E}} = \overline{\sin\theta\cos\varphi C^{2,RS}_{ext,E}} + \overline{\sin\theta\cos\varphi C^{2,RNS}_{ext,E}} \tag{128}$$

$$\overline{\sin\theta\cos\varphi C^{2,R}_{ext,H}} = \overline{\sin\theta\cos\varphi C^{2,RS}_{ext,H}} + \overline{\sin\theta\cos\varphi C^{2,RNS}_{ext,H}} \tag{129}$$

in which:

$$\overline{\sin\theta\cos\varphi C^{2,RS}_{ext,E}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(\mathrm{a}_2)\mathrm{Re}(\mathrm{iX}^{\gamma 2}_{\mathrm{E}}) \tag{130}$$

$$\overline{\sin\theta\cos\varphi C^{2,RNS}_{ext,E}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(\mathrm{a}_2)\mathrm{Re}(\mathrm{X}^{\alpha 2}_{\mathrm{E}} + \mathrm{X}^{\beta 2}_{\mathrm{E}}) \tag{131}$$

and similarly for the magnetic forces $\overline{\sin\theta\cos\varphi C^{2,RS}_{ext,H}}$ and $\overline{\sin\theta\cos\varphi C^{2,RNS}_{ext,H}}$, with a_2 changed to b_2, and X_E to X_H. The distinction between scattering and non-standard forces relies on an argumentation similar to the one used for the z-component. In particular, we let the reader appreciate the structural differences between scattering and non-standard forces.

6.1.3. y-component

Similarly, we use Eq. (98) to distinguish four kinds of forces according to:

$$\overline{\sin\theta\sin\varphi C^{2,I}_{ext,E}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(\mathrm{a}_2)\mathrm{Im}(\mathrm{Y}^2_{\mathrm{E}}) \tag{132}$$

$$\overline{\sin\theta\sin\varphi C^{2,R}_{ext,E}} = \frac{\lambda^2}{2\pi}\mathrm{Im}(\mathrm{a}_2)\mathrm{Re}(\mathrm{Y}^2_{\mathrm{E}}) \tag{133}$$

and similarly for magnetic forces $\overline{\sin\theta\sin\varphi C^{2,I}_{ext,H}}$ and $\overline{\sin\theta\sin\varphi C^{2,R}_{ext,H}}$, with a_2 changed to b_2, and Y^2_E to Y^2_H. The interpretation of these forces run quite similarly as for the previous components, particularly as for the x-component with, however, an interchange between the roles of the superscript I and R. Therefore, $\overline{\sin\theta\sin\varphi C^{2,R}_{ext,E}}$ and $\overline{\sin\theta\sin\varphi C^{2,R}_{ext,H}}$ must be gradient forces, see again Section 6.1.4, while $\overline{\sin\theta\sin\varphi C^{2,I}_{ext,E}}$ and $\overline{\sin\theta\sin\varphi C^{2,I}_{ext,H}}$ are the sum of scattering forces and of non-standard forces according to:

$$\overline{\sin\theta\sin\varphi C^{2,I}_{ext,E}} = \overline{\sin\theta\sin\varphi C^{2,IS}_{ext,E}} + \overline{\sin\theta\sin\varphi C^{2,INS}_{ext,E}} \tag{134}$$

$$\overline{\sin\theta\sin\varphi C^{2,I}_{ext,H}} = \overline{\sin\theta\sin\varphi C^{2,IS}_{ext,H}} + \overline{\sin\theta\sin\varphi C^{2,INS}_{ext,H}} \tag{135}$$

in which:

$$\overline{\sin\theta\sin\varphi C^{2,IS}_{ext,E}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(\mathrm{a}_2)\mathrm{Im}(\mathrm{iY}^{\gamma 2}_{\mathrm{E}}) \tag{136}$$

$$\overline{\sin\theta\sin\varphi C^{2,INS}_{ext,E}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(\mathrm{a}_2)\mathrm{Im}(\mathrm{Y}^{\alpha 2}_{\mathrm{E}} + \mathrm{Y}^{\beta 2}_{\mathrm{E}}) \tag{137}$$

and similarly for magnetic forces $\overline{\sin\theta\sin\varphi C^{2,IS}_{ext,H}}$ and $\overline{\sin\theta\sin\varphi C^{2,INS}_{ext,H}}$, with a_2 changed to b_2, and Y_E to Y_H. Once more, we let the reader appreciate the structural differences between scattering and non-standard forces.

6.1.4. Gradient mixing forces

From the three previous subsections above, we have to consider three electric forces, namely $\overline{\cos\theta C^{2,I}_{ext,E}}$, $\overline{\sin\theta\cos\varphi C^{2,I}_{ext,E}}$, $\overline{\sin\theta\sin\varphi C^{2,R}_{ext,E}}$, and three magnetic forces, namely $\overline{\cos\theta C^{2,I}_{ext,H}}$, $\overline{\sin\theta\cos\varphi C^{2,I}_{ext,H}}$, $\overline{\sin\theta\sin\varphi C^{2,R}_{ext,H}}$. It is the great merit of Zheng et al. [39] to have demonstrated that these forces are indeed mixing gradient forces. For this, they rely on an arXiv paper by Jiang et al.[50], see as well [51]. In these works, the optical forces are deduced by using the Maxwell stress tensor, leading to the first categorization in terms of mixing/recoil forces omitting however to mention that such a categorization was already available nearly four decades ago [7–9], Gouesbet et al. [1]. Other similar omissions seem to have been the motivation for a criticism by Nieto-Vesperinas [52]. Nevertheless, the expressions by Zheng et al., and by Jiang et al., of the gradient mixing forces obtained by coupling field multiple derivatives and an angular spectrum decomposition, represent a genuine advance in the field of optical forces. Furthermore, the agreement between their results and ours is a corroboration of the validity of our results concerning mixing gradient forces. They also obtained expressions for recoil gradient forces but failed to see that recoil gradient forces may be readily expressed in terms of mixing gradient forces with simple formulas. Such simple expressions are given in the present paper for 1-forces and 2-forces and will be generalized in Gouesbet et al. [38] to the case of N-forces, whatever N.

The mixing gradient electric forces $\overline{\cos\theta C^{2,I}_{ext,E}}$, $\overline{\sin\theta\cos\varphi C^{2,I}_{ext,E}}$ and $\overline{\sin\theta\sin\varphi C^{2,R}_{ext,E}}$ above may indeed be shown to be equivalent to the corresponding forces expressed in Eq. (24) of Zheng et al. [39]. However, rather than using the traditional BSCs of GLMT, Zheng et al. introduced so-called partial wave expansion coefficients (PWECs). The translation between PWECs and BSCs is provided in Appendix A. The translation between the electric forces of the present section and those of Eq. (24) in Zheng et al. [39] is provided in Appendix B.

6.1.5. Magnetoelectric quadrupolar 2-forces

These forces are associated with the product $a_2 b_2^*$ of Mie coefficients, and occur in the first forces of Eqs. (81), (94) and (107). Beginning with Eq. (81), we isolate its first force which may be decomposed into two subforces according to:

$$\overline{\cos\theta C^{2,22}_{sca}} = \frac{-2\lambda^2}{\pi}\mathrm{Re}(\mathrm{i}a_2 b_2^* Z^{22}_{\mathrm{EH}}) = \overline{\cos\theta C^{2,\mathrm{R}}_{\mathrm{sca,EH}}} + \overline{\cos\theta C^{2,\mathrm{I}}_{\mathrm{sca,EH}}} \tag{138}$$

in which:

$$\begin{aligned}\overline{\cos\theta C^{2,R}_{sca,EH}} &= \frac{-2\lambda^2}{\pi}\mathrm{Re}(a_2 b_2^*)\mathrm{Re}(\mathrm{i}Z^{22}_{\mathrm{EH}}) \\ &= \frac{-2\lambda^2}{\pi}\mathrm{Re}(a_2 b_2^*)\mathrm{Re}\Big\{\mathrm{i}[\frac{5}{6}(g^1_{2,\mathrm{TM}}g^{1*}_{2,\mathrm{TE}} - g^{-1}_{2,\mathrm{TM}}g^{-1*}_{2,\mathrm{TE}}) \\ &\quad + \frac{20}{3}(g^2_{2,TM}g^{2*}_{2,TE} - g^{-2}_{2,TM}g^{-2*}_{2,TE})]\Big\}\end{aligned} \tag{139}$$

$$\begin{aligned}\overline{\cos\theta C^{2,I}_{sca,EH}} &= \frac{2\lambda^2}{\pi}\mathrm{Im}(a_2 b_2^*)\mathrm{Im}(\mathrm{i}Z^{22}_{\mathrm{EH}}) \\ &= \frac{2\lambda^2}{\pi}\mathrm{Im}(a_2 b_2^*)\mathrm{Im}\Big\{\mathrm{i}[\frac{5}{6}(g^1_{2,\mathrm{TM}}g^{1*}_{2,\mathrm{TE}} - g^{-1}_{2,\mathrm{TM}}g^{-1*}_{2,\mathrm{TE}}) \\ &\quad + \frac{20}{3}(g^2_{2,TM}g^{2*}_{2,TE} - g^{-2}_{2,TM}g^{-2*}_{2,TE})]\Big\}\end{aligned} \tag{140}$$

