UNIVERSIDADE DE SÃO PAULO

UPPER SEMICONTINUITY OF ATTRACTORS AND SYNCHRONIZATION

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NOTAS

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Resumo

Neste artigo provamos que problemas parabólicos semilineares abstratos acoplados por difusão sincronizam. Aplicamos os resultados abstratos obtidos a uma classe de equações diferenciais ordinárias e a problemas de reação e difusão. As técnicas envolvidas consistem em provar que os atratores para as equações diferencias acopladas são semicontínuos superiormente com respeito ao atrator de um problema na diagonal que é exibido explicitamente.

Palavras Chaves: Upper Semicontinuity of Attractors, Synchronization, Diffusively Coupled Systems

Upper Semicontinuity of Attractors and Synchronization

by

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Abstract

In this paper we prove that diffusively coupled abstract semilinear parabolic systems synchronize. We apply the abstract results obtained to a class of ordinary differential equations and to reaction diffusion problems. The technique consists of proving that the attractors for the coupled differential equations are upper semicontinuous with respect to the attractor of a problem, explicitly exhibited, in the diagonal.

1 Introduction

The phenomenon of synchronization is present in many situations of our every-day life. Our heart beat and the menses of room mates are examples of that. The synchronization of clocks has been observed in 1665 by Christian Huygens. In mechanical and electrical coupled oscillators, synchronization is also observed, see for example [2, 4, 5, 10, 15, 20, 22, 23]. In lasers systems it has been observed by Prof. R. Roy and his group, see [12], and in biological oscillators by Mirollo-Strogatz [19] and by Strogatz-Stewart [24]. In [25], Verichev studied synchronization in a coupled system of Lorenz equations. In communications this phenomenon is analyzed by Cuomo-Oppenheim in [11].

In [2], Afraimovich-Verichev-Rabinovich give a mathematical definition of synchronization and a system of coupled nonautonomous nonlinear oscillators is discussed. These concepts were used by Verichev in [25].

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In this paper we deal with mathematical models for physical problems which exhibits such phenomenon. To be more precise consider a physical problem described by the ordinary differential equation

$$\dot{x} = f(x) \tag{1.1}$$

where x is a vector in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field in $\mathbb{R}^n$. When two of such physical problems operate together there may be some interaction between them. As an example let us say that they interact dissipatively; that is

$$\begin{aligned} \dot{x} &= k(y-x) + f(x), \\ \dot{y} &= k(x-y) + f(y). \end{aligned} \tag{1.2}$$

Thus the system x feels the effect of system y through a dissipative coupling and vice versa. In this case if we increase the value of k the effect of the system y on the system x will make the two systems behave alike. In mathematical terms this means that for k large any solution $(x(t), y(t))$ of (1.2) which is globally defined and bounded will satisfy

$$\lim_{t\to\infty} \|x(t) - y(t)\| \to 0, \text{ as } t \to \infty.$$

This property is what we will call synchronization.

Of course there may be more systems involved and the systems involved may be similar but in general are not the same. So it would be more interesting to consider coupling of nonidentical systems as

$$\begin{aligned} \dot{x} &= k(y-x) + f(x), \\ \dot{y} &= k(x-y) + g(y), \end{aligned} \tag{1.3}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is another vector field in $\mathbb{R}^n$ which is not necessarily close to f. In this case we no longer have the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$ as an invariant manifold and therefore we do not expect synchronization as previously defined. However as k gets larger the system x should still start to behave as the system y, more precisely, given ϵ there exists $k_0 > 0$ such that

$$\limsup_{t\to\infty} \|x(t) - y(t)\| \le \epsilon$$

whenever $(x(t), y(t))$ is a globally defined bounded solution of (1.3) with $k \ge k_0$. This is what we call ϵ-synchronization.

Even though this prototype problem is quite helpful for introducing the results we would like to consider more general problems with several systems coupled, unidirectionally or not and not to restrict ourselves to ordinary differential models. To that end, in Section 2 we develop abstract results that apply to ordinary and partial differential equations. In Section 3 we consider examples in ordinary differential equations and in partial differential equations.

Indeed we prove that if the problem (1.3) has a global attractor $\mathcal{A}_k$ and if the problem

$$\dot{z} = \frac{1}{\sqrt{2}} f(\sqrt{2}z) + \frac{1}{\sqrt{2}} g(\sqrt{2}z) \tag{1.4}$$

has a global attractor $\mathcal{A}$ then, under a few more technical hypotheses, we have that the family of attractors $\{\mathcal{A}_k, k \le \infty\}$ is upper semicontinuous at infinity where $\mathcal{A}_\infty = \{(z,z) \in \mathbb{R}^n \times \mathbb{R}^n : z \in \mathcal{A}\}$. This is more then synchronization. It also says that the two coordinates approach a certain set.

To make the results precise let us say that we use the definition of global attractors given in [17] and the concept of upper semicontinuity at infinity for the family $\{\mathcal{A}_k, k \le \infty\}$ is defined as:

Given $\epsilon > 0$ there is a $k_0 > 0$ such that

$$\sup_{u\in\mathcal{A}_k} d(u, \mathcal{A}_\infty) \le \epsilon,$$

for all $k \ge k_0$, where $d(u, \mathcal{A}_\infty)$ is the distance in the phase space $\mathbb{R}^n \times \mathbb{R}^n$ between the point u and the attractor $\mathcal{A}_\infty$.

2 Abstract Results

In this section we introduce some abstract results adapted from [11] which will be used throughout this paper.

Let X be a Banach space and $A : D(A) \subset X \to X$ be the generator of an analytic semigroup of bounded linear operators on X. Then, we can define the fractional powers A^α of A and the associated fractional power spaces X^α, see [18].

Let $\{T(t),\, t \geq 0\}$ be a semigroup (usually nonlinear) on X. A set $B \subset X$ is said to attract a set $C \subset X$ under $T(t)$ if $\text{dist}(T(t)C, B) \to 0$ as $t \to \infty$. A set $S \subset X$ is said to be invariant if $T(t)S = S$ for $t \geq 0$. An invariant set A is said to be a *global attractor* if A is a maximal compact invariant set which attracts each bounded set $B \subset X$.

