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**SET OPERATIONS ON CLOSED
INTERVALS AND THEIR APPLICATIONS
TO THE AUTOMATIC PROGRAMMING
OF MMACH'S**

Junior Barrera, Guillermo Pablo Salas

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Set Operations on closed intervals and their applications to the Automatic Programming of MMach's

Junior Barrera *

Universidade de São Paulo, Departamento de Ciência da Computação,
Cidade Universitária "Armando de Salles Oliveira",
CP: 66281, CEP: 05389-970, São Paulo, SP, Brazil.
E-mail: jb@ime.usp.br

Guillermo Pablo Salas

Universidad Nacional de Mar del Plata, Facultad de Ingeniería,
J. B. Justo 4305, CP 7600, Mar del Plata, Bs As, Argentina.
E-mail: gpsalas@BART.uni-mdp.edu.ar

Abstract

Mathematical Morphology on sets can be understood as a formal language, whose vocabulary are erosions, dilations, complementation, intersection and union. This language is complete, that is, it is enough to perform any set operator. Since the sixties special Machines, called Morphological Machines (MMach's), have been built to implement this language. In the literature, we find hundreds of MMach programs that are used to solve image analysis problems. However, the design of these programs is not an elementary task. Thus, recently much research effort has been addressed to automating the programming of MMach's. A very promising approach to this problem is the description of the target operator by input-output pairs of images and the translation of these data into efficient MMach programs. This approach can be decomposed in two equally important steps: i-learning of the target operator from pairs of images; ii-search of economical representations for the operators learned. The theory presented in this paper is useful in the second step of this procedure. We will present some set operations on collections of closed intervals and give efficient algorithms to perform them. These operations will be used to parallelize MMach programs and to prove the equivalence between distinct MMach programs.

Keywords: Mathematical Morphology, closed interval, canonical decomposition, operator basis.

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1 Introduction

Binary Image Analysis is an important tool for various areas such as industrial process control, office automation, quantitative microscopy, etc.

A natural model for a procedure used in Binary Image Analysis is a set mapping applied on a Discrete Random Set [10]. The set mappings will be called here *set operators*. *Mathematical Morphology* (MM) is a general framework to study set operators [21, 23, 24].

Two simple families of set operators are the so called *erosions* and *dilations*. A central paradigm in MM is the representation of set operators by a concatenation of erosions and dilations through the operations of *composition*, *intersection*, *union* and *complementation*.

This paradigm can be formalized by the use of a formal language, called here the *Morphological Language*, whose vocabulary are erosions, dilations, intersection, union and complementation [4]. This language is complete (i.e., it is enough to describe any set operator) and expressive (i.e., most useful operators can be described by phrases that use relatively few words). A phrase of the Morphological Language will be called here a *Morphological Operator*.

Since the sixties special machines, called here *Morphological Machines* (MMach's), have been built to perform this language [1, 17, 11, 16, 5, 3]. A *program* for a MMach is just an implementation of a Morphological Operator.

In the literature, we find hundreds of applications of Morphological Operators for the solution of Image Analysis Problems [23, 1, 15]. However, the design of Morphological Operators is not an elementary task. Classically this task is performed empirically based just on the experience of the user with the use of Morphological Operators.

The classical approach for the design of morphological operators restrict the universe of MMach programmers just to the experts in MM. In order to overcome this restriction, recently much research effort has been addressed to automate the programming of MMach's [20, 19, 5, 8, 22, 15, 25, 12, 6]. The goal is to find suitable knowledge representation formalisms and to develop tools that translate them into MMach programs. Some of these tools use collections of input-output pairs of images as the knowledge representation formalism.

The problem of designing a Morphological Operator from pairs of images can be decomposed in two equally important steps: i- estimation or learning of the target operator from input-output image pairs; ii-search of economical representations (i.e., that use a minimum number of words) for the set operator estimated. The theory presented in this paper is useful to approach the second step of this procedure.

A particular kind of Morphological Operator used in the automatic programming of MMach's is the *canonical decomposition structure*: union of intersections of erosions and complemented dilations. This decomposition structure is strongly parallel and is enough to perform any set operator. The erosions and dilations used in the canonical

decomposition of a given set operator are characterized by a collection of closed intervals, called the *operator basis*. An interesting property of the operator basis is that it characterizes uniquely the operator, that is, each set operator has a basis that is distinct from the basis of any other set operator [2].

We will present some set operations on collections of closed intervals and give efficient algorithms to perform them. This study will be the background for the definition of a set of rules that permit the generation of Morphological operators that are synonymous of a given Morphological Operator. Particularly, these rules will permit to compute the basis of any set operator from any Morphological Operator that represents this set operator. Hence, they will be useful to parallelize MMach programs, once the basis characterizes a strongly parallel representation, and to prove the equivalence between distinct Morphological Operators, once any representation can be reduced to the basis that determines uniquely a set operator.

Following this introduction, section two reviews some aspects of lattice theory and study some lattice properties of the set of collections of maximal closed intervals. Section three presents some results that permit the computation of set operations on collections of closed intervals. Section four reviews some concepts in MM and presents the generalization of the canonical decomposition to W-operators. In section five, we use the results of section three to show how to compute the basis of the intersection, of the union and of the composition with an erosion (with a dilation or with the complementation) of given operators. In section six, we show how the rules presented in section five can be used to compute the basis of any set operator. Finally, we take some conclusions and propose some future steps in this research.

2 Elements of lattice theory

Let E be a non empty set and let W be a finite subset of E . Let $\mathcal{P}(W)$ be the collection of all subsets of W . Let \subseteq be the usual inclusion relation on sets. The pair $(\mathcal{P}(W), \subseteq)$ is a complete Boolean lattice [9]. The least and the greatest elements of $\mathcal{P}(W)$ are, respectively, \emptyset and W . The intersection and the union of X_1 and X_2 in $\mathcal{P}(W)$ are, respectively, $X_1 \cap X_2$ and $X_1 \cup X_2$. The complementary set of a subset X in $\mathcal{P}(W)$, with respect to W , is denoted X^c , that is, $X^c = \{x \in W : x \notin X\}$.

A particular property of a Boolean Lattice is de Morgan's Law: $\forall X_1, X_2 \in \mathcal{P}(W)$,

$$(X_1^c \cap X_2^c)^c = X_1 \cup X_2.$$

Given $A, B \in \mathcal{P}(W)$, the subcollection $[A, B]$ of $\mathcal{P}(W)$ defined by

$$[A, B] = \{X \in \mathcal{P}(W) : A \subseteq X \subseteq B\}$$

is called a *closed interval* or, simply, an *interval*. If $A \subseteq B$, then the elements A and B are called, respectively, the *left* and the *right extremities* of $[A, B]$. For all pairs (A, B) such that $A \not\subseteq B$, $[A, B]$ represents the empty collection also denoted \emptyset .

In this paper we will denote subcollections of $\mathcal{P}(W)$ by upper case script letters, that is, $\mathcal{A}, \mathcal{B}, \dots, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$. The collections of closed intervals will be denoted by upper case bold face letters, that is, $\mathbf{A}, \mathbf{B}, \dots, \mathbf{X}, \mathbf{Y}, \mathbf{Z}$.

An element of a collection of closed intervals \mathbf{X} is called *maximal in \mathbf{X}* if no other element of \mathbf{X} properly contains it, that is, $\forall [A, B] \in \mathbf{X}$.

$$[A, B] \text{ is maximal in } \mathbf{X} \iff \forall [A', B'] \in \mathbf{X}. [A, B] \subseteq [A', B'] \implies [A, B] = [A', B'].$$

The collection of all maximal closed intervals in \mathbf{X} is denoted $Max(\mathbf{X})$. Of course, if all the intervals in \mathbf{X} are maximal, we have $\mathbf{X} = Max(\mathbf{X})$.

Let \mathcal{X} be a subcollection of $\mathcal{P}(W)$. The collection of all maximal closed intervals contained in \mathcal{X} is denoted $M(\mathcal{X})$, that is,

$$M(\mathcal{X}) = Max(\{[A, B] \subseteq \mathcal{P}(W) : [A, B] \subseteq \mathcal{X}\}).$$

Usually, we will denote a subcollection and the set of maximal intervals contained in it by the same letter. For example, $\mathbf{X} = M(\mathcal{X})$.

We denote by $\cup \mathbf{X}$ the collection of all elements of $\mathcal{P}(W)$ that are elements of closed intervals in \mathbf{X} , that is,

$$\cup X = \{X \in \mathcal{P}(W) : X \in [A, B], [A, B] \in \mathcal{X}\}.$$

Note that, for any $\mathcal{X} \subseteq \mathcal{P}(W)$, $\cup X = \mathcal{X}$. Note also that, once W is finite and $\mathcal{X} \subseteq \mathcal{P}(W)$, for any $[A, B] \subseteq \mathcal{X}$, there exists $[A', B'] \in \mathcal{M}(\mathcal{X})$ such that $[A, B] \subseteq [A', B']$. Hence, any collection \mathcal{X} can be represented by its maximal closed intervals.

For simplicity of notation, we will represent the subsets of W by strings of 0's and 1's, where 0 means that the point does not belong to the subset and 1 means that it does. For example, if W is the set $\{(-1, 0), (0, 0), (1, 0)\}$, the subset $\{(0, 0), (1, 0)\}$ will be represented by 011.

Subsets of finite Boolean Lattices can be represented by diagrams as the ones in Figure 1. The small boxes represent the lattice elements and the line segment (i.e., the edges of the lattice) represent the partial order. The bold faced boxes represent the elements of the subset. The maximal intervals inside a subset are represented by bold face edges. If two maximal intervals of a subset have a common edge, just to be clear, we will represent them in two distinct diagrams.

Examples 2.1 illustrates some of these concepts.

Example 2.1 a) Let $W = \{(-1, 0), (0, 0), (1, 0)\}$ and let

$$\mathcal{X} = \{000, 100, 010, 001, 110, 011\}.$$

The set of maximal intervals contained in \mathcal{X} is:

$$\mathcal{M}(\mathcal{X}) = \{[000, 110], [000, 011]\}.$$

Figure 1a and 1b show \mathcal{X} and $\mathcal{M}(\mathcal{X})$.

b) Let

$$Y = \{[000, 000], [000, 001], [010, 011]\}.$$

The set of maximal intervals in Y is:

$$\text{Max}(Y) = \{[000, 001], [010, 011]\}.$$

Figure 1c shows $\cup Y$ and $\text{Max}(Y)$. □

Note that $\mathcal{M}(\cup X)$ is not necessarily equal to $\text{Max}(X)$. Example 2.1b illustrates this fact, since $\mathcal{M}(\cup Y) = \{[000, 011]\}$.

Let $\mathcal{P}(\mathcal{P}(W))$ be the collection of all subcollections of $\mathcal{P}(W)$. Let \subseteq be the usual inclusion relation on sets. The pair $(\mathcal{P}(\mathcal{P}(W)), \subseteq)$ is a complete Boolean lattice. The least and the greatest elements of $\mathcal{P}(\mathcal{P}(W))$ are, respectively, \emptyset and $\mathcal{P}(W)$. The intersection and the union of \mathcal{X}_1 and \mathcal{X}_2 in $\mathcal{P}(\mathcal{P}(W))$ are, respectively, $\mathcal{X}_1 \cap \mathcal{X}_2$ and $\mathcal{X}_1 \cup \mathcal{X}_2$. The complementary collection of a subcollection \mathcal{X} in $\mathcal{P}(\mathcal{P}(W))$, with respect to $\mathcal{P}(W)$, is denoted \mathcal{X}^c , that is, $\mathcal{X}^c = \{X \in \mathcal{P}(W) : X \notin \mathcal{X}\}$.

Let I_W denote the set $\{M(\mathcal{X}) : \mathcal{X} \subseteq \mathcal{P}(W)\}$. We will define the partial order \leq on the elements of I_W by setting : $\forall X, Y \in I_W$,

$$X \leq Y \iff \forall [A, B] \in X. \exists [A', B'] \in Y : [A, B] \subseteq [A', B'].$$

It is immediate that the relation defined above is reflexive and transitive. This relation is also anti-symmetric, since the collections of closed intervals considered are maximal. In fact, the poset (I_W, \leq) constitutes a complete Boolean lattice, where the infimum, supremum and negation operations are given, respectively, by: $\forall X, Y \in I_W$,

$$X \cap Y = M(\mathcal{X} \cap \mathcal{Y}).$$

$$X \cup Y = M(\mathcal{X} \cup \mathcal{Y}),$$

$$\bar{X} = M(\mathcal{X}^c),$$

where $X = M(\mathcal{X})$ and $Y = M(\mathcal{Y})$.

These expressions follow because the mapping $M(\cdot)$, defined from $\mathcal{P}(\mathcal{P}(W))$ to I_W , is a lattice isomorphism. The inverse of the mapping $M(\cdot)$ is the mapping $U(\cdot)$.

In particular, note that the least and the greatest elements of (I, \leq) are, respectively, $M(\emptyset) = \{\emptyset\}$ and $M(\mathcal{P}(W)) = \{\{\emptyset, W\}\}$.

