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QUASI-ISOMETRIES ON SUBSETS OF $C_0(K)$ AND $C_0^{(1)}(K)$ SPACES WHICH DETERMINE K

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ABSTRACT. We introduce the concept of Banach-Stone subsets of $C_0(K)$ spaces. This allows us to unify and improve several extensions of the classical theorem due to Banach (1933) and Stone (1937). More precisely, we prove that if K and S are locally compact Hausdorff spaces, A and B are Banach-Stone subsets of $C_0(K)$ and $C_0(S)$, respectively, and there exists a map T from A to B (not necessarily injective) with image θ-dense in B for some $\theta > 0$ such that

$$\frac{1}{M}\|f-g\| - L \leq \|T(f)-T(g)\| \leq M\|f-g\| + L,$$

for every $f, g \in A$, then K and S are homeomorphic whenever $L \geq 0$ and $M < \sqrt{2}$. As an application of this more general theorem concerning the quasi-isometries T on subsets of $C_0(K)$ spaces, we show that certain quasi-isometries on $C_0^{(1)}(K)$ spaces also determine the locally compact subspaces K of the real line $\mathbb{R}$ with no isolated points. In turn, this result enables us to prove a unification and improvement of some theorems of Cambern, Pathak, and Vasavada for the first time to the nonlinear case.

1. Introduction

If K is a locally compact Hausdorff space, we denote by $C_0(K)$ the Banach space of real-valued continuous functions vanishing at infinity on K provided with the supremum norm. For any locally compact $K \subset \mathbb{R}$ with no isolated points, we denote by $C_0^{(1)}(K)$ the space of real-valued continuously differentiable functions defined on K such that f and f' both vanish at infinity. This space may be equipped with either the supremum norm $\|\cdot\|_\infty$ or one of the following classic norms: $\|f\|_C = \sup\{|f'(k)| + |f(k)| : k \in K\}$, $\|f\|_M = \max\{\|f'\|_\infty, \|f\|_\infty\}$, and $\|f\|_\Sigma = \|f'\|_\infty + \|f\|_\infty$ for every $f \in C_0^{(1)}(K)$.

The purpose of this paper is twofold: to study quasi-isometries on subsets of $C_0(K)$ spaces and also on subsets of $C_0^{(1)}(K)$ spaces which determine K.

We recall that for any pair of metric spaces (E, d_E) and (F, d_F), a map $T : E \to F$ is said to be a coarse (M, L)-quasi-isometry, for some constants $M \geq 1$ and $L \geq 0$,

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or simply a *quasi-isometry* [4, 15], if the following inequalities hold:

$$\frac{1}{M}d_E(f,g) - L \leq d_F(T(f),T(g)) \leq M d_E(f,g) + L \quad \forall f,g \in E,$$

and there exists $\theta > 0$ such that for all $u \in F$ there is $u' \in T(E)$ such that $d_F(u,u') \leq \theta$. In other words, $T(E)$ is θ*-dense* in F.

This notion includes some important concepts used in the nonlinear classification of Banach spaces [14, 15, 18, 19].

The well-known Banach-Stone theorem asserts that if there is an isometry T from $C_0(K)$ onto $C_0(S)$, then K and S are homeomorphic [3, 22]. Amir [2] and Cambern [5] independently generalized this theorem by showing that if there is a linear isomorphism T from $C_0(K)$ onto $C_0(S)$ satisfying $\|T\|\|T^{-1}\| < 2$, then K and S are homeomorphic. In [6], Cambern showed that 2 is the best constant for this theorem; see also [10]. In [8] Cengiz extended the Amir-Cambern theorem. He said that a closed linear subspace A of $C_0(K)$ is *extremely regular* if for each $k \in K$ and each open neighborhood V of k and $0 < \epsilon < 1$ there exists $f \in A$ such that

$$1 = \|f\| = f(k) > \epsilon > |f(k')| \quad \forall\, k' \notin V; \tag{1.1}$$

see [9], [17, p. 319], and [16, p. 882] for examples of such subspaces. So, he proved:

Theorem 1.1. *Let K and S be locally compact Hausdorff spaces and let A and B be extremely regular subspaces of $C_0(K)$ and $C_0(S)$, respectively. If there is an isomorphism T from A onto B with $\|T\|\|T^{-1}\| < 2$, then K and S are homeomorphic.*

On the other hand, Cambern and Pathak [7] obtained the following version of the Amir-Cambern theorem for spaces of differentiable functions:

Theorem 1.2. *Let K and S be locally compact subsets of $\mathbb{R}$ with no isolated points and the spaces $C_0^{(1)}(K)$ and $C_0^{(1)}(S)$ endowed with the norm $\|\cdot\|_C$. Suppose that T is an isomorphism from $C_0^{(1)}(K)$ onto $C_0^{(1)}(S)$ with $\|T\|\|T^{-1}\| < 2$ and T is an isomorphism with respect to the sup norms. Then K and S are homeomorphic.*

Moreover, Pathak and Vasavada [21] (see [13]) showed that the statement of Theorem 1.2 is still valid when the spaces $C_0^{(1)}(K)$ and $C_0^{(1)}(S)$ are equipped with the norm $\|\cdot\|_M$. Although Theorems 1.1 and 1.2 are in different contexts, we will see that they are closely connected. Indeed, we will provide a unification and extension of them for quasi-isometries. In order to do this, it will be convenient to introduce the notion of *Banach-Stone subsets* of $C_0(K)$ spaces.

Definition 1.3. Let $A \subset C_0(K)$ and $\varepsilon, \delta > 0$. We say that A is an (ε,δ)-subset of $C_0(K)$ if the following conditions are satisfied:

(1) For every open set U of K and $\mathrm{v} \in \mathbb{R}$, there exists $F_{\mathrm{v},U} \in A$ such that
  (a) $|\mathrm{v}| - \varepsilon \leq \|F_{\mathrm{v},U}\| \leq |\mathrm{v}| + \varepsilon$;
  (b) $\inf\{|F_{\mathrm{v},U}(k) - \alpha \mathrm{v}| : \alpha \in [0,1]\} \leq \varepsilon$ for every $k \in U$;
  (c) $|F_{\mathrm{v},U}(k)| \leq \varepsilon$ for every $k \in K \setminus U$.
(2) For every $k, k' \in K$ and $\mathrm{v}, \mathrm{v}' \in \mathbb{R}$ with $k \neq k'$ and $||\mathrm{v}| - |\mathrm{v}'|| < \delta$, there exists $G \in A$ such that

$$|G(k) - \mathrm{v}| \leq \varepsilon, \quad |G(k') - \mathrm{v}'| \leq \varepsilon, \quad \text{and} \quad \|G\| \leq \max\{|\mathrm{v}|, |\mathrm{v}'|\} + \varepsilon.$$

We say that A is a Banach-Stone subset of $C_0(K)$ (in short, BS-subset) if A is an (ε,δ)-subset of $C_0(K)$ for some ε and δ.

As we shall see in section 2, there is a natural way to construct BS-subsets of $C_0(K)$: every θ-dense subset of an extremely regular subspace is a BS-subset of $C_0(K)$. Thus, Theorem 1.1 is an immediate consequence of our main theorem:

Theorem 1.4. *Let K and S be locally compact Hausdorff spaces and let A and B be BS-subsets of $C_0(K)$ and $C_0(S)$, respectively. If there is an (M, L)-quasi-isometry T from A to B with $M < \sqrt{2}$, then K and S are homeomorphic.*

Note that by the above-mentioned results of [6,10], $\sqrt{2}$ is the best number for the formulation of Theorem 1.4. Moreover, Theorem 1.4 in the particular case where $A = C_0(K)$, $B = C_0(S)$, and T is bijective is [11, Theorem 1.2(a)].

The more general statement of Theorem 1.4 by using BS-subsets instead of extremely regular subspace of $C_0(K)$ spaces will allow us to get in section 10 the following improvement of Theorem 1.2. Our result is valid not only when the $C_0^{(1)}(K)$ spaces are equipped with one of the norms $\|\cdot\|_C$, $\|\cdot\|_M$, or $\|\cdot\|_\Sigma$, but with any norm $\|\cdot\|$ satisfying the following inequalities for every f in $C_0^{(1)}(K)$:

$$\|f'\|_\infty \leq \|f\| \leq \|f\|_\Sigma. \tag{1.2}$$

Denote by $\mathcal{D}$ the map $f \mapsto f'$ in both the spaces $C_0^{(1)}(K)$ and $C_0^{(1)}(S)$.