Comparing Eqs. (139), (82) and (119), (117), (121), (124), we obtain:

$$\begin{aligned}\overline{\cos\theta C^{2,R}_{sca,EH}} &= \overline{\cos\theta C^{2,RS}_{sca,EH}} \\ &= 2\frac{\mathrm{Re}(a_2 b_2^*)}{\mathrm{Re}(a_2)}\overline{\cos\theta C^{2,RS}_{ext,E}} = 2\frac{\mathrm{Re}(a_2 b_2^*)}{\mathrm{Re}(b_2)}\overline{\cos\theta C^{2,RS}_{ext,H}}\end{aligned} \tag{141}$$

so that $\overline{\cos\theta C^{2,R}_{sca,EH}}$ is a recoil scattering magnetoelectric force which may be expressed in terms of mixing scattering pure electric and pure magnetic forces. Conversely, $\overline{\cos\theta C^{2,I}_{sca,EH}}$, renamed $\overline{\cos\theta C^{2,INS}_{sca,EH}}$, is interpreted as a recoil non-standard magnetoelectric force, the difference between scattering and non-standard forces being related, in this recoil case, to a change from Re to Im, i.e. involving the same partial waves in both cases (which is different from the discrimination between scattering and non-standard forces in the mixing case, where mixing non-standard forces involve a coupling between different kinds of partial waves).

We now consider the first force of Eq. (94) which may be decomposed into two subforces according to:

$$\begin{aligned}\overline{\sin\theta\cos\varphi C^{2,22}_{sca}} &= \frac{\lambda^2}{\pi}\mathrm{Re}(\mathrm{i}a_2 b_2^* X^{22}_{\mathrm{EH}}) \\ &= \overline{\sin\theta\cos\varphi C^{2,R}_{sca,EH}} + \overline{\sin\theta\cos\varphi C^{2,I}_{sca,EH}}\end{aligned} \tag{142}$$

in which:

$$\overline{\sin\theta\cos\varphi C^{2,R}_{sca,EH}} = \frac{\lambda^2}{\pi}\mathrm{Re}(a_2 b_2^*)\mathrm{Re}(\mathrm{i}X^{22}_{\mathrm{EH}}) \tag{143}$$

$$\overline{\sin\theta\cos\varphi C^{2,I}_{sca,EH}} = -\frac{\lambda^2}{\pi}\mathrm{Im}(a_2 b_2^*)\mathrm{Im}(\mathrm{i}X^{22}_{\mathrm{EH}}) \tag{144}$$

The force $\overline{\sin\theta\cos\varphi C^{2,R}_{sca,EH}}$ may be compared with $\overline{\sin\theta\cos\varphi C^{2,RS}_{ext,E}}$ of Eq. (130), taking account for the fact that $X^{\gamma 2}_{E}$ of Eq. (92) is found to be equal to X^{22}_{EH} of Eq. (95). They may be compared as well with $\overline{\sin\theta\cos\varphi C^{2,RS}_{ext,H}}$ discussed above, taking into account the fact that $X^{\gamma 2}_{H}$ of Eq. (93) is equal to $(-X^{\gamma 2*}_{E})$. We then obtain:

$$\begin{aligned}\overline{\sin\theta\cos\varphi C^{2,R}_{sca,EH}} &= \overline{\sin\theta\cos\varphi C^{2,RS}_{sca,EH}} \\ &= 2\frac{\mathrm{Re}(a_2 b_2^*)}{\mathrm{Re}(a_2)}\overline{\sin\theta\cos\varphi C^{2,RS}_{ext,E}} \\ &= 2\frac{\mathrm{Re}(a_2 b_2^*)}{\mathrm{Re}(b_2)}\overline{\sin\theta\cos\varphi C^{2,RS}_{ext,H}}\end{aligned} \tag{145}$$

showing that $\overline{\sin\theta\cos\varphi C^{2,R}_{sca,EH}}$ is a recoil scattering magnetoelectric force, renamed $\overline{\sin\theta\cos\varphi C^{2,RS}_{sca,EH}}$, while, similarly as previously, $\overline{\sin\theta\cos\varphi C^{2,I}_{sca,EH}}$, renamed $\overline{\sin\theta\cos\varphi C^{2,INS}_{sca,EH}}$, denotes a recoil non-standard magnetoelectric force.

We now consider the first force of Eq. (107) which may again be decomposed into two subforces according to:

$$\begin{aligned}\overline{\sin\theta\sin\varphi C^{2,22}_{sca}} &= \frac{\lambda^2}{\pi}\mathrm{Im}(\mathrm{i}a_2 b_2^* Y^{22}_{\mathrm{EH}}) \\ &= \overline{\sin\theta\sin\varphi C^{2,R}_{sca,EH}} + \overline{\sin\theta\sin\varphi C^{2,I}_{sca,EH}}\end{aligned} \tag{146}$$

in which:

$$\overline{\sin\theta\sin\varphi C^{2,R}_{sca,EH}} = \frac{\lambda^2}{\pi}\mathrm{Im}(a_2 b_2^*)\mathrm{Re}(\mathrm{i}Y^{22}_{\mathrm{EH}}) \tag{147}$$

$$\overline{\sin\theta\sin\varphi C^{2,I}_{sca,EH}} = \frac{\lambda^2}{\pi}\mathrm{Re}(a_2 b_2^*)\mathrm{Im}(\mathrm{i}Y^{22}_{\mathrm{EH}}) \tag{148}$$

Working similarly as for the previous component, we then readily establish that (with an interchange between the superscripts R and I to be noted):

$$\begin{aligned}\overline{\sin\theta\sin\varphi C^{2,I}_{sca,EH}} &= \overline{\sin\theta\sin\varphi C^{2,IS}_{sca,EH}} \\ &= 2\frac{\mathrm{Re}(a_2 b_2^*)}{\mathrm{Re}(a_2)}\overline{\sin\theta\sin\varphi C^{2,IS}_{ext,E}} \\ &= 2\frac{\mathrm{Re}(a_2 b_2^*)}{\mathrm{Re}(b_2)}\overline{\sin\theta\sin\varphi C^{2,IS}_{ext,H}}\end{aligned} \tag{149}$$

meaning that $\overline{\sin\theta\sin\varphi C^{2,I}_{sca,EH}}$ denotes a recoil scattering magnetoelectric force which may be expressed in terms of mixing scattering pure electric and magnetic forces while, conversely, $\overline{\sin\theta\sin\varphi C^{2,R}_{sca,EH}}$, renamed $\overline{\sin\theta\sin\varphi C^{2,RNS}_{sca,EH}}$, denotes a recoil non-standard magnetoelectric force.

6.2. Interpretation of non-quadrupolar 2-forces. General decomposition

These forces are associated with Mie coefficient products $a_2 a_3^*$ and $b_2 b_3^*$ of Eqs. (81), (94) and (107), (in the same way that non-dipolar 1-forces were associated to the products $a_1 a_2^*$ and $b_1 b_2^*$).

