Calculus of Variations

Robust index bounds for minimal hypersurfaces of isoparametric submanifolds and symmetric spaces

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Abstract

We find many examples of compact Riemannian manifolds (M, g) whose closed minimal hypersurfaces satisfy a lower bound on their index that is linear in their first Betti number. Moreover, we show that these bounds remain valid when the metric g is replaced with g' in a neighbourhood of g. Our examples (M, g) consist of certain minimal isoparametric hypersurfaces of spheres, their focal manifolds, the Lie groups $SU(n)$ for $n \leq 17$ and $Sp(n)$ for all n, and all quaternionic Grassmannians.

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1 Introduction

Let (M, g) be a compact Riemannian manifold. We are interested in closed, immersed, minimal hypersurfaces $\Sigma \to M$. The (Morse) index $\text{ind}(\Sigma)$ of such Σ is the maximal dimension of a space of smooth sections of the normal bundle of Σ where the second variation of area is negative-definite. Since Σ is compact, its index is finite. It is natural to ask what is

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the relation between the index and the topology of Σ. In this regard, the following conjecture was first proposed in the 1980s by Schoen, and then rephrased in all dimensions by Marques and Neves in their ICM lectures in 2014 (cf. [14, page 16], or [1, page 3]):

Conjecture *Let (M, g) be a compact Riemannian manifold with positive Ricci curvature, and dimension at least three. Then there exists $C > 0$ such that, for all closed, embedded, orientable, minimal hypersurfaces $\Sigma \to M$, one has*

$$\text{ind}(\Sigma) \geq C b_1(\Sigma)$$

where $b_1(\Sigma)$ denotes the first Betti number of Σ with real coefficients.

Some special cases and related results include: When (M, g) is a flat 3-torus, Ros [17, Theorem 16] has found affine (in the first Betti number) bounds on the index—see also [5] and [2]. For (M, g) a round sphere of any dimension, Savo [18] has given linear bounds on the index. Generalizing Savo's method, Ambrozio, Carlotto, and Sharp [1] have found linear bounds on the index when (M, g) is any compact rank-one symmetric space, or $S^a \times S^b$ for $(a, b) \neq (2, 2)$. Finally, the second and third authors of the present article have proven a linear bound on the index plus nullity when (M, g) is any compact symmetric space [11].

Savo's method, as generalized by Ambrozio–Carlotto–Sharp, relies on the existence of an isometric immersion of (M, g) into a Euclidean space $\mathbb{R}^d$ such that a certain real-valued function, which we call the ACS *quantity*, is everywhere negative. The domain of this function is the total space of the bundle of Stiefel manifolds $V_2(TM)$ of orthonormal 2-frames, and it depends only on the second fundamental form II of $M \to \mathbb{R}^d$, see Definition 1 below for the precise formula. For instance, the standard inclusion $S^n \subset \mathbb{R}^{n+1}$ satisfies ACS < 0. It was already recognized in [1] that this method is flexible in the sense that sometimes the obtained index bound remains valid when the ambient metric g is deformed in certain directions (see [1, Theorems 12 and 13]). We push this idea further, and obtain:

Theorem A *Suppose (M, g) admits a C^∞ isometric immersion into $\mathbb{R}^d$ with negative ACS quantity, and with image contained in a sphere. Let $\lambda \in (0, 1)$. Then there exists $\epsilon > 0$ such that: For any C^∞ metric g' on M with $\|g - g'\|_{C^{2,\lambda}} < \epsilon$ (Hölder norm), and any minimal, closed, immersed hypersurface $\Sigma \subset (M, g')$, one has*

$$\text{ind}(\Sigma) \geq \frac{8}{d(d + 3)(d^2 + 3d - 2)} b_1(\Sigma).$$

Compared to the extrinsic flexibility of the method in [1], Theorem A states that, when the image of the immersion $M \to \mathbb{R}^d$ is contained in a sphere, then the method is actually *intrinsically* flexible: that is, the linear index bound remains valid under any small deformation of the metric itself.

The Proof of Theorem A is based on Günther's approach to the Nash Embedding Theorem. The hypothesis that the image of $M \to \mathbb{R}^d$ is contained in a sphere is satisfied by all known examples (including our new examples described below) of immersions with negative ACS quantity. If one drops this hypothesis from the statement of Theorem A, our proof still yields an open set $\mathcal{U}$ of metrics on M with respect to which the stated index bound holds. Moreover, $\mathcal{U}$ can be taken so that the original metric g belongs to the closure of $\mathcal{U}$ (cf. Remark 8).

Among the ambient symmetric spaces mentioned earlier, Theorem A applies to S^n, $S^a \times S^b$ for $(a, b) \neq (2, 2)$, $\mathbb{H}P^n$, and the Cayley plane. It does not apply to $\mathbb{R}P^n$ and $\mathbb{C}P^n$, because the proof in [1] of the index bound in this cases is less direct, and in particular they do not produce an immersion of these spaces into Euclidean space with ACS < 0.

In the second part of the present paper, we drastically expand the list of spaces to which Theorem A may be applied, and for which, in particular, the conclusion of Conjecture holds. An interesting feature of our new examples is that they all have positive Ricci curvature (see Propositions 21 and 22), while it is possible to find a sequence with minimum sectional curvature tending to $-\infty$ (see Proposition 23), thus providing evidence that the curvature assumption in Conjecture is the correct one. Another novel feature is that infinitely many are not symmetric, and in fact not even homogeneous, although they are all *curvature-homogeneous* (see Remark 24).

Our first class of examples are among *isoparametric* hypersurfaces of round spheres, that is, hypersurfaces with constant principal curvatures; and their focal manifolds. Our motivation to consider isoparametric submanifolds is that they have the simplest extrinsic geometry, making the study of the ACS quantity more manageable. One interesting fact about such submanifolds is that, assuming the multiplicities of the principal curvatures are bigger than one, ACS < 0 *implies* Ric > 0, see Proposition 21.

Isoparametric hypersurfaces of round spheres have been studied for at least a century by many prominent geometers, notably E. Cartan and H. Münzner who proved the first structure results (see Sect. 4.1 below for a short summary, and [3] for a general reference). Nevertheless, they remain a very active area of research, with interesting questions still open.

One particular feature of isoparametric hypersurfaces of round spheres is that the number g of principal curvatures is one of $\{1, 2, 3, 4, 6\}$, and their multiplicities $m_1, \ldots, m_g$ satisfy $m_i = m_{i+2}$, where i runs cyclically from 1 to g (in particular for $g = 3$ all multiplicities coincide). Therefore, all multiplicities are determined by m_1 and m_2, and it is customary to order them by $m_1 \leq m_2$. Furthermore, the set of focal points of any isoparametric hypersurface $M \subset S^{n+1}$ is given by the union of two *focal manifolds* $M_+, M_- \subset S^{n+1}$ of codimension $m_1 + 1$ and $m_2 + 1$ respectively. For any g given, the possible multiplicities (m_1, m_2) have been completely determined, and there is always a finite number of possibilities, except when $g = 4$, which is the case we will concentrate on. We refer the reader to Sect. 4.1 for more details on this theory.

Theorem B *Let $M^n \subset S^{n+1}$ be a minimal isoparametric hypersurface with four principal curvatures, and multiplicities $m_1 \leq m_2$. Let M_+ be the focal manifold of M with codimension $1 + m_1$ in S^{n+1}.*

(a) *If $m_1 \geq 5$, or if $m_1 = 4$ and m_2 is large enough, then M satisfies ACS < 0.*
(b) *If $m_2 > (3m_1 + 10)/4$, then M_+ satisfies ACS < 0.*

There exist infinitely many homogeneous and inhomogeneous isoparametric hypersurfaces $M \subset S^{n+1}$ satisfying the conditions in (a) and (b) of Theorem B (see Sect. 4.3 for precise statements). The homogeneous spaces satisfying (a) are orbits of the group Sp(k) Sp(2) acting on the space of quaternionic $k \times 2$ matrices in the natural way; as well as the isotropy representation of the symmetric space $E_6/ \text{Spin}(10) \text{U}(1)$. The homogeneous focal manifolds satisfying (b) are Stiefel manifolds of 2-frames over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$; and one of the singular orbits of the isotropy representation of the symmetric space $E_6/ \text{Spin}(10) \text{U}(1)$. The inhomogeneous examples satisfying the conditions of Theorem B were constructed using Clifford systems by Ferus–Karcher–Münzner [8], generalizing previous constructions of Ozeki–Takeuchi [15,16].

Our second class of new examples are symmetric spaces:

Theorem C *The following symmetric spaces admit an embedding into some Euclidean space with $ACS < 0$:*

(a) *The quaternionic Grassmannian of d-planes in $\mathbb{H}^n$, for all d, n;*

(b) *The Lie group* $Sp(n)$ *for all* n;

(c) *The Lie group* $SU(n)$ *for* $n \leq 17$.

This article is organized as follows. In Sect. 2, we recall the method of Savo and Ambrozio–Carlotto–Sharp to prove index bounds using an isometric immersion of the ambient manifold into Euclidean space, and in particular define what we call the ACS quantity. Section 3 is devoted to the Proof of Theorem A. Section 4 concerns isoparametric hypersurfaces of the sphere. After some preliminaries, we compute the ACS quantity of such submanifolds, and prove Theorem B. Then, we apply Theorem B to concrete examples, and finish the section with remarks about the geometry of these examples. Finally, in Sect. 5 we study equivariant embeddings of symmetric spaces into Euclidean space, and prove Theorem C: parts (a), (b), and (c) follow from Propositions 34, 30, and 32, respectively.

2 Index bounds

In this section we recall a method due to Ambrozio–Carlotto–Sharp [1] (generalizing previous work, especially [17] and [18]) to prove lower bounds on the index of immersed minimal hypersurfaces.

Consider a complete Riemannian manifold (M, g). Assume M is isometrically immersed into some Euclidean space $\mathbb{R}^d$, and denote by II the second fundamental form of this immersion. Inspired by [1, Proposition 2], we define the following quantity:

Definition 1 The ACS *quantity* associated to the isometric immersion $M \subset \mathbb{R}^d$ at $p \in M$ is defined as

$$\mathrm{ACS}(X, N) = \sum_{k=1}^{n-1} (\|II(e_k, X)\|^2 + \|X\|^2 \|II(e_k, N)\|^2) - \sum_{k=1}^{n-1} R^M(e_k, X, e_k, X)$$
$$- \|X\|^2 \mathrm{Ric}^M(N, N)$$

where $X, N \in T_p M$ are such that $\|N\| = 1$ and $\langle X, N \rangle = 0$; R^M denotes the curvature tensor of M;[1] and $e_1, \ldots e_{n-1}$ is an orthonormal basis of $N^\perp \subset T_p M$.