Theorem 1.5. *Let K and S be locally compact subsets of $\mathbb{R}$ with no isolated points and assume that the norms of all f in $C_0^{(1)}(K)$ or in $C_0^{(1)}(S)$ satisfy (1.2). Suppose that there is an (M, L)-quasi-isometry T from $A \subset C_0^{(1)}(K)$ to $B \subset C_0^{(1)}(S)$, where A and B are $\|\cdot\|_\infty$-bounded and $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are BS-subsets of $C_0(K)$ and $C_0(S)$, respectively. If $M < \sqrt{2}$, then K and S are homeomorphic.*

To show that Theorem 1.5 is in fact an improvement of Theorem 1.2, we will prove in section 11 that it implies the following nonlinear extension of Theorem 1.2.

Corollary 1.6. *Suppose that K and S are locally compact subsets of $\mathbb{R}$ with no isolated points and the norms of all f in $C_0^{(1)}(K)$ or in $C_0^{(1)}(S)$ satisfy (1.2). If there is a bijective (M, L)-quasi-isometry $T : C_0^{(1)}(K) \to C_0^{(1)}(S)$ with $M < \sqrt{2}$ and T is a quasi-isometry with respect to the sup norms, then K and S are homeomorphic.*

The proof method of Theorem 1.4 is a very close adaptation of the method developed in [11]. The difference comes from the fact that the quasi-isometry T in [11] is between the spaces $C_0(K)$ and $C_0(S)$ and in this work the quasi-isometry T is between certain specified subsets $A \subset C_0(K)$ and $B \subset C_0(S)$, that is, BS-subsets which have a density type property; see Definition 1.3. Therefore, in the present case, the proofs of auxiliary results analogous to those of [11] will be more technical. New definitions and small adjustments will be needed in the proofs. For the sake of clarity and completeness, we give the arguments in detail adapted to this more general setting.

The proof will be done in the next eight sections. In section 2 we state some basic properties of BS-subsets of $C_0(K)$ spaces. In section 3, we observe that we may assume that T satisfies some good properties, such as being bijective. Next, the proof goes through a path similar to the techniques in [11]. In order to obtain a homeomorphism $\varphi : K \to S$, we define in section 5 a new class of special sets $\mathcal{A}_{\mathrm{w}}(k, \mathrm{v}) \subset S$ depending on the BS-subset $A \subset C_0(K)$ and show that, under certain conditions, these sets are non-empty (Proposition 5.1). Further, the intuition is that for each $k \in K$ the elements of the sets $\mathcal{A}_{\mathrm{w}}(k, \mathrm{v})$ are candidates to be $\varphi(k)$. Later, in

Proposition 7.1, we state that, under certain conditions, for each $k \in K$ there exists a unique $s \in S$ such that $s \in \mathcal{A}_{\mathrm{w}}(k, \mathrm{v})$. Thus, for each $k \in K$ we define $\varphi(k)$ to be the element s given by Proposition 7.1. Since T is bijective, using the symmetry on the definition of quasi-isometry, we will be able to define $\psi : S \to K$ with respect to T^{-1} in an analogous way to the definition of φ. Finally, in Proposition 9.1, we prove that $\psi = \varphi^{-1}$ and that both φ and ψ are continuous.

2. On Banach-Stone subsets of $C_0(K)$ spaces

In this section we provide some examples and properties of BS-subsets of $C_0(K)$ spaces; see also Proposition 11.2. We shall use the following lemma proved in [20, Proof of Theorem 1]; cf. [1, Lemma 2.4].

Lemma 2.1. *Let A be an extremely regular subspaces of $C_0(K)$, let U be a non-empty open subset of K, and let $0 < \varepsilon < 1$. Then there are $k_0 \in U$ and $f \in A$ such that*

(1) $f(k_0) = 1 \leq \|f\| \leq 1 + \varepsilon$*;*
(2) $-\varepsilon \leq f(k) \leq 1 + \varepsilon$ *for every* $k \in U$*;*
(3) $|f(k)| \leq \varepsilon$ *for every* $k \in K \setminus U$*.*

Proposition 2.2. *If A is an extremely regular subspace of $C_0(K)$, then A is an (ε, δ)-subset of $C_0(K)$ for any ε and δ.*

Proof. Fix $\varepsilon, \delta > 0$. Then,

(1) Put $\theta = \min\{1/2, \varepsilon/|\mathrm{v}|\}$. By Lemma 2.1, there is $f_U \in A$ satisfying:

(i) $1 \leq \|f_U\| \leq 1 + \theta$;
(ii) $-\theta \leq f_U(k) \leq 1 + \theta$ for every $k \in U$;
(iii) $|f_U(k)| \leq \theta$ for every $k \in K \setminus U$.

Define $F_{\mathrm{v},U} = \mathrm{v} \cdot f_U$. Then, it is easy to see that (i), (ii), and (iii) imply, respectively, properties (1a), (1b), and (1c) in the definition of BS-subset.

(2) Let $k \neq k'$ and $\mathrm{v}, \mathrm{v}' \in \mathbb{R}$. Since K is Hausdorff we fix U and U' disjoint open neighborhoods of k and k', respectively. By (1.1), we fix $f \in A$ such that

$$1 = \|f\| = f(k) > \varepsilon/|\mathrm{v}| \geq |f(m)| \quad \forall m \in K \setminus U$$

and $f' \in A$ such that

$$1 = \|f'\| = f'(k') > \varepsilon/|\mathrm{v}'| \geq |f'(m)| \quad \forall m \in K \setminus U'.$$

Define $G = \mathrm{v} \cdot f + \mathrm{v}' \cdot f' \in A$. Then, since $k \in K \setminus U'$,

$$|G(k) - \mathrm{v}| = |\mathrm{v} f(k) + \mathrm{v}' f'(k) - \mathrm{v}| = |\mathrm{v}' f'(k)| \leq \varepsilon.$$

Similarly, we see that $|G(k') - \mathrm{v}'| \leq \varepsilon$. Finally, notice that

$$|G(m)| \leq |\mathrm{v}||f(m)| + |\mathrm{v}'||f'(m)| \leq \varepsilon + |\mathrm{v}'| \quad \forall m \in K \setminus U$$

and similarly that

$$|G(m)| \leq \varepsilon + |\mathrm{v}| \quad \forall m \in K \setminus U'.$$

Then since $(K \setminus U) \cup (K \setminus U') = K$, it follows that

$$\|G\| \leq \max\{|\mathrm{v}|, |\mathrm{v}'|\} + \varepsilon.$$

Since v, v' are arbitrary, such a G exists in the particular case $||\mathrm{v}| - |\mathrm{v}'|| < \delta$, so item (2) in the definition of BS-subset holds. □

The next result follows directly from Definition 1.3 and will be useful later.

Proposition 2.3. *Let A be an (ε,δ)-subset of $C_0(K)$. Then:*

(1) *for each $\alpha>0$, αA is an $(\alpha\varepsilon,\alpha\delta)$-subset of $C_0(K)$;*
(2) *for each $f_0\in C_0(K)$, f_0+A is an $(\varepsilon+\|f_0\|,\delta)$-subset of $C_0(K)$;*
(3) *if A' is θ-dense in A, then A' is a $(\theta+\varepsilon,\delta)$-subset of $C_0(K)$.*

3. Fixing suitable (M,L)-quasi-isometry T and BS-subsets A and B

From now on we will fix K, S, T, A, B, M, and L as in the hypothesis of Theorem 1.4. Our task until section 9 will be to prove that K and S are homeomorphic.

In the present section it will be observed that we may assume that T is bijective, $0\in A$, $T(0)=0$, T and T^{-1} are both bijective $(M,1/2)$-quasi-isometries, and A and B are $(1/(2M),\theta)$-subsets for some $\theta>0$.

