Thermal effective Lagrangian of static gravitational fields

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We compute the effective Lagrangian of static gravitational fields interacting with thermal fields. Our approach employs the usual imaginary time formalism as well as the equivalence between the static and space-time independent external gravitational fields. This allows to obtain a closed form expression for the thermal effective Lagrangian in d space-time dimensions.

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I. INTRODUCTION

The high-temperature limit in thermal field theory has many interesting properties which, in some cases, allow to obtain closed form expressions for quantities like the effective Lagrangians in gauge theories [1]. When the gravitational interactions are taken into account, there are also indications that it may be possible to obtain an effective Lagrangian, though until now this has not been demonstrated in general. Only in the static limit, when the fields are time independent, it is possible to show that all the one-loop thermal Green functions can be generated from an effective Lagrangian which has a simple closed form [2].

More recently, it has been shown that the static limit of thermal Greens functions in gauge theories coincide with the limit when all the external four momenta are equal to zero. This has been explicitly verified for individual thermal Green functions [3] (in another work the long wavelength limit has also been investigated [4]). This result indicates that in configuration space we may make the hypothesis that a static background is equivalent to a space-time independent configuration, in the high-temperature limit. The purpose of the present paper is to further investigate this issue in the context of a gravitational background, using a more general approach which can be easily extended to all orders. In this paper, we say that the background metric is "static" when it does not depend on time, which is less restrictive than the condition of a "static space-time" when, additionally, $g_{0i} = 0$, in some reference frame.

At one-loop order, the effective Lagrangian of the gravitational fields interacting with thermal scalar fields can be written in terms of the functional determinant as follows:

$$\mathcal{L} = \frac{T}{V} \log[\mathrm{Det}(-\beta^2 p_\mu \sqrt{-g} g^{\mu\nu} p_\nu)]_T^{-1/2}, \tag{1}$$

where $p_\mu = -i\partial_\mu$, $g = (-1)^d \det g_{\mu\nu}$ and d is the space-time dimension. This expression is based on the usual approach which is employed in order to obtain the one-loop effective Lagrangians in field theory [5]. Here we are considering that the temperature $T = 1/\beta$ is much bigger than any other mass scale, such as the scalar field mass, and the subscript T is to make explicit that we are considering only the temperature dependent part of the determinant. We will employ the imaginary time formalism [6–8].

In the next section we will consider a perturbative method which expresses $\mathcal{L}$ in terms of powers of the gravitational field $\tilde{h}^{\mu\nu}$ in a Minkowski background. The purpose of the perturbative calculation is to make contact with some known results in the static limit. In Sec. III we derive the effective Lagrangian for a general static background. As a check, we verify that the perturbative results of Sec. II can be obtained from the exact result of Sec. III. Finally, in Sec. IV we discuss the results and perspectives. Some details of the perturbative calculation are left to the Appendix .

II. FIELDS IN A MINKOWSKI BACKGROUND

In order to make contact with some known perturbative results, we define the gravitational field $\tilde{h}^{\mu\nu}$ as [9]

$$\sqrt{-g} g^{\mu\nu} \equiv \tilde{g}^{\mu\nu} = \eta^{\mu\nu} + \tilde{h}^{\mu\nu}, \tag{2}$$

where $\eta^{\mu\nu}$ is the Minkowski metric. Here we have a symmetric tensor field $\tilde{h}^{\mu\nu}$, in a Minkowski background. Inserting Eq. (2) into Eq. (1), yields

$$\mathcal{L} = \frac{T}{V} \log\left[\mathrm{Det}(-\beta^2 p^2)\mathrm{Det}\left(1 + \frac{1}{p^2} p_\mu \tilde{h}^{\mu\nu} p_\nu\right)\right]_T^{-1/2}. \tag{3}$$