Our first result is a converse theorem on existence of a compact attractor for semilinear equations in Banach spaces. Results of this type are standard in the theory of stability and we only give its proof for the sake of completeness

Theorem 2.1 *Consider the differential equation*

$$\dot{x}(t) = Ax(t) + f(x) \tag{2.1}$$

where $f : X^\alpha \to X$ is globally Lipschitz continuous on X^α. Suppose that (2.1) has a global attractor $\mathcal{A}$. Then, given $R > 0$ such that $\mathcal{A} \subset B(0,R) = \{x \in X^\alpha : \|x\|_{X^\alpha} \leq R\}$, there is a function $\Sigma : B(0,R) \to \mathbb{R}^+$ which is Lipschitz and satisfies

- *i) $\Sigma(\phi) = 0, \forall\, \phi \in \mathcal{A}$,*
- *ii) $a(d(x(1,\phi),\mathcal{A})) \leq \Sigma(\phi) \leq b(d(\phi,\mathcal{A}))$, where a is continuous nondecreasing, $a(s) > 0$ if $s > 0$, $b(s)$ is continuous with $b(0) = 0$, $d(\cdot,\mathcal{A})$ is the distance in X^α to $\mathcal{A}$ and $x(t,\phi)$ is the solution of (2.1) satisfying $x(0,\phi) = \phi$,*
- *iii) $\dot{\Sigma}_{(2.1)}(\phi) \leq -\Sigma(\phi)$, where $\dot{\Sigma}_{(2.1)}$ is the right hand derivative of Σ along solutions of (2.1).*

Proof: Let $\phi \in X^\alpha$, $d(\phi,\mathcal{A}) \leq R$. If $x(t,\phi)$ is the solution of (2.1) satisfying $x(0,\phi) = \phi$, there exists a function $\theta(t,R)$ such that

$$d(x(t,\phi),\mathcal{A})) \leq \theta(t,R)$$

where $\theta(t,R)$ is a strictly decreasing in t and a C^1 function. Let $T(\epsilon)$ be its inverse, that is $T(\theta(t,R)) = t$. So $T(\epsilon)$ is continuous on $0 < \epsilon < \theta(0,R)$ and $T(\epsilon) \to \infty$ as $\epsilon \to 0^+$.

From the hypothesis on f, the map $\phi \in X \mapsto x(t,\phi) \in X^\alpha$ is Lipschitz continuous with Lipschitz constant $\leq Le^{Mt}$ for some L, M, $t \geq 1$.

Define

$$g(\epsilon) = e^{-(\beta+M)T(\epsilon)}, \quad g(0) = 0,$$

for some $\beta > 0$. Also

$$G_j(z) = \max\left\{0, z - \frac{1}{j}\right\}, \quad j \geq 1.$$

For $j = 1, 2, 3, \cdots$

$$\Sigma_j(\phi) = g\left(\frac{1}{j+1}\right) \sup_{t\geq 1}\{e^{\beta t} G_j(d(x(t,\phi),\mathcal{A}))\}.$$

Observe that the sup is taken only on $1 \leq t \leq T_j := T\left(\frac{1}{j+1}\right)$ and so

$$0 \leq \Sigma_j(\phi) \leq g\left(\frac{1}{j+1}\right) e^{\beta T_j} \theta(1,R) \leq \theta(1,R),$$

$$
\begin{aligned}
|\Sigma_j(\phi) - \Sigma_j(\psi)| &\leq g\left(\frac{1}{j+1}\right) \sup_{1\leq t\leq T_j} \left\{ L e^{(\beta+M)t} \|\phi-\psi\|_X \right\} \\
&\leq g\left(\frac{1}{j+1}\right) L e^{(\beta+M)T_j} \|\phi-\psi\|_X \leq L\|\phi-\psi\|_X.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\Sigma_j(x(h,\phi)) &= e^{-\beta h} g\left(\frac{1}{j+1}\right) \sup_{t\geq 1+h} \{ e^{\beta t} G_j(d(x(t,\phi),\mathcal{A})) \} \\
&\leq e^{-\beta h} \Sigma_j(\phi),
\end{aligned}
$$

so $\dot{\Sigma}_j(\phi) \leq -\beta \Sigma_j(\phi)$.

If we let

$$
\Sigma(\phi) = \sum_{j=1}^{\infty} 2^{-j} \Sigma_j(\phi),
$$

we have that

$$
|\Sigma(\phi) - \Sigma(\psi)| \leq L\|\phi-\psi\|_X, \qquad \dot{\Sigma}(\phi) \leq -\beta\Sigma(\phi), \qquad \Sigma(\phi) = 0, \ \forall \phi \in \mathcal{A},
$$

and

$$
\Sigma(\phi) \geq \sum_{j=1}^{\infty} 2^{-j} g\left(\frac{1}{j+1}\right) e^{\beta} G_j(d(x(1,\phi),\mathcal{A})) = a(d(x(1,\phi)),\mathcal{A})),
$$

with a Lipschitz continuous, strictly increasing, $a(s) > 0$ if $s > 0$ and $a(s) \to \infty$ as $s \to \infty$. This concludes the proof.□

Let $k > 0$ be a positive parameter, m, n be positive integers, X be a Banach space and $C_k^i : X \to X$, be a bounded linear operator, $1 \leq i \leq m$. Let A, A^α and X^α be as before.

Assume that the semigroup $\{ T_k(t),\, t \geq 0 \}$ generated by $A_k := \operatorname{diag}(A + C_k^1, \cdots, A + C_k^m)$ satisfies

$$
\|T_k(t)w\|_{[X^\alpha]^m} \leq M e^{-\beta(k)t} \|w\|_{[X^\alpha]^m}, \quad t \geq 0,
$$

$$
\|T_k(t)w\|_{[X^\alpha]^m} \leq M t^{-\alpha} e^{-\beta(k)t} \|w\|_{[X]^m}, \quad t > 0,
$$

for any $w \in [X^\alpha]^m$, where $\beta(k) \to \infty$ as $k \to \infty$ and $M \geq 1$ is a constant.

Consider the weakly coupled system

$$
\left.
\begin{aligned}
\dot{x}(t) &= A_k x(t) + f(x,y), \\
\dot{y}(t) &= A y(t) + g(x,y),
\end{aligned}
\right\} \tag{2.2}
$$

where $f : [X^\alpha]^m \times [X^\alpha]^n \to [X]^m$ and $g : [X^\alpha]^m \times [X^\alpha]^n \to [X]^n$ are Lipschitz continuous in bounded subsets of $[X^\alpha]^m \times [X^\alpha]^n$.

Let $h : [X^\alpha]^n \to [X]^n$ be globally Lipschitz continuous on $[X^\alpha]^n$ and assume that

$$
\dot{y}(t) = A y(t) + h(y) \tag{2.3}
$$

has a global attractor $\mathcal{A}$ in $[X^\alpha]^n$. Suppose that there exists a constant $\mathcal{M} > 0$, independent of k, such that the set

$$
\mathcal{B} = \{ u \in [X^\alpha]^m \times [X^\alpha]^n : \|u\|_{[X^\alpha]^m \times [X^\alpha]^n} \leq \mathcal{M} \} \tag{H}
$$

absorbs bounded sets of $[X^\alpha]^m \times [X^\alpha]^n$ under the flow defined by (2.2).

Suppose that there exist nonnegative constants M_f, L_P, such that

$$\|f(\phi,\psi)\|_{[X]^m} \leq M_f, \quad \|P(\phi,\psi)\|_{[X]^n} \leq L_P\|\phi\|_{[X^\alpha]^m}, \tag{2.4}$$

where $P(\phi,\psi) = g(\phi,\psi) - h(\psi)$.