After a now classical decomposition, the corresponding forces generate six kinds of forces reading as:

$$\overline{\cos\theta C^{2}_{sca,E}} = \frac{-2\lambda^2}{\pi}\mathrm{Re}(a_2 a_3^* Z_E^{23}) \tag{150}$$

$$\overline{\cos\theta C^{2}_{sca,H}} = \frac{-2\lambda^2}{\pi}\mathrm{Re}(b_2 b_3^* Z_H^{23}) \tag{151}$$

$$\overline{\sin\theta\cos\varphi C^{2}_{sca,E}} = \frac{\lambda^2}{\pi}\mathrm{Re}(a_2 a_3^* X_E^{23}) \tag{152}$$

$$\overline{\sin\theta\cos\varphi C^{2}_{sca,H}} = \frac{\lambda^2}{\pi}\mathrm{Re}(b_2 b_3^* X_H^{23}) \tag{153}$$

$$\overline{\sin\theta\sin\varphi C^{2}_{sca,E}} = \frac{\lambda^2}{\pi}\mathrm{Im}(a_2 a_3^* Y_E^{23}) \tag{154}$$

$$\overline{\sin\theta\sin\varphi C^{2}_{sca,H}} = \frac{\lambda^2}{\pi}\mathrm{Im}(b_2 b_3^* Y_H^{23}) \tag{155}$$

Each of these forces may be further decomposed in two subforces according to:

$$\overline{\cos\theta C^{2}_{sca}} = \overline{\cos\theta C^{2,R}_{sca,E}} + \overline{\cos\theta C^{2,I}_{sca,E}} \tag{156}$$

in which:

$$\overline{\cos\theta C^{2,R}_{sca,E}} = \frac{-2\lambda^2}{\pi}\mathrm{Re}(a_2 a_3^*)\mathrm{Re}(Z_E^{23}) \tag{157}$$

$$\overline{\cos\theta C^{2,I}_{sca,E}} = \frac{2\lambda^2}{\pi}\mathrm{Im}(a_2 a_3^*)\mathrm{Im}(Z_E^{23}) \tag{158}$$

and:

$$\overline{\cos\theta C^{2}_{sca,H}} = \overline{\cos\theta C^{2,R}_{sca,H}} + \overline{\cos\theta C^{2,I}_{sca,H}} \tag{159}$$

in which:

$$\overline{\cos\theta C^{2,R}_{sca,H}} = \frac{-2\lambda^2}{\pi}\mathrm{Re}(b_2 b_3^*)\mathrm{Re}(Z_H^{23}) \tag{160}$$

$$\overline{\cos\theta C^{2,I}_{sca,H}} = \frac{2\lambda^2}{\pi}\mathrm{Im}(b_2 b_3^*)\mathrm{Im}(Z_H^{23}) \tag{161}$$

and:

$$\overline{\sin\theta\cos\varphi C^{2}_{sca,E}} = \overline{\sin\theta\cos\varphi C^{2,R}_{sca,E}} + \overline{\sin\theta\cos\varphi C^{2,I}_{sca,E}} \tag{162}$$

in which:

$$\overline{\sin\theta\cos\varphi C^{2,R}_{sca,E}} = \frac{\lambda^2}{\pi}\mathrm{Re}(a_2 a_3^*)\mathrm{Re}(X_E^{23}) \tag{163}$$

$$\overline{\sin\theta\cos\varphi C^{2,I}_{sca,E}} = \frac{-\lambda^2}{\pi}\mathrm{Im}(a_2 a_3^*)\mathrm{Im}(X_E^{23}) \tag{164}$$

and:

$$\overline{\sin\theta\cos\varphi C^{2}_{sca}} = \overline{\sin\theta\cos\varphi C^{2,R}_{sca,H}} + \overline{\sin\theta\cos\varphi C^{2,I}_{sca,H}} \tag{165}$$

in which:

$$\overline{\sin\theta\cos\varphi C^{2,R}_{sca,H}} = \frac{\lambda^2}{\pi}\mathrm{Re}(b_2 b_3^*)\mathrm{Re}(X_H^{23}) \tag{166}$$

$$\overline{\sin\theta\cos\varphi C^{2,I}_{sca,H}} = \frac{-\lambda^2}{\pi}\mathrm{Im}(b_2 b_3^*)\mathrm{Im}(X_H^{23}) \tag{167}$$

and:

$$\overline{\sin\theta\sin\varphi C^{2}_{sca,E}} = \overline{\sin\theta\sin\varphi C^{2,R}_{sca,E}} + \overline{\sin\theta\sin\varphi C^{2,I}_{sca,E}} \tag{168}$$

in which:

$$\overline{\sin\theta\sin\varphi C^{2,R}_{sca,E}} = \frac{\lambda^2}{\pi}\mathrm{Im}(a_2 a_3^*)\mathrm{Re}(Y_E^{23}) \tag{169}$$

$$\overline{\sin\theta\sin\varphi C^{2,I}_{sca,E}} = \frac{\lambda^2}{\pi}\mathrm{Re}(a_2 a_3^*)\mathrm{Im}(Y_E^{23}) \tag{170}$$

and:

$$\overline{\sin\theta\sin\varphi C^{2}_{sca,H}} = \overline{\sin\theta\sin\varphi C^{2,R}_{sca,H}} + \overline{\sin\theta\sin\varphi C^{2,I}_{sca,H}} \tag{171}$$

in which:

$$\overline{\sin\theta\sin\varphi C^{2,R}_{sca,H}} = \frac{\lambda^2}{\pi}\mathrm{Im}(b_2 b_3^*)\mathrm{Re}(Y_H^{23}) \tag{172}$$

$$\overline{\sin\theta\sin\varphi C^{2,I}_{sca,H}} = \frac{\lambda^2}{\pi}\mathrm{Re}(b_2 b_3^*)\mathrm{Im}(Y_H^{23}) \tag{173}$$

We shall now demonstrate that, among the twelve kinds of forces so exhibited, six of them generate recoil gradient forces (although supplemented by recoil non-standard forces), namely $\overline{\cos\theta C^{2,I}_{sca,E}}$, $\overline{\cos\theta C^{2,I}_{sca,H}}$, $\overline{\sin\theta\cos\varphi C^{2,I}_{sca,E}}$, $\overline{\sin\theta\cos\varphi C^{2,I}_{sca,H}}$, $\overline{\sin\theta\sin\varphi C^{2,R}_{sca,E}}$, and $\overline{\sin\theta\sin\varphi C^{2,R}_{sca,H}}$, while six of them are pure recoil non-standard forces, namely $\overline{\cos\theta C^{2,R}_{sca,E}}$, $\overline{\cos\theta C^{2,R}_{sca,H}}$, $\overline{\sin\theta\cos\varphi C^{2,R}_{sca,E}}$, $\overline{\sin\theta\cos\varphi C^{2,R}_{sca,H}}$, $\overline{\sin\theta\sin\varphi C^{2,I}_{sca,E}}$ and $\overline{\sin\theta\sin\varphi C^{2,I}_{sca,H}}$.