Remark 3.1. We may assume that T is bijective. Indeed, fix $\delta>ML$ and pick a maximal δ-separated subset $A'\subset A$ (that is, for any distinct $f,g\in A'$, $\|f-g\|\geq\delta$). Then, for any distinct $f,g\in A'$ we obtain

$$\|Tf-Tg\|\geq\frac{1}{M}\|f-g\|-L\geq\frac{1}{M}\delta-L>0.$$

Hence, T is injective in A' and defining $\hat{T}:A'\to T(A')$ to be the restriction of T, we have that $\hat{T}$ is a bijective (M,L)-quasi-isometry. We have only to observe that A' and $T(A')$ are BS-subsets of $C_0(K)$ and $C_0(S)$, respectively. The maximality of A' implies that A' is δ-dense in A. So, by Proposition 2.3(3), A' is a BS-subset of $C_0(K)$. Moreover, the fact that A' is δ-dense in A also implies that

$$T(A')\quad\text{is } (M\delta+L)\text{-dense in } T(A). \tag{3.1}$$

On the other hand, by the definition of quasi-isometries, $T(A)$ is θ-dense in B for some $\theta>0$. Thus according to Proposition 2.3(3),

$$T(A)\quad\text{is a BS-subset of } C_0(S). \tag{3.2}$$

Therefore, again by Proposition 2.3(3), we conclude by (3.1) and (3.2) that $T(A')$ is a BS-subset of $C_0(S)$ and we are done.

Remark 3.2. We may suppose that $0\in A$ and that $T(0)=0$. Indeed, fix $f_0\in A$ and define $A_0=-f_0+A$ and $B_0=-Tf_0+B$. By Proposition 2.3(2), we know that A_0 and B_0 are BS-subsets of $C_0(K)$ and $C_0(S)$, respectively. Define $\hat{T}:A_0\to B_0$ by $\hat{T}(f)=T(f+f_0)-Tf_0$. It is easy to see that $\hat{T}:A_0\to B_0$ is a bijective (M,L)-quasi-isometry and $\hat{T}(0)=0$.

Remark 3.3. We may assume that T and T^{-1} are bijective $(M,1/2)$-quasi-isometries and that A and B are $(1/(2M),\theta)$-subsets of $C_0(K)$ and $C_0(S)$, respectively, for some $\theta>0$. Indeed, if A is an (ε_A,δ_A)-subset of $C_0(K)$ and B is an (ε_B,δ_B)-subset of $C_0(S)$, then by putting $\varepsilon=\max\{\varepsilon_A,\varepsilon_B\}$ and $\delta=\min\{\delta_A,\delta_B\}$, it follows that A and B are (ε,δ)-subsets of $C_0(K)$ and $C_0(S)$, respectively.

On the other hand, it is easy to see that for each $\alpha>0$ the map $\hat{T}:\alpha A\to\alpha B$ defined by $\hat{T}(f)=\alpha T(f/\alpha)$ is a bijective $(M,\alpha L)$-quasi-isometry between $(\alpha\varepsilon,\alpha\delta)$-subsets of $C_0(K)$ and $C_0(S)$, respectively. Hence, by picking $\alpha>0$ such that $\alpha L\leq 1/(2M)$ and $\alpha\varepsilon\leq 1/(2M)$, we conclude that $\hat{T}$ and $\hat{T}^{-1}$ are bijective $(M,1/2)$-quasi-isometries between $(1/(2M),\alpha\delta)$-subsets of $C_0(K)$ and $C_0(S)$, respectively.

4. The numbers $\omega(k, f, \mathrm{v})$ and the BS-subsets of $C_0(K)$ spaces

Following [11, p. 2170], for any locally compact Hausdorff space H, $k \in H$, $f \in C_0(H)$, and $\mathrm{v} \in \mathbb{R}$ we define

$$\omega(k, f, \mathrm{v}) = \max\{\|f\|, |f(k) - \mathrm{v}|\}.$$

In this section we prove the following lemma, which is a generalization of [11, Lemma 2.1] for the BS-subset A.

Lemma 4.1. *Let $k \in K$ and $\mathrm{v} \in \mathbb{R}$. Then, there is a net $(F_i)_{i\in I} \subset A$ such that*

$$|\mathrm{v}| - \frac{1}{2M} \leq \|F_i\| \leq |\mathrm{v}| + \frac{1}{2M} \quad \forall i \in I, \tag{4.1}$$

and for every $f \in C_0(K)$,

$$\limsup_{i\in I} \|f - F_i\| \leq \omega(k, f, \mathrm{v}) + \frac{1}{2M}. \tag{4.2}$$

Proof. Let $\mathcal{V}_k$ denote the set of open neighborhoods of k. Since A is a $(1/(2M), \theta)$-subset, for each $U \in \mathcal{V}_k$ we fix $F_{\mathrm{v},U} \in A$ satisfying:

(i) $|\mathrm{v}| - 1/(2M) \leq \|F_{\mathrm{v},U}\| \leq |\mathrm{v}| + 1/(2M)$;
(ii) $\inf\{|F_{\mathrm{v},U}(k) - \alpha\mathrm{v}| : \alpha \in [0, 1]\} \leq 1/(2M)$ for every $k \in U$;
(iii) $|F_{\mathrm{v},U}(k)| \leq 1/(2M)$ for every $k \in K \setminus U$.

Consider the net $(F_{\mathrm{v},U})_{U\in\mathcal{V}_k}$. Then (4.1) follows immediately by (i), and we only need to prove (4.2). Fix $f \in C_0(K)$. Given $\varepsilon > 0$, take $U_\varepsilon \in \mathcal{V}_k$ such that

$$|f(u) - f(k)| < \varepsilon \quad \forall u \in U_\varepsilon.$$

Now, pick $U \in \mathcal{V}_k$ such that $U \subset U_\varepsilon$. We shall evaluate $\|f - F_{\mathrm{v},U}\|$. If $u \in U$, then

$$|f(u) - F_{\mathrm{v},U}(u)| \leq |f(k) - F_{\mathrm{v},U}(u)| + \varepsilon. \tag{4.3}$$

Since (ii) holds, pick $u' \in \{\alpha\mathrm{v} : \alpha \in [0, 1]\}$ such that

$$|F_{\mathrm{v},U}(u) - u'| \leq \frac{1}{2M} + \varepsilon.$$

Consequently

$$|f(k) - F_{\mathrm{v},U}(u)| \leq |f(k) - u'| + \frac{1}{2M} + \varepsilon \leq \max\{|f(k)|, |f(k) - \mathrm{v}|\} + \frac{1}{2M} + \varepsilon,$$

and therefore by (4.3)

$$|f(u) - F_{\mathrm{v},U}(u)| \leq \max\{|f(k)|, |f(k) - \mathrm{v}|\} + \frac{1}{2M} + 2\varepsilon \quad \forall u \in U. \tag{4.4}$$

On the other hand, it follows by (iii) that

$$|f(u) - F_{\mathrm{v},U}(u)| \leq |f(u)| + \frac{1}{2M} \leq \|f\| + \frac{1}{2M} \quad \forall u \in K \setminus U. \tag{4.5}$$

According to (4.4) and (4.5) we conclude that

$$\|f - F_{\mathrm{v},U}\| \leq \max\{\|f\|, |f(k) - \mathrm{v}|\} + \frac{1}{2M} + 2\varepsilon,$$

and we are done. □

5. Special sets associated to (M, L)-quasi-isometries between BS-subsets

Let $\mathrm{v}, \mathrm{w} \in \mathbb{R}$ with $|\mathrm{v}| \geq M$ and $|\mathrm{w}| = |\mathrm{v}|/M - 1$. Inspired by the definitions of the special sets $\Gamma_{\mathrm{w}}(k, \mathrm{v})$ and $\Gamma_{\mathrm{v}}(s, \mathrm{w})$ given in [11, Definition 3.1], we will introduce two new sets $\mathcal{A}_{\mathrm{w}}(k, \mathrm{v})$ and $\mathcal{B}_{\mathrm{z}}(s, \mathrm{w})$ that in the context of this paper will depend on BS-subsets $A \subset C_0(K)$ and $B \subset C_0(S)$, respectively. So, we set

$$\mathcal{A}_{\mathrm{w}}(k, \mathrm{v}) = \{s \in S \ : \ |Tf(s) - \mathrm{w}| \leq M\omega(k, f, \mathrm{v}) + 1 \quad \forall f \in A\}.$$