Lemma 2.1 *Assume that the flow defined by (2.2) is asymptotically smooth. Then, the problem (2.2) has a global attractor $\mathcal{A}_k$ and given $\eta > 0$, there exists $k_0 > 0$ such that*

$$\|x\|_{[X^\alpha]^m} < \eta, \ \forall (x,y) \in \mathcal{A}_k, \ \forall k \geq k_0.$$

Proof: Since all solutions of (2.2) are globally defined, for any $(x_0,y_0) \in \mathcal{A}_k$ and for any $k > 0$ we have that the solution $(x(t),y(t))$ of (2.2), $(x(0),y(0)) = (x_0,y_0)$ satisfies

$$\begin{aligned} \tfrac{dy}{dt} &= Ay(t) + h(y(t)) + P(x(t),y(t)), \\ x(t) &= T_k(t-t_0)x_0 + \textstyle\int_{t_0}^t T_k(t-s)f(x(s),y(s))ds. \end{aligned} \tag{2.5}$$

for all $t \geq t_0$. Thus, we have that

$$\|x(t)\|_{[X^\alpha]^m} = Me^{-\beta(k)(t-t_0)}\|x_0\|_{[X^\alpha]^m} + MM_f\int_{t_0}^t e^{-\beta(k)(t-t_0)}(t-t_0)^{-\alpha}ds.$$

Letting $t_0 \to -\infty$, we obtain that

$$\|x(t)\|_{[X^\alpha]^m} = MM_f\beta(k)^{\alpha-1}\int_0^\infty e^{-\theta}\theta^{-\alpha}ds$$

and the result follows from the fact that $\beta(k) \to \infty$ as $k \to \infty$. □

Theorem 2.2 *Assume that A is a sectorial operator and that $\beta(k) \to \infty$ as $k \to 0$. Assume also that $[H]$ and (2.4) are satisfied. Then, there exists $k_0 > 0$ such that , for $k \geq k_0$, the problem (2.2) has a global attractor $\mathcal{A}_k$ and the family of attractors $\{\mathcal{A}_k,\ k_0 \leq k \leq \infty\}$ is upper semicontinuous at infinity, where $\mathcal{A}_\infty := \mathcal{A}$.*

Proof: Since (2.3) has a global attractor $\mathcal{A}$ there exists a bounded positively invariant neighborhood C of $\{y \in [X^\alpha]^n : \|y\|_{[X^\alpha]^n} \leq \mathcal{M}\} = P_1\mathcal{B}$ and a globally Lipschitz continuous function $\Sigma : C \to \mathbb{R}^+$ such that for any $\psi \in C$,

- i) $\Sigma(\psi) = 0 \quad \text{if} \quad \psi \in \mathcal{A}$,
- ii) $a(d(x(1,\psi),\mathcal{A})) \leq \Sigma(\psi) \leq b(d(\psi,\mathcal{A}))$, where a is continuous nondecreasing, $a(s) > 0$ if $s > 0$ and $b(s)$ is continuous with $b(0) = 0$,
- iii) $\dot{\Sigma}_{(2.3)}(\psi) \leq -\Sigma(\psi)$, where $\dot{\Sigma}_{(2.3)}$ is the right hand derivative of Σ along the solutions of (2.3).

For $c > 0$, let $\mathcal{B}_c = \{\psi \in C : \Sigma(\psi) < c\}$. Then, from the fact that C is a positively invariant neighborhood of $P_1\mathcal{B}$ we have that $\mathcal{B}_c \supset P_1\mathcal{B}$ for suitably large c.

Suppose $(x(t),y(t))$ is a solution for (2.2) with initial data (ϕ,ψ). Using the variation of constants formula, (2.2) can be rewritten as

$$\begin{aligned} \tfrac{dy}{dt} &= Ay + h(y) + P(x,y), \\ x(t) &= T_k(t)\phi + \textstyle\int_0^t T_k(t-s)f(x(s),y(s))ds. \end{aligned} \tag{2.6}$$

for $t > 0$.

Choose $\eta > 0$ such that

$$c - LL_P\eta > 0,$$

where L is the Lipschitz constant of Σ.

If $y(s) \in \mathcal{B}_c$ and $\|x(s)\|_{[X^\alpha]^m} < \eta$ for $0 \le s \le t$, then

$$\begin{aligned}\dot{\Sigma}(y) &\le -\Sigma(y) + L\|P(x,y)\|_{[X]^n} \\ &\le -\Sigma(y) + LL_P\|x\|_{[X^\alpha]^m} < -\Sigma(y) + LL_P\eta\end{aligned}$$

and

$$\|x(t)\|_{[X^\alpha]^m} \le M\|x_0\|_{[X^\alpha]^m} + \frac{MM_f\Gamma(1-\alpha)}{\beta(k)^{1-\alpha}}. \tag{2.7}$$

Thus, for $\|\phi\|_{[X^\alpha]^m} < \frac{\eta}{2M}$ and k_0 such that

$$\frac{MM_f\Gamma(1-\alpha)}{\beta(k)^{1-\alpha}} < \frac{\eta}{2}, \quad k \ge k_0,$$

(2.7) implies that $\|x(s)\|_{[X^\alpha]^m} < \eta$ and $y(s) \in \mathcal{B}_c$ for all $t \ge 0$.

Therefore, for every $t \ge 0$, $\|\phi\|_{[X^\alpha]^m} < \frac{\eta}{2M}$, $\psi \in \mathcal{B}_c$, (2.7) is satisfied and

$$\dot{\Sigma}(y) \le -\Sigma(y) + LL_P(k)M\frac{M_f\Gamma(1-\alpha)}{\beta(k)^{1-\alpha}}) \tag{2.8}$$

for any $t \ge 0$.

Then, the ω-limit set $\mathcal{A}_k$ of $\mathcal{B}_{\eta,c} = \{(x,y) \in [X^\alpha]^m \times [X]^n : \|x\|_{[X^\alpha]^m} < \frac{\eta}{2M},\ y \in \mathcal{B}_c\}$ attracts $\mathcal{B}_{\eta,c}$. Since from Lemma 2.1 we can assume that $\mathcal{B} \subset \mathcal{B}_{\eta,c}$ we have that $\mathcal{A}_k$ is a global attractor for (2.2).

It remains to show that the family of attractors $\{\mathcal{A}_k, k \ge k_0\}$ is upper semicontinuous at infinity. Consider (2.8) for $\|\phi\|_{[X^\alpha]^n} < \frac{\eta}{2M}$ and $\psi \in \mathcal{B}_c$. Then,

$$\begin{aligned}\frac{d}{dt}(e^t\Sigma(y)) &\le e^t\dot{\Sigma}(y) + e^t\Sigma(y) \\ &\le LL_P(k)M\frac{M_f\Gamma(1-\alpha)}{\beta(k)^{1-\alpha}}e^t\end{aligned}$$

and

$$\Sigma(y) \le \Sigma(\phi)e^{-t} + LL_P(k)M\frac{M_f\Gamma(1-\alpha)}{\beta(k)^{1-\alpha}}(1-e^{-t}). \tag{2.9}$$