6.3. Interpretation of non-quadrupolar 2-forces. Recoil gradient forces

Let us begin with Eq. (158) for $\overline{\cos\theta C^{2,I}_{sca,E}}$. Let us introduce:

$$\Delta Z_E^{23} = \frac{5i}{6}(g_{2,TM}^{-1}g_{2,TE}^{-1*} - g_{2,TM}^{1}g_{2,TE}^{1*}) + \frac{20i}{3}(g_{2,TM}^{-2}g_{2,TE}^{-2*} - g_{2,TM}^{2}g_{2,TE}^{2*}) \tag{174}$$

and:

$$\Delta Z_E^{12} = i(g_{1,TM}^{-1}g_{1,TE}^{-1*} - g_{1,TM}^{1}g_{1,TE}^{1*}) \tag{175}$$

Then using the expression of Eq. (78) for Z_E^2, of Eq. (17) for Z_E^{12}, and of Eq. (174) above, we may rewrite Z_E^{23} as:

$$Z_E^{23} = \Delta Z_E^{23} - Z_E^2 + \frac{3}{2}Z_E^{12*} \tag{176}$$

Also, from Eqs. (175) and (13), we have:

$$Z_E^{12} = Z_E^1 - \Delta Z_E^{12} \tag{177}$$

Inserting Eq. (176) into Eq. (158), we obtain:

$$\overline{\cos\theta C^{2,I}_{sca,E}} = \frac{2\lambda^2}{\pi}\mathrm{Im}(a_2 a_3^*)\mathrm{Im}(\Delta Z_E^{23}) - \frac{2\lambda^2}{\pi}\mathrm{Im}(a_2 a_3^*)\mathrm{Im}(Z_E^2) - \frac{3\lambda^2}{\pi}\mathrm{Im}(a_2 a_3^*)\mathrm{Im}(Z_E^{12}) \tag{178}$$

Now, $\mathrm{Im}(\Delta Z_E^{23})$ may be expressed using Eq. (174), $\mathrm{Im}(Z_E^2)$ may be expressed using Eq. (111), $\mathrm{Im}(Z_E^{12})$ may be expressed in terms of $\mathrm{Im}(Z_E^1)$ and $\mathrm{Im}(\Delta Z_E^{12})$ using Eq. (177), $\mathrm{Im}(Z_E^1)$ may be expressed using Eq. (18) of [32], and $\mathrm{Im}(\Delta Z_E^{12})$ of Eq. (175) using Eq. (45). It is then found that $\overline{\cos\theta C^{2,I}_{sca,E}}$ may be written as the summation of two subforces denoted as $\overline{\cos\theta C^{2,I\alpha}_{sca,E}}$ and $\overline{\cos\theta C^{2,I\beta}_{sca,E}}$ reading as, after rearranging:

$$\overline{\cos\theta C^{2,I\alpha}_{sca,E}} = 2\mathrm{Im}(a_2 a_3^*)[\frac{\overline{\cos\theta C^{2,I}_{ext,E}}}{\mathrm{Im}(a_2)} + \frac{\overline{\cos\theta C^{1,I}_{ext,E}}}{\mathrm{Im}(a_1)}] \tag{179}$$

$$\overline{\cos\theta C^{2,I\beta}_{sca,E}} = \frac{\lambda^2}{\pi}\mathrm{Im}(a_2a_3^*)\mathrm{Im}\Big\{i[3(g^{-1}_{1,TM}g^{-1*}_{1,TE} - g^{1}_{1,TM}g^{1*}_{1,TE}) + \frac{\lambda^2}{\pi}\mathrm{Im}(a_2a_3^*)\mathrm{Im}[\frac{5i}{3}(g^{-1}_{2,TM}g^{-1*}_{2,TE} - g^{1}_{2,TM}g^{1*}_{2,TE}) + \frac{40i}{3}(g^{-2}_{2,TM}g^{-2*}_{2,TE} - g^{2}_{2,TM}g^{2*}_{2,TE})]\Big\} \tag{180}$$

We have demonstrated that $\overline{\cos\theta C^{1,I}_{ext,E}}$ and $\overline{\cos\theta C^{2,I}_{ext,E}}$ are mixing gradient electric forces. Therefore, $\overline{\cos\theta C^{2,I\alpha}_{sca,E}} = \overline{\cos\theta C^{2,IG}_{sca,E}}$ is a recoil gradient electric force which may be expressed in terms of mixing gradient electric forces. Next, using Eqs. (16) for Z^{11}_{EH} and (174) for ΔZ^{23}_E, Eq. (180) may be rewritten as:

$$\overline{\cos\theta C^{2,I\beta}_{sca,E}} = \frac{\lambda^2}{\pi}\mathrm{Im}(a_2a_3^*)\mathrm{Im}(3iZ^{11}_{EH} + 2\Delta Z^{23}_{E}) \tag{181}$$

which, by virtue of Eq. (80) of [32] in which $\mathrm{Im}(iZ^{11}_{EH})$ is expressed in terms of a recoil non-standard magnetoelectric force, $\overline{\cos\theta C^{1,INS}_{sca,EH}}$, is classified as being a recoil non-standard force as well (it may be renamed $\overline{\cos\theta C^{2,INS}_{sca,E}}$), although of the electric kind rather than of a magnetoelectric kind. We now recall that, for $N = 1$, it has been found that $\overline{\cos\theta C^{1,INS}_{sca,E}}$ is proportional to $\overline{\cos\theta C^{1,INS}_{sca,EH}}$, see Eq. (47). Comparing Eqs. (181) for $\overline{\cos\theta C^{2,INS}_{sca,E}}$ and (140) for $\overline{\cos\theta C^{2,INS}_{sca,EH}}$, it is found that this proportionality property does not propagate from $N = 1$ to $N = 2$.

For $\overline{\cos\theta C^{2,I}_{sca,H}}$ of Eq. (161), we introduce:

$$\Delta Z^{23}_H = \frac{5i}{6}(g^{1}_{2,TM}g^{1*}_{2,TE} - g^{-1}_{2,TM}g^{-1*}_{2,TE}) + \frac{20i}{3}(g^{2}_{2,TM}g^{2*}_{2,TE} - g^{-2}_{2,TM}g^{-2*}_{2,TE}) \tag{182}$$

Then, using Eqs. (79) for Z^2_H, (18) for Z^{12}_H and (182) above, we find that Z^{23}_H of Eq. (84) satisfies:

$$Z^{23}_H = -\Delta Z^{23*}_H - Z^2_H + \frac{3}{2}Z^{12*}_H \tag{183}$$

Inserting Eq. (183) into Eq. (161), we obtain:

$$\overline{\cos\theta C^{2,I}_{sca,H}} = \frac{2\lambda^2}{\pi}\mathrm{Im}(b_2b_3^*)\mathrm{Im}(\Delta Z^{23}_H) - \frac{2\lambda^2}{\pi}\mathrm{Im}(b_2b_3^*)\mathrm{Im}(Z^2_H) - \frac{3\lambda^2}{\pi}\mathrm{Im}(b_2b_3^*)\mathrm{Im}(Z^{12}_H) \tag{184}$$

which, similarly as for the corresponding electric force, may be decomposed into two subforces denoted $\overline{\cos\theta C^{2,I\alpha}_{sca,H}}$ and $\overline{\cos\theta C^{2,I\beta}_{sca,H}}$ which, after rearranging, read as:

$$\overline{\cos\theta C^{2,I\alpha}_{sca,H}} = 2\mathrm{Im}(b_2b_3^*)[\frac{\overline{\cos\theta C^{2,I}_{ext,H}}}{\mathrm{Im}(b_2)} + \frac{\overline{\cos\theta C^{1,I}_{ext,H}}}{\mathrm{Im}(b_1)}] \tag{185}$$

$$\overline{\cos\theta C^{2,I\beta}_{sca,H}} = \frac{\lambda^2}{\pi}\mathrm{Im}(b_2b_3^*)\mathrm{Im}(2\Delta Z^{23}_H - 3iZ^{11}_{EH}) \tag{186}$$

Eqs. (185) and (186) indicate (i) that $\overline{\cos\theta C^{2,I\alpha}_{sca,H}}$ (renamed $\overline{\cos\theta C^{2,IG}_{sca,H}}$) is a recoil gradient magnetic force which may be expressed in terms of mixing gradient magnetic forces and (ii) that $\overline{\cos\theta C^{2,I\beta}_{sca,H}}$ (renamed $\overline{\cos\theta C^{2,INS}_{sca,H}}$) is a recoil non-standard magnetic force, similarly as for the corresponding electric force. Also, although $\overline{\cos\theta C^{1,INS}_{sca,H}}$ is proportional to $\overline{\cos\theta C^{1,INS}_{sca,EH}}$, see Eq. (54), it is found that this proportionality property does not propagate from $N = 1$ to $N = 2$.