Let $\{0, 1\}^{\mathcal{P}(W)}$ denote the set of all Boolean functions defined on $\mathcal{P}(W)$. The pair $(\{0, 1\}^{\mathcal{P}(W)}, \leq)$, where \leq is the partial order inherited from the total order in the chain $\{0, 1\}$, constitutes a complete Boolean lattice. This lattice is isomorphic to the lattice (I_W, \leq) , since the mapping F , defined from I_W to $\{0, 1\}^{\mathcal{P}(W)}$ by

$$F(X)(X) = \begin{cases} 1 & \text{if } X \in \cup X \\ 0 & \text{otherwise} \end{cases} \quad (X \in \mathcal{P}(W)),$$

is a lattice isomorphism. The inverse of the mapping F is the mapping F^{-1} , defined by

$$F^{-1}(f) = M(\{X \in \mathcal{P}(W) : f(X) = 1\}) \quad (f \in \{0, 1\}^{\mathcal{P}(W)}).$$

Figure 5 illustrates the lattice isomorphisms between $(\mathcal{P}(\mathcal{P}(W)), \subseteq)$, (I_W, \leq) , and $(\{0, 1\}^{\mathcal{P}(W)}, \leq)$.

3 Set operations on collections of closed intervals

In computational applications usually collections of sets in $\mathcal{P}(W)$ are represented by their maximal closed intervals in order to minimize the storage space. Thus, it is useful to express the operations of intersection, union and complementation of collections of sets in terms of their maximal closed intervals. In this section, we will give a procedure that computes the set of maximal intervals contained in the intersection of two collections of sets from the set of maximal intervals contained in each input collection. Similar results will be given for the operations of complementation and union.

Proposition 3.1 *Let $[A, B]$ and $[C, D]$ be two closed intervals contained in $\mathcal{P}(W)$, then*

$$[A, B] \cap [C, D] = [A \cup C, B \cap D].$$

□

The proof of this result is immediate.

Theorem 3.1 *Let X and Y be two elements of Il_W , then*

$$X \cap Y = \text{Max}(\{[A \cup C, B \cap D] : [A, B] \in X, [C, D] \in Y\}).$$

Proof:

Let denote $X = M(\mathcal{X})$ and $Y = M(\mathcal{Y})$.

$$X \cap Y = M(\mathcal{X} \cap \mathcal{Y})$$

(by the lattice isomorphism between $\mathcal{P}(\mathcal{P}(W))$ and Il_W)

$$= \text{Max}(\{[A, B] \subseteq \mathcal{P}(W) : [A, B] \subseteq \mathcal{X} \cap \mathcal{Y}\})$$

(by the definition of M)

$$= \text{Max}(\{[A, B] \subseteq \mathcal{P}(W) : [A, B] \subseteq \mathcal{X} \text{ and } [A, B] \subseteq \mathcal{Y}\})$$

$$= \text{Max}(\{[A, B] \subseteq \mathcal{P}(W) : \exists [A', B'] \in X, [A, B] \subseteq [A', B']\})$$

and

$$\exists [C', D'] \in Y, [A, B] \subseteq [C', D']\})$$

(by the definition of *Max* with W finite)

$$= \text{Max}(\{[A, B] \subseteq \mathcal{P}(W) : \exists [A', B'] \in X \text{ and } \exists [C', D'] \in Y,$$

$$[A, B] \subseteq [A', B'] \cap [C', D']\})$$

(since \cap is the infimum in $\mathcal{P}(\mathcal{P}(W))$)

$$= \text{Max}(\{[A, B] \subseteq \mathcal{P}(W) : \exists [A', B'] \in X \text{ and } \exists [C', D'] \in Y,$$

$$[A, B] \subseteq [A' \cup C', B' \cap D']\})$$

(by Proposition 3.1)

$$= \text{Max}(\{[A, B] \subseteq \mathcal{P}(W) : \exists [A', B'] \in X \text{ and } \exists [C', D'] \in Y,$$

$$[A, B] = [A' \cup C', B' \cap D']\})$$

(by eliminating the non maximal elements in a chain)

$$= \text{Max}(\{[A' \cup C', B' \cap D'] : [A', B'] \in X \text{ and } [C', D'] \in Y\})$$

□

In the following, we give an example that illustrates the application of this theorem.

Example 3.1 Let $E = \{(-1, 0), (0, 0), (1, 0)\}$. Let

$$X = \{000, 100, 010, 001, 110, 011\}$$

and

$$Y = \{000, 100, 001, 110, 101, 111\}.$$

Figure 2a and 2b show X and Y , respectively.

The maximal closed intervals of these collections are, respectively,

$$X = M(X) = \{[000, 110], [000, 011]\}.$$

and

$$Y = M(Y) = \{[000, 101], [100, 111]\}.$$

Applying Theorem 3.1, we have:

$$\begin{aligned}
X \cap Y &= \text{Max}(\{ [000, 110] \cap [000, 101], [000, 110] \cap \\
&\quad [100, 111], [000, 011] \cap [000, 101], \\
&\quad [000, 011] \cap [100, 111] \}) \\
&= \text{Max}(\{ [000, 100], [100, 110], [000, 001], \emptyset \}) \\
&= \{ [000, 100], [000, 001], [100, 110] \}
\end{aligned}$$

Figure 2c shows $X \cap Y$ and $X \cap Y$. □

Let $[A, B]$ be a closed interval in $\mathcal{P}(W)$. Let $[A, B]^c$ denote the complementary collection of $[A, B]$ in $\mathcal{P}(W)$, that is,

$$[A, B]^c = \{X \in \mathcal{P}(W) : X \not\subseteq [A, B]\}.$$

Now we will study the representation of the subcollection $[A, W]^c$.

Proposition 3.2 *Let A be a non empty subset of W , then*

$$M([A, W]^c) = \{\{\emptyset, \{a\}^c\} : a \in A\}.$$

Proof:

$$\begin{aligned}
M([A, W]^c) &= M(\{X \in \mathcal{P}(W) : X \not\subseteq [A, W]\}) \\
&= M(\{X \in \mathcal{P}(W) : A \not\subseteq X \text{ or } X \not\subseteq W\}) \\
&= M(\{X \in \mathcal{P}(W) : A \not\subseteq X\}) \\
&= M(\{X \in \mathcal{P}(W) : \exists a \in A : a \notin X\}) \\
&= M(\cup\{\{\emptyset, \{a\}^c\} : a \in A\}) \\
&= \{\{\emptyset, \{a\}^c\} : a \in A\}
\end{aligned}$$

The last step holds, since, for any $a \in A$, $\{\emptyset, \{a\}^c\} \subseteq \cup\{\{\emptyset, \{a\}^c\} : a \in A\}$ and $[\emptyset, W]$, the only interval contained in $\mathcal{P}(W)$ that contains properly $\{\emptyset, \{a\}^c\}$, is not included in $[A, W]^c$. □

Now we will study the representation of the subcollection $[\emptyset, B]^c$.

Proposition 3.3 Let B be a non empty subset of W , distinct of W , then

$$M([\emptyset, B]^c) = \{[\{b\}, W] : b \in B^c\}.$$

□

This result is dual of the result of Proposition 3.2. Now we will study the representation of the subcollection $[.A, B]^c$.

Proposition 3.4 Let A and B be two non empty subsets of W , such that $A \subseteq B$ and B is distinct of W . The set of maximal closed intervals contained in $[A, B]^c$ is given by the following expression:

$$M([.A, B]^c) = \{[\emptyset, \{a\}^c] : a \in A\} \cup \{[\{b\}, W] : b \in B^c\}.$$

Proof:

$$M([.A, B]^c) = M([\{.A, W\} \cap [\emptyset, B]^c])$$

$$= M([\{.A, W\}^c \cup [\emptyset, B]^c])$$

(by de Morgan's law)

$$= M(\cup\{[\emptyset, \{a\}^c] : a \in A\} \cup \cup\{[\{b\}, W] : b \in B^c\})$$

(by Propositions 3.2 ; id 3.3)

$$= \{[\emptyset, \{a\}^c] : a \in A\} \cup \{[\{b\}, W] : b \in B^c\}$$

The last step holds, since, for any $a \in A$ and $b \in B^c$, $[\emptyset, \{a\}^c]$ and $[\{b\}, W]$ are maximal elements contained, respectively, in $[.A, W]^c$ and $[\emptyset, B]^c$, and $[\emptyset, W]$, the only set that is bigger than $[\emptyset, \{a\}^c]$ or $[\{b\}, W]$, is not included in $[.A, B]^c$.

□

Obviously, when $[.A, B] = \mathcal{P}(W)$ and $[.A, B] = \emptyset$ we have, respectively, $M([.A, B]^c) = \{\emptyset\}$ and $M([.A, B]^c) = \{[\emptyset, W]\}$.

We will apply Theorem 3.1 and Propositions 3.2 to 3.4 for finding a procedure to compute \overline{X} .

Theorem 3.2 Let X be an element of H_W , then

$$\overline{X} = \cap\{M([.A, B]^c) : [.A, B] \in X\}.$$

Proof:

$$\overline{X} = M((\cup X)^c)$$

(by the lattice isomorphism between $(\mathcal{P}(\mathcal{P}(W)), \subseteq)$, and (\mathcal{H}_W, \leq))

$$\begin{aligned} &= M(\{X \in \mathcal{P}(W) : X \not\subseteq \cup X\}) \\ &= M(\{X \in \mathcal{P}(W) : X \not\subseteq [A, B], \forall [A, B] \in X\}) \\ &= M(\{X \in \mathcal{P}(W) : X \in [A, B]^c, \forall [A, B] \in X\}) \\ &= M(\cap \{X \in \mathcal{P}(W) : X \in [A, B]^c, [A, B] \in X\}) \\ &= M(\cap \{[A, B]^c : [A, B] \in X\}) \\ &= \cap \{M([A, B]^c) : [A, B] \in X\} \end{aligned}$$

(by the lattice isomorphism between $(\mathcal{P}(\mathcal{P}(W)), \subseteq)$ and (\mathcal{H}_W, \leq)).

□

In the following, we give an example that illustrates the application of this theorem.

Example 3.2 Let $W = \{(-1, 0), (0, 0), (1, 0)\}$ and let

$$X = \{ [100, 110], [001, 011] \}.$$

Applying Theorem 3.2, we have:

$$\begin{aligned} \overline{X} &= M([100, 110]^c) \cap M([001, 011]^c) \\ &= \{ [000, 011], [001, 111] \} \cap \{ [000, 110], [100, 111] \} \\ &= \text{Max}(\{ [000, 010], \emptyset, \emptyset, [101, 111] \}) \\ &= \{ [000, 010], [101, 111] \}. \end{aligned}$$

Figure 3 shows $\cup X$ and \overline{X} .

□

In the following, we show how Theorems 3.1 and 3.2 can be applied to compute the supremum of two maximal collections of closed intervals.

Corollary 3.1 Let X and Y be elements of II_W , then

$$X \cup Y = \overline{X \cap Y}.$$

Proof:

This corollary is de Morgan's law for the complete Boolean lattice (II_W, \leq) . □

Example 3.3 gives an application of this Corollary.

Example 3.3 Let $E = \{(-1,0), (0,0), (1,0)\}$ and let

$$X = \{ [100, 110], [010, 110] \}$$

and

$$Y = \{ [100, 101], [101, 111], [011, 111] \}.$$

Figure 4a shows $\cup X$ and X , while Figure 4b shows $\cup Y$ and Y .

We apply Corollary 3.1 to compute $X \cup Y$. First we compute \overline{X} .

$$\begin{aligned} \overline{X} &= M([100, 110]^c) \cap M([010, 110]^c) \\ &= \{ [000, 011], [001, 111] \} \cap \{ [000, 101], [001, 111] \} \\ &= \text{Max}(\{ [000, 001], [001, 011], [001, 101], [001, 111] \}) \\ &= \{ [000, 001], [001, 111] \}. \end{aligned}$$

Now we compute \overline{Y} .

$$\begin{aligned} \overline{Y} &= M([100, 101]^c) \cap M([101, 111]^c) \cap M([011, 111]^c) \\ &= \{ [000, 011], [010, 111] \} \cap \{ [000, 110], [000, 011] \} \\ &\quad \cap \{ [000, 101], [000, 110] \} \\ &= \text{Max}(\{ [000, 010], [000, 011], [010, 110], [010, 011] \}) \\ &\quad \cap \{ [000, 101], [000, 110] \} \\ &= \{ [000, 011], [010, 110] \} \cap \{ [000, 101], [000, 110] \} \end{aligned}$$

$$\begin{aligned}
&= \text{Max}(\{ [000, 001], [000, 010], \emptyset, [010, 110] \}) \\
&= \{ [000, 001], [000, 010], [010, 110] \}
\end{aligned}$$

Now we compute $\overline{X \cap Y}$.

$$\begin{aligned}
\overline{X \cap Y} &= \{ [000, 001], [001, 111] \} \cap \{ [000, 001], [000, 010], \\
&\quad [010, 110] \} \\
&= \text{Max}(\{ [000, 001], [000, 000], \emptyset, [001, 001], \emptyset, \emptyset \}) \\
&= \{ [000, 001] \}.
\end{aligned}$$

Finally, we compute $\overline{\overline{X \cap Y}}$.

$$\overline{\overline{X \cap Y}} = M([000, 001]^c) = \{ [100, 111], [010, 111] \}.$$

Figure 4c and 4d show $\cup(X \sqcup Y)$ and $X \sqcup Y$. □

The results of Theorems 3.1 and 3.2 were implemented by efficient algorithms. The performance of these algorithms in some applications are given in Tables 1 and 2. These experiments were performed in a SPARC Station 2 and the processing time was measure in seconds. For more details about these algorithms see [7].