We now make use of the hypothesis that the zero momentum limit can give us information about the static limit. This makes it possible to write

$$\begin{aligned}\mathcal{L}^{\text{stat.}} &= T \log\left[\mathrm{Det}(-\beta^2 p^2)\mathrm{Det}\left(1 + \tilde{h}^{\mu\nu} \frac{p_\mu p_\nu}{p^2}\right)\right]_T^{-1/2}. \\ &= \mathcal{L}^{(0)} - \frac{1}{2}\frac{T}{V} \log \mathrm{Det}\left(1 + \tilde{h}^{\mu\nu} \frac{p_\mu p_\nu}{p^2}\right),\end{aligned} \tag{4}$$

where $\mathcal{L}^{(0)}$ is given by [6–8]

$$\mathcal{L}^{(0)} = \frac{\Gamma[d]\zeta(d)}{2^{d-2}\pi^{(d-1)/2}\Gamma(\frac{d-1}{2})(d-1)} T^d. \tag{5}$$

Γ is the Gamma function and ζ is the Riemann zeta function. Equation (5) is simply the Stefan-Boltzmann pressure of the free gas in d space-time dimensions, so that the second term in Eq. (4) can also be interpreted as the corrections to the pressure due to the interaction with the gravitational field.

Let us now consider the quantity

$$\mathcal{L}^I = -\frac{1}{2}\frac{T}{V}\log \mathrm{Det}\left(1 + \tilde{h}^{\mu\nu}\frac{p_\mu p_\nu}{p^2}\right). \tag{6}$$

Using the relation $\log \mathrm{Det}A = \mathrm{Tr}\log A$, the imaginary time formalism leads to

$$\mathcal{L}^I = -\frac{T}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\log\left(1 + \tilde{h}^{\mu\nu}\frac{p_\mu p_\nu}{p^2}\right), \tag{7}$$

where $p_0 = i\omega_n = i2\pi nT$ and $p^2 = \eta^{\mu\nu}p_\mu p_\nu = p_0^2 - |\vec{p}|^2 = -(2\pi nT)^2 - |\vec{p}|^2$. It is understood that we are using the reference frame where the heat bath is at rest.

Let us now investigate the properties of Eq. (6) employing a perturbative expansion in powers of $\tilde{h}^{\mu\nu}$. Upon using the expansion $\log(1+x) = x - x^2/2 + x^3/3 + \cdots$, Eq. (6) can be written as

$$\begin{aligned}\mathcal{L}^I &= -\frac{T}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^{d-1}p}{(2\pi)^{d-1}} \\ &\quad\times\left(\tilde{h}^{\mu\nu}\frac{p_\mu p_\nu}{p^2} - \frac{\tilde{h}^{\mu\nu}\tilde{h}^{\alpha\beta}}{2}\frac{p_\mu p_\nu p_\alpha p_\beta}{p^4} + \cdots\right) \\ &= \tilde{h}^{\mu\nu}I_{\mu\nu} + \frac{1}{2}\tilde{h}^{\mu\nu}\tilde{h}^{\alpha\beta}I_{\mu\nu\alpha\beta} + \cdots,\end{aligned} \tag{8}$$

where

$$I_{\mu\nu} = -\frac{T}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{p_\mu p_\nu}{p^2} \tag{9}$$

and

$$I_{\mu\nu\alpha\beta} = \frac{T}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{p_\mu p_\nu p_\alpha p_\beta}{p^4}. \tag{10}$$

Each individual term in Eq. (8) is promptly identified as the n-point one-loop Feynman diagram with vanishing external momentum, contracted with n fields $\tilde{h}^{\mu\nu}$. The calculation of the first two terms in Eq. (8) can be done in a straightforward way. The details of this calculation is presented in the Appendix .

Combining the Eqs. (4), (8), (A2), (A6), (A8), and (A14), the effective Lagrangian can be written as follows:

$$\mathcal{L}^{\,\mathrm{stat.}} = \mathcal{L}^{(0)} + \tilde{\Gamma}_{\mu\nu}\tilde{h}^{\mu\nu} + \frac{1}{2}\tilde{\Pi}_{\mu\nu\alpha\beta}\tilde{h}^{\mu\nu}\tilde{h}^{\alpha\beta} + \cdots, \tag{11}$$

where

$$\tilde{\Gamma}_{\mu\nu} = \frac{\mathcal{L}^{(0)}}{2}(du_\mu u_\nu - \eta_{\mu\nu}) \tag{12}$$

and

$$\begin{aligned}\tilde{\Pi}_{\mu\nu\alpha\beta} = \mathcal{L}^{(0)}\Big[&\tilde{\Gamma}_{\mu\nu}\tilde{\Gamma}_{\alpha\beta} + \tilde{\Gamma}_{\mu\alpha}\tilde{\Gamma}_{\nu\beta} + \tilde{\Gamma}_{\mu\beta}\tilde{\Gamma}_{\nu\alpha} \\ &- \frac{d(d-1)}{2}u_\mu u_\nu u_\alpha u_\beta\Big].\end{aligned} \tag{13}$$

Both results in Eqs. (12) and (13) are exactly the same as one would obtain for the static limit of the one-loop Feynman diagrams.