Calculations for the other cases are similar, and we shall be content to present the results, omitting many details which are parallel to the ones for the z-components above. Then, $\overline{\sin\theta\cos\varphi C^{2,I}_{sca,E}}$ may be decomposed in two subforces denoted according to $\overline{\sin\theta\cos\varphi C^{2,I\alpha}_{sca,E}}$ and $\overline{\sin\theta\cos\varphi C^{2,I\beta}_{sca,E}}$, reading as:

$$\overline{\sin\theta\cos\varphi C^{2,I\alpha}_{sca,E}} = 2\mathrm{Im}(a_2a_3^*)[\frac{\overline{\sin\theta\cos\varphi C^{2,I}_{ext,E}}}{\mathrm{Im}(a_2)} + \frac{\overline{\sin\theta\cos\varphi C^{1,I}_{ext,E}}}{\mathrm{Im}(a_1)}] \tag{187}$$

$$\overline{\sin\theta\cos\varphi C^{2,I\beta}_{sca,E}} = \frac{\lambda^2}{\pi}\mathrm{Im}(a_2a_3^*)\mathrm{Im}[i(\frac{3}{2}X^{11}_{EH} + X^{\gamma 2}_{E})] \tag{188}$$

with $\overline{\sin\theta\cos\varphi C^{2,I\alpha}_{sca,E}}$ (renamed $\overline{\sin\theta\cos\varphi C^{2,IG}_{sca,E}}$) being a recoil gradient electric force and $\overline{\sin\theta\cos\varphi C^{2,I\beta}_{sca,E}}$ (renamed $\overline{\sin\theta\cos\varphi C^{2,INS}_{sca,E}}$) being a recoil non-standard electric force. Again, there is a proportionality relation for $N = 1$ between $\overline{\sin\theta\cos\varphi C^{1,INS}_{sca,E}}$ and $\overline{\sin\theta\cos\varphi C^{1,INS}_{sca,EH}}$, see Eq. (65), but upon inspection it is found that this relation does not propagate from $N = 1$ to $N = 2$.

Similarly, $\overline{\sin\theta\cos\varphi C^{2,I}_{sca,H}}$ may be decomposed in two subforces denoted $\overline{\sin\theta\cos\varphi C^{2,I\alpha}_{sca,H}}$ and $\overline{\sin\theta\cos\varphi C^{2,I\beta}_{sca,H}}$, reading as:

$$\overline{\sin\theta\cos\varphi C^{2,I\alpha}_{sca,H}} = 2\mathrm{Im}(b_2b_3^*)[\frac{\overline{\sin\theta\cos\varphi C^{2,I}_{ext,H}}}{\mathrm{Im}(b_2)} + \frac{\overline{\sin\theta\cos\varphi C^{1,I}_{ext,H}}}{\mathrm{Im}(b_1)}] \tag{189}$$

$$\overline{\sin\theta\cos\varphi C^{2,I\beta}_{sca,H}} = \frac{\lambda^2}{\pi}\mathrm{Im}(b_2b_3^*)\mathrm{Im}[i(X^{\gamma 2}_{H} - \frac{3}{2}X^{11}_{EH})] \tag{190}$$

with $\overline{\sin\theta\cos\varphi C^{2,I\alpha}_{sca,H}}$ (renamed $\overline{\sin\theta\cos\varphi C^{2,IG}_{sca,H}}$) being a recoil gradient magnetic force and $\overline{\sin\theta\cos\varphi C^{2,I\beta}_{sca,H}}$ (renamed $\overline{\sin\theta\cos\varphi C^{2,INS}_{sca,H}}$) being a recoil non-standard magnetic force. Again, there is a proportionality relation for $N = 1$ between $\overline{\sin\theta\cos\varphi C^{1,INS}_{sca,H}}$ and $\overline{\sin\theta\cos\varphi C^{1,INS}_{sca,EH}}$, see Eq. (70), but upon inspection it is found that this relation does not propagate from $N = 1$ to $N = 2$.

Next, $\overline{\sin\theta\sin\varphi C^{2,R}_{sca,E}}$ can be decomposed into two subforces denoted $\overline{\sin\theta\sin\varphi C^{2,R\alpha}_{sca,E}}$ and $\overline{\sin\theta\sin\varphi C^{2,R\beta}_{sca,E}}$ reading as:

$$\overline{\sin\theta\sin\varphi C^{2,R\alpha}_{sca,E}} = 2\mathrm{Im}(a_2a_3^*)[\frac{\overline{\sin\theta\sin\varphi C^{2,R}_{ext,E}}}{\mathrm{Im}(a_2)} + \frac{\overline{\sin\theta\sin\varphi C^{1,R}_{ext,E}}}{\mathrm{Im}(a_1)}] \tag{191}$$

$$\overline{\sin\theta\sin\varphi C^{2,R\beta}_{sca,E}} = \frac{-\lambda^2}{\pi}\mathrm{Im}(a_2a_3^*)\mathrm{Re}[i(\frac{3}{2}Y^{11}_{EH} + Y^{\gamma 2}_{E})] \tag{192}$$

with $\overline{\sin\theta\sin\varphi C^{2,R\alpha}_{sca,E}}$ (renamed $\overline{\sin\theta\sin\varphi C^{2,RG}_{sca,E}}$) being a recoil gradient electric force and $\overline{\sin\theta\sin\varphi C^{2,R\beta}_{sca,E}}$ (renamed $\overline{\sin\theta\sin\varphi C^{2,RNS}_{sca,E}}$) being a recoil non-standard force electric force. Again, there is a proportionality relation for $N = 1$ between $\overline{\sin\theta\sin\varphi C^{1,RNS}_{sca,E}}$ and $\overline{\sin\theta\sin\varphi C^{1,INS}_{sca,EH}}$, see Eq. (73), but which does not propagate from $N = 1$ to $N = 2$.

Next and finally, we deal with $\overline{\sin\theta\sin\varphi C^{2,R}_{sca,H}}$ which is decomposed into two subforces reading as:

$$\overline{\sin\theta\sin\varphi C^{2,R\alpha}_{sca,H}} = 2\mathrm{Im}(b_2b_3^*)[\frac{\overline{\sin\theta\sin\varphi C^{2,R}_{ext,H}}}{\mathrm{Im}(b_2)} + \frac{\overline{\sin\theta\sin\varphi C^{1,R}_{ext,H}}}{\mathrm{Im}(b_1)}] \tag{193}$$

$$\overline{\sin\theta\sin\varphi C^{2,R\beta}_{sca,H}} = \frac{\lambda^2}{\pi}\mathrm{Im}(b_2b_3^*)\mathrm{Re}[i(\frac{3}{2}Y^{11}_{EH} + Y^{\gamma 2*}_{H})] \tag{194}$$

with $\overline{\sin\theta\sin\varphi C^{2,R\alpha}_{sca,H}}$ (renamed $\overline{\sin\theta\sin\varphi C^{2,RG}_{sca,H}}$) being a recoil gradient magnetic force and $\overline{\sin\theta\sin\varphi C^{2,R\beta}_{sca,H}}$ (renamed

$\overline{\sin\theta\sin\varphi C^{2,RNS}_{sca,H}}$) being a recoil non-standard magnetic force. Again, there is a proportionality relation for $N=1$ between $\overline{\sin\theta\sin\varphi C^{1,RNS}_{sca,H}}$ and $\overline{\sin\theta\sin\varphi C^{1,RNS}_{sca,EH}}$, see Eq. (76), but which does not propagate to $N=2$.