4 Canonical decomposition of set operators

The set E is assumed to be an *Abelian group* with respect to a binary operation denoted by $+$. The zero element of $(E, +)$ is denoted by o . This zero element is also called the origin of E and is represented by a bold face character in the string description of a subset of E . For example, let E be the set $\{(-1, 0), (0, 0), (1, 0)\}$ and let X and Y be, respectively, the subsets $\{(-1, 0), (0, 0)\}$ and $\{(-1, 0), (1, 0)\}$. The subsets X and Y will be represented, respectively, by the strings 110 and 101.

Let X^t be the transpose of a subset X , that is, $X^t = \{y \in E : y = -x, x \in X\}$.

For any $h \in E$ and $X \subseteq E$, the set $X + h = \{x \in E : x - h \in X\}$ is called the translation of X by h . In particular, $X_o = X$.

A *set operator* is any mapping defined from $\mathcal{P}(E)$ into itself. The set Ψ of all the operators from $\mathcal{P}(E)$ to $\mathcal{P}(E)$ inherits the complete lattice structure of $(\mathcal{P}(E), \subseteq)$ by setting, $\forall \psi_1, \psi_2 \in \Psi$,

$$\psi_1 \leq \psi_2 \iff \psi_1(X) \subseteq \psi_2(X) \quad (X \in \mathcal{P}(E)).$$

The supremum and infimum of a subset Θ of the complete lattice (Ψ, \leq) verify

$$(\vee \Theta)(X) = \cup\{\theta(X) : \theta \in \Theta\} \quad (X \in \mathcal{P}(E))$$

and

$$(\wedge \Theta)(X) = \cap\{\theta(X) : \theta \in \Theta\} \quad (X \in \mathcal{P}(E)),$$

respectively.

A set operator ψ is called *translation invariant* (t.i.) iff, $\forall h \in E$,

$$\psi(X + h) = \psi(X) + h \quad (X \in \mathcal{P}(E)).$$

Let W be a finite subset of E . A set operator ψ is called *locally defined within a window* W iff, $\forall h \in E$,

$$h \in \psi(X) \iff h \in \psi(X \cap (W + h)) \quad (X \in \mathcal{P}(E)).$$

Let Ψ_W denote the collection of t.i. operators locally defined within a window W . The elements of Ψ_W are called W -operators. The pair (Ψ_W, \leq) constitutes a sublattice of the lattice (Ψ, \leq) .

The Kernel $K_W(\psi)$ of a W -operator ψ is the subcollection of $\mathcal{P}(W)$ defined by

$$K_W(\psi) = \{X \in \mathcal{P}(W) : o \in \psi(X)\}.$$

Proposition 4.1 *The mapping K_W from Ψ_W to $\mathcal{P}(\mathcal{P}(W))$ defined by*

$$K_W(\psi) = \{X \in \mathcal{P}(W) : o \in \psi(X)\} \quad (\psi \in \Psi_W)$$

constitutes a lattice isomorphism between the lattices (Ψ_W, \leq) and $(\mathcal{P}(\mathcal{P}(W)), \subseteq)$. The inverse of the mapping K_W is the mapping K_W^{-1} defined by

$$K_W^{-1}(X)(X) = \{x \in E : (X - x) \cap W \in \mathcal{X}\} \quad (X \in \mathcal{P}(E)).$$

Proof:

Let us prove the bijection between Ψ_W and $\mathcal{P}(\mathcal{P}(W))$. At first, we are going to prove that K_W is injective.

$$\begin{aligned} K_W^{-1}(K_W(\psi))(X) &= \{x \in E : (X - x) \cap W \in K_W(\psi)\} && \text{(by the definition of } K_W^{-1}) \\ &= \{x \in E : o \in \psi((X - x) \cap W)\} && \text{(by the definition of } K_W) \\ &= \{x \in E : o \in \psi((X - x) \cap (W + o))\} \\ &= \{x \in E : o \in \psi(X - x)\} && \text{(since } \psi \text{ is locally defined)} \\ &= \{x \in E : x \in \psi(X)\} && \text{(since } \psi \text{ is t.i.)} \\ &= \psi(X), \end{aligned}$$

that is, $\psi = K_W^{-1}(K_W(\psi))$ and K_W is injective.

Now we are going to prove that K_W is surjective.

$$\begin{aligned}
\mathcal{K}_W(\mathcal{K}_W^{-1}(\mathcal{X})) &= \{X \in \mathcal{P}(W) : \exists \sigma \in \mathcal{K}_W^{-1}(\mathcal{X})(X)\} \quad (\text{by the definition of } \mathcal{K}_W) \\
&= \{X \in \mathcal{P}(W) : \exists \sigma \in \{x \in E : (X - x) \cap W \in \mathcal{X}\}\} \quad (\text{by the definition of } \mathcal{K}_W^{-1}) \\
&= \{X \in \mathcal{P}(W) : X \cap W \in \mathcal{X}\} \\
&= \{X \in \mathcal{P}(W) : X \in \mathcal{X}\} \\
&= \mathcal{X},
\end{aligned}$$

that is, $\mathcal{X} = \mathcal{K}_W(\mathcal{K}_W^{-1}(\mathcal{X}))$ and so \mathcal{K}_W is surjective. Hence, \mathcal{K}_W is injective and surjective, consequently, it is bijective.

Let us now prove that \mathcal{K}_W preserves the partial order.

$$\begin{aligned}
\psi_1 \leq \psi_2 &\iff \forall X \in \mathcal{P}(E), \psi_1(X) \subseteq \psi_2(X) \\
&\iff \forall X \in \mathcal{P}(E), x \in \psi_1(X) \implies x \in \psi_2(X) \\
&\iff \forall X \in \mathcal{P}(E), \sigma \in \psi_1(X - x) \implies \sigma \in \psi_2(X - x)
\end{aligned}$$

(since ψ_1 and ψ_2 are t.i.)

$$\iff \forall X \in \mathcal{P}(E), \sigma \in \psi_1((X - x) \cap W) \implies \sigma \in \psi_2((X - x) \cap W)$$

(since ψ_1 and ψ_2 are locally defined)

$$\iff \forall X \in \mathcal{P}(E), (X - x) \cap W \in \mathcal{K}_W(\psi_1) \implies (X - x) \cap W \in \mathcal{K}_W(\psi_2)$$

(by the definition of \mathcal{K}_W)

$$\iff \forall Y \in \mathcal{P}(W), Y \in \mathcal{K}_W(\psi_1) \implies Y \in \mathcal{K}_W(\psi_2)$$

$$\iff \mathcal{K}_W(\psi_1) \subseteq \mathcal{K}_W(\psi_2).$$

Therefore,

$$\psi_1 \leq \psi_2 \iff \mathcal{K}_W(\psi_1) \subseteq \mathcal{K}_W(\psi_2), \quad \forall \psi_1, \psi_2 \in \Psi_W.$$

Hence, \mathcal{K}_W is a bijection that preserves the partial order, so it is a lattice isomorphism. \square

As a consequence of Proposition 1.1, the following equalities hold: $\forall \psi_1, \psi_2 \in \Psi_W$,

$$\mathcal{K}_W(\psi_1 \wedge \psi_2) = \mathcal{K}(\psi_1) \cap \mathcal{K}(\psi_2)$$

and

$$\mathcal{K}_{IV}(\psi_1 \vee \psi_2) = \mathcal{K}(\psi_1) \cup \mathcal{K}(\psi_2).$$

The set operators ι and ν defined by

$$\iota(X) = X \quad (X \in \mathcal{P}(E))$$

and

$$\nu(X) = X^c \quad (X \in \mathcal{P}(E)).$$

are called, respectively, the identity and the negation operators.

Let $A, B \in \mathcal{P}(E)$. The operations

$$A \oplus B = \bigcup_{b \in B} A + b \text{ and } A \ominus B = \bigcap_{b \in B} A - b$$

are called, respectively, *Minkowski addition and subtraction*.

Let $B \in \mathcal{P}(IV)$. The t.i set operators δ_B and ε_B defined by

$$\delta_B(X) = X \oplus B \quad (X \in \mathcal{P}(E))$$

and

$$\varepsilon_B(X) = X \ominus B \quad (X \in \mathcal{P}(E))$$

are called, respectively, *dilation and erosion by B*. The parameter B , that characterizes a dilation or an erosion, is called a structural element.

Let $A, B \in \mathcal{P}(IV)$, such that $A \subseteq B$. The t.i set operators $\lambda_{[A,B]}^W$ and $\mu_{[A,B]}^W$ defined by

$$\lambda_{[A,B]}^W(X) = \{x \in E : A \subseteq (X - x) \cap W \subseteq B\} \quad (X \in \mathcal{P}(E))$$

and

$$\mu_{[A,B]}^W(X) = \{x \in E : (X - x) \cap A^c \neq \emptyset \text{ or } ((X - x) \cap W^c) \cup B^c \neq W^c\} \quad (X \in \mathcal{P}(E)),$$

are called, respectively, sup-generating and inf-generating operators.

Note that these operators are locally defined within, respectively, W and W^c . The sup-generating and the inf-generating operators can be decomposed in terms of erosions and dilations, respectively, by

$$\lambda_{[A,B]}^W(X) = \varepsilon_A(X) \cap \nu \delta_{B^c}(X) \quad (X \in \mathcal{P}(E))$$

and

$$\mu_{[A,B]}^W(X) = \delta_A(X) \cup \nu \varepsilon_{B^c}(X) \quad (X \in \mathcal{P}(E)),$$

where the complement of B is taken relatively to W .

Proposition 4.2 *The Kernel of a sup-generating operator is a closed interval, that is,*

$$\mathcal{K}_W(\lambda_{[A,B]}^W) = [A, B].$$

proof:

$$\begin{aligned} \mathcal{K}(\lambda_{[A,B]}^W) &= \{X \in \mathcal{P}(W) : \sigma \in \lambda_{[A,B]}^W(X)\} \\ &= \{X \in \mathcal{P}(W) : \sigma \in \{x \in E : A \subseteq (X - x) \cap (W) \subseteq B\}\} \\ &= \{X \in \mathcal{P}(W) : A \subseteq X \cap W \subseteq B\} \\ &= [A, B]. \end{aligned}$$

□

Theorem 4.1 *Let ν be a W -operator, then*

$$\nu(X) = \bigcup \{\lambda_{[A,B]}^W(X) : [A, B] \subseteq \mathcal{K}_W(\nu)\} \quad (X \in \mathcal{P}(E)).$$

Proof:

$$\mathcal{K}_W(\psi) = \bigcup \{ [A, B] \subseteq \mathcal{P}(W) : [A, B] \subseteq \mathcal{K}_W(\psi) \}$$

(since any subset of a complete lattice can be built by the union of its closed intervals)

$$\mathcal{K}_W(\psi) = \bigcup \{ \mathcal{K}_W(\lambda_{[A, B]}^W) : [A, B] \subseteq \mathcal{K}_W(\psi) \}$$

(by Proposition 4.2)

$$\psi = \bigvee \{ \lambda_{[A, B]}^W : [A, B] \subseteq \mathcal{K}_W(\psi) \}$$

(since \mathcal{K}_W is a lattice isomorphism between (Ψ_W, \leq) and $(\mathcal{P}(\mathcal{P}(W)), \subseteq)$).

□

The dual operator of the operator ψ , denoted by ψ^* , is

$$\psi^* = \nu \circ \psi.$$

An example of dual operators are $\lambda_{[A, B]}^W$ and $\mu_{[A^c, B^c]}^{W^c}$, that is, $\mu_{[A^c, B^c]}^{W^c} = (\lambda_{[A, B]}^W)^*$.

Theorem 4.2 *Let ψ be a W -operator, then*

$$\psi(X) = \bigcap \{ \mu_{[A^c, B^c]}^{W^c}(X) : [A, B] \subseteq \mathcal{K}_W(\psi^*) \} \quad (X \in \mathcal{P}(E)).$$

Proof:

By Theorem 4.1, we have:

$$\psi^*(X) = \bigcup \{ \lambda_{[A, B]}^W(X) : [A, B] \subseteq \mathcal{K}_W(\psi^*) \} \quad (X \in \mathcal{P}(E)).$$

By the definition of ψ^* we have:

$$\psi = \nu \circ \psi^*.$$

Therefore, $\forall X \in \mathcal{P}(E)$,

$$\begin{aligned} \psi(X) &= \nu(\bigcup \{ \lambda_{[A, B]}^W(X) : [A, B] \subseteq \mathcal{K}_W(\psi^*) \}) \\ &= \bigcap \{ \nu \lambda_{[A, B]}^W(X) : [A, B] \subseteq \mathcal{K}_W(\psi^*) \} \\ &= \bigcap \{ \mu_{[A^c, B^c]}^{W^c}(X) : [A, B] \subseteq \mathcal{K}_W(\psi^*) \}. \end{aligned}$$

□

The decompositions of Theorems 4.1 and 4.2 are called, respectively, canonical sup-decomposition and canonical inf-decomposition.

Though these decompositions are quite general, they may lead to inefficient computational representations for most W -operators, in the sense that a smaller family of sup-generating or inf-generating operators may be sufficient to represent the same operator.

The set $B_W(\psi)$ of all maximal closed intervals contained in $\mathcal{K}_W(\psi)$ is called the *basis* of ψ , that is, $B_W(\psi) = M(\mathcal{K}_W(\psi))$.