Similarly, for $s \in S$, $\mathrm{w}, \mathrm{z} \in \mathbb{R}$ with $|\mathrm{w}| \geq M$ and $|\mathrm{z}| = |\mathrm{w}|/M - 1$, we also set

$$\mathcal{B}_{\mathrm{z}}(s, \mathrm{w}) = \left\{k \in K \ : \ |T^{-1}g(k) - \mathrm{z}| \leq M\omega(s, g, \mathrm{w}) + 1 \quad \forall g \in B\right\}.$$

The objective of this section is to prove that under certain conditions the sets $\mathcal{A}_{\mathrm{w}}(k, \mathrm{v})$ and $\mathcal{B}_{\mathrm{z}}(s, \mathrm{w})$ are not empty (Proposition 5.1). First of all, notice that since it is assumed in the definition of $\mathcal{A}_{\mathrm{w}}(k, \mathrm{v})$ that $|\mathrm{v}| \geq M$ and $|\mathrm{w}| = |\mathrm{v}|/M - 1$, these restrictions over $|\mathrm{v}|$ and $|\mathrm{w}|$ will be implicit in every statement involving the sets $\mathcal{A}_{\mathrm{w}}(k, \mathrm{v})$. Also, the restrictions $|\mathrm{w}| \geq M$ and $|\mathrm{z}| = |\mathrm{w}|/M - 1$ will be implicit in any statement involving the sets $\mathcal{B}_{\mathrm{z}}(s, \mathrm{w})$.

Notice that since both T and T^{-1} are bijective $(M, 1/2)$-quasi-isometries between $(1/(2M), \theta)$-subsets, any result involving the sets $\mathcal{A}_{\mathrm{w}}(k, \mathrm{v})$ will also hold for the sets $\mathcal{B}_{\mathrm{z}}(s, \mathrm{w})$. We shall refer to these results in either case in what follows.

The following proposition is a weak version of [11, Proposition 3.2]. In [11] we were able to use Urysohn's Lemma to fix functions $f_0 \in C_0(K)$ such that $\|f_0\| = |\mathrm{v}|/2$ and $f_0(k) = \mathrm{v}/2$ for any $\mathrm{v} \in \mathbb{R}$ and $k \in K$. In our current context, the hypothesis on the set A only guarantees the existence of $f_0 \in A$ such that $\|f_0\| \leq |\mathrm{v}|/2 + 1/2M$ and $|f_0(k) - \mathrm{v}/2| \leq 1/2M$. So we had to take care of several details and thought it wise to write them down.

Proposition 5.1. *There exists $r_0 \geq M$, depending only on M, such that, for all $k \in K$ and $\mathrm{v} \in \mathbb{R}$ with $|\mathrm{v}| \geq r_0$, there exists $\mathrm{w} \in \mathbb{R}$ such that $\mathcal{A}_{\mathrm{w}}(k, \mathrm{v}) \neq \emptyset$.*

Proof. Fix $(F_i)_{i \in I}$ satisfying Lemma 4.1 for k and v. Then, for each $i \in I$, we fix $s_i \in S$ such that $|T(F_i)(s_i)| = \|T(F_i)\|$. Since T is an $(M, 1/2)$-quasi-isometry, $T(0) = 0$, and (4.1) holds, we conclude that

$$\frac{|\mathrm{v}|}{M} - \frac{1}{2M^2} - \frac{1}{2} \leq \liminf_{i \in I} |T(F_i)(s_i)| \leq \limsup_{i \in I} |T(F_i)(s_i)| \leq |\mathrm{v}|M + 1.$$

Then, the net $(T(F_i)(s_i))_{i \in I}$ admits a convergent subnet, and we may assume that $T(F_i)(s_i) \to \mathrm{w}_0$ for some $\mathrm{w}_0 \in \mathbb{R}$ satisfying

$$|\mathrm{w}_0| \geq \frac{|\mathrm{v}|}{M} - \frac{1}{2M^2} - \frac{1}{2} \geq \frac{|\mathrm{v}|}{M} - 1. \tag{5.1}$$

Now, let us see that $(s_i)_{i \in I}$ admits a convergent subnet. By (4.2) we see that

$$\begin{aligned} \limsup_{i \in I} |Tf(s_i) - T(F_i)(s_i)| &\leq \limsup_{i \in I} \|Tf - TF_i\| \\ &\leq M \limsup_{i \in I} \|f - F_i\| + 1/2 \\ &\leq M\omega(k, f, \mathrm{v}) + 1 \ \forall f \in A. \end{aligned} \tag{5.2}$$

Since A is a $(1/(2M), \theta)$-subset, by Definition 1.3(2), we may fix $f_0 \in A$ such that

$$|f_0(k) - \mathrm{v}/2| \leq \frac{1}{2M} \quad \text{and} \quad \|f_0\| \leq \frac{|\mathrm{v}|}{2} + \frac{1}{2M}.$$

Then

$$\omega(k, f_0, \mathrm{v}) \leq \frac{|\mathrm{v}|}{2} + \frac{1}{2M},$$

and by (5.1) and (5.2) we have

$$\begin{aligned}\liminf_{i\in I} |Tf_0(s_i)| &\geq \liminf_{i\in I} |T(F_i)(s_i)| - \limsup_{i\in I} |Tf_0(s_i) - T(F_i)(s_i)| \\ &\geq |\mathrm{w}_0| - (M\omega(k, f_0, \mathrm{v}) + 1) \\ &\geq |\mathrm{v}|/M - 1 - (M|\mathrm{v}|/2 + 3/2) \\ &= |\mathrm{v}|(1/M - M/2) - 5/2.\end{aligned}$$

Since $M < \sqrt{2}$, we see that $1/M - M/2 > 0$. Then, fixing $r_0 \geq M$ such that

$$r_0(1/M - M/2) - 5/2 \geq 0, \tag{5.3}$$

r_0 depends only on M, and for every $|\mathrm{v}| > r_0$ we have

$$\liminf_{i\in I} |Tf_0(s_i)| > 0.$$

Since Tf_0 vanishes at infinity, it follows that $(s_i)_{i\in I}$ admits a convergent subnet, so we may assume that $s_i \to s$ for some $s \in S$. By (5.2), we infer that

$$|Tf(s) - \mathrm{w}_0| \leq M\omega(k, f, \mathrm{v}) + 1 \quad \forall f \in A. \tag{5.4}$$

Put $\alpha_0 = (|\mathrm{v}|/M - 1)/|\mathrm{w}_0|$ and $\mathrm{w} = \alpha_0 \mathrm{w}_0$. So $|\mathrm{w}| = |\mathrm{v}|/M - 1$, and by (5.1), $\alpha_0 \leq 1$.

We will conclude the proof by showing that (5.4) is also satisfied for w instead of w_0. Given $f \in A$, notice that

$$|Tf(s)| \leq \|Tf\| \leq M\|f\| + 1/2 \leq M\omega(k, f, \mathrm{v}) + 1.$$

Then by (5.4)

$$|Tf(s) - \mathrm{w}| \leq \alpha_0 |Tf(s) - \mathrm{w}_0| + (1 - \alpha_0)|Tf(s)| \leq M\omega(k, f, \mathrm{v}) + 1.$$

□

Henceforth we consider r_0 given by Proposition 5.1 to be fixed. Since r_0 depends only on M, this same constant works for sets $\mathcal{B}_{\mathrm{z}}(s, \mathrm{w})$.

6. The special sets $\mathcal{A}_{\mathrm{w}}(k, \mathrm{v})$ when $M^2 < 2$

This section is devoted to proving Corollary 6.2. It concerns the special sets $\mathcal{A}_{\mathrm{w}}(k, \mathrm{v})$ associated to the $(M, 1/2)$-quasi-isometry T. This result is a mid-step in the proof of Proposition 7.1 and will allow us to define a function $\varphi : K \to S$ which as we shall see in section 9 will be a homeomorphism between K and S. The proposition and the corollary below are versions of [12, Proposition 4.1] and [12, Corollary 4.2], respectively, for the sets $\mathcal{A}_{\mathrm{w}}(k, \mathrm{v})$.