From (2.7) and (2.9), for every $(\phi,\psi) \in \mathcal{A}_k$,

$$\|\phi\|_{[X^\alpha]^m} \le \frac{2MM_f\Gamma(1-\alpha)}{\beta(k)^{1-\alpha}}, \quad \Sigma(\psi) \le LL_PM\frac{M_f\Gamma(1-\alpha)}{\beta(k)^{1-\alpha}}$$

and from property *ii)* of Σ

$$\lim_{k\to\infty} \sup_{(\phi,\psi)\in\mathcal{A}_k} \operatorname{dist}((\phi,\psi),(0,\mathcal{A})) = 0,$$

and Theorem 2.2 is proved. □

3 Examples:

3.1 Examples in Ordinary Differential Equations

Consider the problem

$$\dot{w} = -kAw + g(w) \tag{3.1}$$

where $w = (w_1, \cdots, w_\ell)^\top \in \mathbb{R}^\ell$, $g : \mathbb{R}^\ell \to \mathbb{R}^\ell$ and $g(w) = (g_1(w), \cdots, g_\ell(w))^\top$ with g_i satisfying

$$g_i(w)w_i < 0, \quad |w| \geq \xi, \tag{3.2}$$

for some $\xi > 0$. The matrix A is assumed to satisfy:

$$A = M^{-1}B, \quad M = \operatorname{diag}(d_1, \cdots, d_\ell),$$

$d_i > 0$, $1 \leq i \leq \ell$ and

$$-B = \begin{bmatrix} a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1\ell} \\ -a_{21} & a_{22} & -a_{23} & \cdots & -a_{2\ell} \\ -a_{31} & -a_{32} & a_{33} & \cdots & -a_{3\ell} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{\ell 1} & -a_{\ell 2} & -a_{\ell 3} & \cdots & a_{\ell\ell} \end{bmatrix},$$

$a_{ij} \geq 0, \ 1 \leq i, j \leq \ell.$

The matrix A is said *dissipative* if in addition it satisfies

$$a_{ii} \geq \sum_{\substack{j=1 \\ j \neq i}}^{\ell} a_{ij}, \ 1 \leq i \leq \ell.$$

Observe that this problem consists of ℓ coupled scalar ordinary differential equations. The linear coupling depends on a positive parameter k and the nonlinear coupling appears in the function g. If each g_i depends only upon w_i we say that the problems are linearly-coupled. We also observe that with similar treatment one can consider coupled systems of ordinary differential equations.

Next we address the question of how to prove that the solution operator $\{T(t) : t \in \mathbb{R}^+\}$ defined (3.1) has a global attractor. This is done using La Salle's invariance theory, which we briefly explain next.

We say that a function $V : \mathbb{R}^\ell \to \mathbb{R}$ is a *Liapunov function* for (3.1) if it is continuous and satisfy

$$\dot{V}_{(3.1)}(w) \leq 0, \ \forall w \in \mathbb{R}^\ell$$

where $\dot{V}_{(3.1)}(w) = \limsup_{t\to 0^+} \frac{V(T(t)w)-V(w)}{t}$ is the derivative along solutions of (3.1) at w. We call $\mathcal{E}$ the set of points w where $\dot{V}_{(3.1)}(w) = 0$.

Theorem 3.1 *Let $V : \mathbb{R}^\ell \to \mathbb{R}$ be a Liapunov function for (3.1), $\mathcal{E} = \{w \in \mathbb{R}^\ell : \dot{V}_{(3.1)}(w) = 0\}$ and $\mathcal{M}$ be the maximal invariant subset of $\mathcal{E}$. If $\{T(t)w : t \geq 0\}$ ($\{T(-t)w : t \geq 0\}$) is a bounded subset of $\mathbb{R}^\ell$; then, $T(t)w \to \mathcal{M}$ ($T(-t)w \to \mathcal{M}$) as $t \to \infty$.*

Proof: $V(T(t)w)$ is nonincreasing for $t \geq 0$ by hypothesis and bounded below from the fact that V is continuous and $\gamma^+(w)$ is bounded. Thus $\ell = \lim_{t\to\infty} V(T(t)w)$ exists. If $y \in \omega(w)$, then $V(y) = \ell$ and also $V(T(t)y) = \ell$, $t \geq 0$. This implies that $\dot{V}_{(3.1)}(y) = 0$. Hence $\omega(w) \subset \mathcal{E}$ and $\omega(w) \subset \mathcal{M}$. It follows from the fact that $\omega(w)$ attracts w that $T(t)w \to \mathcal{M}$ as $t \to \infty$ and the result is proved. □

We see that rectangles of the form $R_s = (1+s)[-\xi,\xi]^\ell$, with $s > 0$ are such that the vector field g points strictly inward at its boundary. To construct a Liapunov function for (3.1) we choose a function p_{R_0} that has level sets ∂R_s. This is done as follows.

Let $R_0 = [-\xi,\xi]^\ell$ and R_s as before $s > 0$, then $\{R_s,\ s \geq 0\}$ covers $\mathbb{R}^\ell$. Let $p_{R_0} : \mathbb{R}^\ell \to \mathbb{R}^+$ defined by

$$p_{R_0}(w) = \inf\{s \geq 0 : w \in R_s\}. \tag{3.3}$$

Definition 3.1 *A bounded rectangle $R \subset \mathbb{R}^\ell$ is contracting for the vector field g, if for every point w which belongs to a (closed) face of R, we have that $g(w)\cdot \mathbf{n}(w) < 0$, where $\mathbf{n}(w)$ is the outward normal to R at w (normal to the face that contains w, and if w belongs to more than one such faces this condition has to be satisfied for all of them).*

Theorem 3.2 *Suppose that $w(t) \in \mathbb{R}^\ell$ is a solution of (3.1) and that g satisfies (3.2). Then, for any $T > 0$ for which $p_{R_0}(w(T)) = \tau > 0$, there exists $\eta > 0$ such that*

$$\limsup_{h\to 0} \frac{p_{R_0}(w(T+h)) - p_{R_0}(w(T))}{h} \leq -\eta. \tag{3.4}$$

Proof: Let $w = (w_1,\cdots,w_\ell)$, $g = (g_1,\cdots,g_\ell)^\top$. If $p_{R_0}(w(T)) = \tau$ and R_τ, we have that $-(1+\tau)\xi \leq w_i(T) \leq (1+\tau)\xi$, $1 \leq i \leq n$. If $w(T) \in \partial R_\tau$, $w(T)$ is in one of the faces of R_τ, say the right-hand j^{th} face of R_τ, then $w_j(T) = (1+\tau)\xi$. Since R_τ is a contracting rectangle for g, there is an $\eta > 0$ such that for all $w \in \partial(R_\tau)_j^r$,

$$g(w)\cdot \mathbf{n}(w) < -\eta\xi,$$

where $\mathbf{n}(w)$ is the outward-pointing normal to R_τ at w. Since $w(T)$ is in the right-hand j^{th} face of R_τ, we have that

$$\begin{aligned}\dot{w}_j &= k(d_j)^{-1}\sum_{\substack{k=1\\k\neq j}}^{\ell} a_{jk} - k(d_j)^{-1}a_{jj}(\xi+\tau) + g_j(w)\\ &\leq k(d_j)^{-1}\sum_{\substack{k=1\\k\neq j}}^{\ell} a_{jk}(w_k - \xi - \tau) + g_j(w) \leq g_j(w) < -\eta\xi,\end{aligned}$$