6.4. Interpretation of non-quadrupolar 2-forces. Recoil non-standard forces

We end the formal part of this paper with recoil non-standard forces listed at the end of section 6.2, namely $\overline{\cos\theta C^{2,R}_{sca,E}}$, $\overline{\cos\theta C^{2,R}_{sca,H}}$, $\overline{\sin\theta\cos\varphi C^{2,R}_{sca,E}}$, $\overline{\sin\theta\cos\varphi C^{2,R}_{sca,H}}$, $\overline{\sin\theta\sin\varphi C^{2,I}_{scaE}}$ and $\overline{\sin\theta\sin\varphi C^{2,I}_{sca,H}}$. To deal with the recoil force $\overline{\cos\theta C^{2,R}_{sca,E}}$, we begin with a return to the mixing force $\overline{\cos\theta C^{2,R}_{ext,E}}$ of Eq. (112) which may be separated in two subforces according to Eqs. (119) and (120). Then $\overline{\cos\theta C^{2,RNS}_{ext,E}}$ of Eq. (120) may be decomposed again in two subforces denoted as $\overline{\cos\theta C^{2,RNS1}_{ext,E}}$ and $\overline{\cos\theta C^{2,RNS2}_{ext,E}}$ reading as:

$$\overline{\cos\theta C^{2,RNS1}_{ext,E}} = \frac{3\lambda^2}{2\pi}\mathrm{Re}(a_2)\mathrm{Re}(Z^{12}_E) \tag{195}$$

$$\overline{\cos\theta C^{2,RNS2}_{ext,E}} = \frac{-\lambda^2}{\pi}\mathrm{Re}(a_2)\mathrm{Re}(Z^{23}_E) \tag{196}$$

in which, from Eqs. (17), (83) and (118), we have used:

$$Z^{2,NS}_E = \frac{3}{2}Z^{12*}_E - Z^{23}_E \tag{197}$$

Comparing Eqs. (157) and (196), we obtain:

$$\overline{\cos\theta C^{2,R}_{sca,E}} = 2\frac{\mathrm{Re}(a_2a_3^*)}{\mathrm{Re}(a_2)}\overline{\cos\theta C^{2,RNS2}_{ext,E}} \tag{198}$$

so that $\overline{\cos\theta C^{2,R}_{sca,E}}$ (renamed $\overline{\cos\theta C^{2,RNS}_{sca,E}}$) is now interpreted as a recoil non-standard electric force expressed versus a mixing non-standard electric force, similarly as for $\overline{\cos\theta C^{1,R}_{sca,E}}$ of Eq. (42). Similarly, we establish that:

$$\overline{\cos\theta C^{2,R}_{sca,H}} = 2\frac{\mathrm{Re}(b_2b_3^*)}{\mathrm{Re}(b_2)}\overline{\cos\theta C^{2,RNS2}_{ext,H}} \tag{199}$$

in which:

$$\overline{\cos\theta C^{2,RNS2}_{ext,H}} = \frac{-\lambda^2}{\pi}\mathrm{Re}(b_2)\mathrm{Re}(Z^{23}_H) \tag{200}$$

so that $\overline{\cos\theta C^{2,R}_{sca,H}}$ (renamed $\overline{\cos\theta C^{2,RNS}_{sca,H}}$) is now interpreted as a recoil non-standard magnetic force expressed versus a mixing non-standard magnetic force, similarly as for $\overline{\cos\theta C^{1,R}_{sca,H}}$ of Eq. (50).

For $\overline{\sin\theta\cos\varphi C^{2,R}_{sca,E}}$ of Eq. (163), we use Eqs. (96), (88), (90) and (21) to establish:

$$X^{23}_E = X^{\alpha 2}_E + X^{\beta 2}_E - \frac{3}{2}X^{12*}_E \tag{201}$$

so that we obtain:

$$\overline{\sin\theta\cos\varphi C^{2,R}_{sca,E}} = \frac{\lambda^2}{\pi}\mathrm{Re}(a_2a_3^*)[\mathrm{Re}(X^{\alpha 2}_E + X^{\beta 2}_E) - \frac{3}{2}\mathrm{Re}(X^{12}_E)] \tag{202}$$

In the r.h.s. of Eq. (202), the first force is a non-standard force by virtue of Eq. (131) while the second force is a non-standard force as well by virtue of Eq. (71) of [32], so that $\overline{\sin\theta\cos\varphi C^{2,R}_{sca,E}}$ (renamed $\overline{\sin\theta\cos\varphi C^{2,RNS}_{sca,E}}$) is a recoil non-standard electric force which may be expressed in terms of mixing non-standard electric forces according to:

$$\overline{\sin\theta\cos\varphi C^{2,R}_{sca,E}} = \overline{\sin\theta\cos\varphi C^{2,RNS}_{sca,E}} = 2\mathrm{Re}(a_2a_3^*)\left[\frac{\overline{\sin\theta\cos\varphi C^{2,RNS}_{ext,E}}}{\mathrm{Re}(a_2)} - \frac{\overline{\sin\theta\cos\varphi C^{1,RNS}_{ext,E}}}{\mathrm{Re}(a_1)}\right] \tag{203}$$

Similarly, we obtain:

$$\overline{\sin\theta\cos\varphi C^{2,R}_{sca,H}} = \overline{\sin\theta\cos\varphi C^{2,RNS}_{sca,H}} = 2\mathrm{Re}(b_2b_3^*)\left[\frac{\overline{\sin\theta\cos\varphi C^{2,RNS}_{ext,H}}}{\mathrm{Re}(b_2)} - \frac{\overline{\sin\theta\cos\varphi C^{1,RNS}_{ext,H}}}{\mathrm{Re}(b_1)}\right] \tag{204}$$

$$\overline{\sin\theta\sin\varphi C^{2,I}_{sca,E}} = \overline{\sin\theta\sin\varphi C^{2,INS}_{sca,E}} = 2\mathrm{Re}(a_2a_3^*)\left[\frac{\overline{\sin\theta\sin\varphi C^{2,INS}_{ext,E}}}{\mathrm{Re}(a_2)} - \frac{\overline{\sin\theta\sin\varphi C^{1,INS}_{ext,E}}}{\mathrm{Re}(a_1)}\right] \tag{205}$$

$$\overline{\sin\theta\sin\varphi C^{2,I}_{sca,H}} = \overline{\sin\theta\sin\varphi C^{2,INS}_{sca,H}} = 2\mathrm{Re}(b_2b_3^*)\left[\frac{\overline{\sin\theta\sin\varphi C^{2,INS}_{ext,H}}}{\mathrm{Re}(b_2)} - \frac{\overline{\sin\theta\sin\varphi C^{1,INS}_{ext,H}}}{\mathrm{Re}(b_1)}\right] \tag{206}$$

Let us note that the above relationships between recoil and mixing non-standard forces exhibit an internal coherency which is a supplementary justification for the introduction of the third-level categorization. The summary of Section 7 will emphasize this coherency.

7. Summary of results and classifications

It is now convenient to summarize the classification of forces developed in the present paper, based on a three-level categorization and a two-level decomposition. This classification will exhibit the fact that there are a few differences between 1-forces and 2-forces as already noticed in the bulk of the paper. It is however expected that the structure and properties of 2-forces will likely propagate to the case of N -forces, $N > 2$ [38]. In this section, we do not distinguish between dipolar and non-dipolar forces, nor between quadrupolar and non-quadrupolar forces. Such distinctions will not be emphasized for $N > 2$. We shall consider (i) mixing forces, (ii) recoil forces which are not expressed in terms of mixing forces (recoil forces in their own right) and (iii) recoil forces which are expressed in terms of mixing forces (recoil forces mixing-force dependent). In the case of recoil forces, we may have to distinguish the cases $N=1$ and $N=2$. This classification is displayed in the best of our present knowledge and understanding, although it might have possibly to be modified by further investigations.

7.1. Mixing forces

The classification of mixing forces, valid for both $N=1$ and $N=2$, has been obtained as follows.

(i) Mixing gradient electric forces $\overline{\cos\theta C^{N,IG}_{ext,E}}$, $\overline{\sin\theta\cos\varphi C^{N,IG}_{ext,E}}$, $\overline{\sin\theta\sin\varphi C^{N,RG}_{ext,E}}$.

(ii) Mixing gradient magnetic forces $\overline{\cos\theta C^{N,IG}_{ext,H}}$, $\overline{\sin\theta\cos\varphi C^{N,IG}_{ext,H}}$, $\overline{\sin\theta\sin\varphi C^{N,RG}_{ext,H}}$.

(iii) Mixing scattering electric forces $\overline{\cos\theta C^{N,RS}_{ext,E}}$, $\overline{\sin\theta\cos\varphi C^{N,RS}_{ext,E}}$, $\overline{\sin\theta\sin\varphi C^{N,IS}_{ext,E}}$.

(iv) Mixing non-standard electric forces $\overline{\cos\theta C^{N,RNS}_{ext,E}}$, $\overline{\sin\theta\cos\varphi C^{N,RNS}_{ext,E}}$, $\overline{\sin\theta\sin\varphi C^{R,INS}_{ext,E}}$.