The basis of the sup-generating operator $\lambda_{[A, B]}^W$ is the unitary collection $\{[A, B]\}$. Particularly, the basis of the erosion operator ε_A is $\{[A, W]\}$ and the basis of the identity operator is $\{[o, W]\}$.

The basis of the dilation operator δ_A is the collection $\{[a, W] : a \in A^t\}$.

Theorem 4.3 *Let ψ be a W -operator, then*

$$\psi(X) = \bigcup \{ \lambda_{[A, B]}^W(X) : [A, B] \in B_W(\psi) \} \quad (X \in \mathcal{P}(E)).$$

Proof:

Once W is finite, we can build $\mathcal{K}_W(\psi)$ in terms of its maximal closed intervals, that is,

$$\mathcal{K}_W(\psi) = \bigcup \{ [A, B] \subseteq \mathcal{P}(W) : [A, B] \in B_W(\psi) \}.$$

The result follows from the same arguments used to prove Theorem 4.1. □

As a consequence of Theorem 4.3, the decomposition of Theorem 4.2 can also be simplified.

Theorem 4.4 *Let ψ be a W -operator, then*

$$\psi(X) = \bigcap \{ \mu_{[A, B]}^{W'}(X) : [A, B] \in B_W(\psi^*) \} \quad (X \in \mathcal{P}(E))$$

□

Proof:

This result can be proved by the same arguments used in the proof of Theorem 4.2, with Theorem 4.3 taken the role of Theorem 4.1. \square

We have seen that the W -operators can be represented by their Kernel or their Basis. We will study now a third way of representing these operators: the equivalent Boolean functions.

Let T be the mapping between Ψ_W and $\{0, 1\}^{\mathcal{P}(W)}$ defined by

$$T(\psi)(X) = \begin{cases} 1 & \text{if } v \in \psi(X) \\ 0 & \text{otherwise} \end{cases} \quad (X \in \mathcal{P}(W))$$

The mapping T constitutes a lattice isomorphism between the complete lattices (Ψ_W, \leq) and $(\{0, 1\}^{\mathcal{P}(W)}, \leq)$ and its inverse T^{-1} is defined by

$$T^{-1}(f)(X) = \{x \in E : f((X - x) \cap W) = 1\} \quad (X \in \mathcal{P}(E)).$$

We should note that under all these decomposition results are the isomorphisms between the lattices (Ψ_W, \leq) , $(\mathcal{P}(\mathcal{P}(W)), \subseteq)$, (H_W, \leq) and $(\{0, 1\}^{\mathcal{P}(W)}, \leq)$. Figure 5 presents a picture that describes graphically these four basic isomorphisms.

Figure 6 illustrates the application of the decomposition theory presented in this section, showing the representation of the vertical edges detector by its Boolean function, Kernel and Basis. An application of this operator to a particular image is also presented.

Finally, we should observe that if $W = E$ the class of W -operators is itself the class of t.i. operators and, under this condition, the decomposition results presented in this section reduce to the ones presented by Banon and Barrera [2].

5 Composition of operators in the canonical form

In this section, we use the results of the previous sections to show how to compute the basis of the intersection, of the union and of the composition with an erosion (with a dilation or with the complementation) of given operators, from the basis of these operators.

The results presented in this section extend the work of Jones [13, 14], who has studied the particular case of increasing t.i. operators.

Let \mathcal{X} be a subcollection of $\mathcal{P}(W)$. We denote by $\mathcal{X} + h$ the collection built by the translation of the elements of \mathcal{X} by the vector h , that is,

$$\mathcal{X} + h = \{X \in \mathcal{P}(W + h) : X - h \in \mathcal{X}\}.$$

Particularly, $[A, B] + h$ denotes the translation of all the elements of the interval $[A, B]$ by the vector h , that is,

$$[A, B] + h = \{X \in \mathcal{P}(W + h) : X - h \in [A, B]\}.$$

Let \mathbf{X} be a collection of closed intervals contained in $\mathcal{P}(W)$. We denote by $\mathbf{X} + h$ the collection of intervals built by the translation of the elements of \mathbf{X} by the vector h , that is,

$$\mathbf{X} + h = \{[A, B] \subseteq \mathcal{P}(W + h) : [A, B] - h \in \mathbf{X}\}.$$

Let \mathcal{X} be a subcollection of $\mathcal{P}(W)$, with $\mathbf{X} = \mathbf{M}(\mathcal{X})$. For $h \in E$, we have:

$$\mathcal{X} + h = \cup(\mathbf{X} + h) = (\cup\mathbf{X}) + h.$$

Now we will study some properties of compositions of locally defined operators within distinct windows.

Proposition 5.1 *If a set operator is locally defined within a window W , then it is locally defined within any window $W' \supseteq W$.*

Proof:

Let $W' \supseteq W$. For any $X \in \mathcal{P}(E)$ and $h \in E$,

$$h \in \psi(X \cap (W' + h)) \iff h \in \psi((X \cap (W' + h)) \cap (W + h))$$

(since ψ is locally defined within W)

$$\iff h \in \psi(X \cap (W' + h) \cap (W + h))$$

$$\iff h \in \psi(X \cap (W + h))$$

(since $W \subseteq W'$)

$$\iff h \in \psi(X)$$

(since ψ is locally defined within W)

Hence, ψ is locally defined within W' . □

Note that to change the representation of the basis of an operator ψ locally defined within a window W , from this window to another window $W' \supseteq W$, it is enough to complete the right extremities of the intervals with the complement of W relatively to W' , that is,

$$\mathcal{B}_{W'}(\psi) = \{[A', B'] \subseteq \mathcal{P}(W') : A' = A \text{ and } B' = B \cup W^c, [A, B] \in \mathcal{B}_W(\psi)\}.$$

Of course, $\mathcal{B}_W(\psi)$ and $\mathcal{B}_{W'}(\psi)$ are equivalent representation of ψ . Note that for each $[A', B'] \in \mathcal{B}_{W'}(\psi)$, there exists $[A, B] \in \mathcal{B}_W(\psi)$ such that $A = A'$ and $\{x \in W : x \notin B\} = \{x \in W' : x \notin B'\}$. The next example illustrates this change of representation for the operator presented in Figure 6.

Example 5.1 Let $W = 111$ and $W' = \begin{matrix} 1 \\ 111 \\ 1 \end{matrix}$. Let ψ be the operator that performs vertical edge detection. The basis of ψ in the windows W and W' are, respectively,

$$\mathcal{B}_W(\psi) = \left\{ \left[\begin{array}{cc} 010 & 110 \end{array} \right], \left[\begin{array}{cc} 010 & 011 \end{array} \right] \right\}$$

and

$$\mathcal{B}_{W'}(\psi) = \left\{ \left[\begin{array}{cc} 0 & 1 \\ 010 & 110 \\ 0 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 010 & 011 \\ 0 & 1 \end{array} \right] \right\}. \quad \square$$

Proposition 5.2 If ψ_1 and ψ_2 are two locally defined operators within windows, respectively, W_1 and W_2 , then the operators $\psi_1 \wedge \psi_2$ and $\psi_1 \vee \psi_2$ are locally defined within the window $W_1 \cup W_2$.

Proof:

$\forall h \in E$ and $X \in \mathcal{P}(E)$.

$$h \in (\psi_1 \wedge \psi_2)(X) \iff h \in \psi_1(X) \cap \psi_2(X)$$

$$\iff h \in \psi_1(X) \text{ and } h \in \psi_2(X)$$

$$\iff h \in \psi_1(X \cap (W_1 + h)) \text{ and } h \in \psi_2(X \cap (W_2 + h))$$

(since ψ_1 and ψ_2 are locally defined operators, respectively, within W_1 and W_2)

$$\iff h \in \psi_1(X \cap ((W_1 \cup W_2) + h)) \text{ and } h \in \psi_2(X \cap ((W_1 \cup W_2) + h))$$

(since ψ_1 and ψ_2 are locally defined within any $W \supseteq W_1 \cup W_2$)

$$\iff h \in (\psi_1 \wedge \psi_2)(X \cap ((W_1 \cup W_2) + h)).$$

Hence, $\psi_1 \wedge \psi_2$ is locally defined within $W_1 \cup W_2$.

By similar arguments, we prove the result for $\psi_1 \vee \psi_2$.

□

For any $u \in E$, the set operator τ_u , defined by

$$\tau_u(X) = X + u \quad (X \in \mathcal{P}(E)),$$

is called the *translation operator by u* .

Proposition 5.3 *If ψ is a locally defined operator within a window W and, for any $u \in E$, τ_u is the translation operator by u , then $\tau_u\psi$ is a locally defined operator within the window $W - u$.*

Proof:

$\forall h \in E$ and $X \in \mathcal{P}(E)$.

$$h \in \tau_u\psi(X) \iff h \in \psi(X) + u$$

$$\iff h - u \in \psi(X)$$

$$\iff h - u \in \psi(X \cap (W + (h - u)))$$

(since ψ is locally defined within W)

$$\iff h \in \nu(X \cap ((W - u) + h)) + u$$

$$\iff h \in \tau_u \nu(X \cap ((W - u) + h))$$

(by the definition of τ_u).

Hence, $\tau_u \nu$ is locally defined within the window $W - u$.

□

Proposition 5.4 *If ν is an operator locally defined within a window W , then the operators $\delta_B \nu$ and $\varepsilon_B \nu$ are locally defined, respectively, within the windows $W \oplus B^t$ and $W \oplus B$.*

Proof:

$\forall h \in E$ and $X \in \mathcal{P}(E)$.

$$h \in \varepsilon_B \nu(X) \iff h \in \cap_{b \in B} \nu(X) - b$$

$$\iff h \in (\wedge_{b \in B} \tau_{-b} \nu)(X)$$

$$\iff h \in (\wedge_{b \in B} \tau_{-b} \nu)(X \cap ((\cup_{b \in B} W + b) + h))$$

(by Proposition 5.2 and 5.3)

$$\iff h \in (\wedge_{b \in B} \tau_{-b} \nu)(X \cap ((W \oplus B) + h))$$

$$\iff h \in \cap_{b \in B} \nu(X \cap ((W \oplus B) + h)) - b$$

$$\iff h \in \varepsilon_B \nu(X \cap ((W \oplus B) + h))$$

Hence, $\varepsilon_B \nu$ is locally defined within $W \oplus B$.

A similar proof can be written for $\delta_B \nu$.

□

Note that the operators ν and ι are neutral operators in relation to the size of the window, that is, if ν is a locally defined operator within the window W then $\nu \nu$, $\nu \iota$, $\iota \nu$, and $\iota \iota$ are also locally defined within W .

Corollary 5.1 *If ν_1 and ν_2 are two locally defined l.i. set operators within windows, respectively, W_1 and W_2 , then the operator $\nu_2 \nu_1$ is a locally defined operator within the window $W_1 \oplus W_2$.*

Proof:

Let $[A, B] \in B_{W_2}(\psi_2)$. The operator $\varepsilon_A \psi_1$ is locally defined within the window $W_1 \oplus A$ (by Proposition 5.4), then it is locally defined within $W_1 \oplus W_2$, since $A \subseteq W_2$. The operator $\nu \delta_{B^c} \psi_1$ is locally defined within $W_1 \oplus (B^c)^c = W_1 \oplus B^c$ (by Proposition 5.4 and because ν is neutral in relation to the window), then it is locally defined within $W_1 \oplus W_2$, since $B^c \subseteq W_2$. Consequently, the operator $\varepsilon_A \psi_1 \wedge \nu \delta_{B^c} \psi_1$ is locally defined within $W_1 \oplus W_2$. Hence, the operator $\psi_2 \psi_1 = \vee \{ \varepsilon_A \psi_1 \wedge \nu \delta_{B^c} \psi_1 : [A, B] \in B_{W_2}(\psi_2) \}$ is locally defined within $W_1 \oplus W_2$. □

Note that Proposition 5.4 and Corollary 5.1 give worst case conditions for the increase of the window in composition of operators. It may even happen that the composition of two specific operators locally defined with the same window produces a new morphological operator that also depends on this window. An example of this fact is the following composition: $(\delta_A \varepsilon_A)(\delta_A \varepsilon_A)$.

In the following, we will show how to compute the basis of the infimum of two given W -operators, from the basis of the given operators.

Proposition 5.5 *If ψ_1 and ψ_2 are two W -operators, respectively, within the windows W_1 and W_2 and with basis $B_{W_1}(\psi_1)$ and $B_{W_2}(\psi_2)$, then*

$$B_{W_1 \cup W_2}(\psi_1 \wedge \psi_2) = B_{W_1 \cup W_2}(\psi_1) \cap B_{W_1 \cup W_2}(\psi_2).$$

Proof:

This result is an immediate consequence of Proposition 5.2 and the lattice isomorphism between $(\Psi_{W_1 \cup W_2}, \leq)$ and $(H_{W_1 \cup W_2}, \leq)$. □

The next example gives an application of this property.

Example 5.2 *Let ψ_1 and ψ_2 be two W -operators, with basis, respectively,*

$$B_W(\psi_1) = \left\{ \left[\begin{array}{cc} 010 & , & 111 \end{array} \right] \right\}$$

and

$$B_W(\psi_2) = \left\{ \left[\begin{array}{cc} 000 & , & 011 \end{array} \right], \left[\begin{array}{cc} 000 & , & 101 \end{array} \right], \left[\begin{array}{cc} 000 & , & 110 \end{array} \right] \right\},$$

where $W = 111$.