and $w_j(T+h) < (1+\tau)\xi - \eta h\xi$ for small h. Therefore

$$p_{R_0}(w(T+h)) \leq \tau - \eta h$$

and

$$\frac{p_{R_0}(w(T+h)) - p_{R_0}(w(T))}{h} \leq -\eta.$$

and the result is proved. □

Corollary 3.1 *Under the above hypothesis R_s is positively invariant for any $s \geq 0$ and orbits of any bounded subset of $\mathbb{R}^\ell$ are bounded. Furthermore, the problem (3.1) with g satisfying (3.2) has a global attractor $\mathcal{A}_k$ satisfying*

$$\mathcal{A}_k \subset R_0.$$

Proof: The only part that remains to be proved is the fact that R_0 contains the attractor $\mathcal{A}_k$. This is done in the following way. Given any point $w \in \mathcal{A}_k$ its α-limit set is contained in the set of points where $\dot{p}_{R_0(3.1)} = 0$. The later is contained in R_0 and therefore $\text{dist}(T(-t)w, R_0) \to 0$ as $t \to \infty$. Thus, given $\epsilon > 0$ there is a t_ϵ such that $T(-t)w \in R_\epsilon$, for all $t > t_\epsilon$. Since R_ϵ is positively invariant we have that $w \in R_\epsilon$. Since $\epsilon > 0$ was taken arbitrarily we have that $w \in R_0$ and the inclusion follows. □

These results imply that we may assume, without loss of generality, that g is globally Lipschitz and globally bounded.

3.1.1 Totally Coupled First Order Systems

We will call a dissipative matrix A a *total coupling dissipative matrix* if it is self adjoint and satisfies

$$a_{ii} = \sum_{\substack{j=1 \\ j \neq i}}^{\ell} a_{ij} > 0, \ 1 \leq i \leq \ell.$$

If we consider the space $\mathbb{R}^\ell$ with the inner product $\langle \cdot, \cdot \rangle_M := \langle M \cdot, \cdot \rangle$, a total coupling matrix A defines in it a self adjoint nonnegative operator. The first eigenvalue of A, $\lambda_1 = 0$, is simple and an associated normalized eigenvector is $z_1 = c(1, \cdots, 1)^\top \in \mathbb{R}^\ell$, where $c = (d_1 + \cdots + d_\ell)^{-\frac{1}{2}}$. Let $\lambda_2, \cdots, \lambda_\ell$ be the remaining eigenvalues of A and $z_i = (z_i^1, \cdots, z_i^\ell)^\top$, $2 \leq i \leq \ell$, be associated normalized eigenvectors, that is $\langle z_i, z_j \rangle_M = \delta_{ij}$ and

$$\langle A z_i, z_i \rangle_M = -\lambda_i.$$

This implies that

$$-Z \Lambda Z^\top M = A, \quad Z^{-1} = Z^\top M, \quad (Z^\top)^{-1} = MZ,$$

where $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_\ell)$. Thus if $v = Z^{-1}w$, the system (3.1) becomes

$$\begin{aligned} \dot{v}_j &= -k\lambda_j v_j + f_j(v_1, \cdots, v_\ell), \\ f_j(v_1, \cdots, v_l) &= \textstyle\sum_{q=1}^{\ell} g_q \left(\sum_{i=1}^{\ell} z_i^1 v_i, \cdots, \sum_{i=1}^{\ell} z_i^\ell v_i \right) z_j^q, \quad 1 \leq j \leq \ell. \end{aligned} \tag{3.5}$$

This problem has a global attractor $\tilde{\mathcal{A}}_k$ and can now be rewritten in the abstract form

$$\begin{aligned} \dot{x} &= A_k x + h_1(x, y), \\ \dot{y} &= h_2(x, y), \end{aligned} \tag{3.6}$$

where $A_k = \text{diag}(-k\lambda 2, \cdots, -k\lambda_\ell)$, $h_1 : [\mathbb{R}]^{\ell-1} \times \mathbb{R} \to \mathbb{R}^{\ell-1}$ and $h_2 : [\mathbb{R}]^{\ell-1} \times \mathbb{R} \to \mathbb{R}$ are given by

$$h_1 = \begin{pmatrix} f_2 \\ \vdots \\ f_\ell \end{pmatrix},$$

and

$$h_2(v_1, \cdots, v_\ell) = \sum_{q=1}^{\ell} g_q \left(\sum_{i=1}^{\ell} z_i^1 v_i, \cdots, \sum_{i=1}^{\ell} z_i^\ell v_i \right) z_1^q.$$

Theorem 3.3 *The family of attractors $\{\tilde{\mathcal{A}}_k, k \leq \infty\}$ is upper semicontinuous at infinity, where $\tilde{\mathcal{A}}_\infty = \{(v, 0, \cdots, 0) \in [\mathbb{R}]^{\ell-1} : v \in \mathcal{A}\}$ and $\mathcal{A}$ is the attractor for the problem*

$$\dot{v} = c \sum_{q=1}^{\ell} \tilde{g}_q(cv),$$

where $\tilde{g}_q(s) = g_q(s, \cdots, s)$.

The proof of this result follows from Theorem 2 and it is easily seen that h_1, h_1 and P satisfy the required hypotheses.

Changing back the variables v_i to u_i we have the following result:

Theorem 3.4 *The family of attractors $\{\mathcal{A}_k, k \leq \infty\}$ is upper semicontinuous at infinity where $\mathcal{A}_\infty = \{(u, \cdots, u) : u \in \mathcal{A}\}$ and $\mathcal{A}$ is the global attractor of the problem*

$$\dot{u} = c^2 \sum_{q=1}^{\ell} \tilde{g}_q(u). \tag{3.7}$$

This theorem and the definition of upper semicontinuity of attractors immediately imply the following result.

Corollary 3.2 *Given $\epsilon > 0$ there is a $k_0 > 0$ such that, for $k \geq k_0$ and for any $(u_{10}, \cdots, u_{\ell 0}) \in [\mathbb{R}]^\ell$ the solution $(u_1(t), \cdots, u_\ell(t))$ of (3.1), with $(u_1(0), \cdots, u_\ell(0)) = (u_{10}, \cdots, u_{\ell 0})$, satisfy*

$$\limsup_{t \to \infty} \sum_{q=2}^{\ell} \|u_{q-1}(t) - u_q(t)\|_{\mathbb{R}} \leq \epsilon.$$

3.1.2 Partially Coupled First Order Systems

We will call a dissipative matrix A a *partial coupling dissipative matrix* if it is self adjoint and satisfies

$$a_{ij} = 0, \; 1 \leq j \leq \ell$$

for some, but not all, $1 \leq i \leq \ell$. This describes the situation when a particular coordinate has no coupling with the other coordinates of the systems (a line in the coupling matrix contains only zero elements). In this case, reordering the coordinates, the system (3.1) can be written as:

$$\dot{w} = -k\tilde{A}w + g(w), \tag{3.8}$$

where $w = (w_1, \cdots, w_{k+\ell})^\top \in \mathbb{R}^{k+\ell}$. The matrix $\tilde{A}$ is given by

$$\tilde{A} = \begin{bmatrix} 0\, I_{k\times k} & 0\, I_{k\times \ell} \\ 0\, I_{\ell\times k} & A \end{bmatrix}$$

with A as in the previous section and with $g : \mathbb{R}^{k+\ell} \to \mathbb{R}^{k+\ell}$ satisfying (3.2). Then, the problem (3.8) has a global attractor $\mathcal{A}_k$.