The geometric significance of the ACS quantity stems from the following result, which easily follows from [1, Theorem A] (and by slight abuse of language, we will attribute to them):

Theorem 2 (Ambrozio–Carlotto–Sharp) *Suppose* (M, g) *admits an isometric immersion into a Euclidean space* $\mathbb{R}^d$ *such that, for all* $p \in M$, *and all* $X, N \in T_p M$ *with* $\|X\| = \|N\| = 1$ *and* $\langle X, N \rangle = 0$, *one has* $\mathrm{ACS}(X, N) < 0$. *Then every closed immersed minimal hypersurface* $\Sigma \subset M$ *satisfies*

$$\mathrm{ind}(\Sigma) \geq \binom{d}{2}^{-1} b_1(M).$$

In particular, (M, g) *satisfies the conclusion of Conjecture with* $C = \binom{d}{2}^{-1}$.

Remark 3 The main result of [1] is more general, because it only requires to check $\mathrm{ACS}(X, N) < 0$ for N normal to a minimal hypersurface in M. Furthermore, the proof of that Theorem applies in even more generality, since it only requires, for any 2-sided hypersurface Σ with normal vector N and any vector field X dual to a harmonic 1-form, that $\int_\Sigma \mathrm{ACS}(X, N) < 0$ (cf. Proposition 2 of [1]). The pointwise condition, albeit more

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[1] We use the sign convention for R such that $\sec(v \wedge w) = R(v, w, v, w)/\|v \wedge w\|^2$.

restrictive, is still satisfied by the examples provided in [1], and is precisely what allows the extra flexibility required for Theorem A.

From now on, we will consider the quantity $\text{ACS}(X, N)$ for *orthonormal* vectors X, N. It will be convenient to rewrite the ACS quantity in terms of II only.

Lemma 4 *In the notation of Definition* 1 *above, and for X, N orthonormal,*

$$
\begin{aligned}
\text{ACS}(X, N) = &- \langle H, II(X, X) + II(N, N) \rangle + 2\|II(X, \cdot)\|^2 + 2\|II(N, \cdot)\|^2 \\
&+ \langle II(X, X), II(N, N) \rangle - 2\|II(X, N)\|^2 - \|II(N, N)\|^2
\end{aligned}
\tag{1}
$$

where H denotes the mean curvature vector of $M \subset \mathbb{R}^d$; $\|II(X, \cdot)\|$ denotes the Frobenius norm of the linear map $Y \mapsto II(X, Y)$; and similarly for $\|II(N, \cdot)\|$.

Proof First note that

$$
\sum_{k=1}^{n-1} \|II(e_k, X)\|^2 = \|II(X, \cdot)\|^2 - \|II(X, N)\|^2
$$

and similarly for $\sum_{k=1}^{n-1} \|II(e_k, N)\|^2$. Next, use the Gauss equation to write

$$
\text{Ric}^M(N, N) = \langle II(N, N), H \rangle - \|II(N, \cdot)\|^2
$$

$$
\begin{aligned}
\sum_{k=1}^{n-1} R^M(e_k, X, e_k, X) &= \text{Ric}^M(X, X) - R^M(N, X, N, X) \\
&= \langle II(X, X), H - II(N, N) \rangle - \|II(X, \cdot)\|^2 + \|II(X, N)\|^2
\end{aligned}
$$

Putting these terms together yields the desired formula. □

3 Robust index bound

The goal of this section is to prove Theorem A. The main ingredient of our proof is also the main ingredient of the proof of the Nash embedding theorem. For convenience we will use the simplification of Nash's proof due to Günther [10].

Following Gromov–Rohlin [9], we define the class of *free* immersions:

Definition 5 A smooth immersion $u : M \to \mathbb{R}^d$ with second fundamental form II is called *free* if, for any point $p \in M$, and any basis $\{e_1, \ldots e_n\}$ of $T_p M$, the normal vectors $II(e_i, e_j)$ for $1 \leq i \leq j \leq n$, are linearly independent.

Note that if $N \to M$ is an immersion, and $M \to \mathbb{R}^d$ is a free immersion, then the composite immersion $N \to \mathbb{R}^d$ is free.

Fix a "Hölder exponent" λ with $0 < \lambda < 1$, and denote by $\| \cdot \|_s$ the Hölder norm of a real-valued function on the open unit ball $B \subset \mathbb{R}^n$, given by

$$
\|u\|_s = \sum_{|\alpha| \leq s} \sup_{x \in U} |D^\alpha u(x)| + \sum_{|\alpha| = s} \sup_{x \neq y \in U} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\lambda}
\tag{2}
$$

Fixing an atlas of M and a partition of unity, there is an extension of the definition above to smooth functions on M, and sections of any vector bundle on M, all of which are still

denoted by $\|\cdot\|_s$. We note that Günther uses a different definition (see [10, page 70]) of the Hölder norm, but it is equivalent to the more common definition (2) above.

Let $u : M \to \mathbb{R}^d$ be a free immersion. Günther defines a map from the space of smooth symmetric 2-tensors $C^\infty(M, \mathrm{Sym}^2 T^*M)$ to the space of smooth normal sections, denoted by $f \mapsto E(u)(0, f)$, in the following way. For any $p \in M$, since u is free, there exists a normal $v \in \nu_p M$ such that $\langle II(X, Y), v \rangle = f(X, Y)$ for all $X, Y \in T_p M$. Such v is not unique, but selecting at every point the unique v with minimal norm yields the normal vector field $E(u)(0, f)$. Moreover, there exist constants K, depending only on the fixed atlas and partition of unity, and $D(u)$, depending on these and the free immersion u, such that, for all $f \in C^\infty(M, \mathrm{Sym}^2 T^*M)$, the following inequality is satisfied (see [10, Equation (34)]):

$$\|E(u)(0, f)\|_2 \le K D(u) \|f\|_2. \tag{3}$$

Theorem 6 [10] *Let M be a compact manifold with fixed atlas and partition of unity as above. Then, there exists $\theta > 0$ such that, for any free immersion $u : M \to \mathbb{R}^d$, and $f \in C^\infty(M, \mathrm{Sym}^2 T^*M)$ such that $D(u)\|E(u)(0, f)\|_2 \le \theta$, there exists $v \in C^\infty(M, \mathbb{R}^d)$ with $\|v\|_2 \le \|E(u)(0, f)\|_2$ such that $u + v$ is an isometric immersion with respect to the metric $g' = g + f$, where g is the metric induced by u.*

The following statement is an immediate consequence of Theorem 6 and (3).

Lemma 7 *Let (M, g) be a compact Riemannian manifold, and $u : M \to \mathbb{R}^d$ a smooth free isometric immersion. Then, for any $\delta > 0$, there is $\epsilon > 0$ such that, for all $f \in C^\infty(M, \mathrm{Sym}^2 T^*M)$ with $\|f\|_2 < \epsilon$, there is $v \in C^\infty(M, \mathbb{R}^d)$ with $\|v\|_2 < \delta$ such that $u + v$ is an isometric immersion with respect to the metric $g' = g + f$.*

Proof Let $\delta > 0$. Let $\theta > 0$ satisfying the conclusion of Theorem 6. Take

$$\epsilon = \min\left\{\frac{\delta}{K D(u)}, \frac{\theta}{K D(u)^2}\right\}.$$

Then, for any $f \in C^\infty(M, \mathrm{Sym}^2 T^*M)$ with $\|f\|_2 < \epsilon$, (3) implies that

$$D(u)\|E(u)(0, f)\|_2 < \theta.$$

Thus, by Theorem 6, there exists $v \in C^\infty(M, \mathbb{R}^d)$ with $\|v\|_2 < \|E(u)(0, f)\|_2$ such that the immersion $u + v$ induces the metric $g + f$. By (3), $\|v\|_2 < \delta$. □

Proof of Theorem A We may assume, without loss of generality, that the image of the isometric immersion $u : (M, g) \to \mathbb{R}^d$ is contained in the unit sphere S^{d-1} centered at the origin.

Let $\mathrm{Sym}^2 \mathbb{R}^d \simeq \mathbb{R}^{\binom{d+1}{2}}$ denote the space of symmetric $d \times d$ matrices with inner product $\langle A, B \rangle = \mathrm{tr}\, AB$, and consider for $\theta > 0$ the "Veronese" embedding

$$V_\theta : \mathbb{R}^d \to \mathbb{R}^d \times \mathrm{Sym}^2 \mathbb{R}^d$$
$$V_\theta(y) = (y, \theta y y^t).$$

The orthogonal group $O(d)$ acts by linear isometries on $\mathbb{R}^d \times \mathrm{Sym}^2 \mathbb{R}^d$, via

$$A \cdot (y, X) = (Ay, A X A^t).$$

With respect to this action, V_θ is:

- $O(d)$-equivariant: in fact, for $A \in O(d)$,

$$V_\theta(Ay) = (Ay, \theta Ay(Ay)^t) = (Ay, A(\theta y y^t)A^t) = A \cdot (y, \theta y y^t) = A \cdot V_\theta(y)$$

- A free immersion: in fact, by $O(d)$-equivariance it is enough to check this for $y = re_1$, where $e_1, \ldots e_d$ is the standard orthonormal basis of $\mathbb{R}^d$. Let $E_{ij}, 1 \leq i, j \leq d$, denote the symmetric matrix with $(E_{ij})_{ij} = (E_{ij})_{ji} = 1$ and 0 otherwise. The differential and second fundamental form of V_θ at y are then:

$$d_y V_\theta(e_i) = (e_i, \theta r E_{1i}), \qquad\qquad i = 1, \ldots, d,$$
$$II(e_i, e_j) = (0, \theta E_{ij}), \qquad\qquad 2 \leq i < j \leq d,$$
$$II(e_i, e_i) = (0, 2\theta E_{ii}), \qquad\qquad 2 \leq i \leq d,$$
$$II(e_1, e_j) = pr_{V_\theta(\mathbb{R}^d)^\perp}(0, \theta E_{1j})$$
$$= (0, \theta E_{1j}) - \frac{2\theta^2 r}{1 + 2\theta^2 r^2}(e_j, \theta r E_{1j})$$
$$= \frac{\theta}{1 + 2\theta^2 r^2}(-2\theta r e_j, E_{1j}) \qquad\qquad 2 \leq j \leq d$$
$$II(e_1, e_1) = pr_{V_\theta(\mathbb{R}^d)^\perp}(0, 2\theta E_{11})$$
$$= (0, 2\theta E_{11}) - \frac{2\theta^2 r}{1 + \theta^2 r^2}(e_1, \theta r E_{11})$$
$$= \frac{2\theta}{1 + \theta^2 r^2}(-\theta r e_j, E_{1j})$$

Clearly the vectors $II(e_i, e_j)$ are linearly independent.