One can verify that there are Weyl identities which relate $\tilde{\Pi}_{\mu\nu\alpha\beta}$ with $\tilde{\Gamma}_{\mu\nu}$. Indeed,

$$\eta^{\mu\nu}\tilde{\Gamma}_{\mu\nu} = 0, \tag{14a}$$

$$\eta^{\mu\nu}\tilde{\Pi}_{\mu\nu\alpha\beta} = -\tilde{\Gamma}_{\mu\nu}. \tag{14b}$$

These identities are a consequence of the conformal symmetry under the transformation $\sqrt{-g}g^{\mu\nu} \to (1+\epsilon)\times\sqrt{-g}g^{\mu\nu}$, which is equivalent to $\tilde{h}^{\mu\nu} \to \tilde{h}^{\mu\nu} + \epsilon\tilde{h}^{\mu\nu} + \epsilon\eta^{\mu\nu}$, with ϵ infinitesimal. Even in the nonstatic case it is known that the Weyl identities are satisfied by the high-temperature thermal amplitudes [10]. This is important information which constrains the general form of the Lagrangian and may help to obtain a closed form in terms of the exact metric tensor, although this has not been achieved yet.

One can also show that $\mathcal{L}^{\mathrm{stat}}$ is independent of the representation of the graviton field. As an example, alternatively one could define the graviton field as

$$\bar{h}^{\mu\nu} = g^{\mu\nu} - \eta^{\mu\nu}. \tag{15}$$

The relation between $\tilde{h}^{\mu\nu}$ and $\bar{h}^{\mu\nu}$ can be readily found to be

$$\tilde{h}^{\mu\nu} = \bar{h}^{\mu\nu} - \eta^{\mu\nu}\left(\frac{\bar{h}}{2} - \frac{1}{4}\bar{h}_{\alpha\beta}\bar{h}^{\beta\alpha} - \frac{\bar{h}^2}{8}\right) - \frac{\bar{h}}{2}\bar{h}^{\mu\nu} + \mathcal{O}(\bar{h}^3), \tag{16}$$

where the raising and lowering of indices are performed with the Minkowski metric. Inserting Eq. (16) into Eq. (11) and using the Weyl identities in Eq. (14) one can see that $\mathcal{L}^{\mathrm{stat.}}$ has the same form when the graviton field is defined as $\bar{h}^{\mu\nu}$. This independence on the graviton field parametrization is expected for a physical quantity like the effective Lagrangian even in more general cases, when the field transformation is not induced by a simple rescaling of the metric, as in the previous example. In general, both the fields and the amplitudes would change in such a way to preserve the invariance of the Lagrangian [10]. For instance, transforming to the graviton representation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, we obtain

$$\mathcal{L}^{\,\mathrm{stat.}} = \mathcal{L}^{(0)} + \Gamma^{\mu\nu}h_{\mu\nu} + \frac{1}{2}\Pi^{\mu\nu\alpha\beta}h_{\mu\nu}h_{\alpha\beta} + \cdots, \tag{17}$$

where

$$\Gamma^{\mu\nu} = -\frac{\mathcal{L}^{(0)}}{2}(du^\mu u^\nu - \eta^{\mu\nu}) \tag{18}$$

and

$$\Pi^{\mu\nu\alpha\beta} = \mathcal{L}^{(0)}\Big[\Gamma^{\mu\nu}\Gamma^{\alpha\beta} - \frac{1}{4}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}) + \frac{d}{2}u^\mu u^\nu u^\alpha u^\beta\Big] \tag{19}$$

which are both in agreement with the static result obtained in [2]. Also, $\Gamma^{\mu\nu}$ and $\Pi^{\mu\nu\alpha\beta}$ satisfy the identities (14). We remark that while the result in Eq. (19) exhibits only the Bosonic symmetry and the symmetry associated with the metric, Eq. (13) has a larger tensor symmetry, as could be anticipated from Eq. (10).