(v) Mixing scattering magnetic forces $\overline{\cos\theta C^{N,RS}_{ext,H}}$, $\overline{\sin\theta\cos\varphi C^{N,RS}_{ext,H}}$, $\overline{\sin\theta\sin\varphi C^{N,IS}_{ext,H}}$.

(vi) Mixing non-standard magnetic forces $\overline{\cos\theta C^{N,RNS}_{ext,H}}$, $\overline{\sin\theta\cos\varphi C^{N,RNS}_{ext,H}}$, $\overline{\sin\theta\sin\varphi C^{N,INS}_{ext,H}}$.

7.2. Type-1 recoil forces, for $N = 1$

Type-1 recoil forces are recoil forces that we could not express in terms of mixing forces. For the sake of clarity, we distinguish the cases $N = 1$ and $N = 2$ which do not behave exactly in the same way. The corresponding classification is then as follows.

(viia) Recoil non-standard magnetoelectric forces $\overline{\cos\theta C^{1,INS}_{sca,EH}}$, $\overline{\sin\theta\cos\varphi C^{1,INS}_{sca,EH}}$, $\overline{\sin\theta\sin\varphi C^{1,RNS}_{sca,EH}}$.

(viiia) Recoil non-standard electric force $\overline{\cos\theta C^{1,INS}_{sca,E}}$, $\overline{\sin\theta\cos\varphi C^{1,INS}_{sca,E}}$, $\overline{\sin\theta\sin\varphi C^{1,RNS}_{sca,E}}$. These forces may be expressed in terms of recoil non-standard magnetoelectric forces of item (viia) above, see Eqs. (47), (65) and (73).

(ixa) Recoil non-standard magnetic forces $\overline{\cos\theta C^{1,INS}_{sca,H}}$, $\overline{\sin\theta\cos\varphi C^{1,INS}_{sca,H}}$, $\overline{\sin\theta\sin\varphi C^{1,RNS}_{sca,H}}$. These forces may be expressed in terms of recoil non-standard magnetoelectric forces of item (viia) above, see Eqs. (54), (70) and (76).

7.3. Type-1 recoil forces, for $N = 2$

The difference between this subsection and the previous one is that the properties of the forces of items (viiia) and (ixa), namely to be expressed in terms of recoil non-standard magnetoelectric forces, do not propagate from $N = 1$ to $N = 2$. The corresponding classification is then as follows.

(viib) Recoil non-standard magnetoelectric forces $\overline{\cos\theta C^{1,INS}_{sca,EH}}$, $\overline{\sin\theta\cos\varphi C^{1,INS}_{sca,EH}}$, $\overline{\sin\theta\sin\varphi C^{1,RNS}_{sca,EH}}$.

(viiib) Recoil non-standard electric force $\overline{\cos\theta C^{1,INS}_{sca,E}}$, $\overline{\sin\theta\cos\varphi C^{1,INS}_{sca,E}}$, $\overline{\sin\theta\sin\varphi C^{1,RNS}_{sca,E}}$.

(ixb) Recoil non-standard magnetic forces $\overline{\cos\theta C^{1,INS}_{sca,H}}$, $\overline{\sin\theta\cos\varphi C^{1,INS}_{sca,H}}$, $\overline{\sin\theta\sin\varphi C^{1,RNS}_{sca,H}}$.

7.4. Type-2 recoil forces

Type-2 recoil forces may be expressed in terms of mixing forces. The classification is then as follows.

(x) Recoil scattering magnetoelectric forces $\overline{\cos\theta C^{N,RS}_{sca,EH}}$, $\overline{\sin\theta\cos\varphi C^{N,RS}_{sca,EH}}$, $\overline{\sin\theta\sin\varphi C^{N,IS}_{sca,EH}}$. These forces may be expressed in terms of mixing scattering pure electric and magnetic forces, see Eqs. (35), (36), (37) for $N = 1$ and (141), (145), (149) for $N = 2$.

(xi) Recoil non-standard electric forces $\overline{\cos\theta C^{N,RNS}_{sca,E}}$, $\overline{\sin\theta\cos\varphi C^{N,RNS}_{sca,E}}$, $\overline{\sin\theta\sin\varphi C^{N,INS}_{sca,E}}$. These forces may be expressed in terms of mixing non-standard electric forces, see Eqs. (42), (59), (71) for $N = 1$ and (198), (203), (205) for $N = 2$.

(xii) Recoil gradient electric forces $\overline{\cos\theta C^{N,IG}_{sca,E}}$, $\overline{\sin\theta\cos\varphi C^{N,IG}_{sca,E}}$, $\overline{\sin\theta\sin\varphi C^{N,RG}_{sca,E}}$. These forces may be expressed in terms of mixing gradient electric forces, see Eqs. (46), (63), (72) for $N = 1$ and (179), (187), (191) for $N = 2$.

(xiii) Recoil non-standard magnetic force $\overline{\cos\theta C^{N,RNS}_{sca,H}}$, $\overline{\sin\theta\cos\varphi C^{N,RNS}_{sca,H}}$, $\overline{\sin\theta\sin\varphi C^{N,INS}_{sca,H}}$. These forces may be expressed in terms of mixing non-standard magnetic forces, see Eqs. (50), (66), (74) for $N = 1$ and (199), (204), (206) for $N = 2$.

(xiv) Recoil gradient magnetic forces $\overline{\cos\theta C^{N,IG}_{sca,H}}$, $\overline{\sin\theta\cos\varphi C^{N,IG}_{sca,H}}$, $\overline{\sin\theta\sin\varphi C^{N,RG}_{sca,H}}$. These forces may be expressed in terms of mixing gradient magnetic forces, see Eqs. (53), (69), (75) for $N = 1$ and (185), (189), (193) for $N = 2$.

8. Conclusion

The present paper discussed the partition of optical forces exerted by EM arbitrary shaped beams on quadrupoles (therefore in particular on dipoles) in the framework of the generalized Lorenz–Mie theory. The partition first relies on a first-level categorization between mixing and recoil forces already published nearly four decades ago in early works devoted to GLMT. A second-level categorization distinguishes gradient forces and non-gradient (non-conservative) forces. Although non-gradient (non-conservative) forces are usually named scattering forces, we rely on the existence of non-standard forces uncovered in Gouesbet [30] (where they were called axicon forces in an inappropriate way) to introduce a third-level categorization in terms of scattering and non-standard forces. A parallel two-level decomposition distinguishes between (i) N-forces, N from 1 to ∞ and (ii) electric, magnetic and magnetoelectric forces. The study of dipoles relies on the use of 1-forces while the study of quadrupoles has to consider 2-forces as well. All the forces in the different partitions are expressed in terms of BSCs which encode the description of the illuminating beam in the GLMT framework (and of Mie coefficients). This paper emphasizes the fact that 1-forces and 2 -forces, although they are sharing similar properties, are however sufficiently different to warrant the publication of optical forces restricted to these forces which can then be both compared in their similarities and opposed in their differences, the general case for $N > 1$ being postponed to a future paper. One of the most appealing results is that most of the recoil forces may be expressed in terms of mixing forces. In particular, all recoil gradient forces may be expressed in terms of mixing gradient forces. Also, It is to be noted that the researcher who would like to avoid the third-level categorization could deal with a two-level characterization between mixing and recoil forces, and between gradient (conservative) and non-gradient (non-conservative) forces, the last ones being obtained as a summation of scattering and non-standard forces. Finally, the GLMT which expresses the forces in terms of BSCs (and of Mie coefficients) focuses the attention on the necessary and sufficient quantities involved. For instance, in the case of dipoles, forces are expressed in terms of partial waves of orders 1 and 2 while the dipole theory of forces, although equivalent, is expressed in terms of total fields and therefore deals as well with an infinite number of partial waves, from orders 3 to infinity, which actually have a contribution equal to 0.

Data Availability

No data was used for the research described in the article.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Gérard Gouesbet: Conceptualization, Writing – original draft. **V.S. De Angelis:** Writing – review & editing. **Leonardo André Ambrosio:** Writing – review & editing.

Data Availability

No data was used for the research described in the article.