Proposition 6.1. *There exists $r_1 \geq r_0$, depending only on M and θ, such that, for all $k \in K$, $\mathrm{v} \in \mathbb{R}$, and $\mathrm{v}' \in \mathbb{R}$ with $|\mathrm{v}|, |\mathrm{v}'| > r_1$, and $|\mathrm{v} + \mathrm{v}'| \leq M\theta$, if $s \in \mathcal{A}_{\mathrm{w}}(k, \mathrm{v})$ and $s' \in \mathcal{A}_{\mathrm{w}'}(k, \mathrm{v}')$ for some $\mathrm{w}, \mathrm{w}' \in \mathbb{R}$, then $s = s'$.*

Proof. Suppose that $s \neq s'$. We may assume that $|\mathrm{v}'| = \max\{|\mathrm{v}|, |\mathrm{v}'|\}$. Then $|\mathrm{w}'| = \max\{|\mathrm{w}|, |\mathrm{w}'|\}$. Note that

$$||\mathrm{w}| - |\mathrm{w}'|| = \frac{1}{M}||\mathrm{v}| - |\mathrm{v}'|| \leq \frac{1}{M}|\mathrm{v} + \mathrm{v}'| \leq \theta.$$

Then, since B is a $(1/(2M),\theta)$-subset, by Definition 1.3(2), take $g \in B$ such that

$$|g(s)+\mathrm{w}| \le \frac{1}{2M}, \quad |g(s')+\mathrm{w}'| \le \frac{1}{2M}, \quad \text{and} \quad \|g\| \le |\mathrm{w}'| + \frac{1}{2M}. \tag{6.1}$$

By applying the definitions of the sets $\mathcal{A}_{\mathrm{w}}(k,\mathrm{v})$ and $\mathcal{A}_{\mathrm{w}'}(k,\mathrm{v}')$ to $T^{-1}g$, we get respectively the following inequalities:

$$2|\mathrm{w}| - \frac{1}{2M} \le |\mathrm{w}+\mathrm{w}| - |g(s)+\mathrm{w}| \le |T(T^{-1}g)(s) - \mathrm{w}| \le M\omega(k, T^{-1}g, \mathrm{v}) + 1 \tag{6.2}$$

and

$$2|\mathrm{w}| - \frac{1}{2M} \le 2|\mathrm{w}'| - \frac{1}{2M} \le |T(T^{-1}g)(s') - \mathrm{w}'| \le M\omega(k, T^{-1}g, \mathrm{v}') + 1. \tag{6.3}$$

Since $|\mathrm{w}| = |\mathrm{v}|/M - 1$, by (6.2) and (6.3), we obtain that

$$\frac{2|\mathrm{v}|}{M} \le M\omega(k, T^{-1}g, \mathrm{v}) + 3 + \frac{1}{2M} \tag{6.4}$$

and

$$\frac{2|\mathrm{v}|}{M} \le M\omega(k, T^{-1}g, \mathrm{v}') + 3 + \frac{1}{2M}. \tag{6.5}$$

Since $|\mathrm{v}+\mathrm{v}'| \le M\theta$, we have that

$$\begin{aligned} \omega(k, T^{-1}g, \mathrm{v}') &= \max\{\|T^{-1}g\|, |T^{-1}g(k) - \mathrm{v}'|\} \\ &\le \max\{\|T^{-1}g\|, |T^{-1}g(k) + \mathrm{v}|\} + |\mathrm{v}+\mathrm{v}'| \\ &\le \omega(k, T^{-1}g, -\mathrm{v}) + M\theta. \end{aligned}$$

Therefore, according to (6.5) we deduce that

$$\frac{2|\mathrm{v}|}{M} \le M\omega(k, T^{-1}g, -\mathrm{v}) + 3 + M^2\theta + \frac{1}{2M}. \tag{6.6}$$

Thus, putting $\Delta = 3 + M^2\theta + \dfrac{1}{2M}$, it follows from (6.4) and (6.6) that

$$\frac{2|\mathrm{v}|}{M} \le M \min\left\{\omega(k, T^{-1}g, \mathrm{v}), \omega(k, T^{-1}g, -\mathrm{v})\right\} + \Delta.$$

That is, $2|\mathrm{v}|/M$ is less than or equal to

$$M \min\left\{\max\{\|T^{-1}g\|, |T^{-1}g(k) - \mathrm{v}|\}, \max\{\|T^{-1}g\|, |T^{-1}g(k) + \mathrm{v}|\}\right\} + \Delta.$$

Then, by using the following identity of real numbers a, b, and c,

$$\min\{\max\{a,b\}, \max\{a,c\}\} = \max\{a, \min\{b,c\}\},$$

with

$$a = \|T^{-1}g\|, \quad b = |T^{-1}g(k) - \mathrm{v}|, \quad \text{and} \quad c = |T^{-1}g(k) + \mathrm{v}|,$$

we have

$$\begin{aligned} \frac{2|\mathrm{v}|}{M} &\le M \max\left\{\|T^{-1}g\|, \min\{|T^{-1}g(k) + \mathrm{v}|, |T^{-1}g(k) - \mathrm{v}|\}\right\} + \Delta \\ &= M \max\left\{\|T^{-1}g\|, ||T^{-1}g(k)| - |\mathrm{v}||\right\} + \Delta. \end{aligned} \tag{6.7}$$

Having in mind (6.1), note that

$$|T^{-1}g(k)| \le \|T^{-1}g\| \le M\|g\| + 1/2 \le |\mathrm{v}'| - M + 1 \le |\mathrm{v}| + 2,$$

so

$$\max\left\{\|T^{-1}g\|, ||T^{-1}g(k)| - |\mathrm{v}||\right\} \le |\mathrm{v}| + 2,$$

and then, by (6.7), we infer that

$$|\mathrm{v}|\left(\frac{2}{M}-M\right)\leq 2M+\Delta.$$

Since $M<\sqrt{2}$ it follows that $2/M-M>0$. Then, there exists $r_1\geq r_0$, depending only on M and θ, such that this is a contradiction if $|\mathrm{v}|>r_1$. □

We consider r_1 given by the Proposition 6.1 to be fixed. Since it depends only on M and θ, this same constant works for the sets $\mathcal{B}_{\mathrm{z}}(s,\mathrm{w})$.

Corollary 6.2. *For all $k\in K$, $s,s'\in S$, and $\mathrm{v},\mathrm{v}'\in\mathbb{R}$, with $|\mathrm{v}|,|\mathrm{v}'|>r_1$ and $|\mathrm{v}-\mathrm{v}'|<M\theta$, if $s\in\mathcal{A}_{\mathrm{w}}(k,\mathrm{v})$ and $s'\in\mathcal{A}_{\mathrm{w}'}(k,\mathrm{v}')$ for some $\mathrm{w},\mathrm{w}'\in\mathbb{R}$, then $s=s'$.*

Proof. Since $|-\mathrm{v}|>r_1\geq r_0$, by Proposition 5.1 there exists $\mathrm{w}''\in\mathbb{R}$ such that $\mathcal{A}_{\mathrm{w}''}(k,-\mathrm{v})\neq\emptyset$. Take $s''\in\mathcal{A}_{\mathrm{w}''}(k,-\mathrm{v})$.

Now observe that since $s''\in\mathcal{A}_{\mathrm{w}''}(k,-\mathrm{v})$ and $s\in\mathcal{A}_{\mathrm{w}}(k,\mathrm{v})$, it follows by Proposition 6.1 that $s''=s$. Moreover, since $s''\in\mathcal{A}_{\mathrm{w}''}(k,-\mathrm{v})$ and $s'\in\mathcal{A}_{\mathrm{w}'}(k,\mathrm{v}')$, again by Proposition 6.1 we infer that $s''=s'$. Hence $s=s'$. □

7. The functions $\varphi:K\to S$ and $\psi:S\to K$

In this section, we will begin to construct a homeomorphism between K and S via the following proposition. The following proposition and its proof is similar to [12, Proposition 5.1].