III. GENERAL STATIC BACKGROUNDS

Let us now consider a more general physical scenario described by a background metric which is not necessarily close to the Minkowski metric. Using again the relation $\log \det A = \mathrm{Tr} \log A$ in the context of the imaginary time formalism, Eq. (1) yields

$$\mathcal{L} = -\frac{T}{2}\sum_n \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \log(-\beta^2 p_\mu \tilde{g}^{\mu\nu} p_\nu). \tag{20}$$

Using the hypothesis of a space-time independent metric, the Lagrangian can be written as follows:

$$\mathcal{L}^{\text{stat.}} = -\frac{T}{2}\sum_n \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \times \log[-\beta^2(g^{00}p_0^2 + 2g^{0i}p_0 p_i + g^{ij}p_i p_j)], \tag{21}$$

where we have split the metric into its space-time components and neglected terms which are temperature independent.

Let us now perform the change of variables

$$p_i \to p_i' = M_i^j p_j + f_i p_0, \tag{22}$$

where M is symmetric. Upon imposing the condition

$$p_i' p_i' = g^{ij}p_i p_j + 2g^{0i}p_0 p_i + f_i f_i p_0^2, \tag{23}$$

we obtain

$$M_i^j M_i^k = g^{jk} \qquad f^i M^{ij} = g^{0j}. \tag{24}$$

Therefore, the effective Lagrangian can be written as

$$\mathcal{L}^{\text{stat.}} = -\frac{T}{2}\frac{1}{\sqrt{-\det \mathbf{g}}}\sum_n \int \frac{d^{d-1}p'}{(2\pi)^{d-1}} \times \log[-\beta^2((g^{00} - f^i f^i)p_0^2 - p_i' p_j')], \tag{25}$$

where the entries of the matrix $\mathbf{g}$ are g^{ij}. Performing the transformation

$$p_i \to \sqrt{g^{00} - f^j f^j}\, p_i \tag{26}$$

we readily obtain

$$\mathcal{L}^{\text{stat.}} = \mathcal{L}^{(0)} \frac{(g^{00} - (\mathbf{g}^{-1})^{ij} g^{0i} g^{0j})^{(d-1)/2}}{\sqrt{-\det \mathbf{g}}}. \tag{27}$$

A straightforward calculation shows that the perturbative results obtained in the previous section can be generated from the Lagrangian (27). Indeed, using Eq. (2), the first-order contribution from Eq. (27) is simply

$$\mathcal{L}^{(1)} = \frac{\mathcal{L}^{(0)}}{2}(d\tilde{h}^{00} - \tilde{h}), \tag{28}$$

which is the same as the first-order term in (11). Proceeding similarly, the second-order contribution from (27) produces

$$\mathcal{L}^{(2)} = \frac{1}{8}\mathcal{L}^{(0)}[d(d-2)(\tilde{h}^{00})^2 - 2d\tilde{h}^{00}\tilde{h} + \tilde{h}^2 + 2(\tilde{h}_{\mu\nu}\tilde{h}^{\mu\nu})_{\tilde{h}_{0i}=0} - 4(d-1)\tilde{h}_{0i}\tilde{h}^{0i}], \tag{29}$$

which is also in agreement with the second-order contribution from Eq. (11). We recall that, as a consequence of conformal invariance, the results are unchanged under the transformation $g^{\mu\nu} \to \tilde{g}^{\mu\nu} \equiv \sqrt{-g}\, g^{\mu\nu}$.