In particular, the metric on S^{d-1} induced by the embedding $V_\theta|_{S^{d-1}} : S^{d-1} \to \mathbb{R}^d \times \text{Sym}^2 \mathbb{R}^d$ is $O(d)$-invariant, hence a constant scalar multiple $c = c(\theta)$ of the original (round) metric, because the isometric action of $O(d)$ on S^{d-1} takes any 2-plane to any other 2-plane. From the computation of dV_θ it is easy to check that $c(\theta) = 1 + 2\theta^2$.

Therefore, the composition $V_\theta \circ u : M \to \mathbb{R}^d \times \text{Sym}^2 \mathbb{R}^d$ is free, and induces a constant scalar multiple cg of the metric g, where $c = c(\theta) = 1 + 2\theta^2$. Moreover, if θ is small enough, $V_\theta \circ u$ has negative ACS quantity.

The maximum of the ACS quantity depends continuously on the immersion, with respect to the Hölder norm $\| \cdot \|_2$ [cf. (1)]. Thus there is $\delta > 0$ such that, if $v : M \to \mathbb{R}^d \times \text{Sym}^2 \mathbb{R}^d$ satisfies $\|v\|_2 < \delta$, then the immersion $V_\theta \circ u + v$ has negative ACS quantity. By Lemma 7, there is $\epsilon > 0$ such that, for all metrics g' on M with $\|g - g'\|_2 < \epsilon$, there is $v : M \to \mathbb{R}^d \times \text{Sym}^2 \mathbb{R}^d$ such that $V_\theta \circ u + v$ induces cg' and has negative ACS quantity. By Theorem 2, if g' is such a metric, and Σ is a closed immersed minimal hypersurface in (M, g'), then

$$\text{ind}(\Sigma) \geq \binom{d + \binom{d+1}{2}}{2}^{-1} b_1(\Sigma) = \frac{8}{d(d+3)(d^2 + 3d - 2)} b_1(\Sigma).$$

$\square$

Remark 8 In the case that the image of $u : (M, g) \to \mathbb{R}^d$ is not contained in the unit sphere (but it still has $\text{ACS}(X, N) < 0$ for every X, N orthonormal), the family of metrics $\{g_\theta\}_{\theta \in (0, \infty)}$ on M induced by $V_\theta \circ u : M \to \mathbb{R}^d \times \text{Sym}^2 \mathbb{R}^d$ converge smoothly to g as $\theta \to 0$. In particular, there is some θ_0 such that for every $\theta \in (0, \theta_0)$, the free isometric embedding $V_\theta \circ u : (M, g_\theta) \to \mathbb{R}^d \times \text{Sym}^2 \mathbb{R}^d$ still has $\text{ACS} < 0$. By the Proof of Theorem A, for any such θ there is a neighbourhood $\mathcal{U}_\theta$ of g_θ in the space of $C^{2,\alpha}$ metrics on M, such that for every $g' \in \mathcal{U}_\theta$ and every minimal hypersurface $\Sigma \subseteq (M, g')$ one has

$$\mathrm{ind}(\Sigma) \geq \frac{8}{d(d+3)(d^2+3d-2)} b_1(\Sigma).$$

Taking

$$\mathcal{U} = \bigcup_{\theta \in (0,\theta_0)} \mathcal{U}_\theta$$

It then follows that:

- $\mathcal{U}$ is open, and contains g in its closure.
- For every $g' \in \mathcal{U}$ and every minimal hypersurface $\Sigma \subset (M, g')$, one has

$$\mathrm{ind}(\Sigma) \geq \frac{8}{d(d+3)(d^2+3d-2)} b_1(\Sigma).$$

4 Isoparametric examples

4.1 Preliminaries

We start by recalling some basic definitions and facts, as well as fixing the notation. We refer the reader to Section 2.9 and Chapter 4 of [3] for a complete treatment.

A submanifold M of Euclidean space (or the sphere, or hyperbolic space) is called *isoparametric* if it has flat normal bundle, and constant principal curvatures along any locally defined parallel normal field. Using the Ricci equation, this implies that the tangent bundle TM decomposes as the orthogonal direct sum of *curvature distributions* E_i, for $i = 1, \ldots g$, which are the common eigenspaces for the shape operators. The eigenvalues are encoded in a family of parallel sections of the normal bundle, called the *curvature normals* ξ_i: For any normal vector ξ, the shape operator A_ξ in the direction of ξ has eigenvalues $\langle \xi, \xi_i \rangle$ with eigenspaces E_i. It follows that

$$II(x_i, y_j) = \langle x_i, y_j \rangle \xi_i \qquad (4)$$

for $x_i \in E_i$ and $y_j \in E_j$. The dimensions m_i of the curvature distributions E_i are called *multiplicities* [3, pp. 139–140].

Given a parallel normal field ξ, the set $M_\xi = \{p + \xi(p) \mid p \in M\}$ is a smooth manifold. If $\dim(M_\xi) = \dim(M)$, M_ξ is again an isoparametric submanifold, and is called a *parallel* manifold to M. If $\dim(M_\xi) < \dim(M)$, M_ξ is called a *focal* manifold to M. It still has constant principal curvatures along any parallel normal field, but the normal bundle is no longer flat. The set $\mathcal{F} = \{M_\xi\}$ of all parallel and focal manifolds of M forms a singular Riemannian foliation of Euclidean space, called an *isoparametric foliation* [3, pp. 141–142].

Given an isoparametric submanifold M of Euclidean space, let V be the normal space of M at $p \in M$. In this setting, V is sometimes called a *section* of the isoparametric foliation. This section intersects all the parallel and focal submanifolds of $\mathcal{F}$ perpendicularly. Furthermore, Terng ([19], see [3, pp. 142–144]) has shown there is a discrete group W of isometries of V associated to M, called *Coxeter group*, with the following properties:

- W is generated by reflections along hyperplanes;
- the W-orbit of any point $q \in V$ coincides with the intersection between V and the parallel or focal manifold M_ξ through q.
- the union of all the hyperplanes fixed by some element of W, coincides with the intersection between V and the focal manifolds M_ξ in the isoparametric foliation.

Fig. 1 Section through $p \in M$, when $M \subset \mathbb{R}^{n+2}$ is an isoparametric submanifold of codimension 2 and $g = 4$ principal curvatures. The origin coincides with the origin of $\mathbb{R}^{n+2}$. The picture shows the reflection lines L_i, as well as the principal curvature vectors $\xi_i(p)$, $i = 1, \ldots 4$. The dots represent the W-orbit of p, which coincides with the intersection between M and V in $\mathbb{R}^{n+2}$
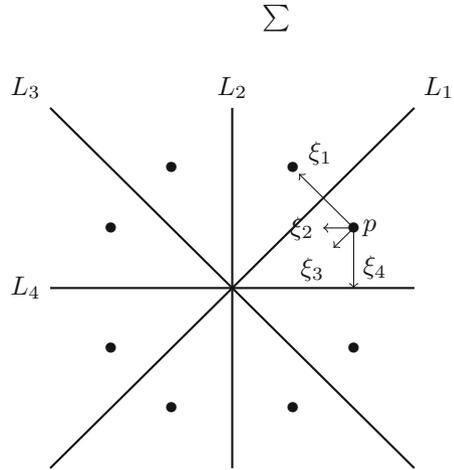
We will consider the case where $M^n \subset S^{n+1} \subset \mathbb{R}^{n+2}$ [3, Example 4.2.1 and section 4.2.5]. Münzner [12,13] has shown that, in this case, the possible values for the number g of principal curvatures are 1, 2, 3, 4, 6. We will consider the case $g = 4$ only, as it contains the richest class of examples. In this case, any section V through $p \in M$ is a two-dimensional subspace of $\mathbb{R}^{n+2}$ with $p \in V$, and the Coxeter group W is a dihedral group with $2g = 8$ elements, corresponding to four reflection lines denoted L_1, L_2, L_3, L_4. Again, each parallel and focal manifold intersects V in the orbit of a point under W, and the focal submanifolds correspond to points in $L_1 \cup L_2 \cup L_3 \cup L_4$.

Choose an orthonormal basis of V so that L_i is the line orthogonal to the vector α_i for all i, where

$$\alpha_1 = (1, -1) \quad \alpha_2 = (1, 0) \quad \alpha_3 = (1, 1) \quad \alpha_4 = (0, 1).$$

The multiplicities satisfy $m_1 = m_3$ and $m_2 = m_4$, because W acts on $\{L_1, L_2, L_3, L_4\}$ with orbits $\{L_1, L_3\}$ and $\{L_2, L_4\}$. Because of this, in the literature it is customary to say that such an isoparametric hypersurface *has multiplicities* (m_1, m_2).

The curvature normals at p of the isoparametric submanifold M are given by (Fig. 1)

$$\xi_i = -\frac{\alpha_i}{\langle \alpha_i, p \rangle}. \tag{5}$$

(Compare [3, Example 2.7.1] for the homogeneous case.) Note that $\langle \xi_i, p \rangle = -1$ for every i. Recall formula (4) for the second fundamental formula. In particular, the mean curvature vector is given by $H = \sum_i m_i \xi_i$.

We will be interested in minimal (in the sphere) isoparametric submanifolds:

Lemma 9 *In the notation above, let M be the isoparametric submanifold through $p = (\cos(\theta), \sin(\theta)) \in V$ where $0 < \theta < \pi/4$. The following are equivalent:*

(a) *M is minimal in S^{n+1}.*
(b) *$H = -np$.*
(c) *The volume of M is maximal among its parallel hypersurfaces in the sphere.*
(d) *$\theta = (1/2) \arctan(\sqrt{m_2/m_1})$.*

Proof The mean curvature vector of M in $\mathbb{R}^{n+2}$ is $H = \sum_i m_i \xi_i = -np + H^S$, where H^S denotes the mean curvature vector of M in the sphere. Thus, (a) and (b) are equivalent.

We claim that, up to a constant, the function $f(p) = \mathrm{Vol}(M)$ is given by

$$f(p) = \langle \alpha_1, p \rangle^{m_1} \langle \alpha_2, p \rangle^{m_2} \langle \alpha_3, p \rangle^{m_1} \langle \alpha_4, p \rangle^{m_2}$$

Indeed, given two parallel isoparametric submanifolds M, M', through points p, $p' \in V$, let $\eta = p' - p$, and extend it to a parallel normal field to M, also called η. Then the endpoint map $\phi : M \to M'$ given by $x \mapsto x + \eta(x)$ is a diffeomorphism. Its differential is block diagonal with $d\phi|_{E_i} = (1 - \langle \eta, \xi_i \rangle) \, \mathrm{Id}$. In particular,

$$\frac{\mathrm{Vol}(M')}{\mathrm{Vol}(M)} = \pm \det d\phi = \pm \prod_i (1 - \langle \eta, \xi_i \rangle)^{m_i} = \pm \prod_i (1 + \langle p, \xi_i \rangle - \langle p', \xi_i \rangle)^{m_i}$$

$$= \pm \frac{\prod_i \langle p', \alpha_i \rangle^{m_i}}{\prod_i \langle p, \alpha_i \rangle^{m_i}}$$

thus finishing the proof of the claim.