Keeping the first k variables fixed $W_1 = (w_1, \cdots, w_k)$ and proceeding as in the previous sections with the last ℓ variables $W_2 = (w_{k+1}, \cdots, k_{k+\ell})$; that is, making $V^\top := (v_1, \cdots, v_\ell)^\top := Z^{-1}W_2^\top$, the system (3.8) becomes

$$\begin{aligned}
&\dot{W}_1 = G(W_1, V) := (g_1(W_1, V), \cdots, g_k(W_1, V)),\\
&\dot{v}_j = -k\lambda_j v_j + f_j(W_1, V),\\
&f_j(W_1, V) = \textstyle\sum_{q=1}^{\ell} g_q\left(W_1, \sum_{i=1}^{\ell} z_i^1 v_i, \cdots, \sum_{i=1}^{\ell} z_i^{\ell} v_i\right) z_j^q, \quad 1 \leq j \leq \ell.
\end{aligned} \tag{3.9}$$

This problem has a global attractor $\tilde{A}_k$ and can now be rewritten in the abstract form

$$\begin{aligned}
&\dot{x} = A_k x + h_1(x, y),\\
&\dot{y} = h_2(x, y),
\end{aligned} \tag{3.10}$$

where A_k is as in the previous section, $h_1 : [\mathbb{R}]^{\ell-1} \times \mathbb{R}^{k+1} \to \mathbb{R}^{\ell-1}$ and $h_2 : [\mathbb{R}]^{\ell-1} \times \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ are given by

$$h_1 = \begin{pmatrix} f_2 \\ \vdots \\ f_\ell \end{pmatrix},$$

and

$$h_2(W_1, V) = \left(G(W_1, V), \sum_{q=1}^{\ell} g_q \left(W_1, \sum_{i=1}^{\ell} z_i^1 v_i, \cdots, \sum_{i=1}^{\ell} z_i^{\ell} v_i\right) z_1^q\right).$$

Theorem 3.5 *The family of attractors $\{\tilde{A}_k, k \leq \infty\}$ is upper semicontinuous at infinity, where $\tilde{A}_\infty = \{(W_1, v, 0, \cdots, 0) \in [\mathbb{R}]^{k+1} \times [\mathbb{R}]^{\ell-1} : v \in \mathcal{A}\}$ and $\mathcal{A}$ is the attractor in $[\mathbb{R}]^{k+1}$ of the problem*

$$\begin{aligned}
&\dot{W}_1 = G(W_1, v, \cdots, v),\\
&\dot{v} = c \textstyle\sum_{q=1}^{\ell} \tilde{g}_q(W_1, cv),
\end{aligned}$$

where $\tilde{g}_q(s) = g_q(W_1, s, \cdots, s)$.

The proof of this result follows from Theorem 2 and it is easily seen that h_1, h_1 and P satisfy the required hypotheses.

Changing back the variables v_i to w_i we have the following result:

Theorem 3.6 *The family of attractors $\{A_k, k \leq \infty\}$ is upper semicontinuous at infinity where $\mathcal{A}_\infty = \{(W_1, u, \cdots, u) : (W_1, u) \in \mathcal{A}\}$ and $\mathcal{A}$ is the global attractor of the problem*

$$\begin{aligned}
&\dot{W}_1 = G(W_1, w, \cdots, w),\\
&\dot{w} = c^2 \textstyle\sum_{q=1}^{\ell} \tilde{g}_q(w).
\end{aligned} \tag{3.11}$$

This theorem and the definition of upper semicontinuity of attractors immediately imply the following result.

Corollary 3.3 *Given $\epsilon > 0$ there is a $k_0 > 0$ such that, for $k \geq k_0$ and for any $(u_{10}, \cdots, u_{(\ell+k)0}) \in [\mathbb{R}]^{\ell+k}$ the solution $(u_1(t), \cdots, u_{\ell+k}(t))$ of (3.8), with $(u_1(0), \cdots, u_{\ell+k}(0)) = (u_{10}, \cdots, u_{(\ell+k)0})$, satisfy*

$$\limsup_{t\to\infty} \sum_{q=2}^{\ell} \|u_{q-1}(t) - u_q(t)\|_{\mathbb{R}} \leq \epsilon, \ \ k+1 < q \leq \ell + k.$$

Application to Partially, Linearly-Coupled Lorenz Equations

Consider the Lorenz equations:

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y,\\ \dot{y} &= -y - x z + r x,\\ \dot{z} &= -b z + x y.\end{aligned}$$

If we couple two of such systems on the first variable, with the coupling term depending on the difference of the corresponding variables we obtain

$$\begin{aligned}\dot{x}_1 &= -\sigma x_1 + \sigma y_1 - k(x_1 - x_2),\\ \dot{y}_1 &= -y_1 - x_1 z_1 + r x_1,\\ \dot{z}_1 &= -b z_1 + x_1 y_1,\\[1em] \dot{x}_2 &= -\sigma x_2 + \sigma y_2 + k(x_1 - x_2),\\ \dot{y}_2 &= -y_2 - x_2 z_2 + r x_2,\\ \dot{z}_2 &= -b z_2 + x_2 y_2.\end{aligned}$$

where $k > 0$ is the coupling parameter.

If we reorder the system above and write it in the matrix notation we obtain

$$\dot{\zeta} = -k A \zeta + f(\zeta)$$

where

$$\zeta = \begin{pmatrix} x_1\\ x_2\\ y_1\\ y_2\\ z_1\\ z_2 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0\\ -1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad f(\zeta) = \begin{pmatrix} -\sigma x_1 + \sigma y_1\\ -\sigma x_2 + \sigma y_2\\ -y_1 - x_1 z_1 + r x_1\\ -y_2 - x_2 z_2 + r x_2\\ -b z_1 + x_1 y_1\\ -b z_2 + x_2 y_2. \end{pmatrix}$$

To proceed we need to further analyze the matrix A. Note that this matrix has 0 (zero) as an eigenvalue with multiplicity five and the sixth eigenvalue is 2 (two). Let us consider the associated eigenvectors.