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Appendix A. Translation between PWECs and BSCs

Subscripts G and Z may be used respectively to denote quantities having the same symbolic notations in "Gouesbet"-work and in "Zheng"-work but which are defined differently. In particular, we have:

$$[P_n^m(\cos\theta)]_G = (-1)^m [P_n^m(\cos\theta)]_Z \tag{207}$$

in which $P_n^m(\cos\theta)$ are the associated Legendre functions. To see this, we may for instance compare Eq. (5) of [39] and Eq.(2.75) of [2]. Next, we compare the expressions for the Vector Spherical Wave Functions (VSWFs) by comparing Eqs. (1) and (2), p. 266 of [2], or Eqs. (1) and (2) of [53], and Eq. (3) of [39]. In doing so, we remark that the π_{mn} in Eq. (4) of [39] contains a "m" which is not present in the definition of π_n^m used in GLMT. Taking also account for Eq. (207), we then establish:

$$[\mathbf{M}_{mn}^{(1)}]_G = [\mathbf{M}_{mn}^{(1)}]_Z \tag{208}$$

$$[\mathbf{N}_{mn}^{(1)}]_G = [\mathbf{N}_{mn}^{(1)}]_Z \tag{209}$$

We next compare the electric field $\mathbf{E}$ of Eqs. (1) and (2) in [39] and Eqs. (6)–(8), p. 267–268 of [2], see also [53]. We do not need to deal with the magnetic field. Using the expression for c_n^{pw}, e.g. Eq. (3.3) of [2], the expansion coefficients a_{mn} and b_{mn} of Eqs. (7) and (8), pp.267–268, of [2] become:

$$a_{mn} = E_0(-1)^{m+1}(-1)^{(m-|m|)/2}(-i)^n \frac{2n+1}{n(n+1)}\frac{(n-m)!}{(n-|m|)!} g_{n,TE}^m \tag{210}$$

$$b_{mn} = E_0(-1)^{m}(-1)^{(m-|m|)/2}(-i)^{n+1} \frac{2n+1}{n(n+1)}\frac{(n-m)!}{(n-|m|)!} g_{n,TM}^m \tag{211}$$

Next, comparing the expression for $\mathbf{E}$ in Eq. (1) of [39] and Eq. (6), p.267 of [2], one has:

$$a_{mn} = -E_0 i C_{mn} q_{mn} \tag{212}$$

$$b_{mn} = -E_0 i C_{mn} p_{mn} \tag{213}$$

in which C_{mn} is given by Eq. (2) of [39] according to:

$$C_{mn} = i^n \left[\frac{(2n+1)(n-m)!}{n(n+1)(n+m)!}\right]^{1/2} \tag{214}$$

Eqs. (212) –(214) then lead to:

$$q_{mn} = i(-1)^{n+m+1}(-1)^{(m-|m|)/2}\sqrt{\frac{2n+1}{n(n+1)}}\frac{\sqrt{(n-m)!(n+m)!}}{(n-|m|)!} g_{n,TE}^m \tag{215}$$

$$p_{mn} = (-1)^{n+m}(-1)^{(m-|m|)/2}\sqrt{\frac{2n+1}{n(n+1)}}\frac{\sqrt{(n-m)!(n+m)!}}{(n-|m|)!} g_{n,TM}^m \tag{216}$$

It may be interesting to separate the cases $m \geq 0$ and $m < 0$. We then obtain:

$$q_{mn} = i(-1)^{n+m+1}\sqrt{\frac{2n+1}{n(n+1)}}\sqrt{\frac{(n+m)!}{(n-m)!}} g_{n,TE}^m \text{ for } m \geq 0 \tag{217}$$

$$q_{mn} = i(-1)^{n+1}\sqrt{\frac{2n+1}{n(n+1)}}\sqrt{\frac{(n-m)!}{(n+m)!}} g_{n,TE}^m \text{ for } m < 0 \tag{218}$$

$$p_{mn} = (-1)^{n+m}\sqrt{\frac{2n+1}{n(n+1)}}\sqrt{\frac{(n+m)!}{(n-m)!}} g_{n,TM}^m \text{ for } m \geq 0 \tag{219}$$

$$p_{mn} = (-1)^{n}\sqrt{\frac{2n+1}{n(n+1)}}\sqrt{\frac{(n-m)!}{(n+m)!}} g_{n,TM}^m \text{ for } m < 0 \tag{220}$$

Appendix B. Comparing mixing gradient forces

According to Eq. (24) in Zheng et al. [39], the mixing gradient electric force, specified for 2-forces, here denoted as $\mathbf{F}_E^{2,mix,G}$, reads as:

$$\mathbf{F}_E^{2,mix,G} = -2\pi\varepsilon\frac{|E_0|^2}{k^2}[\mathrm{Im}(\mathrm{a}_2)\mathrm{Im}(\mathbf{A}_2^* + \mathbf{A}_1 + \mathbf{U}_2)] \tag{221}$$

This Appendix will deal with the longitudinal component only. The transverse components would be studied quite similarly and, for the sake of saving room, this is left to the reader. Eq. (221) then reduces to:

$$F_{E,z}^{2,mix,G} = -2\pi\varepsilon\frac{|E_0|^2}{k^2}[\mathrm{Im}(\mathrm{a}_2)\mathrm{Im}(\mathrm{A}_{2z}^* + \mathrm{A}_{1z} + \mathrm{U}_{2z})] \tag{222}$$

From Eqs. (9), (10) in Zheng et al. [39], we may evaluate A_{2z}^*, A_{1z}, and U_{2z} in terms of PWECs, and PWECs may afterward be expressed in terms of BSCs using the translation formulae of Appendix A. After a bit of straightforward although tedious calculations, this process leads to:

$$A_{2z}^* = Z_E^{23},\ A_{1z} = -\frac{3}{2}Z_E^{12*},\ U_{2z} = -Z_E^{2,S} \tag{223}$$

so that, as a whole, we establish:

$$A_{2z}^* + A_{1z} + U_{2z} = -Z_E^2 \tag{224}$$

Therefore, Eq. (222) becomes:

$$F_{E,z}^{2,mix,G} = +2\pi\varepsilon\frac{|E_0|^2}{k^2}\mathrm{Im}(\mathrm{a}_2)\mathrm{Im}(\mathrm{Z}_\mathrm{E}^2) \tag{225}$$

to be compared to Eq. (111). This is sufficient to claim that $\overline{\cos\theta C_{ext,E}^{2,I}}$ is indeed a mixing gradient force. The difference in the prefactors is due to the fact that cross-sections are forces expressed in square meters. For the relationship between cross-sections and genuine forces (in newtons), we may rely on Eqs. (3.106) and (3.144) of [2], i.e. to $\mathbf{F}_{ext,E}^{2,I} = I_0\overline{\cos\theta C_{ext,E}^{2,I}}/c$, with $c = 1/\sqrt{\varepsilon\mu}$, to obtain:

$$\mathbf{F}_{ext,E}^{2,I} = -2\pi\varepsilon\frac{|E_0|^2}{k^2}\mathrm{Im}(\mathrm{a}_2)\mathrm{Im}(\mathrm{Z}_\mathrm{E}^2) \tag{226}$$

which differs from Eq. (225) by a sign difference. This sign difference is due to the fact that the time-harmonic convention used by Zheng et al. [39] is of the form $\exp(-i\omega t)$ in contrast with the time convention in GLMT which is of the form $\exp(i\omega t)$, implying that we have to change a_2 to a_2^*, i.e. $\mathrm{Im}(a_2)$ to $-\mathrm{Im}(a_2)$.This demonstrates that $\overline{\cos\theta C_{ext,E}^{2,I}}$ is indeed a gradient force.

The same process is used to similarly deal with the other electric components $\overline{\sin\theta\cos\varphi C_{ext,E}^{2,I}}$ and $\overline{\sin\theta\sin\varphi C_{ext,E}^{2,R}}$. It may be used as well for the magnetic components $\overline{\cos\theta C_{ext,H}^{2,I}}$, $\overline{\sin\theta\cos\varphi C_{ext,H}^{2,I}}$, $\overline{\sin\theta\sin\varphi C_{ext,H}^{2,R}}$ although it is more expedient to invoke a duality between electric and magnetic components, as already discussed in Gouesbet et al. [32].

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