Proposition 7.1. *For all $k\in K$ there exists $s\in S$ such that for all $\mathrm{v}\in\mathbb{R}$ with $|\mathrm{v}|>r_1$ and $\mathrm{w}\in\mathbb{R}$ either $\mathcal{A}_{\mathrm{w}}(k,\mathrm{v})=\{s\}$ or $\mathcal{A}_{\mathrm{w}}(k,\mathrm{v})=\emptyset$.*

Proof. Take $k\in K$ and put $D=(-\infty,-r_1)\cup(r_1,+\infty)$. It suffices to show that for any $\mathrm{v},\mathrm{v}'\in D$, if $s\in\mathcal{A}_{\mathrm{w}}(k,\mathrm{v})$ and $s'\in\mathcal{A}_{\mathrm{w}'}(k,\mathrm{v}')$ for some $\mathrm{w},\mathrm{w}'\in\mathbb{R}$, then $s=s'$. Suppose thus that $s\in\mathcal{A}_{\mathrm{w}}(k,\mathrm{v})$ and $s'\in\mathcal{A}_{\mathrm{w}'}(k,\mathrm{v}')$ for some $\mathrm{w},\mathrm{w}'\in\mathbb{R}$. Fix w'' such that $\mathcal{A}_{\mathrm{w}''}(k,-\mathrm{v})\neq\emptyset$. Then, by Proposition 6.1 we have

$$\mathcal{A}_{\mathrm{w}''}(k,-\mathrm{v})=\mathcal{A}_{\mathrm{w}}(k,\mathrm{v})=\{s\}.$$

Consequently, in order to prove that $s=s'$, we may suppose that either $\mathrm{v},\mathrm{v}'\in(-\infty,-r_1)$ or $\mathrm{v},\mathrm{v}'\in(r_1,+\infty)$. In both these cases, we may find points $\mathrm{u}_0,\ldots,\mathrm{u}_n$ in D such that $\mathrm{u}_0=\mathrm{v}'$, $\mathrm{u}_n=\mathrm{v}$, and $|\mathrm{u}_j-\mathrm{u}_{j-1}|\leq M\theta$ for all $1\leq j\leq n$. Put $s_0=s'$ and $s_n=s$. Moreover, according to Proposition 5.1, for each $1\leq j\leq n-1$, there exist $s_j\in S$ and $\mathrm{w}_j\in\mathbb{R}$ such that $s_j\in\mathcal{A}_{\mathrm{w}_j}(k,\mathrm{u}_j)$.

For each $1\leq j\leq n$, since $|\mathrm{u}_j-\mathrm{u}_{j-1}|\leq M\theta$, Corollary 6.2 implies that $s_j=s_{j-1}$. By using this fact repeatedly, we conclude that $s'=s_1=\cdots=s_{n-1}=s$. □

Thus, we are able to define the function $\varphi:K\to S$ where $\varphi(k)$ is the element s given by Proposition 7.1. By symmetry, we may also define a function $\psi:S\to K$ such that $\psi(s)$ is the element k given by the symmetric version of Proposition 7.1.

We will show that φ is a homeomorphism by proving that φ and ψ are continuous and $\psi=\varphi^{-1}$. First we will prove another property of the sets $\mathcal{A}_{\mathrm{w}}(k,\mathrm{v})$.

8. A fundamental property of the sets $\mathcal{A}_{\mathrm{w}}(k,\mathrm{v})$ when $M^2<2$

The next proposition is a version of [11, Proposition 3.3] for the sets $\mathcal{A}_{\mathrm{w}}(k,\mathrm{v})$ and will help us prove that the functions φ and ψ defined in the previous section are homeomorphisms provided that we change r_1 in the statement of Proposition 7.1 by a convenient number greater than it. See Proposition 9.1.

Proposition 8.1. *There exists $r_2 \geq r_1$, depending only on M and θ, such that, for all $k \in K$ and $\mathrm{v} \in \mathbb{R}$ with $|\mathrm{v}| > r_2$, if $s \in \mathcal{A}_{\mathrm{w}}(k, \mathrm{v})$ for some $\mathrm{w} \in \mathbb{R}$ and $\mathcal{B}_{\mathrm{z}}(s, \mathrm{w}) \neq \emptyset$ for some $\mathrm{z} \in \mathbb{R}$, then $\mathcal{B}_{\mathrm{z}}(s, \mathrm{w}) = \{k\}$.*

Proof. Pick $k' \in \mathcal{B}_{\mathrm{z}}(s, \mathrm{w})$, and we must show that $k' = k$. Suppose the contrary and fix $f_0 \in A$ such that

$$\left| f_0(k) - \frac{\mathrm{v}}{2} \right| \leq \frac{1}{2M}, \quad \left| f_0(k') + \frac{|\mathrm{v}|}{2|\mathrm{z}|}\mathrm{z} \right| \leq \frac{1}{2M}, \quad \text{and} \quad \|f_0\| \leq \frac{|\mathrm{v}|}{2} + \frac{1}{2M}. \tag{8.1}$$

Thus,

$$\omega(k, f_0, \mathrm{v}) \leq \frac{|\mathrm{v}|}{2} + \frac{1}{2M}.$$

Applying the definition of $\mathcal{A}_{\mathrm{w}}(k, \mathrm{v})$ to f_0, we see that

$$|Tf_0(s) - \mathrm{w}| \leq M\omega(k, f_0, \mathrm{v}) + 1 \leq \frac{M}{2}|\mathrm{v}| + 3/2.$$

Moreover, since

$$\|Tf_0\| \leq M\|f_0\| + 1/2 \leq \frac{M}{2}|\mathrm{v}| + 1,$$

it follows that

$$\omega(s, Tf_0, \mathrm{w}) \leq \frac{M}{2}|\mathrm{v}| + 3/2.$$

So, by applying the definition of $\mathcal{B}_{\mathrm{z}}(s, \mathrm{w})$ to Tf_0, we have

$$|f_0(k') - \mathrm{z}| = |T^{-1}(Tf_0)(k') - \mathrm{z}| \leq M\omega(s, Tf_0, \mathrm{w}) + 1 \leq \frac{M^2}{2}|\mathrm{v}| + \frac{3M}{2} + 1. \tag{8.2}$$

On the other hand, since $|\mathrm{w}| = |\mathrm{v}|/M - 1$ and $|\mathrm{z}| = |\mathrm{w}|/M - 1$ we obtain that

$$|\mathrm{z}| = \left(\frac{|\mathrm{v}|}{M} - 1 \right) \frac{1}{M} - 1 = \frac{|\mathrm{v}|}{M^2} - \frac{1}{M} - 1.$$

Furthermore, according to (8.1),

$$|f_0(k') - \mathrm{z}| \geq \left| \frac{|\mathrm{v}|}{2|\mathrm{z}|}\mathrm{z} + \mathrm{z} \right| - \frac{1}{2M} = \frac{|\mathrm{v}|}{2} + |\mathrm{z}| - \frac{1}{2M} = \frac{|\mathrm{v}|}{2} + \frac{|\mathrm{v}|}{M^2} - \frac{3}{2M} - 1. \tag{8.3}$$

Therefore, putting $\Delta' = \frac{3M}{2} + 2 + \frac{3}{2M}$, by (8.2) and (8.3) we conclude that

$$\left(\frac{1}{2} + \frac{1}{M^2} - \frac{M^2}{2} \right) |\mathrm{v}| \leq \Delta'. \tag{8.4}$$

Since $M^2 < 2$, it can be easily seen that

$$\frac{1}{2} + \frac{1}{M^2} - \frac{M^2}{2} > 0.$$

So, there exists $r_2 \geq r_1$ depending only on M and θ such that the inequality (8.4) fails to be true for $\mathrm{v} \in \mathbb{R}$ with $|\mathrm{v}| > r_2$, and the proposition is proved. □

As we did to r_0 and r_1, we may fix r_2 given by the Proposition 8.1, and it is clear that this constant also works for the sets $\mathcal{B}_{\mathrm{z}}(s, \mathrm{w})$.

9. The homeomorphism between K and S

Observe that the statements of Proposition 5.1, Corollary 6.2, Proposition 7.1, and Proposition 8.1 remain true if we change r_0 and r_1 to r_2. Thus consider φ and ψ as defined at the end of section 7. To complete the proof of Theorem 1.4, we prove Proposition 9.1. Its proof is essentially the same as the reasoning in [11, Section 4].