A set of eigenvectors associated to the eigenvalue 0 (zero) is;

$$\begin{aligned}\zeta_1 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, 0)\\ \zeta_2 &= (0, 0, 1, 0, 0, 0)\\ \zeta_3 &= (0, 0, 0, 1, 0, 0)\\ \zeta_4 &= (0, 0, 0, 0, 1, 0)\\ \zeta_5 &= (0, 0, 0, 0, 0, 1)\end{aligned}$$

and one eigenvector associated to the eigenvalue 2 (two) is;

$$\zeta_6 = (\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0).$$

Next we consider the following change of variables

$$v = \begin{pmatrix} v_1\\ v_2\\ v_3\\ v_4\\ v_5\\ v_6 \end{pmatrix} \qquad v = Z\zeta, \qquad Z = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}$$

which diagonalizes the matrix A. Then we have that v satisfy the following equation

$$\dot{v} = -k \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} v + g(v)$$

where g is the function

$$g(v) = \begin{pmatrix} -\sigma v_1 + \sigma \frac{v_2+v_3}{2} \\ -v_2 - (v_1+v_6)v_4 + r(v_1+v_6) \\ -v_3 - (v_1-v_6)v_5 + r(v_1-v_6) \\ -bv_4 + (v_1+v_6)v_2 \\ -bv_5 + (v_1-v_6)v_3 \\ -\sigma v_6 + \sigma \frac{v_2-v_3}{2} \end{pmatrix}$$

The uniform bounds for the attractors can be found in [21]. Note that we are now in condition to apply Theorem (3.6) with the limiting problem being

$$\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} -\sigma w_1 + \sigma \frac{w_2+w_3}{2} \\ -w_2 - w_1 w_4 + r w_1 \\ -w_3 - w_1 w_5 + r w_1 \\ -bw_4 + w_1 w_2 \\ -bw_5 + w_1 w_3 \end{pmatrix},$$

and if we now change variables to the original system and make $x_1 = x_2 = x$, we have that

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y_1, \\ \dot{y}_1 &= -y_1 - x z_1 + r x, \\ \dot{y}_2 &= -y_2 - x z_2 + r x, \\ \dot{z}_1 &= -b z_1 + x y_1, \\ \dot{z}_2 &= -b z_2 + x y_2. \end{aligned}$$

For this particular case we can show synchronization in the other two variables as well. Note that if $(x(t), y_1(t), y_2(t), z_1(t), z_2(t))$ is a solution of the above problem then the coordinate y_1 and y_2 (z_1 and z_2) satisfy the same equation and therefore if $y_1(0) = y_2(0)$ ($z_1(0) = z_2(0)$) they stay the same for $t \geq 0$. On the other hand if they start different, we have that they satisfy

$$\begin{aligned} \dot{y}_1 &= -y_1 - x(t)\, z_1 + r\, x(t), \\ \dot{y}_2 &= -y_2 - x(t)\, z_2 + r\, x(t), \\ \dot{z}_1 &= -b\, z_1 + x(t)\, y_1, \\ \dot{z}_2 &= -b\, z_2 + x(t)\, y_2. \end{aligned} \tag{3.12}$$

if $\eta = y_1 - y_2$ and $\rho = z_1 - z_2$, lets prove that (η, ρ) tends to zero exponentially. In fact, they satisfy the following system of ordinary differential equations

$$\begin{aligned} \dot{\eta} &= -\eta - x(t)\, \rho, \\ \dot{\rho} &= -b\, \rho + x(t)\eta. \end{aligned}$$

By taking the Liapunov function $V(\eta, \rho) = \eta^2 + \rho^2$ one easily sees that they approach zero exponentially. This implies that $\{(t, y_1, y_2, z_1, z_2) \in \mathbb{R}^4 : y_1 = y_2,\ z_1 = z_2\}$ is an exponentially attracting invariant manifold for the problem (3.12).

3.2 Examples in Parabolic Equations

3.2.1 Coupled Scalar Parabolic Equations

We are now able to apply the abstract result of Theorem 2 to a system of reaction-diffusion equations of the form:

$$\begin{aligned} u_t &= \Delta u - k(u-v) + f_1(u), \\ v_t &= \Delta v + k(u-v) + f_2(v), \end{aligned} \tag{3.13}$$

in $(0,\infty)\times\Omega$, $\Omega\subset \mathbb{R}^n$ bounded, smooth, subjected to Dirichlet initial-boundary conditions:

$$\begin{aligned} &u(t,x) = 0 \text{ and } v(t,x) = 0, \text{ for } (t,x)\in(0,\infty)\times\partial\Omega, \\ &u(0,x) = u_0(x),\ v(0,x) = v_0(x), \text{ for } x\in\Omega. \end{aligned} \tag{3.14}$$

We assume the dissipativeness condition on the functions f_1 and f_2,

$$f_i(z)z < 0, \text{ for } |z| > \xi_i,\ i = 1,2. \tag{3.15}$$

Under these conditions the problem (3.13) has a global attractor $\mathcal{A}_k$ in X^α, for $\alpha > \frac{n}{4}$, see [7].

Using the Moser iteration technique we will obtain an $L^\infty(\Omega)$ a priori estimate for the solutions (u,v) of (3.13). Multiplying the first equation in (3.13) by u^{r-1}, the second by v^{r-1}, $r = 1,2,3,\cdots$, integrating over Ω and adding, we get the equality:

$$\begin{aligned} \int_\Omega (u_t u^{2r-1} + v_t v^{2r-1})dx &= \int_\Omega (\Delta u u^{2r-1} + \Delta v v^{2r-1})dx \\ &\quad + k\int_\Omega ((v-u)u^{2r-1} - (v-u)v^{2r-1})dx \\ &\quad + \int_\Omega (f_1(u)uu^{2r-2} + f_2(v)vv^{2r-2})dx. \end{aligned} \tag{3.16}$$

Now we check that

$$\int_\Omega u_t u^{2r-1}dx = \frac{1}{2r}\frac{d}{dt}\int_\Omega u^{2r}dx,$$

$$\int_\Omega \Delta u u^{2r-1}dx = -\frac{2r-1}{r^2}\frac{d}{dt}\int_\Omega |\nabla u^r|^2 dx,$$

$$\int_\Omega (-u^{2r} + vu^{2r-1} - v^{2r} + uv^{2r-1})dx \le 0,$$

since, by Young inequality

$$vu^{2r-1} \le \frac{2r-1}{2r}u^{2r} + \frac{1}{2r}v^{2r}.$$

Moreover

$$\begin{aligned} \int_\Omega (f_1(u)uu^{2r-2} + f_2(v)vv^{2r-2})dx &\le \int_{\{x\in\Omega:|u|\le\xi_1 \text{ and } |v|\le\xi_2\}} (f_1(u)u^{2r-1} + f_2(v)v^{2r-1})dx \\ &\le M|\Omega|[\xi_1^{2r-1} + \xi_2^{2r-1}], \end{aligned}$$

where we set $M := \max_{i=1,2}\max_{|s|\le\xi_i}|f_i(s)|$.