Plugging in $p = (\cos(\theta), \sin(\theta))$ yields

$$f(p) = (\cos(\theta) - \sin(\theta))^{m_1} \cos^{m_2}(\theta)(\cos(\theta) + \sin(\theta))^{m_1} \sin^{m_2}(\theta)$$
$$= \cos^{m_1}(2\theta) \sin^{m_2}(2\theta)/2^{m_2}.$$

Differentiating with respect to θ and setting equal to zero gives us

$$\cos^{m_1-1}(2\theta) \sin^{m_2-1}(2\theta)\big(- m_1 \sin^2(2\theta) + m_2 \cos^2(2\theta)\big) = 0.$$

So the unique critical point in the open interval $(0, \pi/4)$ is $(1/2) \arctan(\sqrt{m_2/m_1})$. It is the maximum because the volume goes to zero as $\theta \to 0$ or $\pi/4$, thus proving the equivalence of (c) and (d).

Finally, to prove that (b) and (c) are equivalent, note that

$$\nabla(\log f) = \sum_i m_i \langle \alpha_i, p \rangle^{-1} \alpha_i = -H$$

so that f is maximum subject to $\|p\|^2 = 1$ if and only if H is parallel to p, that is, if and only if $H = -np$. $\qquad\square$

4.2 The ACS quantity of isoparametric submanifolds and their focal manifolds

Since the second fundamental form of an isoparametric submanifold is easy to write down [cf. (4)], we get a very explicit formula for the ACS quantity. The exact second fundamental form of a focal manifold is more subtle, but we find estimates that suffice for our purposes. We state these formulas and estimates only in the special situation we are considering, namely minimal isoparametric hypersurfaces of the sphere, with 4 principal curvatures, but similar formulas hold for isoparametric submanifolds of general codimension, number of principal curvatures, and multiplicities.

Lemma 10 *Assume $M \subset S^{n+1}$ is a minimal (cf. Lemma 9) isoparametric hypersurface with four principal curvatures, and $p \in M$. Let $X, N \in T_p M$ with $\|X\| = \|N\| = 1$ and $\langle X, N \rangle = 0$. Write $X = \sum_i x_i$ and $N = \sum_i y_i$, where $x_i, y_i \in E_i$. Then*

$$\text{ACS}(X, N) = -2n + \sum_{i,j=1}^{4} \left((\|x_i\|^2 - \|y_i\|^2)\|y_j\|^2 - 2\langle x_i, y_i\rangle \langle x_j, y_j\rangle \right) \langle \xi_i, \xi_j\rangle$$

$$+ 2\sum_{i=1}^{4} \left(\|x_i\|^2 + \|y_i\|^2 \right) \|\xi_i\|^2. \tag{6}$$

Proof We use the expression for the ACS quantity given in Lemma 4, together with the formula for the second fundamental form in (4). Since M is minimal in the sphere, the first term in Lemma 4, namely $-\langle H, II(X, X) + II(N, N)\rangle$, equals $-2n$.

We claim that the next term $2\|II(X, \cdot)\|^2$ equals $2\sum_i \|x_i\|^2 \|\xi_i\|^2$, and similarly for $2\|II(N, \cdot)\|^2$. Indeed, take an orthonormal basis $e_1, \ldots e_n$ of $T_p M$ where each e_k belongs to $E_{i(k)}$ for some (unique) index $i(k)$. Then,

$$\|II(X, \cdot)\|^2 = \sum_k \|II(X, e_k)\|^2 = \sum_k \langle x_{i(k)}, e_k\rangle^2 \|\xi_{i(k)}\|^2$$

$$= \sum_i \sum_{k, i(k)=i} \langle x_i, e_k\rangle^2 \|\xi_i\|^2 = \sum_i \|x_i\|^2 \|\xi_i\|^2.$$

Since $II(X, N) = \sum_i \langle x_i, y_i\rangle \xi_i$, it follows that

$$-2\|II(X, N)\|^2 = -2\sum_{i,j} \langle x_i, y_i\rangle \langle x_j, y_j\rangle \langle \xi_i, \xi_j\rangle,$$

and similarly for the two remaining terms $\langle II(X, X), II(N, N)\rangle$ and $-\|II(N, N)\|^2$. Adding everything we obtain the expression in the statement of the lemma. $\square$

For simplicity we make a change of variables:

Lemma 11 *Let $M \subset S^{n+1}$ be a minimal isoparametric hypersurface with four principal curvatures, all of whose multiplicities are larger than one, and $p \in M$. Then the maximum of $\text{ACS}(X, N)$ over all pairs $X, N \in T_p M$ with $\|X\| = \|N\| = 1$ and $\langle X, N\rangle = 0$ is equal to the maximum of $\text{ACS}'(s, t)$ over all $(s, t) \in \Delta^3 \times \Delta^3$, where Δ^3 is the standard 3-simplex*

$$\Delta^3 = \{u \in \mathbb{R}^4 \mid u_i \geq 0 \,\forall i, \ u_1 + u_2 + u_3 + u_4 = 1\}$$

and $\text{ACS}'(s, t)$ is defined by

$$\text{ACS}'(s, t) = -2n + \sum_{i,j=1}^{4} (s_i - t_i)t_j \langle \xi_i, \xi_j\rangle + 2\sum_{i=1}^{4} (s_i + t_i)\|\xi_i\|^2. \tag{7}$$

Proof Given $X, N \in T_p M$ with $\|X\| = \|N\| = 1$ and $\langle X, N\rangle = 0$, write $X = \sum_i x_i$ and $N = \sum_i y_i$, where $x_i, y_i \in E_i$. Define $s_i = \|x_i\|^2$, $t_i = \|y_i\|^2$, $s = (s_1, s_2, s_3, s_4)$, and $t = (t_1, t_2, t_3, t_4)$. Note that $(s, t) \in \Delta^3 \times \Delta^3$. Moreover, by Lemma 10,

$$\text{ACS}(X, N) = \text{ACS}'(s, t) - 2\|II(X, N)\|^2 \leq \text{ACS}'(s, t).$$

This shows that $\max \text{ACS} \leq \max \text{ACS}'$.

To prove the reverse inequality, let $(s, t) \in \Delta^3 \times \Delta^3$. Then, since $\dim(E_i) > 1$ for all i, there exists (X, N) such that $s_i = \|x_i\|^2$, $t_i = \|y_i\|^2$, and $\langle x_i, y_i\rangle = 0$, for every i. In particular, $\langle X, N\rangle = 0$, and

$$\|II(X, N)\|^2 = \sum_{i,j} \langle x_i, y_i\rangle \langle x_j, y_j\rangle \langle \xi_i, \xi_j\rangle = 0.$$

Thus $\mathrm{ACS}'(s, t) = \mathrm{ACS}(X, N)$, completing the proof. $\square$

Remark 12 $\mathrm{ACS}'(s, t)$ is linear in s, so that the maximum over all s, given a fixed t, must occur for s at one of the four vertices of Δ^3. On the other hand, if one fixes s, then $\mathrm{ACS}'(s, t)$ is quadratic and *concave* in t. This means that given explicit values of the curvature normals ξ_i, the maximum of the ACS quantity may be efficiently computed by solving four convex quadratic optimization problems over the 3-simplex. In practice this can be done with interior-point methods, implemented in e.g. the CVXOPT package for Python.

We start with a simple upper bound for the ACS quantity, which depends only on the multiplicities:

Lemma 13 *In the notations and assumptions of Lemma 11,*

$$\mathrm{ACS} \leq -2n + \frac{10(m_1 + m_2)}{m_1} \left(1 + \sqrt{\frac{m_2}{m_1 + m_2}}\right).$$

Proof First we claim that $\mathrm{ACS}' \leq -2n + 5 \max_k \|\xi_k\|^2$. Indeed,

$$\mathrm{ACS}' \leq -2n + \sum_{i,j=1}^{4} s_i t_j \langle \xi_i, \xi_j \rangle + 2 \sum_{i=1}^{4} (s_i + t_i) \|\xi_i\|^2$$

$$\leq -2n + \left(\sum_{i,j=1}^{4} s_i t_j + 2 \sum_{i=1}^{4} (s_i + t_i)\right) \max_k \|\xi_k\|^2 = -2n + 5 \max_k \|\xi_k\|^2.$$

Since we are assuming $m_2 \geq m_1$, we have $\theta \geq \pi/8$, so that ξ_1 is the curvature normal with the largest norm. Computing directly from (5), we obtain

$$\|\xi_1\|^2 = \frac{2}{(\cos(\theta) - \sin(\theta))^2} = \frac{2}{1 - \sin(2\theta)} = \frac{2}{1 - \sqrt{\frac{m_2}{m_1 + m_2}}}$$

$$= \frac{2(m_1 + m_2)}{m_1} \left(1 + \sqrt{\frac{m_2}{m_1 + m_2}}\right).$$

Therefore $\mathrm{ACS} \leq \mathrm{ACS}' \leq -2n + \frac{10(m_1 + m_2)}{m_1}\left(1 + \sqrt{\frac{m_2}{m_1 + m_2}}\right).$ $\square$

Proof of Theorem B part (a) First assume $m_1 \geq 5$. Then, by Lemma 13,

$$\mathrm{ACS} \leq -4(m_1 + m_2) + \frac{10(m_1 + m_2)}{m_1}\left(1 + \sqrt{\frac{m_2}{m_1 + m_2}}\right)$$

$$< 4(m_1 + m_2)\left(-1 + \frac{5}{m_1}\right) \leq 0.$$

Now let $m_1 = 4$. By Lemma 9, since M is minimal, it contains the point $p = (\cos(\theta), \sin(\theta)) \in V$, where $\theta = (1/2)\arctan(\sqrt{m_2/4})$. As m_2 goes to infinity, the curvature normals ξ_2, ξ_3, ξ_4 converge, while the norm of ξ_1 goes to infinity. More precisely, by the Proof of Lemma 13,

$$\|\xi_1\|^2 = \frac{4 + m_2}{2}\left(1 + \sqrt{\frac{m_2}{4 + m_2}}\right).$$

It is enough to show that the maximum of the $\mathrm{ACS}'(s, t)$ quantity of M over $(s, t) \in \Delta^3 \times \Delta^3$ is negative when m_2 is large (see Lemma 11). Moreover, since the maximum

occurs for s at one of the vertices of Δ^3, we need to show that, for m_2 large and $i = 1, 2, 3, 4$, the maximum of $\mathrm{ACS}'(e_i, t)$ over $t \in \Delta^3$ is negative.