We can also express the exact result in terms of the covariant metric components. Using the identity (this follows from $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$)

$$g^{00} - (\mathbf{g}^{-1})^{ij} g^{0i} g^{0j} = (g_{00})^{-1}, \tag{30}$$

Eq. (27) can be written as

$$\mathcal{L}^{\text{stat.}} = \mathcal{L}^{(0)} \frac{\sqrt{-g_{00}\, \det \mathbf{g}^{-1}}}{(g_{00})^{d/2}}, \tag{31}$$

Expanding the determinant of $g^{\mu\nu}$ in terms of cofactors and using the identity $g_{i0} = -\mathbf{g}^{-1}_{ij} g^{j0} g_{00}$ as well as (30) we can show that

$$g_{00}\, \det \mathbf{g}^{-1} = \frac{1}{\det g^{\mu\nu}} = g. \tag{32}$$

Therefore, Eq. (31) yields

$$\mathcal{L}^{\text{stat.}} = \mathcal{L}^{(0)} \frac{\sqrt{-g}}{(g_{00})^{d/2}}, \tag{33}$$

which is in agreement with the known result when $d = 4$ [2]. This static effective Lagrangian can also be obtained using a much more involved approach in terms of the heat-kernel technique restricted to a static space-time, in a reference frame such that $g_{0i} = 0$ [11]. Since the heat bath breaks the invariance under general coordinate transformations, as it is evident due to the presence of the Matsubara sum in Eq. (20), it is essential to perform the calculation for general values of g_{0i}. Physically one must impose that the heat bath is freely moving in a time-like geodesic, so that in the heat bath frame g_{0i} vanishes only in very special cases, even for static space-times.

IV. DISCUSSION

In this paper we have presented a simple method which allows to obtain the effective Lagrangian of static gravitational fields interacting with thermal fields. A key ingredient in this analysis was the hypothesis that the effective Lagrangian can be obtained from a space-time independent background. Conversely, one can also claim, using the perturbative results of Sec. II, further support to this hypothesis.

It is not very difficult to extend the present analysis to include spinor and gauge thermal fields. A straightforward calculation shows that the only modification is the replacement of $\mathcal{L}^{(0)}$ by the corresponding free contributions of fermions or gauge bosons.

We point out that the perturbative expansion of the static effective Lagrangian given by Eq. (33), is in agreement with the known static limit of Feynman amplitudes. On the other hand, we have demonstrated in this paper that (33) can be directly derived from the functional determinant given by Eq. (1), when the metric is space-time independent. This is consistent with the analytical behavior of individual thermal Feynman amplitudes [3]. Therefore, the present approach may be a suitable starting point towards the analysis of more general backgrounds. This would be useful in order to obtain a closed form expression for other background configurations, such as the long-wavelength limit, or even more general gravitational backgrounds, using the known symmetries which are characteristic of the high-temperature limit.

The result given by Eq. (33) may also be viewed as the pressure of a weakly interacting gas subjected to an external gravitational field. An obvious extension of the present analysis would be to consider the contributions which are higher than one-loop, so that the effects of interactions between the gas particles would be taken into account, and more realistic applications could be considered. This may be interesting in the context of stellar evolution or cosmology.

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APPENDIX A

In this appendix we compute the first- and second-order terms in Eq. (8). Let us first notice that the space-time trace of $I_{\mu\nu}$ in Eq. (9),

$$\eta_{\mu\nu} I^{\mu\nu} = -\frac{T}{2}\sum_{n=-\infty}^{\infty}\int \frac{d^{d-1}p}{(2\pi)^{d-1}}, \tag{A1}$$

can be set equal to zero in the context of the dimensional regularization technique. Therefore, the most general form of $I_{\mu\nu}$ can be written as

$$I_{\mu\nu} = a(du_\mu u_\nu - \eta_{\mu\nu}), \tag{A2}$$

where u_μ is such that $\eta^{\mu\nu}u_\mu u_\nu = 1$ and can be identified with the heat bath four-velocity (in the rest frame of the heat bath $u_\mu = (1, 0, 0, \cdots, 0)$). Projecting both sides of Eq. (A2) along the tensor $(du_\mu u_\nu - \eta_{\mu\nu})$ and using Eq. (9) yields

$$a = -\frac{T}{2}\frac{1}{d-1}\sum_{n=-\infty}^{\infty}\int \frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{p_0^2}{p^2}. \tag{A3}$$