Proposition 9.1. *The functions $\varphi : K \to S$ and $\psi : S \to K$ are continuous and $\psi = \varphi^{-1}$.*

Proof. First we will show that $\psi = \varphi^{-1}$. Fix $k \in K$. By the definition of $\varphi(k)$ there are $\mathrm{v}, \mathrm{w} \in \mathbb{R}$ with $|\mathrm{v}| > (r_2 + 1)M$ such that $\varphi(k) \in \mathcal{A}_{\mathrm{w}}(k, \mathrm{v})$. Thus, $|\mathrm{w}| > r_2$, and by Proposition 5.1 there exists $\mathrm{z} \in \mathbb{R}$ satisfying $\mathcal{B}_{\mathrm{z}}(\varphi(k), \mathrm{w}) \neq \emptyset$. Then, according to Proposition 8.1 we know that $\mathcal{B}_{\mathrm{z}}(\varphi(k), \mathrm{w}) = \{k\}$. Therefore, it follows by the definition of ψ that $\psi(\varphi(k)) = k$. Hence $\psi \circ \varphi = \mathrm{Id}_K$. Analogously we deduce that $\varphi \circ \psi = \mathrm{Id}_S$.

We now pass to proving that φ is continuous. The proof that ψ is continuous is analogous. Observe that it suffices to prove that for each net $(k_j)_{j \in J}$ of K converging to $k \in K$, the net $(\varphi(k_j))_{j \in J}$ admits a subnet converging to $\varphi(k)$.

Assume then that $(k_j)_{j \in J}$ is a net of K converging to k. For all $j \in J$ take v_j and w_j such that $|\mathrm{v}_j| = c$, for some $c > r_2$, and

$$\varphi(k_j) \in \mathcal{A}_{\mathrm{w}_j}(k_j, \mathrm{v}_j). \tag{9.1}$$

Since the nets $(\mathrm{v}_j)_{j \in J}$ and $(\mathrm{w}_j)_{j \in J}$ are limited, we may assume that there are $\mathrm{v}, \mathrm{w} \in \mathbb{R}$ such that $\mathrm{v}_j \to \mathrm{v}$ and $\mathrm{w}_j \to \mathrm{w}$. For each $f \in A$ we have

$$\omega(k_j, f, \mathrm{v}_j) \to \omega(k, f, \mathrm{v}), \tag{9.2}$$

and according to (9.1),

$$|Tf(\varphi(k_j)) - \mathrm{w}_j| \leq M\omega(k_j, f, \mathrm{v}_j) + 1 \quad \forall j \in J. \tag{9.3}$$

Fix $f_0 \in A$ satisfying

$$\|f_0\| \leq |\mathrm{v}|/2 + \frac{1}{2M} \quad \text{and} \quad |f_0(k) - \mathrm{v}/2| \leq \frac{1}{2M}.$$

Then (9.3) implies that

$$|Tf_0(\varphi(k_j))| \geq |\mathrm{w}_j| - |Tf_0(\varphi(k_j)) - \mathrm{w}_j| \geq \frac{c}{M} - M\omega(k_j, f_0, \mathrm{v}_j) - 2$$

for every $j \in J$. Notice that

$$\omega(k, f_0, \mathrm{v}) \leq |\mathrm{v}|/2 + \frac{1}{2M} = c/2 + \frac{1}{2M}.$$

So, by (9.2) we have

$$\liminf_{j \in J} |Tf_0(\varphi(k_j))| \geq \left(\frac{1}{M} - \frac{M}{2}\right) c - 5/2,$$

and since $c > r_2 \geq r_0$, observing (5.3), we obtain

$$\liminf_{j \in J} |Tf_0(\varphi(k_j))| > 0.$$

Since Tf_0 vanishes at infinity, this implies that $(\varphi(k_j))_{j \in J}$ admits a subnet converging to some $s \in S$, so we assume that $\varphi(k_j) \to s$. Hence, by (9.2) and (9.3),

$$|Tf(s) - \mathrm{w}| \leq M\omega(k, f, \mathrm{v}) + 1 \quad \forall f \in A,$$

which means that $s \in \mathcal{A}_{\mathrm{w}}(k, \mathrm{v}) = \{\varphi(k)\}$. Consequently $s = \varphi(k)$. □

10. On quasi-isometries on subsets of $C_0^{(1)}(K)$ spaces

The purpose of this section is to present the proof of Theorem 1.5. It follows immediately from Theorem 1.4 and the following proposition.

Proposition 10.1. *Let K and S be locally compact subsets of $\mathbb{R}$ with no isolated points and assume that the inequalities (1.2) hold for all f in $C_0^{(1)}(K)$ or in $C_0^{(1)}(S)$. Suppose that there exists an (M,L)-quasi-isometry T from $A \subset C_0^{(1)}(K)$ to $B \subset C_0^{(1)}(S)$, where A and B are $\|\cdot\|_\infty$-bounded and $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are BS-subsets of $C_0(K)$ and $C_0(S)$, respectively. Then there exists an (M,L')-quasi-isometry $\hat{T}: D \to E$, where D and E are BS-subsets of $C_0(K)$ and $C_0(S)$, respectively.*

Proof. Since A and B are $\|\cdot\|_\infty$-bounded, fix $c > 0$ such that $\|h\|_\infty \leq c$ for all $h \in A$ or $h \in B$. Now, notice that since T is a quasi-isometry, $T(A)$ is θ-dense in B for some $\theta > 0$. Then, by (1.2), for every $f \in B$ there is $g \in T(A)$ such that $\|f' - g'\|_\infty \leq \|f - g\| \leq \theta$. Thus, $\mathcal{D}(T(A))$ is θ-dense in $\mathcal{D}(B)$. Therefore, since $\mathcal{D}(B)$ is a BS-subset of $C_0(S)$, it follows by Proposition 2.3(3) that

$$\mathcal{D}(T(A)) \quad \text{is a BS-subset of } C_0(S). \tag{10.1}$$

Next, according to (1.2), for every $f, g \in A$ we have

$$\begin{aligned}\|(Tf)' - (Tg)'\|_\infty \leq \|Tf - Tg\| &\leq M\|f - g\| + L \\ &\leq M\|f' - g'\|_\infty + M\|f - g\|_\infty + L \\ &\leq M\|f' - g'\|_\infty + 2Mc + L\end{aligned}$$

and

$$\begin{aligned}\|(Tf)' - (Tg)'\|_\infty \geq \|Tf - Tg\| - \|Tf - Tg\|_\infty &\geq \frac{1}{M}\|f - g\| - L - 2c \\ &\geq \frac{1}{M}\|f' - g'\|_\infty - L - 2c.\end{aligned}$$

Hence putting $L' = 2Mc + L$, it follows that

$$\frac{1}{M}\|f' - g'\|_\infty - L' \leq \|(Tf)' - (Tg)'\|_\infty \leq M\|f' - g'\|_\infty + L' \quad \forall f, g \in A. \tag{10.2}$$

Since (10.1) and (10.2) hold, if we could define $\hat{T}: \mathcal{D}(A) \to \mathcal{D}(T(A))$ by

$$\hat{T}(f') = (Tf)', \tag{10.3}$$

then $\hat{T}$ would be an (M,L')-quasi-isometry between BS-subsets, as we wished. However, since $\mathcal{D}$ is not necessarily injective, we cannot guarantee that the $\hat{T}$ given by (10.3) is well defined. To fix this, we pick a set of representatives $C \subset A$ of the equivalence relation $f \sim g$ if and only if $f' = g'$ $\forall f, g \in A$.

Then surely $\mathcal{D}$ is injective in C and moreover the $\hat{T}$ given by (10.3) is well defined if we put $\hat{T}: \mathcal{D}(C) \to \mathcal{D}(T(C))$. We know that $\mathcal{D}(C) = \mathcal{D}(A)$; thus $\mathcal{D}(C)$ is a BS-subset of $C_0(K)$. Furthermore, by (10.2) the map $\hat{T}$ is an (M,L')-quasi-isometry. We have only to show that $\mathcal{D}(T(C))$ is a BS-subset of $C_0(S)$. Notice by (10.2) that if $f, g \in A$ and $f \sim g$, then $\|(Tf)' - (Tg)'\|_\infty \leq L'$.