Since $n = 2(4 + m_2)$, one has

$$\mathrm{ACS}'(s, t) = -4(4 + m_2) + [(s_1 - t_1)t_1 + 2(s_1 + t_1)]\|\xi_1\|^2 + O(\sqrt{m_2})$$

$$= \frac{4 + m_2}{2}\left(-8 + [(s_1 - t_1)t_1 + 2(s_1 + t_1)]\left(1 + \sqrt{\frac{m_2}{4 + m_2}}\right)\right) + O(\sqrt{m_2})$$

If $s = e_i$ for $i \neq 1$, that is, if $s_1 = 0$, then $(s_1 - t_1)t_1 + 2(s_1 + t_1) = -t_1^2 + 2t_1 \leq 1$. This implies that $\max_t \mathrm{ACS}'(e_i, t) \to -\infty$ as $m_2 \to \infty$.

Assume now that $s = e_1$. We claim that, for large m_2, the maximum of $\mathrm{ACS}'(e_1, t)$ occurs at $t = e_1$. This will finish the proof, because $\mathrm{ACS}'(e_1, e_1) = -2n + 4\|\xi_1\|^2 < 0$.

To prove the claim, it is enough to show that (when m_2 is large) the gradient of $\mathrm{ACS}'(e_1, t)$ at $t = e_1$ has negative inner product with the vectors $e_2 - e_1$, $e_3 - e_1$, and $e_4 - e_1$. This is because $\mathrm{ACS}'(e_1, t)$ is concave, the simplex Δ^3 is convex, and its tangent cone at $t = e_1$ is the cone over the convex hull of $e_2 - e_1$, $e_3 - e_1$, and $e_4 - e_1$. But it is clear from the formula for ACS' that

$$\nabla \mathrm{ACS}'(e_1, t)|_{t=e_1} = \|\xi_1\|^2 e_1 + O(\sqrt{m_2})$$

thus finishing the proof. $\qquad\square$

Remark 14 Based on numerical evidence (cf. Remark 12), we believe that the conclusion of Theorem B(a) also holds without the hypothesis "m_2 is large enough".

Proof of Theorem B(b) The focal submanifold M_+ contains the point $p = (1, 1)/\sqrt{2}$, and the tangent space at p is $T_p M_+ = E_2 \oplus E_3 \oplus E_4$ (compare [3, Example 3.4.1]). Write the normal space as an orthogonal direct sum $\nu_p M_+ = U \oplus V$, where $U = E_1$, and V is the same section as before, described in Sect. 4.1. Denote by II^U and II^V the components of the second fundamental form in the directions of U and V.

By the Tube Formula ([3, Lemma 3.4.7], II^V is given by the same formula as in the isoparametric case, see Eqs. (5) and (4). More precisely, for $i, j \in \{2, 3, 4\}$ and $x_i \in E_i$ and $y_j \in E_j$, one has $II^V(x_i, y_j) = \langle x_i, y_j \rangle \xi_i$, where

$$\xi_2 = (-\sqrt{2}, 0), \quad \xi_3 = -(1, 1)/\sqrt{2} = -p, \quad \xi_4 = (0, -\sqrt{2}).$$

As for II^U, one can say that, for every $v \in U$ with $\|v\| = 1$, the shape operators of M_+ in the directions of v and $v_0 = (1, -1)/\sqrt{2} \in V$ are conjugate. Indeed, for small ϵ, $p + \epsilon v$ belongs to an isoparametric manifold parallel to M, whose normal space V' contains v. Then, on V' one has the same picture as in V, with v playing the role of v_0, so the result follows from the Tube Formula again. Explicitly, these shape operators have eigenvalues $\langle v_0, \xi_i \rangle$, that is, $-1, 0, 1$, with multiplicities m_2, m_1, m_2, respectively. In particular, M_+ is minimal in the sphere, so that the ACS quantity of M_+ satisfies

$$\mathrm{ACS}(X, N) + 2(m_1 + 2m_2) \leq 2\|II(X, \cdot)\|^2 + 2\|II(N, \cdot)\|^2 + \langle II(X, X), II(N, N)\rangle.$$

The right-hand side equals the sum of the analogous expressions with II replaced with II^U and II^V, respectively. The latter is at most $5 \max_{i=2,3,4} \|\xi_i\|^2 = 10$, by the same argument as in the Proof of Lemma 13. Thus

$$\mathrm{ACS} \leq -2(m_1 + 2m_2) + 10 + 2\|II^U(X, \cdot)\|^2 + 2\|II^U(N, \cdot)\|^2 + \left\langle II^U(X, X), II^U(N, N)\right\rangle.$$

Since the shape operator in the direction of any $v \in U$ with $\|v\| = 1$ has largest eigenvalue (in absolute value) equal to 1, we have $\|II^U(X, \cdot)\|^2 \leq \dim U = m_1$, and analogously for the other terms, so that

$$\text{ACS} \leq -2(m_1 + 2m_2) + 10 + 5m_1.$$

Therefore, the ACS quantity is negative provided that $m_2 > (3m_1 + 10)/4$. □

4.3 Examples

Isoparametric hypersurfaces in spheres have been almost completely classified through the work of several mathematicians (see [3, Section 2.9.6]). The homogeneous ones are precisely the principal orbits of isotropy representations of rank two symmetric spaces. All known inhomogeneous isoparametric hypersurfaces in spheres have 4 principal curvatures. They were constructed in [8] using Clifford systems, and are usually called of *FKM-type* (see [6, section 3.9] for a detailed construction). We will identify some of these isoparametric foliations whose multiplicities satisfy the conditions in Theorem B.

Starting with the homogeneous examples, given $k \geq 3$ let $G = \text{SO}(k)\,\text{SO}(2)$ (respectively $S(\text{U}(k)\,\text{U}(2))$, $\text{Sp}(k)\,\text{Sp}(2)$) act on the space of $k \times 2$ matrices with coefficients in $\mathbb{R}$ (respectively $\mathbb{C}$, $\mathbb{H}$), by $(A, B)C = ACB^{-1}$. These are equivalent to the isotropy representations of the Grassmannians of two-planes over the reals, complex numbers, or quaternions). The generic G-orbits are isoparametric hypersurfaces with multiplicities $(m_1, m_2) = (1, k - 2)$ (respectively $(2, 2k - 3)$, $(4, 4k - 5)$) (see Sect. 4.1 for the notation on multiplicities and [3, p. 86] for the values of the multiplicities). The singular G-orbit with codimension $m_1 + 1$ in the sphere is the Stiefel variety of 2-planes in k-space.

Applying Theorem B(a) immediately yields:

Corollary 15 *For large k, the unique principal orbit of the representation of* $\text{Sp}(k)\,\text{Sp}(2)$ *on $k \times 2$ matrices with coefficients in $\mathbb{H}$ that is minimal in the sphere has negative* ACS *quantity.*

Applying Theorem B(b) immediately yields:

Corollary 16 *The Stiefel variety of 2-frames in $\mathbb{R}^k$ (respectively $\mathbb{C}^k$, $\mathbb{H}^k$), with metric induced from the embedding in the sphere as the orbit of the matrix*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$$

under the G-action described above, satisfies ACS < 0 *provided that $k \geq 6$ (respectively $k \geq 4$, $k \geq 3$).*

We can also apply Theorem B to one isolated example, which has multiplicities $(6, 9)$ [3, p. 87] and immediately obtain:

Corollary 17 *The unique minimal (in S^{31}) principal orbit of the isotropy representation of the symmetric space $E_6/\text{Spin}(10)\,\text{U}(1)$ has* ACS < 0. *The singular orbit with codimension 7 in S^{31} also has* ACS < 0.

Now we turn to isoparametric hypersurfaces of FKM-type [8]. Recall that a Clifford system on $\mathbb{R}^{2l}$ is a set of $m + 1$ symmetric $2l \times 2l$ matrices $C = (P_0, \ldots, P_m)$ such that

Table 1 Dimensions of irreducible Clifford systems

$m =$	1	2	3	4	5	6	7	8	...	$m' + 8$	...
$\delta(m) =$	1	2	4	4	8	8	8	8	...	$16\delta(m')$	...

$P_i^2 = I$ for all i, and $P_i P_j = -P_j P_i$ for $i \neq j$. Then l needs to be of the form $l = k\delta(m)$, where $\delta(m)$ is described in Table 1 (see [8, page 483]). Conversely, given m and k, there do exist Clifford systems as above (see [8] for a discussion of different equivalence relations of Clifford systems, and classification results). An isoparametric foliation of S^{2l-1} is defined by the level sets of the polynomial $H(x)$ on $\mathbb{R}^{2l}$ given by

$$H(x) = \sum_{i=0}^{m} (x^T P_i x)^2,$$

where x is regarded as a column vector, and x^T denotes its transpose. The regular leaves are isoparametric with multiplicities $(m, l - m - 1)$. The level set $H^{-1}(0)$ is one of the singular leaves, and it has codimension $1 + m$ in S^{2l-1}. It is a quadric, because it can also be described as $\{x \in S^{2l-1} \mid x^T P_i x = 0 \ \forall i\}$, and it is sometimes called a *Clifford–Stiefel variety*. The other singular leaf is $H^{-1}(1)$, and it has codimension $l - m$. In all but finitely many cases,[2] $m \leq l - m - 1$, so that, in our notation, $m_1 = m$, $m_2 = l - m - 1$, and $M_+ = H^{-1}(0)$.

From Theorem B(a) we immediately deduce:

Corollary 18 *Let $C = (P_0, \ldots, P_m)$ be a Clifford system on $\mathbb{R}^{2l}$, with $l = k\delta(m)$. Then the unique regular leaf with maximal volume satisfies* ACS < 0 *provided:* $m, l - m - 1 \geq 5$; *or* $m = 4$ *and k is large enough.*

Applying Theorem B(b), we obtain:

Corollary 19 *Let $C = (P_0, \ldots, P_m)$ be a Clifford system on $\mathbb{R}^{2l}$, with $l = k\delta(m)$. Assume $k > \frac{7m+14}{4\delta(m)}$. Then the Clifford–Stiefel variety M_+ satisfies* ACS < 0.

Proof We claim that $m \leq l - m - 1$. Indeed, assuming $m > l - m - 1$, we obtain $k < \frac{2m+1}{\delta(m)}$, which, together with $k > \frac{7m+14}{4\delta(m)}$, implies $m > 10$. But then, by Table 1, $\frac{2m+1}{\delta(m)}$ is less than one, contradicting the fact that $k \geq 1$ and proving the claim.

Thus $m_1 = m$ and $m_2 = k\delta(m) - m - 1$, so that $k > \frac{7m+14}{4\delta(m)}$ implies $m_2 > (3m_1 + 10)/4$, and we may apply Theorem B(b).   □

Note that Corollary 19 applies to all but finitely many FKM-type isoparametric foliations.