We apply Proposition 5.5 to compute $B_W(\psi_1 \wedge \psi_2)$ from $B_W(\psi_1)$ and $B_W(\psi_2)$.

$$\begin{aligned}
B_W(\psi_1 \wedge \psi_2) &= B_W(\psi_1) \cap B_W(\psi_2) \\
&= \left\{ \begin{bmatrix} 010 & . & 111 \end{bmatrix} \right\} \cap \left\{ \begin{bmatrix} 000 & . & 011 \end{bmatrix}, \begin{bmatrix} 000 & . & 101 \end{bmatrix}, \right. \\
&\quad \left. \begin{bmatrix} 000 & . & 110 \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} 010 & . & 011 \end{bmatrix}, \begin{bmatrix} 010 & . & 110 \end{bmatrix} \right\}.
\end{aligned}$$

□

Figure 7 shows two possible representations of the operator presented in Example 5.2. In this paper, one and zero pixels of images are represented, respectively, by black and white. The pixels outside of the image window are supposed to be zero. In Figure 7 and in others, we use the following notation: $\Gamma_{B(\psi)} = \vee \{ \varepsilon_A \wedge \delta_{B'} : [A, B] \in B(\psi) \}$.

In the following, we will show how to compute the basis of the composition of a W-operator with the negation operator, from the input operator basis.

Proposition 5.6 *If ψ is a W-operator with basis $B_W(\psi)$, then*

$$B_W(\nu\psi) = \overline{B}_W(\psi).$$

Proof:

This result is an immediate consequence of the lattice isomorphism between (Ψ_W, \leq) and (H_W, \leq) .

□

The next example gives an application of Proposition 5.6.

Example 5.3 *Let ψ be a W-operator, with basis*

$$B_W(\psi) = \left\{ \begin{bmatrix} 100 & . & 111 \\ 000 & . & 111 \\ 000 & . & 111 \end{bmatrix}, \begin{bmatrix} 000 & . & 111 \\ 010 & . & 111 \\ 000 & . & 111 \end{bmatrix}, \begin{bmatrix} 000 & . & 111 \\ 000 & . & 111 \\ 001 & . & 111 \end{bmatrix} \right\}.$$

$$\text{with } W = \begin{matrix} 111 \\ 111 \\ 111 \end{matrix}.$$

The basis of the operator $\nu\psi$ will be

$$B_W(\nu\psi) = \left\{ \begin{bmatrix} 000 & . & 011 \\ 000 & . & 101 \\ 000 & . & 110 \end{bmatrix} \right\}.$$

□

Figure 8 shows two possible representations of the operator presented in Example 5.3.

In the following, we will show how to compute the basis of the supremum of two given W -operators, from the basis of the given operators.

Proposition 5.7 *If ψ_1 and ψ_2 are two W -operators, respectively, within the windows W_1 and W_2 and with basis $B_{W_1}(\psi_1)$ and $B_{W_2}(\psi_2)$, then*

$$B_{W_1 \cup W_2}(\psi_1 \vee \psi_2) = B_{W_1 \cup W_2}(\psi_1) \sqcup B_{W_1 \cup W_2}(\psi_2).$$

Proof:

This result is an immediate consequence of Proposition 5.2 and the lattice isomorphism between $(\Psi_{W_1 \cup W_2}, \leq)$ and $(\mathcal{H}_{W_1 \cup W_2}, \leq)$. □

The next example gives an application of Proposition 5.7.

Example 5.4 *Let ψ_1 and ψ_2 be two W -operators, with basis, respectively,*

$$B_W(\psi_1) = \left\{ \left[\begin{array}{cc} 010 & 111 \\ 111 & 111 \\ 010 & 111 \end{array} \right] \right\}$$

and

$$B_W(\psi_2) = \left\{ \left[\begin{array}{cc} 010 & 111 \\ 101 & 101 \\ 010 & 111 \end{array} \right] \right\}.$$

$$\text{with } W = \begin{array}{cc} 111 \\ 111 \\ 111 \end{array}.$$

Applying Proposition 5.7, we compute the basis of $\psi_1 \vee \psi_2$:

$$\begin{aligned} B_W(\psi_1 \vee \psi_2) &= \left\{ \left[\begin{array}{cc} 010 & 111 \\ 111 & 111 \\ 010 & 111 \end{array} \right] \right\} \sqcup \left\{ \left[\begin{array}{cc} 010 & 111 \\ 101 & 101 \\ 010 & 111 \end{array} \right] \right\} \\ &= \left\{ \left[\begin{array}{cc} 010 & 111 \\ 101 & 111 \\ 010 & 111 \end{array} \right] \right\}. \end{aligned}$$

□

Figure 9 shows two alternative ways to represent the operator presented in Example 5.4.

Now, we will show how to compute the basis of the composition of a W -operator with a dilation or with an erosion, from the operator basis.

Proposition 5.8 *If ψ is a W -operator with basis $B_W(\psi)$ and, $\forall b \in E$, τ_b is the translation operator by the vector b , then*

$$B_{W-b}(\tau_b\psi) = B_W(\psi) - b.$$

Proof:

$$B_{W-b}(\tau_b\psi) = M(\mathcal{K}_{W-b}(\tau_b\psi))$$

(by the definition of B_W)

$$= M(\{X \in \mathcal{P}(W-b) : o \in (\tau_b\psi)(X)\})$$

(by the definition of \mathcal{K}_W)

$$= M(\{X \in \mathcal{P}(W-b) : o \in \psi(X) + b\})$$

(by the definition of τ_b)

$$= M(\{X \in \mathcal{P}(W-b) : o \in \psi(X+b)\})$$

(since ψ is t.i.)

$$= M(\{X \in \mathcal{P}(W) : o \in \psi(X)\} - b)$$

(by a property of the translation)

$$= M(\mathcal{K}_W(\psi) - b)$$

(by the definition of \mathcal{K}_W)

$$= B_W(\psi) - b$$

(by the definition of B_W and a property of the translation). □

Proposition 5.9 *If ψ is a W -operator with basis $B_W(\psi)$ and δ_B is the dilation by B , then*

$$B_{W \circ B}(\delta_B\psi) = \cup_{b \in B} B_{W \circ W}(\tau_b\psi).$$

Proof:

$$\mathbf{B}_{W \oplus B'}(\delta_B \psi) = M(\mathcal{K}_{W \oplus B'}(\delta_B \psi))$$

(by the definition of \mathbf{B}_W and Proposition 5.4)

$$= M(\{X \in \mathcal{P}(W \oplus B') : \psi \in (\delta_B \psi)(X)\})$$

(by the definition of \mathcal{K}_W)

$$= M(\{X \in \mathcal{P}(W \oplus B') : \psi \in \cup_{b \in B} \psi(X) + b\})$$

(by the definition of δ_B)

$$= M(\{X \in \mathcal{P}(W \oplus B') : \exists b \in B : \psi \in \psi(X) + b\})$$

(by a property of \cup)

$$= M(\{X \in \mathcal{P}(W \oplus B') : \exists b \in B : \psi \in \tau_b \psi(X)\})$$

(by the definition of τ_b)

$$= M(\cup_{b \in B} \{X \in \mathcal{P}(W \oplus B') : \psi \in \tau_b \psi(X)\})$$

(by a property of \cup)

$$= M(\cup_{b \in B} \mathcal{K}_{W \oplus B'}(\tau_b \psi))$$

(by the definition of \mathcal{K}_W and because $\tau_b \psi$ is locally defined within $W \oplus B'$, since $W - b \subseteq W \oplus B'$)

$$= \cup_{b \in B} \mathbf{B}_{W \oplus B'}(\tau_b \psi)$$

(by the lattice isomorphism between $(\mathcal{P}(\mathcal{P}(W \oplus B')), \subseteq)$ and $(H_{W \oplus B'}, \leq)$).

□

Example 5.5 gives an application of Proposition 5.9.

Example 5.5 Let ψ be a W -operator, with basis

$$\mathbf{B}_{W_1}(\psi) = \left\{ \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right], \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right], \left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right] \right\},$$

where $W_1 = 111$, and let $B = 011$. Applying Proposition 5.9, we compute the basis of $\delta_B \psi$:

$$\begin{aligned}
B_W(\delta_B \psi) &= \{ [0000, 1001], [0001, 1011], [0011, 1111] \} \\
&\cup \{ [0000, 0011], [0010, 0111], [0110, 1111] \} \\
&= \{ [0010, 0111], [0000, 0011], [0110, 1111] \\
&\quad [0000, 1001], [0011, 1111], [0001, 1011] \},
\end{aligned}$$

where $W = W_1 \oplus B^t = 1111$.

□

Figure 10 shows two alternative ways to represent the operator presented in Example 5.5.

Proposition 5.10 *If ψ is a W -operator with basis $B_W(\psi)$ and ε_B is an erosion by B , then*

$$B_{W \oplus B}(\varepsilon_B \psi) = \cap_{b \in B} B_{W \oplus B}(\tau_b \psi).$$

Proof:

$$B_{W \oplus B}(\varepsilon_B \psi) = M(\mathcal{K}_{W \oplus B}(\varepsilon_B \psi))$$

(by the definition of $B_{W \oplus B}$ and Proposition 5.1)

$$= M(\{X \in \mathcal{P}(W \oplus B) : o \in (\varepsilon_B \psi)(X)\})$$

(by the definition of \mathcal{K}_W)

$$= M(\{X \in \mathcal{P}(W \oplus B) : o \in \cap_{b \in B} \psi(X) - b\})$$

(by the definition of ε_B)

$$= M(\{X \in \mathcal{P}(W \oplus B) : o \in \psi(X) - b, \forall b \in B\})$$

(by a property of \cap)

$$= M(\cap_{b \in B} \{X \in \mathcal{P}(W \oplus B) : o \in \psi(X) - b\})$$

(by a property of \cap)

$$= M(\cap_{b \in B} \{X \in \mathcal{P}(W \oplus B) : o \in \tau_{-b} \psi(X)\})$$

(by the definition of τ_b)

$$= M(\cap_{b \in B} \mathcal{K}_{W \oplus B}(\tau_b v))$$

(by the definition of \mathcal{K}_U and because $\tau_b v$ is locally defined within $W \oplus B$, since $W + b \subseteq W \oplus B$)

$$= \cap_{b \in B} B_{W \oplus B}(\tau_b v)$$

(by the lattice isomorphism between $(\mathcal{P}(\mathcal{P}(W \oplus B)), \subseteq)$ and $(H_{W \oplus B}, \leq)$).

□

Example 5.6 gives an application of Proposition 5.10.

Example 5.6 Let v be a W -operator, with basis

$$B_{W_1}(v) = \left\{ \left[\begin{array}{cc} 1 & 0 \\ & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ & 1 \end{array} \right] \right\},$$

where $W_1 = 11$, and let $B = 110$. Applying Proposition 5.10, we compute the basis of the operator $\varepsilon_B v$:

$$\begin{aligned} B_W(\varepsilon_B v) &= \left\{ \left[\begin{array}{cc} 0 & 1 \\ & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 0 \\ & 1 \end{array} \right] \right\} \cap \left\{ \left[\begin{array}{cc} 1 & 0 \\ & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ & 1 \end{array} \right] \right\} \\ &= \left\{ \left[\begin{array}{cc} 0 & 1 \\ & 1 \end{array} \right], \left[\begin{array}{cc} 1 & 0 \\ & 1 \end{array} \right] \right\}, \end{aligned}$$

where $W = W_1 \oplus B = 111$.

□

Figure 11 shows two alternative ways to represent the operator presented in Example 5.6.

6 Incremental computation of operator basis

The results given in the last section are the key to compute the basis of any W -operator for which is known a representation in the Morphological Language. The idea is to use recursively these results to compute incrementally the basis of the operator. In this process, the initial condition is the trivial basis of the identity operator.

Figure 12a shows the icon which represents the basis of the identity operator. Figure 12b shows the icons which represent the elementary operations used to compute basis incrementally. We use these icons in diagrams that illustrate graphically the computation of basis. These diagrams will be graphs that describe the known representation of operators in the Morphological Language.

Now, we will give some examples that illustrate the incremental computation of basis.