Consequently $\mathcal{D}(T(C))$ is L'-dense in $\mathcal{D}(T(A))$. Hence by (10.1) and Proposition 2.3(3) $\mathcal{D}(T(C))$ is a BS-subset of $C_0(S)$, and the proof is finished. □

11. A nonlinear extension of Cambern, Pathak, and Vasavada theorems

In this last section we will show that in fact Corollary 1.6 is a consequence of Theorem 1.5. First we need to prove Lemma 11.1 and Proposition 11.2.

Lemma 11.1. *Let K be a locally compact subspace of $\mathbb{R}$ with no isolated points. Let $U \subset K$ be an open set with $k \in U$. Then for any $0 < \varepsilon \leq 1$ there exists $f \in C_0^{(1)}(K)$ such that $f(K \setminus U) = f'(K \setminus U) = \{0\}$, $\|f\|_\infty \leq \varepsilon$, $f'(k) = \|f'\|_\infty = 1$, and $f'(u) \geq -\varepsilon\ \forall u \in U$.*

Proof. Since K is locally compact, take $\theta > 0$ such that $[k-\theta, k+\theta] \cap K$ is compact and contained in U. Now, let $a, b \in \mathbb{R}$ such that $k \in (a, b) \subset (k-\theta, k+\theta)$. We fix three continuous functions as follows. Take $g_1 : [k-\theta, a] \to \mathbb{R}$ such that

(i) $-\varepsilon \leq g_1(u) \leq 0\ \forall u \in [k-\theta, a]$ and $g_1(k-\theta) = g_1(a) = 0$;
(ii) $\int g_1(t)dt = -\varepsilon$.

Then, take $g_2 : [a, b] \to \mathbb{R}$ such that

(iii) $0 \leq g_2(u) \leq 1\ \forall u \in [a, b]$, $g_2(a) = g_2(b) = 0$, and $g_2(k) = 1$;
(iv) $\int_a^k g_2(t)dt = \int_k^b g_2(t)dt = \varepsilon$.

Finally, take $g_3 : [b, k+\theta] \to \mathbb{R}$ such that

(v) $-\varepsilon \leq g_3(u) \leq 0\ \forall u \in [b, k+\theta]$ and $g_3(b) = g_3(k+\theta) = 0$;
(vi) $\int g_3(t)dt = -\varepsilon$.

Let $g \in C_0(\mathbb{R})$ vanish outside $[k-\theta, k+\theta]$ and be identical to each of the functions above at their domain. Next, define $G : \mathbb{R} \to \mathbb{R}$ by

$$G(x) = \int_{k-\theta}^{x} g(t)dt.$$

Then G is continuous and it follows by (ii), (iv), and (vi) that $\|G\|_\infty \leq \varepsilon$ and G vanishes outside $[k-\theta, k+\theta]$. Finally, define $f = G|_K$. Then, of course, f is differentiable and $f' \equiv g|_K$. Moreover, since both G and g vanish outside $[k-\theta, k+\theta]$, we see that f and f' vanish in $K \setminus [k-\theta, k+\theta]$. Hence $f(K \setminus U) = f'(K \setminus U) = \{0\}$. Since $[k-\theta, k+\theta] \cap K$ is compact, we deduce that $f \in C_0^{(1)}(K)$.

Since $\|G\|_\infty \leq \varepsilon$, we obtain that $\|f\|_\infty \leq \varepsilon$ and by (i), (iii), and (v), we conclude that $f'(k) = \|f'\|_\infty = 1$ and $f'(u) \geq -\varepsilon\ \forall u \in U$. □

Next, it will be convenient to use the following notation with $H = K$ or $H = S$:

$$B_\infty[f, r] = \{g \in C_0^{(1)}(H) : \|f - g\|_\infty \leq r\} \quad \forall f \in C_0^{(1)}(H) \quad \text{and} \quad r > 0.$$

Proposition 11.2. *$\mathcal{D}(B_\infty[0, r])$ is a BS-subset of $C_0(K)$ for all $r > 0$.*

Proof. Since $\mathcal{D}(B_\infty[0, r]) = r\mathcal{D}(B_\infty[0, 1])$, by Proposition 2.3(1), it suffices to prove that $\mathcal{D}(B_\infty[0, 1])$ is a BS-subset of $C_0(K)$.

Given $k \in K$ and $\mathrm{v} \in \mathbb{R}$, we fix, for each $U \in \mathcal{V}_k$, $f_{k,U} \in C_0^{(1)}(K)$ given by Lemma 11.1 for U, k, and $\varepsilon = \dfrac{1}{2|\mathrm{v}|}$ and define $F_{k,\mathrm{v},U} = \mathrm{v} \cdot f_{k,U}$.

It is easily seen that the following properties of the functions $F_{k,\mathrm{v},U}$ are valid:

(a) $F'_{k,\mathrm{v},U}(K \setminus U) = \{0\}$, $F'_{k,\mathrm{v},U}(k) = \mathrm{v}$, and $\|F'_{k,\mathrm{v},U}\|_\infty = |\mathrm{v}|$;

(b) $\inf\{|F'_{k,\mathrm{v},U}(t) - \alpha\mathrm{v}| : t \in K \text{ and } \alpha \in [0,1]\} \leq \dfrac{1}{2}$;

(c) $\|F_{k,\mathrm{v},U}\|_\infty \leq \dfrac{1}{2}$.

Now, let $W = \{F_{k,\mathrm{v},U} : k \in K,\ \mathrm{v} \in \mathbb{R},\ U \in \mathcal{V}_k\}$ and define $Z = W + W$. It is easy to verify by (a), (b), and (c) that $\mathcal{D}(Z) = \mathcal{D}(W) + \mathcal{D}(W)$ is a $(1/2, 1)$-subset of $C_0(K)$. Moreover, according to (c) we have

$$\|F_{k,\mathrm{v},U} + F_{k',\mathrm{v}',U'}\|_\infty \leq \|F_{k,\mathrm{v},U}\|_\infty + \|F_{k',\mathrm{v}',U'}\|_\infty \leq 1.$$

Thus $Z \subset B_\infty[0,1]$, and hence $\mathcal{D}(B_\infty[0,1]) \supset \mathcal{D}(Z)$ is a BS-subset of $C_0(K)$. □

We are now ready to prove the main result of this section.

Proof of Corollary 1.6. Let K and S be locally compact subsets of $\mathbb{R}$ with no isolated points and suppose that the inequalities (1.2) hold for every $f \in C_0^{(1)}(K)$ or $f \in C_0^{(1)}(S)$. Assume there is a bijective (M, L)-quasi-isometry $T : C_0^{(1)}(K) \to C_0^{(1)}(S)$ with $M < \sqrt{2}$ and that T is a quasi-isometry with respect to the sup norms.

Pick $M' \geq 1$ and $L' \geq 0$ such that $T : (C_0^{(1)}(K), \|\cdot\|_\infty) \to (C_0^{(1)}(S), \|\cdot\|_\infty)$ is an (M', L')-quasi-isometry. Put $r = M'(L' + 1)$ and $A = B_\infty[0, r]$ in $C_0^{(1)}(K)$.

Now, consider $\hat{T} : A \to T(A)$ the restriction of T to A. It follows that $\hat{T}$ is an (M, L)-quasi-isometry with $M < \sqrt{2}$ and by Proposition 11.2 that $\mathcal{D}(A)$ is a BS-subset of $C_0(K)$. Moreover, $T(A)$ is $\|\cdot\|_\infty$-bounded because

$$\|T(f) - T(0)\|_\infty \leq M'\|f\|_\infty + L' \leq M'r + L' \quad \forall f \in A.$$

Next, we will show that $\mathcal{D}(T(A))$ is a BS-subset of $C_0(S)$. Observe that it suffices to prove that $B_\infty[T(0), 1] \subset T(A)$. Thus, let $g \in B_\infty[T(0), 1]$. Then

$$\frac{1}{M'}\|T^{-1}(g)\|_\infty - L' \leq \|T(T^{-1}(g)) - T(0)\|_\infty \leq 1,$$

which implies that $\|T^{-1}(g)\|_\infty \leq r$, i.e., $T^{-1}(g) \in A$ or equivalently $g \in T(A)$.

Therefore, $\hat{T} : A \to T(A)$ satisfies the hypothesis of Theorem 1.5, which implies that K and S are homeomorphic. This completes the proof.

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