4.4 Remarks about the geometry of the examples

In this subsection we collect a few remarks about the curvature and homogeneity of the isoparametric examples described above.

We start by relating the Ricci curvature and the ACS quantity of general isoparametric submanifolds of Euclidean space.

Lemma 20 *Let M be an isoparametric submanifold of Euclidean space, with curvature distributions E_i and curvature normals ξ_i, for $i = 1, \ldots g$. Then the Ricci tensor of M has E_i as eigenspaces, with respective eigenvalues $\langle \xi_i, H \rangle - \|\xi_i\|^2$.*

---

[2] More precisely, the values of (m, k) such that $0 < l - m - 1 < m$ are $(2, 2)$, $(4, 2)$, $(5, 1)$, $(6, 1)$, $(8, 2)$, $(9, 1)$.

Proof Given $X = \sum_i x_i$ with $x_i \in E_i$, it follows from the Gauss equation that

$$\text{Ric}(X) = \sum_k R(X, e_k, X, e_k) = \sum_k \langle II(X, X), II(e_k, e_k) \rangle - \sum_k \|II(X, e_k)\|^2$$

$$= \langle II(X, X), H \rangle - \|II(X, \cdot)\|^2 = \sum_i \big(\langle \xi_i, H \rangle - \|\xi_i\|^2 \big) \|x_i\|^2$$

where we have used the identity $\|II(X, \cdot)\|^2 = \sum_i \|x_i\|^2 \|\xi_i\|^2$, see the Proof of Lemma 10.
$\square$

Proposition 21 *Let M be an isoparametric submanifold of Euclidean space, and assume the multiplicities $m_i = \dim E_i$ are larger than one. Then* ACS < 0 *implies* Ric > 0.

Proof We compute the ACS quantity as in Lemma 10. Let $p \in M$, $X, N \in T_p M$ with $\|X\| = \|N\| = 1$ and $\langle X, N \rangle = 0$. Write $X = \sum_i x_i$ and $N = \sum_i y_i$, where $x_i, y_i \in E_i$. Then

$$\text{ACS}(X, N) = \sum_{i,j=1}^g \left((\|x_i\|^2 - \|y_i\|^2) \|y_j\|^2 - 2 \langle x_i, y_i \rangle \langle x_j, y_j \rangle \right) \langle \xi_i, \xi_j \rangle$$

$$+ \sum_{i,j=1}^g -m_i (\|x_j\|^2 + \|y_j\|^2) \langle \xi_i, \xi_j \rangle + 2 \sum_{i=1}^g \left(\|x_i\|^2 + \|y_i\|^2 \right) \|\xi_i\|^2.$$

As in Lemma 11, the assumption that the multiplicities are greater than one implies that the maximum of the ACS quantity is equal to the maximum of

$$\text{ACS}'(s, t) = \sum_{i,j=1}^g \left(-m_i (s_j + t_j) + (s_i - t_i)t_j \right) \langle \xi_i, \xi_j \rangle + 2 \sum_{i=1}^g (s_i + t_i) \|\xi_i\|^2$$

for $(s, t) \in \Delta^{g-1} \times \Delta^{g-1}$, where Δ^{g-1} denotes the standard $(g - 1)$-simplex.

Let $k = 1, \ldots, g$. Setting $s_i = t_i = \delta_{ik}$ for all i in the equation above, we get ACS$'(s, t) = -2 \langle H, \xi_k \rangle + 4\|\xi_k\|^2$, where $H = \sum_i m_i \xi_i$ is the mean curvature vector. This is negative by assumption, so that, in particular, $\langle H, \xi_k \rangle - \|\xi_k\|^2 > 0$. By Lemma 20, these are the eigenvalues of the Ricci tensor. $\square$

Thus, isoparametric hypersurfaces $M^n \subset S^{n+1}$ satisfying the conditions of Theorem B(a), and in particular all isoparametric examples listed in Sect. 4.3, have positive Ricci curvature. Similarly, one has:

Proposition 22 *The focal manifolds M_+ satisfying the conditions of Theorem B(b) have positive Ricci curvature.*

Proof We will freely use the notations and facts established in the Proof of Theorem B(b). Let $X = x_2 + x_3 + x_4 \in T_p M_+ - \{0\}$ with $x_i \in E_i$ for $i = 2, 3, 4$. Then, by the Gauss equation,

$$\text{Ric}(X, X) = \langle II(X, X), H \rangle - \|II(X, \cdot)\|^2$$

$$= (m_1 + 2m_2)\|X\|^2 - \|II^U(X, \cdot)\|^2 - \|II^V(X, \cdot)\|^2$$

because M_+ is minimal in the sphere.

But $\|II^V(X, \cdot)\|^2 = \sum \|\xi_i\|^2 \|x_i\|^2 = 2\|x_2\|^2 + \|x_3\|^2 + 2\|x_4\|^2$, while $\|II^U(X, \cdot)\|^2 \leq m_1 \|X\|^2$. Therefore, Ric$(X, X) > 0$. $\square$

Proposition 23 *Let M_i be a sequence of minimal isoparametric hypersurfaces of spheres with four principal curvatures, with m_1 fixed, and $m_2 \to \infty$. Then the minimum of the sectional curvatures of M_i diverges to $-\infty$, while the diameter is bounded from below by π.*

Proof The diameter of M_i is at least π because it is contained in the sphere, and is invariant under the antipodal map. By the Gauss equation, the sectional curvature of a plane of the form $x \wedge y$, for $x \in E_1$ and $y \in E_4$ is:

$$\sec(x \wedge y) = R(x, y, x, y) = \langle II(x, x), II(y, y) \rangle - \|II(x, y)\|^2 = \langle \xi_1, \xi_4 \rangle < 0.$$

Moreover, as $m_2 \to \infty$, $\xi_4 \to (0, -\sqrt{2})$, so that $\sec(x \wedge y)$ is asymptotic to

$$-\|\xi_1\| = -\sqrt{\frac{2(m_1 + m_2)}{m_1}\left(1 + \sqrt{\frac{m_2}{m_1 + m_2}}\right)} \simeq -Cm_2^{1/2},$$

for a positive constant C (see Proof of Lemma 13). In particular, the minimum value of the sectional curvature of M diverges to $-\infty$ as $m_2 \to \infty$. $\square$

Remark 24 It has been determined in [8] exactly which isoparametric hypersurfaces of FKM-type are extrinsically homogeneous. In particular, when $m \le l - m - 1$, so that, in our notation, $m_1 = m$ and $m_2 = l - m - 1$, they prove that M is extrinsically homogeneous if and only if $m = 1, 2$; or $m = 4$ and $P_0 P_1 P_2 P_3 P_4 = \pm I$.

Moreover, for isoparametric hypersurfaces $M \subset S^{n+1}$ with four principal curvatures, extrinsic and intrinsic homogeneity are equivalent. Indeed, the rank of the shape operator in the sphere is constant and ≥ 2. Thus, we may apply [7, Theorem 2] to conclude that the embedding is rigid, so that, in particular, every isometry of M extends to an isometry of S^{n+1}.

On the other hand, any isoparametric submanifold M is curvature-homogeneous, by the Gauss equation and the fact that the second fundamental form is "the same" everywhere. More precisely, given $p, q \in M$, any linear isometry $T_p M \to T_q M$ that sends each curvature distribution $E_i(p)$ to $E_i(q)$ maps the curvature operator at p to the one at q.

5 Symmetric examples

5.1 Embeddings of symmetric spaces

The goal of this section is to prove Theorem C, whose parts (a), (b), and (c) correspond to Propositions 34, 30, and 32, respectively. First we recall some well-known facts about symmetric spaces and their equivariant embeddings into Euclidean spaces. References for this material are [20], [4, Chapter 7].

Let $K \subset G$ be compact Lie groups. Recall that (G, K) is called a *symmetric pair*, and G/K a (compact) *symmetric space*, if there is an order two automorphism $\tau : G \to G$ such that $(G^\tau)_0 \subset K \subset G^\tau$. Here $G^\tau = \{g \in G \mid \tau(g) = g\}$, and $(G^\tau)_0$ denotes the connected component of G^τ.

For example, every compact Lie group G is diffeomorphic to the symmetric space $(G \times G)/\Delta G$. Here the automorphism τ is given by $\tau(a, b) = (b, a)$, and $\Delta G = (G \times G)^\tau = \{(a, a) \mid a \in G\}$. The diffeomorphism is given by $a \in G \mapsto (a, e)K \in G \times G/\Delta G$, where $e \in G$ denotes the identity element.

Let (G, K) be a symmetric pair with G, K compact, and denote by $\mathfrak{k} \subset \mathfrak{g}$ their Lie algebras. Since the K-action $C_K : K \to \text{Aut}(G)$ by conjugation fixes the neutral element $e \in G$, the differential of C_K at e induces a map $\text{Ad}_K = d_e C_k : K \to \text{GL}(\mathfrak{g})$. Let $\mathfrak{m} \subset \mathfrak{g}$ be

the Ad_K-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$ given by the (-1)-eigenspace of the differential of τ at the identity. Then $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$. The tangent space of G/K at eK can be identified with $\mathfrak{m}$. The G-invariant metrics on G/K correspond to the Ad_K-invariant inner products on $\mathfrak{m}$. When G is semi-simple, a natural choice for such a metric is $(-B)|_{\mathfrak{m}}$, where $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is the Cartan–Killing form, defined by $B(X, Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$.

Lemma 25 *Let (G, K) be a symmetric pair of compact Lie groups with G semi-simple, and let B denote the Cartan–Killing form on $\mathfrak{g}$.*

(a) *$(\mathcal{N}, g) = (G, -B)$ is Einstein with $\mathrm{Ric} = (1/4)g$.*
(b) *$(\mathcal{N}, g) = (G/K, -B|_{\mathfrak{m}})$ is Einstein with $\mathrm{Ric} = (1/2)g$.*

Proof For the symmetric space, see [4, Theorem 7.73]. For the Lie group, note that the diffeomorphism $G \to G \times G/\Delta G$ is an isometry with respect to the metrics $-B$ and $-2(B \oplus B)$. Since the Ricci tensor is scale-invariant, it follows that

$$\mathrm{Ric}^{(G,-B)} = \mathrm{Ric}^{(G \times G/\Delta G, -2(B \oplus B))} = -(1/2)B \oplus B = -(1/4)B.$$

$\square$

Now we consider G-equivariant embeddings of G/K into Euclidean space:

Lemma 26 *Let (G, K) be a symmetric pair of compact Lie groups. Let $\rho : G \to O(V)$ an orthogonal representation of G on the Euclidean space V, and let $p \in V$ with isotropy $K = G_p$. Denote by II the second fundamental form of the embedding of G/K as the G-orbit $G \cdot p \subset V$ given by $aK \in G/K \mapsto \rho(a)p \in V$. Then*

$$II(d\rho(X), d\rho(Y)) = d\rho(X)d\rho(Y)p$$

for all $X, Y \in \mathfrak{m}$.