Example 6.1 *Let us study the operator that perform the extraction of interval edges. Let us call this operator ψ and adopt its usual representation, that is, $\psi = \iota \wedge \nu \varepsilon_{B^t}$, with*

$$B = \begin{array}{c} \downarrow \\ 111 \\ \downarrow \\ 1 \end{array}$$

We compute incrementally the basis of ψ from the basis of the identity operator:

$$\begin{aligned} B_W(\psi) &= B_W(\iota \wedge \nu \varepsilon_{B^t}) \\ &= B_W(\iota) \cap B_W(\nu \varepsilon_{B^t}) \\ &= B_W(\iota) \cap \overline{B}_W(\varepsilon_{B^t}) \\ &= \left\{ \left[\begin{array}{cc} 0 & 0 \\ 010 & 111 \\ 0 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 010 & 011 \\ 0 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 010 & 110 \\ 0 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 010 & 111 \\ 0 & 0 \end{array} \right] \right\}. \end{aligned}$$

$$\text{where } W = \{o\} \cup (\{o\} \oplus (B \oplus \{o\})) = \begin{array}{c} \downarrow \\ 111 \\ \downarrow \\ 1 \end{array}$$

Figure 13 shows a graph representing this procedure. □

Example 6.2 *Let us study the operator that performs the directional thinning. Let us call this operator ψ and adopt its usual representation, that is, $\psi = \iota \wedge \nu(\varepsilon_{\downarrow} \wedge \nu \delta_{B^t})$, with*

$$A = \begin{array}{cc} 111 & 111 \\ 010 & 000 \\ 000 & 000 \end{array} \text{ and } B = \begin{array}{cc} 111 & \\ 000 & \\ 000 & \end{array}$$

We compute incrementally the basis of ψ from the basis of the identity operator:

$$\begin{aligned}
 B_W(\psi) &= B_W(\iota \wedge \nu(\varepsilon_{A'} \wedge \nu \delta_{B'})) \\
 &= B_W(\iota) \cap B_W(\nu(\varepsilon_{A'} \wedge \nu \delta_{B'})) \\
 &= B_W(\iota) \cap \overline{B_W(\varepsilon_{A'} \wedge \nu \delta_{B'})} \\
 &= B_W(\iota) \cap \overline{B_W(\varepsilon_{A'}) \cap B_W(\nu \delta_{B'})} \\
 &= B_W(\iota) \cap \overline{B_W(\varepsilon_{A'})} \cap \overline{B_W(\nu \delta_{B'})} \\
 &= \left\{ \left[\begin{array}{cc} 000 & 111 \\ \mathbf{1} & \mathbf{1} \\ 001 & 111 \end{array} \right], \left[\begin{array}{cc} 000 & 111 \\ \mathbf{1} & \mathbf{1} \\ 010 & 111 \end{array} \right], \left[\begin{array}{cc} 000 & 111 \\ \mathbf{1} & \mathbf{1} \\ 100 & 111 \end{array} \right], \left[\begin{array}{cc} 000 & 110 \\ \mathbf{1} & \mathbf{1} \\ 000 & 111 \end{array} \right] \right. \\
 &\quad \left. \left[\begin{array}{cc} 000 & 101 \\ \mathbf{1} & \mathbf{1} \\ 000 & 111 \end{array} \right], \left[\begin{array}{cc} 000 & 011 \\ \mathbf{1} & \mathbf{1} \\ 000 & 111 \end{array} \right] \right\}.
 \end{aligned}$$

$$\text{where } W = \{o\} \cup (\{o\} \oplus ((1 \oplus \{o\}) \cup ((\{o\} \oplus B^t) \oplus \{o\}))) = \begin{array}{c} 111 \\ \mathbf{1} \\ 111 \end{array}.$$

Figure 14 shows a graph representing this procedure. □

Let $I = \{1, 2, 3, \dots, n\}$ be a set of indices. Now we give a result that shows how to compute the basis of an operator from the basis of its dual.

Proposition 6.1 *If ψ is a W -operator with basis $B_W(\psi) = \{[\lambda_i, B_i] : i \in I\}$, then the basis of its dual operator ψ^* is*

$$B_W(\psi^*) = \overline{\cup \{ \{ [B_i^c, \lambda_i^c] : i \in I \} \}}.$$

Proof:

$$\begin{aligned}
 \psi^* &= \nu \psi \nu \quad (\text{by the definition of } \psi^*) \\
 &= \nu(\vee \{ \lambda_{[A, B_i]}^W : i \in I \}) \nu \quad (\text{by the canonical sup-decomposition of } \psi) \\
 &= \nu(\vee \{ \varepsilon_{A'} \wedge \nu \delta_{B_i'} : i \in I \}) \nu \quad (\text{by the decomposition of } \lambda_{[A, B_i]}^W) \\
 &= \nu(\vee \{ \varepsilon_{A'} \wedge \nu \delta_{B_i'} \nu : i \in I \}) \\
 &= \nu(\vee \{ \nu \delta_{A_i'} \wedge \varepsilon_{B_i'} : i \in I \}) \quad (\text{since } \varepsilon_{A'} \nu = \nu \delta_{A_i'} \text{ and } \nu \delta_{B_i'} \nu = \varepsilon_{B_i'}).
 \end{aligned}$$

From the lattice isomorphisms between (Ψ_W, \leq) and (H_W, \leq) , we have:

$$\begin{aligned} B_W(\psi^*) &= \overline{B_W(\bigvee \{\nu\delta_{A^c} \wedge \varepsilon_{B^c} : i \in I\})} \\ &= \overline{\sqcup \{B_W(\nu\delta_{A^c} \wedge \varepsilon_{B^c}) : i \in I\}}. \end{aligned}$$

Furthermore,

$$\begin{aligned} B_W(\nu\delta_{A^c} \wedge \varepsilon_{B^c}) &= B_W(\nu\delta_{A^c}) \cap B_W(\varepsilon_{B^c}) \\ &= \overline{\{[a, W] : a \in A\}} \cap \{[B^c, W]\} \end{aligned}$$

(since $B_W(\delta_A) = \{[a, W] : a \in A\}$ and $B_W(\varepsilon_B) = \{[B, W]\}$)

$$\begin{aligned} &= \{[\emptyset, A^c]\} \cap \{[B^c, W]\} \quad (\text{by Theorem 3.2}) \\ &= \{[B^c, A^c]\} \quad (\text{by Theorem 3.1}). \end{aligned}$$

Hence,

$$B_W(\psi^*) = \overline{\sqcup \{[B_i^c, A_i^c] : i \in I\}}$$

□

Now we will give a result that shows how to compute the basis of the composition of two operators from the basis of these operators.

Proposition 6.2 *If ψ_1 and ψ_2 are W -operators, respectively, with windows W_1 and W_2 and the basis of ψ_2 is $B_{W_2}(\psi_2) = \{[A_i, B_i] : i \in I\}$, then*

$$B_{W_1 \circ W_2}(\psi_2 \psi_1) = \sqcup \{B_{W_1 \circ W_2}(\varepsilon_{A_i} \psi_1) \cap \overline{B_{W_1 \circ W_2}(\delta_{B_i^c} \psi_1)} : i \in I\}.$$

Proof:

$$\psi_2 \psi_1 = (\bigvee \{\lambda_{[A_i, B_i]}^{W_2} : i \in I\}) \psi_1$$

(by the canonical sup-decomposition of ψ_2)

$$= (\bigvee \{(\varepsilon_{A_i} \wedge \nu\delta_{B_i^c}) : i \in I\}) \psi_1$$

(by the decomposition of $\lambda_{[A_i, B_i]}^{W_2}$)

$$= \bigvee \{\varepsilon_{A_i} \psi_1 \wedge \nu\delta_{B_i^c} \psi_1 : i \in I\}$$

From the lattice isomorphism between (Ψ_W, \leq) and (H_W, \leq) and Corollary 5.1, we have:

$$B_{W_1 \oplus W_2}(\psi_2 \psi_1) = \sqcup \{ B_{W_1 \oplus W_2}(\epsilon_i, \psi_1) \cap \bar{B}_{W_1 \oplus W_2}(\delta_{B^t} \psi_1) : i \in I \}.$$

□

One practical application of the incremental computation of the basis is in the automatic proof of the equivalence between morphological operators, since two W -operators are equal if and only if their basis (represented in the window W) are equal. The next example illustrates this procedure. Figure 15 shows the icon used to represent the comparison of equality between the basis of two operators.

Example 6.3 Let the operators ψ_1 and ψ_2 be defined by $\delta_{B \in B^t}$ and $\delta_{B \in B} \delta_{B \in B^t}$, respectively, with $B = 111$.

Computing the basis of ψ_1 , we have:

$$B_{W_1}(\psi_1) = \left\{ \left[\begin{array}{cc} 11100 & . & 11111 \end{array} \right], \left[\begin{array}{cc} 01110 & . & 11111 \end{array} \right], \left[\begin{array}{cc} 00111 & . & 11111 \end{array} \right] \right\},$$

$$\text{with } W_1 = B^t \oplus (B \oplus \{o\}) = 11111$$

Computing the basis of ψ_2 , we have:

$$B_{W_2}(\psi_2) = \left\{ \left[\begin{array}{cc} 001110000 & . & 111111111 \end{array} \right], \left[\begin{array}{cc} 000111000 & . & 111111111 \end{array} \right], \right. \\ \left. \left[\begin{array}{cc} 000011100 & . & 111111111 \end{array} \right] \right\}.$$

$$\text{with } W_2 = B^t \oplus (B \oplus (B^t \oplus (B \oplus \{o\}))) = 111111111$$

Note that these basis are equivalents. Figure 16 shows a graph representing this procedure. □

Another application of the incremental computation of basis is in the parallelization of morphological operators. This technique is useful in the programming of highly parallel MMach's.

In the parallelization of a morphological operator we could choose between the two canonical decompositions. Figure 17 shows the highly parallel structure of these decompositions.

Note that the number of sup-generating operators used in the sup-decomposition depends on the basis of the operator, while the number of inf-generating operators in the inf-decomposition depends on the basis of the dual operator. Thus, to choose the best parallel representation we must compare the cardinality of the operator basis with the cardinality of the dual operator basis.

The next examples illustrate the procedure of parallelization of MMach's programs.

Example 6.4 Let the operator ψ be defined by $\psi = \iota \wedge \nu (\epsilon_{\iota} \wedge \nu \delta_{\beta \iota})$, with $A = \begin{matrix} 111 \\ 010 \\ 000 \end{matrix}$ and

$B = \begin{matrix} 111 \\ 000 \\ 000 \end{matrix}$. This representation of ψ is illustrated in Figure 18a. $B_W(\psi)$ was computed in Example 6.2. Computing the basis of ψ^* we have:

$$B(\psi^*) = \left\{ \left[\begin{array}{cc} 000 & 000 \\ 000 & 111 \\ 111 & 111 \end{array} \right], \left[\begin{array}{cc} 000 & 111 \\ 010 & 111 \\ 000 & 111 \end{array} \right] \right\},$$

that is used in the inf-decomposition. This representation of ψ is illustrated in Figure 18b. Note that this last representation is more efficient than the first, since it has less levels of data processing.

□

Example 6.5 Let the operator ψ be defined by $\psi = (\iota \wedge \nu \epsilon_{\beta \iota}) \vee (\nu \iota \wedge \delta_{\beta \iota})$, with $B = \begin{matrix} 010 \\ 111 \\ 010 \end{matrix}$.

This representation of ψ is illustrated in Figure 19a. The basis of ψ has twenty elements. Computing the basis of ψ^* we have:

$$B_W(\psi^*) = \left\{ \left[\begin{array}{cc} 000 & 101 \\ 000 & 000 \\ 000 & 101 \end{array} \right], \left[\begin{array}{cc} 010 & 111 \\ 111 & 111 \\ 010 & 111 \end{array} \right] \right\},$$

that is used in the inf-decomposition. This representation of ψ is illustrated in Figure 19b. Note that this last representation is more efficient than the first, since it has less data processing levels.

□

7 Conclusion

In this paper, we have presented some fast algorithms to compute set operations on collections of closed intervals. We have also showed how to apply these algorithms to solve a problem related with the automatic programming of **MMach**'s: the incremental computation of basis of W -operators.

The algorithms proposed are supported by a sound Mathematical theory that studies operations on collections of sets, represented by collections of closed intervals. A fundamental fact in this theory is that, for a given finite subset W of a set E , equipped with an Abelian group $+$, the posets $(\mathcal{P}(\mathcal{P}(W)), \subseteq)$, (H_W, \leq) , $(\{0, 1\}^{\mathcal{P}(W)}, \leq)$ and (Ψ_W, \leq) are isomorphic complete Boolean lattices.

As erosions and dilations are built from set operations, their basis are built from the corresponding closed intervals operations. Once the set operators can be decomposed in terms of erosions and dilations, their basis can be computed also from the basis of these elementary operators. Thus, the algorithms that perform operations on collections of closed intervals constitute the kernel of a software system for the incremental computation of the basis of set operators.

The incremental computation of the basis can be applied in practice for the parallelization of **MMach** programs or for the automatic proof of the equivalence between distinct morphological operators.

Another contribution of this paper was the derivation of the canonical decomposition expressions for the family of W -operators, that generalizes the known results for the family of t.i. operators.

The computational results presented in this paper were implemented in a software system for the automatic design of programs for **MMach**'s. This software was implemented as a Toolbox of the *khoro*s system [18] and has routines for the extraction of examples from pairs of images, learning of Boolean functions from the examples extracted, application of operators learned on images and incremental computation of operator basis. Some applications of this system into real world problems are given in [6].

Finally, this work should be the basis for the study of the transformation of the canonical representations into other representation structures that use smaller numbers of erosions and dilations. This problem, that can be understood as the inverse of the incremental computation of the basis, is extremely complex, but should play an important role in the design of a system that performs the automatic design of efficient programs for **MMach**'s.

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Figure 15 - Comparison of equality between two operators.

Figure 16 - Verification of the equivalence of ψ_1 and ψ_2 .

Figure 17 - Highly parallel decomposition structures. a) Canonical sup-decomposition. b) Canonical inf-decomposition.

Figure 18 - Two representations of the operator ψ . a) Definition of ψ . b) Inf-decomposition of ψ .

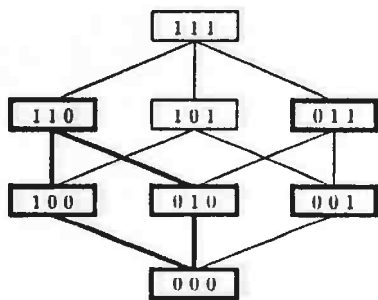
Figure 19 - Two representations of the operator ψ . a) Definition of ψ . b) Inf-decomposition of ψ .