Using $p_0^2 = p^2 + |\vec{p}|^2$ and the dimensional regularization prescription for the temperature independent terms,

$$a = \frac{1}{2}\frac{1}{d-1}\int \frac{d\Omega}{(2\pi)^d}\int_0^\infty |\vec{p}|^d S^1(|\vec{p}|)d|\vec{p}|, \tag{A4}$$

where

$$\begin{aligned} S^1(|\vec{p}|) &= T\sum_{n=-\infty}^{\infty}\frac{1}{(2\pi nT)^2 + |\vec{p}|^2} \\ &= \frac{1}{2|\vec{p}|} + \frac{1}{|\vec{p}|}\frac{1}{e^{|\vec{p}|/T} - 1}. \end{aligned} \tag{A5}$$

Substituting Eq. (A5). into Eq. (A4) and using again the dimensional regularization prescription for the T-independent term, we finally obtain

$$a = \frac{1}{2}\frac{1}{d-1}\int \frac{d\Omega}{(2\pi)^d}\int_0^\infty \frac{|\vec{p}|^{d-1}}{e^{|\vec{p}|/T} - 1}d|\vec{p}| = \frac{\mathcal{L}^{(0)}}{2}, \tag{A6}$$

where in the previous expression we have identified the result for the effective Lagrangian of free scalar fields. From Eqs. (4) and (8), (A2) and (A6) we obtain the following result up to the first order

$$\begin{aligned} \mathcal{L}^{\text{stat.}} &= \mathcal{L}^{(0)}\left[1 + \frac{1}{2}(du_\mu u_\nu - \eta_{\mu\nu})\tilde{h}^{\mu\nu}\right] + \cdots \\ &= \mathcal{L}^{(0)}\left[1 + \frac{1}{2}(d\tilde{h}^{00} - \tilde{h})\right] + \cdots. \end{aligned} \tag{A7}$$

In the second line of the previous equation the effective Lagrangian is explicitly expressed in the rest frame of the thermal bath.

Let us now consider the second order contribution to $\mathcal{L}^{\text{stat}}$. From the tensorial symmetry of the Eq. (10) one can see that the result of the integration can be written in terms of three independent tensors as follows:

$$\begin{aligned} I^{\mu\nu\alpha\beta} = b_1 u^\mu u^\nu u^\alpha u^\beta &+ b_2(\eta^{\mu\nu}u^\alpha u^\beta + \text{symm.}) \\ &+ b_3(\eta^{\mu\nu}\eta^{\alpha\beta} + \text{symm.}). \end{aligned} \tag{A8}$$

Projecting both sides of Eq. (A8) along each of the three tensors and solving the resulting system, yields

$$b_1 = -(d+2)\mathcal{L}^{(0)} + (d+2)(d+4)b \tag{A9a}$$

$$b_2 = \frac{\mathcal{L}^{(0)}}{2} - (d+2)b \tag{A9b}$$

$$b_3 = b, \tag{A9c}$$

where

$$b = \frac{1}{2}\frac{1}{(d-1)(d+1)}\int \frac{d\Omega}{(2\pi)^d}\int_0^\infty |\vec{p}|^{d+2}S^2(|\vec{p}|)d|\vec{p}|, \tag{A10}$$

and

$$S^2(|\vec{p}|) = T\sum_{n=-\infty}^{\infty}\frac{1}{[(2\pi nT)^2 + |\vec{p}|^2]^2} \tag{A11}$$

Using the identity

$$\frac{1}{[(2\pi nT)^2 + |\vec{p}|^2]^2} = -\frac{1}{2|\vec{p}|}\frac{d}{d|\vec{p}|}\frac{1}{(2\pi nT)^2 + |\vec{p}|^2} \tag{A12}$$

and performing integration by parts, one can proceed similarly to the first-order calculation, yielding the result

$$b = \frac{\mathcal{L}^{(0)}}{4}. \tag{A13}$$

Substituting Eq. (A13) into Eq. (A9) we obtain the following result for the three structure constants in Eq. (A8).

$$b_1 = \frac{d(d+2)}{4}\mathcal{L}^{(0)} \tag{A14a}$$

$$b_2 = -\frac{d}{4}\mathcal{L}^{(0)} \tag{A14b}$$

$$b_3 = \tfrac{1}{4}\mathcal{L}^{(0)}. \tag{A14c}$$

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