Proof Let $\xi \in T_p(G \cdot p)$ be a normal vector to the orbit $G \cdot p$ at $p \in V$, and $X, Y \in \mathfrak{m}$. Extend ξ to a vector field along the curve $t \mapsto \rho(e^{tX})p$ by the formula $\hat{\xi}(t) = \rho(e^{tX})\xi$, so that

$$\nabla^V_{d\rho(X)}\hat{\xi} = \frac{d}{dt}\Big|_{t=0} \rho(e^{tX})\xi = d\rho(X)\xi.$$

Then

$$\begin{aligned}
\langle II(X, Y), \xi \rangle &= \langle S_\xi(d\rho(X)), d\rho(Y)p \rangle \\
&= -\langle \nabla^V_{d\rho(X)}\hat{\xi}, d\rho(Y)p \rangle \\
&= -\langle d\rho(X)\xi, d\rho(Y)p \rangle \\
&= \langle d\rho(X)d\rho(Y)p, \xi \rangle \qquad \text{because } d\rho(X)^T = -d\rho(X)
\end{aligned}$$

It remains to show that $d\rho(X)d\rho(Y)p$ is normal to the orbit, or, equivalently, that the trilinear tensor $\eta : \mathfrak{m} \times \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ defined by

$$\eta(X, Y, Z) = \langle (d\rho(X)d\rho(Y)p \, , \, d\rho(Z)p \rangle$$

vanishes identically.

Since (G, K) is a symmetric pair, $[X, Y] \in \mathfrak{k}$, and in particular $d\rho([X, Y])p = 0$. This means that

$$
\begin{aligned}
\eta(X, Y, Z) &= \langle d\rho(X)d\rho(Y)p, d\rho(Z)p \rangle \\
&= -\langle d\rho(X)d\rho(Z)p, d\rho(Y)p \rangle \\
&= -\eta(X, Z, Y) \\
&= -\eta(Z, X, Y)
\end{aligned}
$$

Since the permutation $(X, Y, Z) \mapsto (Z, X, Y)$ has order three, we conclude $\eta = -\eta$, that is, $\eta = 0$. $\qquad\square$

5.2 Rewriting the ACS quantity

We start with an equivalent reformulation of the ACS quantity.

Lemma 27 *Denoting by H the mean curvature vector of the embedding $M \subset \mathbb{R}^d$,*

$$
\begin{aligned}
\mathrm{ACS} = &-2\,\mathrm{Ric}(X, X) - 2\,\mathrm{Ric}(N, N) + \langle H, II(X, X) + II(N, N) \rangle \\
&- 2\|II(X, N)\|^2 - \|II(N, N)\|^2 + \langle II(N, N), II(X, X) \rangle.
\end{aligned} \tag{8}
$$

Proof Use Lemma 4 and the formula $\mathrm{Ric}(X, X) = \langle II(X, X), H \rangle - \|II(X, \cdot)\|^2$, which is a consequence of the Gauss equation. $\qquad\square$

Now assume (M, g) is Einstein with $\mathrm{Ric} = E.g$ and the embedding $M \subset \mathbb{R}^d$ is minimal into a sphere $S(r) \subset \mathbb{R}^d$. Then:

$$
\mathrm{ACS} = -4E + \frac{2\dim(M)}{r^2} - 2\|II(X, N)\|^2 - \|II(N, N)\|^2 + \langle II(N, N), II(X, X) \rangle. \tag{9}
$$

We will refer to the term $-4E + 2\dim(M)/r^2$ in (9) as the *constant term*.

Remark 28 The coordinates of the embedding $M \subset \mathbb{R}^d$ are eigenfunctions of the Laplace–Beltrami operator with eigenvalue $\lambda = \dim(M)/r^2$ (see [20, Cor. 5.2]). By Lichnerowicz's Theorem, $\lambda \geq E\dim(M)/(\dim(M) - 1)$, so that the constant term satisfies

$$
-4E + \frac{2\dim(M)}{r^2} \geq \frac{-2E(\dim(M) - 2)}{\dim(M) - 1}.
$$

5.3 Unitary groups

Let $G = \mathrm{SU}(n)$, $\mathrm{Sp}(n)$, and consider their natural embedding into $V = \mathbb{C}^{n \times n}$, $\mathbb{H}^{n \times n}$, as $n \times n$ complex-unitary and quaternionic-unitary matrices. We endow G with the metric given by the negative of the Cartan–Killing form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, and extend it to the inner product $\langle \cdot, \cdot \rangle$ on V defined by

$$
\langle X, Y \rangle = c_n \Re(\mathrm{tr}(XY^*)) \tag{10}
$$

where c_n equals $2n$, $4(n + 1)$ in the complex and quaternionic cases, respectively, and $\Re$ denotes the real part.

Lemma 29 *With the notations above, the* ACS *quantity of the isometric embedding $G \subset V$ is given by:*

$$
\mathrm{ACS} = \begin{cases} -\frac{1}{n^2} - \langle NX, XN \rangle - \|N^2\|^2 & \text{in the complex case} \\ -\frac{1}{2(n+1)} - \langle NX, XN \rangle - \|N^2\|^2 & \text{in the quaternionic case} \end{cases}
$$

where $X, N \in \mathfrak{g}$ such that $\|X\| = \|N\| = 1$ and $\langle X, N \rangle = 0$.

Proof The image of $G \subset V$ is contained in the sphere of radius $r = \sqrt{nc_n}$.

The group $G \times G$ acts orthogonally on V through the representation $\rho(A, B)Z = AZB^{-1}$, whose derivative is given by $d\rho(X, Y)Z = XZ - ZY$. The point $p = I \in V$ has isotropy ΔG, and the embedding $G \subset V$ factors as $G = G \times \{e\} \to G \times G/\Delta G \to V$, with the last map given by $(A, B)\Delta G \mapsto \rho(A, B)p$. By Lemma 26, the second fundamental form is given by

$$II(X, Y) = d\rho \left(\frac{(X, -X)}{2} \right) d\rho \left(\frac{(Y, -Y)}{2} \right) p = d\rho \left(\frac{(X, -X)}{2} \right) Y = \frac{XY + YX}{2}.$$

It follows from an easy computation that the embedding of G in the sphere of radius $r = \sqrt{nc_n}$ is minimal.

The constant term in (9) is:

$$-4E + 2 \frac{\dim(G)}{r^2} = \begin{cases} -1 + 2\frac{n^2-1}{nc_n} = -\frac{1}{n^2} & \text{in the complex case} \\ -1 + 2\frac{(2n+1)n}{nc_n} = -\frac{1}{2(n+1)} & \text{in the quaternionic case} \end{cases}$$

Let $X, N \in \mathfrak{g}$ be a pair of orthogonal unit vectors. The non-constant term in (9) is:

$$-2\|II(X, N)\|^2 - \|II(N, N)\|^2 + \langle II(N, N), II(X, X) \rangle$$
$$= -2\|(XN + NX)/2\|^2 - \|N^2\|^2 + \langle N^2, X^2 \rangle$$
$$= -\langle NX, XN \rangle - \|N^2\|^2.$$

We have used the identities $\|NX\|^2 = \|XN\|^2 = \langle N^2, X^2 \rangle$, which follow from the definition of the inner product and the assumption that X, N are skew-Hermitian. $\square$

Proposition 30 *The standard isometric embedding* $(\mathrm{Sp}(n), -B)$ *into* $\mathbb{H}^{n \times n}$ *satisfies* ACS < 0. *In particular, every closed embedded minimal hypersurface* $M \subset \mathrm{Sp}(n)$ *satisfies*

$$\mathrm{ind}(M) \geq \binom{4n^2}{2}^{-1} b_1(M).$$

Proof Note that the so-called Frobenius inner product, given by $\langle \cdot, \cdot \rangle_F = (4n + 4)^{-1} \langle \cdot, \cdot \rangle$, is sub-multiplicative, which implies that

$$\langle NX, XN \rangle \geq -\|NX\|^2 \geq -\frac{\|X\|^2\|N\|^2}{4n + 4} = -\frac{1}{4n + 4}$$

Therefore, using Lemma 29, we conclude that ACS < 0:

$$-\frac{1}{2(n + 1)} - \langle NX, XN \rangle - \|N^2\|^2 \leq -\frac{1}{4n + 4} < 0.$$

The stated bound for the index of M follows from Theorem 2. $\square$

To treat the case $G = \mathrm{SU}(n)$ we need a lemma:

Lemma 31 *Let* $n \geq 2$. *The function* $\mathrm{tr}((XN)^2 + N^4)$ *is real-valued on the set* $\{(X, N) \in \mathfrak{su}(n)^2 \mid \mathrm{tr}(X^2) = \mathrm{tr}(N^2) = -1, \; \mathrm{tr}(XN) = 0\}$. *Let* a_n *denote its minimum value in this set. Then:*

(a) *If n is even, then* $a_n = \frac{2-n}{8n}$.

(b) *If n is odd, $\frac{3-n}{8(n-1)} = a_{n-1} \geq a_n \geq a_{n+1} = \frac{1-n}{8(n+1)}$.*

Proof The functions $\operatorname{tr}((XN)^2)$ and $\operatorname{tr}(N^4)$ are real-valued because X, N are skew-Hermitian.

(a) Since $\operatorname{tr}((XN)^2 + N^4)$ is invariant under simultaneous conjugation of X and N by $SU(n)$, we may assume that $N = i \operatorname{diag}(z_1, \ldots z_n)$, where $z_j \in \mathbb{R}$, $z_1 + \cdots + z_n = 0$, and $z_1^2 + \cdots + z_n^2 = 1$.

Fixing such N, the (real-valued) function $X \mapsto \operatorname{tr}((XN)^2)$ is quadratic, hence achieves its minimum at an eigenvector associated to the smallest eigenvalue of the map $X \mapsto -NXN$. Thus

$$\min_X \operatorname{tr}((XN)^2 + N^4) = \min z_i z_j + \sum_k z_k^4$$
$$\qquad\qquad\qquad\qquad\qquad i < j$$

Since the minimum of $z_i z_j + \sum_k z_k^4$ does not depend on i, j, it suffices to show that one of them, say $z_1 z_2 + \sum_k z_k^4$, has minimum $\frac{2-n}{8n}$ with the constraints that $z_1 + \cdots + z_n = 0$ and $z_1^2 + \cdots + z_n^2 = 1$.