$ X $	$ Y $	$ X \cap Y $	$ W $	Comparisons	Time (s)
808	1,210	961	37	27,807,299	102.7
1,210	808	961	37	85,738,988	240.0
117	117	52	41	209,783	1.7
658	658	2738	11	26,132,636	71.0
2,113	2,113	2,766	39	219,718,589	720.7

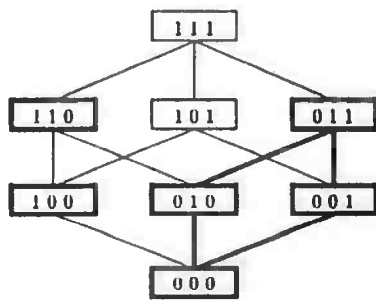
Table 1:

$ X $	$ \bar{X} $	$ W $	Comparisons	Time (s)
26	28	9	1,193	.2
13	1561	20	3,098,112	8.1
1561	13	20	28,916,001	63.8
729	9	15	155,695	1.2
337	136	77	1,536,591	1.1

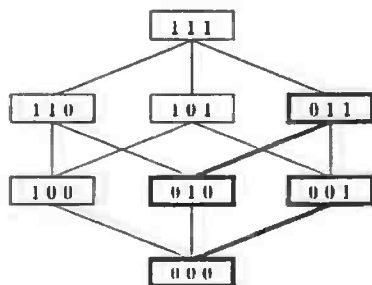
Table 2:



a)

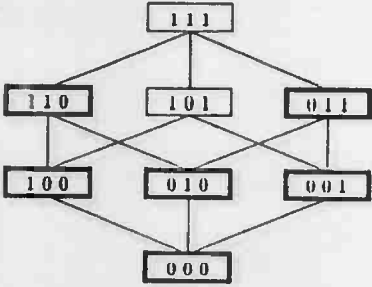


b)

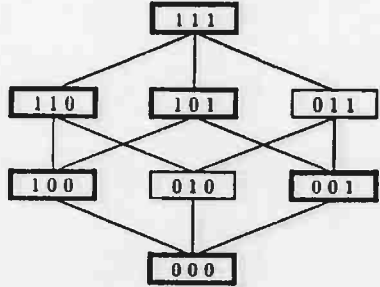


c)

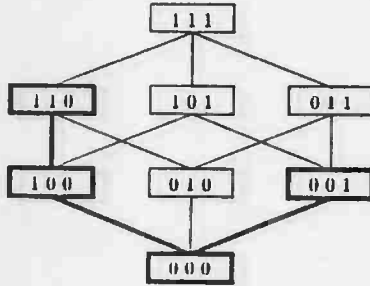
Figure 1:



a)



b)



c)

Figure 2:

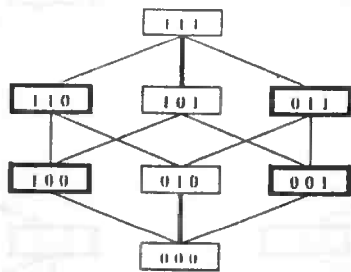


Figure 3:

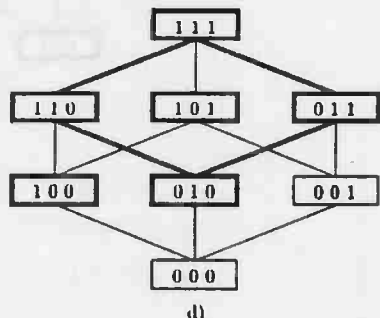
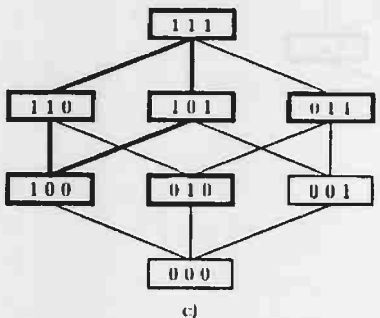
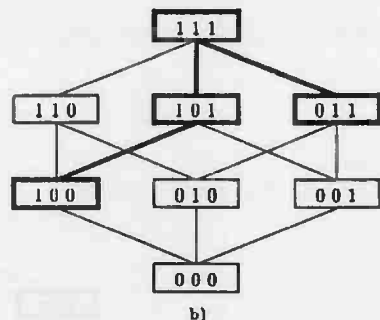
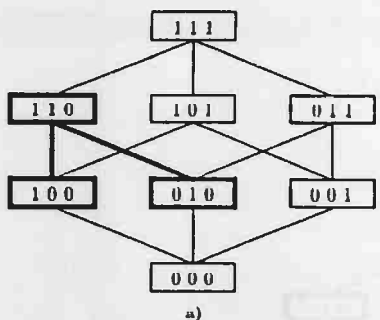


Figure 1:

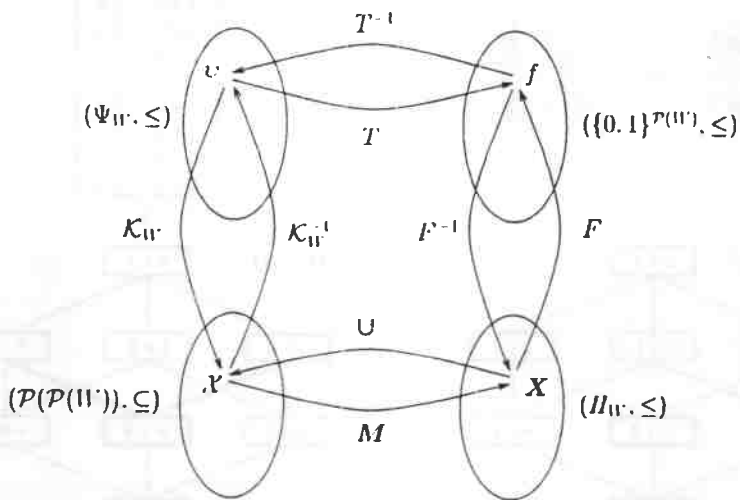
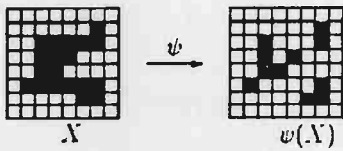


Figure 5:



$x_1x_2x_3$	f_ψ
000	0
001	0
010	1
011	1
100	0
101	0
110	1
111	0

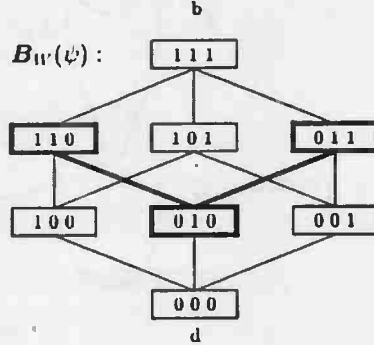
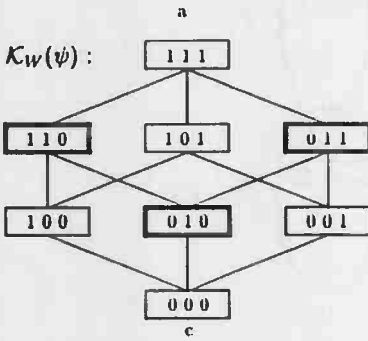


Figure 6:

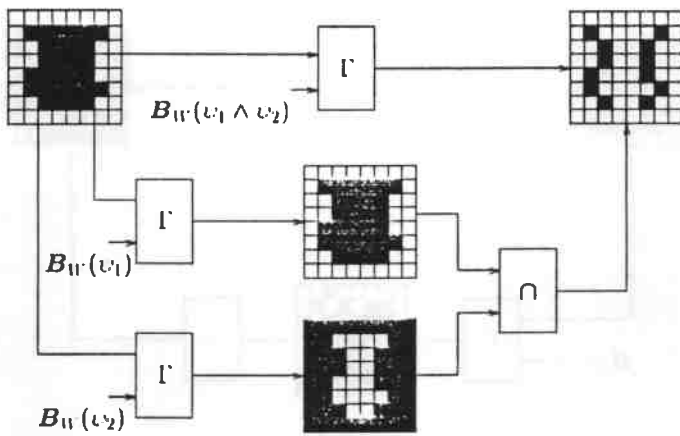


Figure 7:

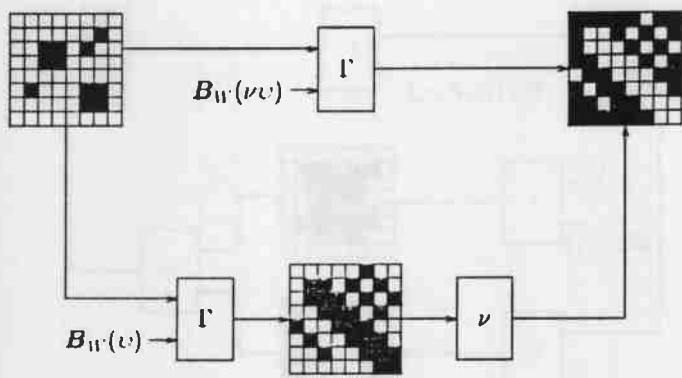


Figure 8:

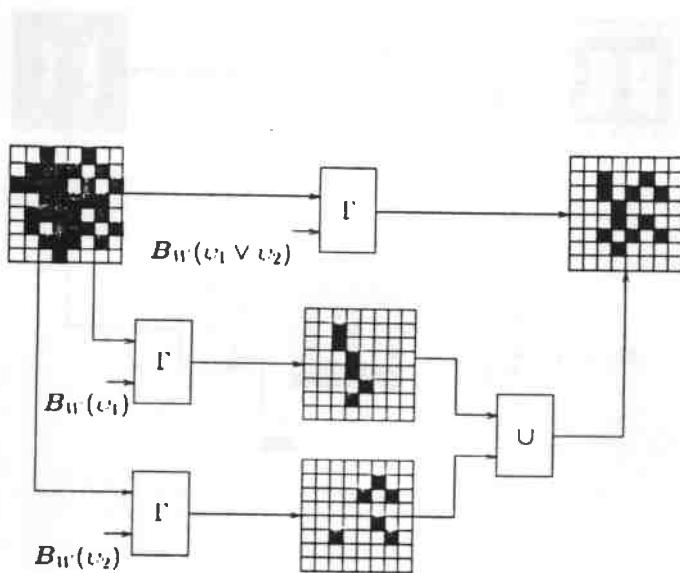


Figure 9:

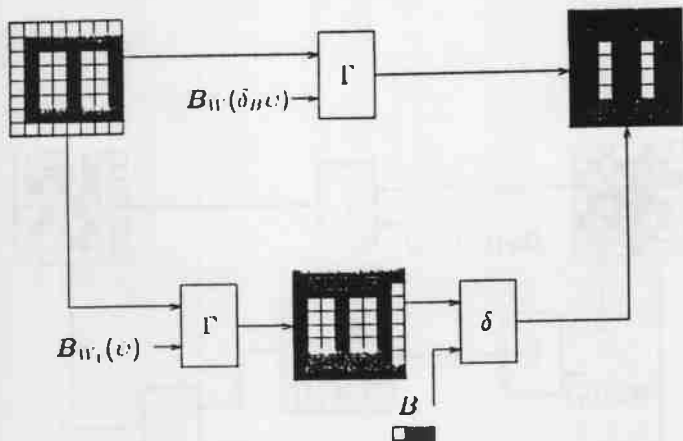


Figure 10:

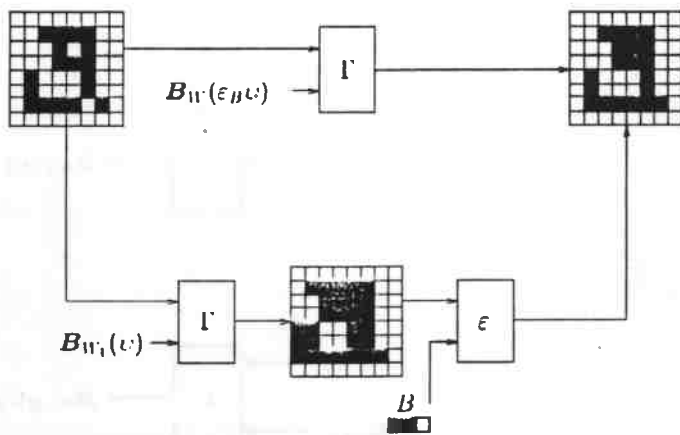


Figure 11:

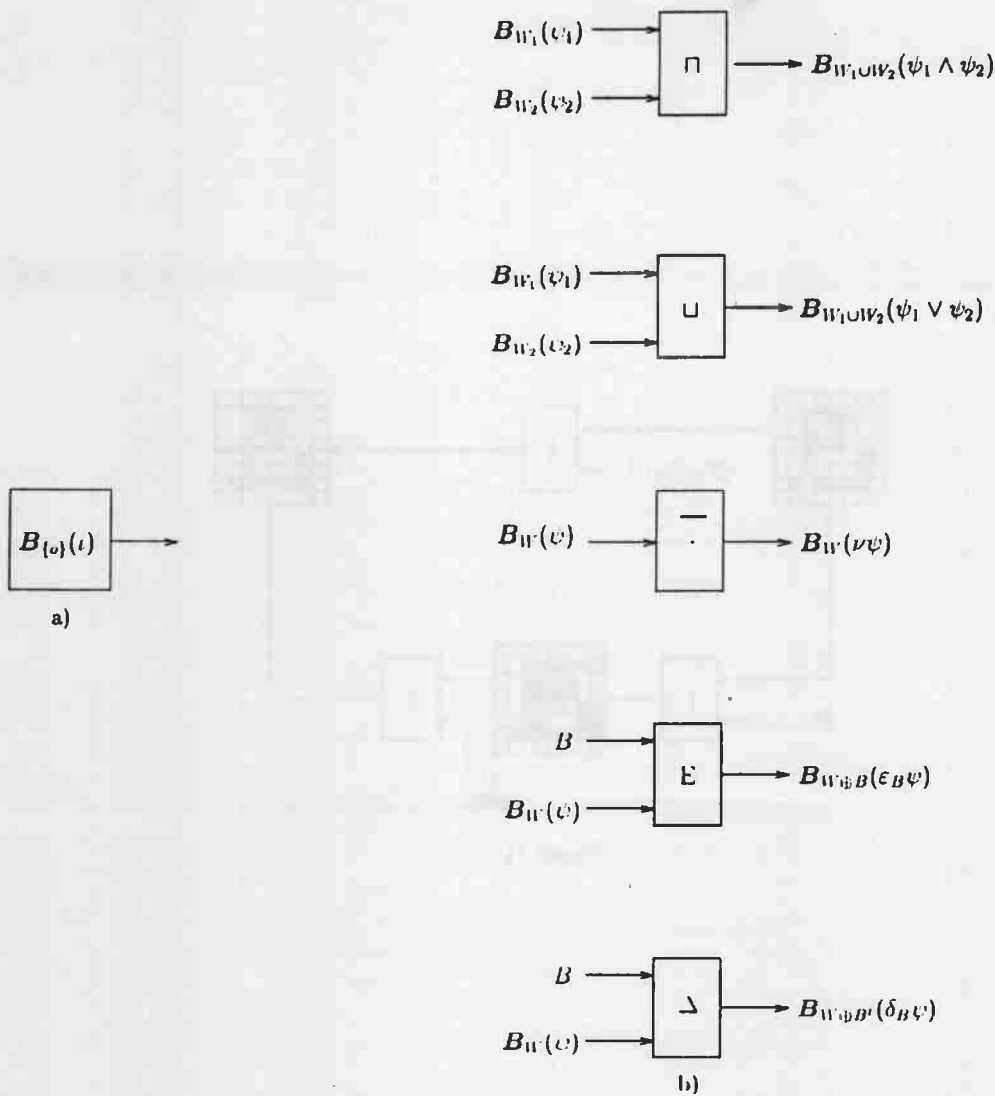


Figure 12:

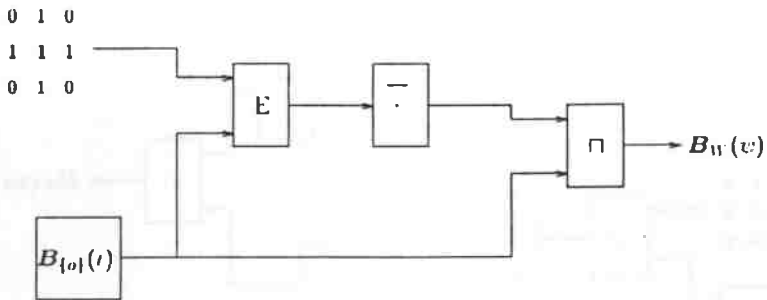


Figure 13:

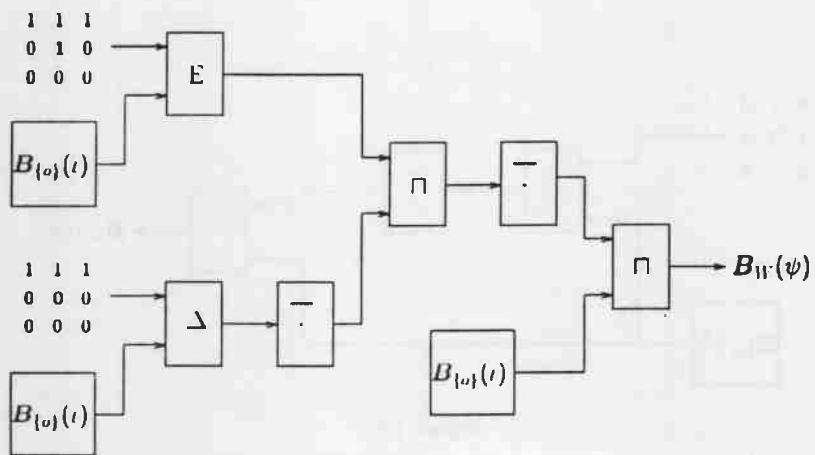


Figure 14:

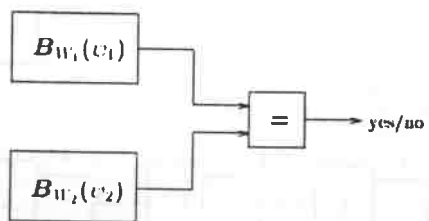


Figure 15:

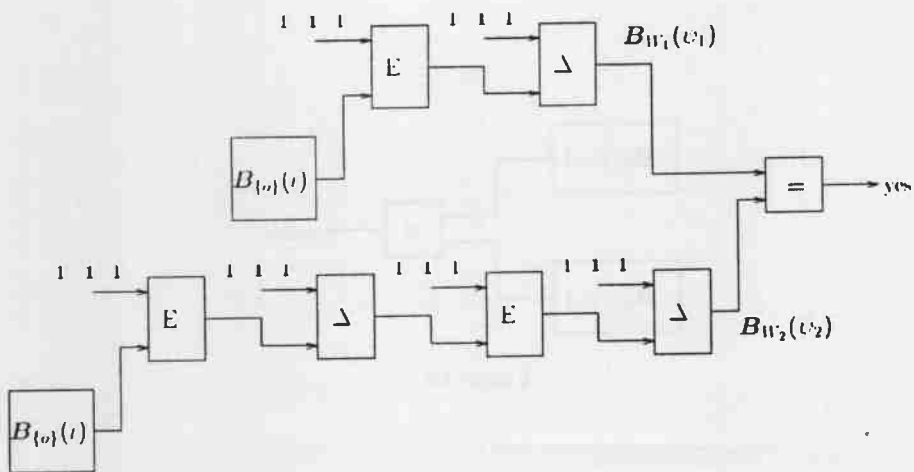
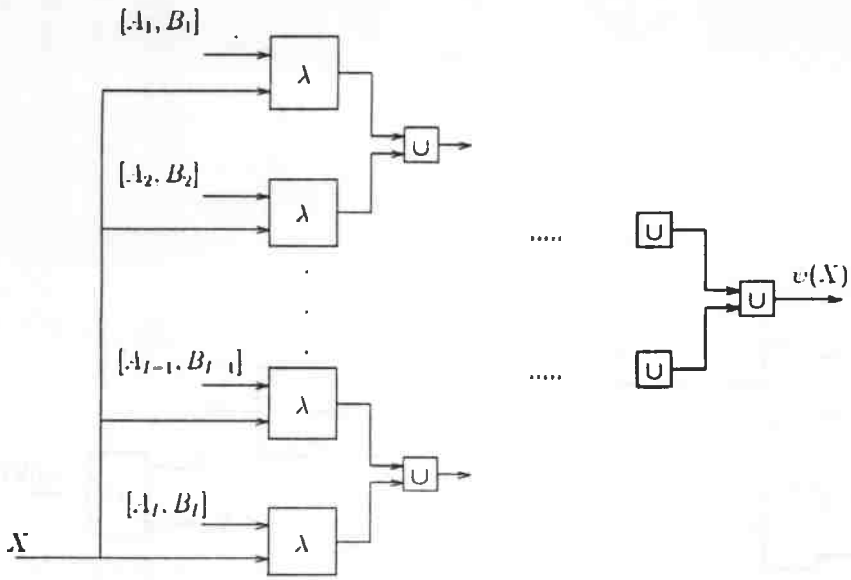
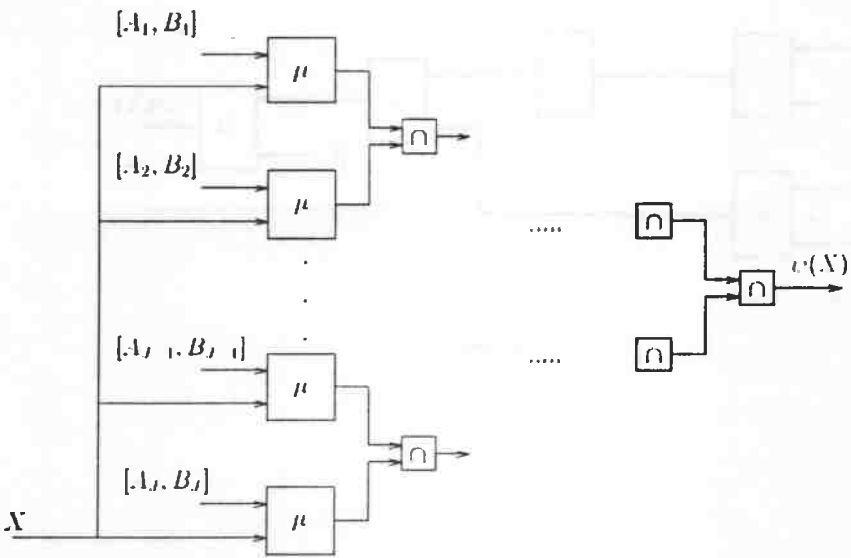


Figure 16:

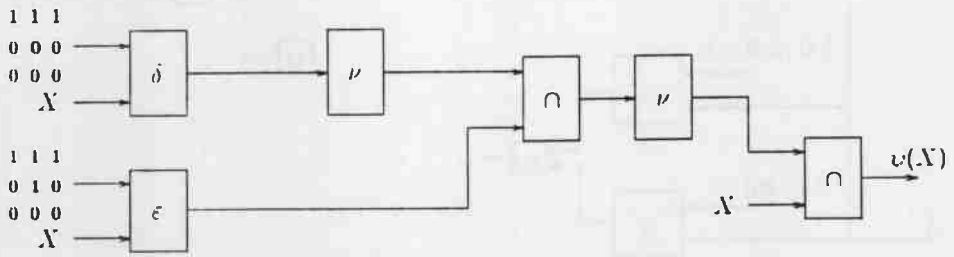


a)

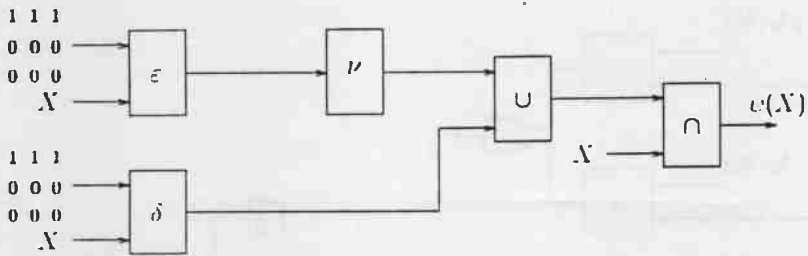


b)

Figure 17:



a)



b)

Figure 18:

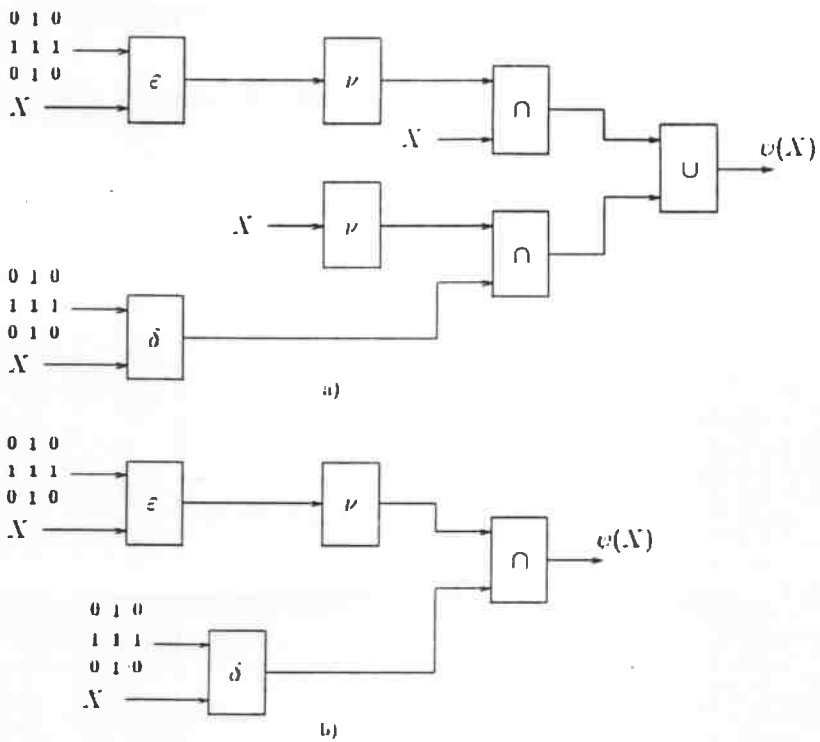


Figure 19:



RELATÓRIOS TÉCNICOS
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Instituto de Matemática e Estatística da USP

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