Moreover, since n is even, it is enough to prove the Claim below. Indeed, the minimum of $z_1 z_2 + \sum_k z_k^4$ will then be achieved at

$$(z_1, z_2, \ldots, z_n) = \left(-\sqrt{\frac{n+2}{4n}}, +\sqrt{\frac{n+2}{4n}}, -\sqrt{\frac{1}{2n}}, +\sqrt{\frac{1}{2n}}, \ldots, -\sqrt{\frac{1}{2n}}, +\sqrt{\frac{1}{2n}} \right),$$

because, by the Claim, this is the point where the minimum of $z_1 z_2 + \sum_k z_k^4$ subject only to $z_1^2 + \cdots + z_n^2 = 1$ is achieved, and this point happens to also satisfy the other constraint $z_1 + \cdots + z_n = 0$.

Claim The minimum of $f = -\sqrt{w_1 w_2} + \sum_j w_j^2$ subject to $\sum_j w_j = 1$ and $w_j \geq 0 \,\forall j$ equals $\frac{2-n}{8n}$ and is achieved at

$$w_1 = w_2 = \frac{n+2}{4n} \quad w_j = \frac{1}{2n}, \; j = 3, \ldots n.$$

We prove the Claim by induction on n. The base case $n = 2$ is straightforward. Assume $n > 2$. We use Lagrange multipliers:

$$\nabla f = \left(2w_1 - \frac{w_2}{2\sqrt{w_1 w_2}}, 2w_2 - \frac{w_1}{2\sqrt{w_1 w_2}}, 2w_3, 2w_4, \ldots, 2w_n \right)$$

The equation $\nabla f = a(1, \ldots 1)$ for some $a \in \mathbb{R}$ implies that

$$w_1 - w_2 = -4(w_1 - w_2)\sqrt{w_1 w_2}$$

and therefore $w_1 = w_2 = \frac{2a+1}{4}$ and $w_j = \frac{a}{2}$ for $j = 3, \ldots n$. Since $\sum_j w_j = 1$, we have exactly one critical point in the interior of the region defined by $w_j \geq 0$ for all j, namely

$$w_c = \left(\frac{n+2}{4n}, \frac{n+2}{4n}, \frac{1}{2n}, \ldots, \frac{1}{2n} \right)$$

Note that $f(w_c) = \frac{2-n}{8n}$.

On the other hand, assume $w = (w_1, \ldots w_n)$ lies on the boundary, that is, $w_j = 0$ for some j. If $j = 1, 2$, then $f(w) \geq 0 > \frac{2-n}{8n}$. If $j > 2$, then by the inductive hypothesis we have $f(w) \geq \frac{2-(n-1)}{8(n-1)} > \frac{2-n}{8n}$. This concludes the proof of the Claim.

(b) It is true for all $n \geq 2$ that $a_n \geq a_{n+1}$. Indeed, the sets

$$S_n = \{(X, N) \in \mathfrak{su}(n)^2 \mid \operatorname{tr}(X^2) = \operatorname{tr}(N^2) = -1, \ \operatorname{tr}(XN) = 0\}$$

satisfy $S_n \subset S_{n+1}$, and the function $\operatorname{tr}((XN)^2 + N^4)$ on S_n is the restriction to S_n of the corresponding function on S_{n+1}. The stated result then follows from (a).

$\square$

Proposition 32 *Consider the standard isometric embedding of* $(\mathrm{SU}(n), -B)$ *into* $\mathbb{C}^{n \times n}$, *where B denotes the Cartan–Killing form.*

(a) *Suppose $n < 18$. Then the embedding satisfies* ACS < 0. *In particular, every closed embedded minimal hypersurface* $M \subset \mathrm{SU}(n)$ *satisfies*

$$\operatorname{ind}(M) \geq \binom{2n^2}{2}^{-1} b_1(M).$$

(b) *If $n > 18$, the embedding* $\mathrm{SU}(n) \subset \mathbb{C}^{n \times n}$ *does not satisfy* ACS < 0.

(c) *The embedding* $\mathrm{SU}(18) \subset \mathbb{C}^{18 \times 18}$ *satisfies* ACS ≤ 0.

Proof By Lemma 29, it is enough to determine the sign of

$$b_n = \min \left(\frac{1}{n^2} + \langle NX, XN \rangle + \|N^2\|^2 \right)$$

where the minimum is taken over all $X, N \in \mathfrak{su}(n)$ such that $\|X\| = \|N\| = 1$ and $\langle X, N \rangle = 0$.

We claim that $b_n = \frac{1}{n^2} + \frac{a_n}{2n}$, where a_n is defined in Lemma 31. Indeed, letting $X' = \sqrt{2n}X$ and $N' = \sqrt{2n}N$, it follows that $\|X\| = 1$ if and only if $\operatorname{tr}((X')^2) = -1$, and similarly for N, N'. Thus

$$b_n = \frac{1}{n^2} + 2n \min \left(\operatorname{tr}(NXNX) + \operatorname{tr}(N^4) \right)$$

$$= \frac{1}{n^2} + \frac{1}{2n} \min \left(\operatorname{tr}(N'X'N'X') + \operatorname{tr}((N')^4) \right) = \frac{1}{n^2} + \frac{a_n}{2n}$$

where the first minimum is taken over $X, N \in \mathfrak{su}(n)$ such that $\|X\| = \|N\| = 1$ and $\langle X, N \rangle = 0$, while the second minimum is taken over $X', N' \in \mathfrak{su}(n)$ such that $\operatorname{tr}((X')^2) = \operatorname{tr}((N')^2) = -1$, and $\operatorname{tr}(X'N') = 0$. This finishes the proof of the claim.

If n is even, then by Lemma 31, $a_n = \frac{2-n}{8n}$, so that $b_n = \frac{18-n}{16n^2}$. Therefore (c) and the statements in (a), (b) with n even follow.

If n is odd, then Lemma 31 implies that

$$\frac{1}{n^2} + \frac{a_{n+1}}{2n} = \frac{-n^2 + 17n + 16}{16n^2(n+1)} \leq b_n \leq \frac{-n^2 + 19n - 16}{16n^2(n-1)} = \frac{1}{n^2} + \frac{a_{n-1}}{2n}$$

In particular, $n < 18$ implies $b_n > 0$ and $n > 18$ implies $b_n < 0$, proving (a), (b) for n odd.

$\square$

5.4 Quaternionic Grassmannians

Given $d \leq n$, consider the Grassmannian manifold of d-planes in $\mathbb{H}^n$. It is a symmetric space which we will write as

$$G/K = \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(d) \times \mathrm{Sp}(n-d)}.$$

We endow G/K with the metric induced from the Killing form on G (see (10)), so that G/K is Einstein with constant $1/2$, see Lemma 25. The Ad_K-invariant complement $\mathfrak{m}$ of $\mathfrak{k}$ in $\mathfrak{g}$ consists of the matrices

$$\hat{X} = \begin{bmatrix} 0 & X \\ -X^* & 0 \end{bmatrix}$$

where X is a $d \times (n-d)$ matrix with entries in $\mathbb{H}$.

Let V be the space of traceless Hermitian $n \times n$ matrices, and endow V with the "same" metric as $\mathfrak{g}$, given by (10). The group G acts on V by conjugation, and the orbit through $p \in V$ is an isometric embedding of G/K into V, with the metrics defined above, where

$$p = \frac{1}{n} \begin{bmatrix} (n-d)I_d & 0 \\ 0 & -dI_{n-d} \end{bmatrix}$$

(and I_k denotes the $k \times k$ identity matrix).

Lemma 33 *With the notations above, the* ACS *quantity of the isometric embedding $G/K \subset V$ is given by:*

$$\mathrm{ACS} = -\frac{2}{n+1} - 8c_n \Re \, \mathrm{tr}(XN^*XN^* + NN^*NN^*)$$

where $c_n = 4(n+1)$, and $X, N \in \mathbb{H}^{d \times (n-d)}$, such that

$$\mathrm{tr}(XN^*) = 0, \quad \mathrm{tr}(XX^*) = \mathrm{tr}(NN^*) = \frac{1}{2c_n}.$$

Proof The image of $G/K \subset V$ is contained in the sphere of radius r, where

$$r^2 = c_n \frac{d(n-d)}{n}.$$

By Lemma 26, the second fundamental form is given by

$$II(X, Y) = d\rho(\hat{X})d\rho(\hat{N})p = -\begin{bmatrix} XN^* + NX^* & 0 \\ 0 & -(X^*N + N^*X) \end{bmatrix}$$

From this, an easy computation shows that the embedding $G/K \subset V$ is minimal. The constant term in (9) is:

$$-4E + 2\frac{\dim(G/K)}{r^2} = -2 + 2\frac{4d(n-d)}{c_n d(n-d)/n} = -\frac{2}{(n+1)}$$

Let $X, N \in \mathfrak{m}$ be a pair of orthogonal unit vectors. A straight-forward computation yields the non-constant term in (9):

$$
\begin{aligned}
&- 2\|II(X,N)\|^2 - \|II(N,N)\|^2 + \langle II(N,N), II(X,X) \rangle \\
&= -c_n \Re \operatorname{tr} \begin{bmatrix} 2(XN^* + NX^*)^2 & 0 \\ 0 & 2(X^*N + N^*X)^2 \end{bmatrix} \\
&\quad - c_n \Re \operatorname{tr} \begin{bmatrix} 4(NN^*)^2 & 0 \\ 0 & 4(N^*N)^2 \end{bmatrix} \\
&\quad + c_n \Re \operatorname{tr} \begin{bmatrix} 4NN^*XX^* & 0 \\ 0 & 4N^*NX^*X \end{bmatrix} \\
&= -8c_n \Re \operatorname{tr}(XN^*XN^* + NN^*NN^*)
\end{aligned}
$$

Adding the constant and non-constant terms we arrive at the stated formula for ACS.   □

Proposition 34 *Let $d \le n$, and let (M, g) be the quaternionic Grassmannian of d-planes in n-space. The standard embedding of (M, g) into the space of traceless Hermitian $n \times n$ matrices satisfies* ACS < 0 *for every d, n.*

Proof We use the formula for ACS stated in Lemma 33. Recall that the "Frobenius" norm $\|A\|_F^2 = \Re \operatorname{tr}(AA^*)$ on matrices is submultiplicative. Thus

$$
-\frac{2}{n+1} - 8c_n \Re \operatorname{tr}(XN^*XN^* + NN^*NN^*) \le -\frac{2}{n+1} + 8c_n \frac{1}{4c_n^2} = -\frac{3}{2(n+1)} < 0
$$

because $c_n = 4(n+1)$ in the quaternionic case.   □

Remark 35 The natural embeddings of the group $SO(n)$, and the real and complex Grassmannians, analogous to the embeddings of $SU(n)$, $Sp(n)$, and the quaternionic Grassmannians we have considered in this section, do *not* satisfy ACS < 0.

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