Putting all the estimates together into (3.16) we get a differential inequality of the form:

$$\begin{aligned}\frac{1}{2r}\frac{d}{dt}\int_\Omega (u^{2r}+v^{2r})dx \le & -\frac{2r-1}{r^2}\int_\Omega (|\nabla u^r|^2+|\nabla v^r|^2)dx \\ & + M|\Omega|[\xi_1^{2r-1}+\xi_2^{2r-1}],\end{aligned} \tag{3.17}$$

$r=1,2,3,\cdots$. Next, by the Poincaré inequality $\|\phi\|^2_{L^2(\Omega)} \le c\|\nabla\phi\|^2_{L^2(\Omega)}$ applied to u^r and v^r we obtain

$$\begin{aligned}\frac{1}{2r}\frac{d}{dt}\int_\Omega (u^{2r}+v^{2r})dx \le & -\frac{2r-1}{r^2}c^{-1}\int_\Omega (|u|^{2r}+|v|^{2r})dx \\ & + M|\Omega|[\xi_1^{2r-1}+\xi_2^{2r-1}],\end{aligned} \tag{3.18}$$

$r=1,2,3,\cdots$, or, if $y(t) := \int_\Omega (u^{2r}+v^{2r})dx$, we have

$$\frac{1}{2r}y'(t) \le -\frac{2r-1}{r^2}c^{-1}y(t) + M|\Omega|[\xi_1^{2r-1}+\xi_2^{2r-1}]. \tag{3.19}$$

The solutions of such differential inequality satisfies an estimate of the form

$$\sup_t y(t) \le \max\{y(0), M|\Omega|\,r\,c\,[\xi_1^{2r-1}+\xi_2^{2r-1}]\}, \tag{3.20}$$

and also the asymptotic estimate

$$\limsup_{t\to\infty} y(t) \le M|\Omega|\,r\,c\,[\xi_1^{2r-1}+\xi_2^{2r-1}], \tag{3.21}$$

Taking the $2r$ roots and letting $r\to\infty$ in the above estimates we get the $L^\infty(\Omega)$ bounds

$$\sup_t(\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)}) \le \max\{\|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)}, \xi_1+\xi_2\}, \tag{3.22}$$

and

$$\limsup_{t\to\infty}(\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)}) \le \xi_1+\xi_2. \tag{3.23}$$

With these $L^\infty(\Omega)$ bounds we can cut the nonlinearities f_1 and f_2 outside the interval $[-\xi_1-\xi_2, \xi_1+\xi_2]$ in such a way that the attractors $\mathcal{A}_k$ of the problem (3.13) are not changed and such that the new nonlinearities are globally Lipschitz and globally bounded. This implies that we can assume without loss of generality that the original nonlinearities f_1 and f_2 are globally Lipschitz and globally bounded.

Now it is convenient to rewrite the system (3.13) in a more suitable form to apply Theorem 2. For this we introduce the new unknown

$$w := \frac{u-v}{\sqrt{2}},\ z := \frac{u+v}{\sqrt{2}},$$

and obtain the equations

$$\begin{aligned} w_t &= \Delta w - 2kw + \tfrac{1}{\sqrt{2}}f_1(\tfrac{w+z}{\sqrt{2}}) - \tfrac{1}{\sqrt{2}}f_2(\tfrac{z-w}{\sqrt{2}}), \\ z_t &= \Delta z + \tfrac{1}{\sqrt{2}}f_1(\tfrac{w+z}{\sqrt{2}}) + \tfrac{1}{\sqrt{2}}f_2(\tfrac{z-w}{\sqrt{2}}). \end{aligned} \tag{3.24}$$

This problem, with homogeneous Dirichlet boundary condition, has a global attractor $\tilde{\mathcal{A}}_k$ can be rewritten in the abstract form

$$\begin{aligned} \dot{w} &= A_k w + f^e(w,z), \\ \dot{z} &= Az + g^e(w,z), \end{aligned} \tag{3.25}$$

where A is the Laplacian with homogeneous Neumann boundary condition, $A_k := A - 2k$, $f^e, g^e : X^\alpha \times X^\alpha \to X$,

$$f^e(w,z)(x) = \frac{1}{\sqrt{2}} f_1\Big(\frac{w(x)+z(x)}{\sqrt{2}}\Big) - \frac{1}{\sqrt{2}} f_2\Big(\frac{w(x)-z(x)}{\sqrt{2}}\Big),$$

and

$$g^e(w,z)(x) = \frac{1}{\sqrt{2}} f_1\Big(\frac{w(x)+z(x)}{\sqrt{2}}\Big) + \frac{1}{\sqrt{2}} f_2\Big(\frac{w(x)-z(x)}{\sqrt{2}}\Big).$$

This is now in the form (2.2) and we can apply Theorem 2 to obtain the upper semicontinuity of the attractors $\{\tilde{\mathcal{A}}_k, k \le \infty\}$ at infinity where $\tilde{\mathcal{A}}_\infty = \{(0,z) : z \in \mathcal{A}\}$ and $\mathcal{A}$ is the global attractor of the problem

$$z_t = \Delta z + \tfrac{1}{\sqrt{2}} f_1\big(\tfrac{z}{\sqrt{2}}\big) + \tfrac{1}{\sqrt{2}} f_2\big(\tfrac{z}{\sqrt{2}}\big). \tag{3.26}$$

Changing back the variables to u and v the previous considerations imply the following:

Theorem 3.7 *The family of attractors $\{\mathcal{A}_k, k \le \infty\}$ is upper semicontinuous at infinity where $\mathcal{A}_\infty = \{(u,u) : u \in \mathcal{A}\}$ and $\mathcal{A}$ is the global attractor of the problem*

$$u_t = \Delta u + \tfrac{1}{2} f_1(u) + \tfrac{1}{2} f_2(u). \tag{3.27}$$

This theorem and the definition of upper semicontinuity of attractors immediately imply the following result.

Corollary 3.4 *Given $\epsilon > 0$ there is a $k_0 > 0$ such that, for $k \ge k_0$ and for any $(u_0, v_0) \in X^\alpha \times X^\alpha$ the solution $(u(t), v(t))$ of (3.13), with $(u(0), v(0)) = (u_0, v_0)$, satisfy*

$$\limsup_{t\to\infty} \|u(t) - v(t)\|_{X^\alpha} \le \epsilon.$$

3.2.2 Coupled Systems of Parabolic Equations

Now we study a system of $2m$ equations generalizing the previous example. Let Ω be a bounded smooth domain in $\mathbb{R}^n$. Consider the homogeneous Dirichlet problem for a system:

$$\begin{aligned} u_t &= Lu - k(u-v) + f_1(u), \quad x \in \Omega,\ t > 0, \\ v_t &= Lv - k(v-u) + f_2(v), \quad x \in \Omega,\ t > 0, \end{aligned} \tag{3.28}$$

where $u = (u_1, \cdots, u_m)$, $v = (v_1, \cdots, v_m)$ and $f_i \in C^1(\mathbb{R}^m, \mathbb{R}^m)$, $i = 1,2$. Here L is a second order diagonal matrix operator in the divergence form:

$$L = \begin{bmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A \end{bmatrix}_{m\times m}, \qquad A = \operatorname{div}(a(x)\cdot\nabla), \tag{3.29}$$

with $a \in C^1(\bar{\Omega}, \mathbb{R}^m)$, $a_j(x) \ge a_0 > 0$, $j = 1, \cdots, n$. The dissipativeness condition for the functions $f_i = (f_i^1, \cdots, f_i^m)$:

$$f_i^k(u_1, \cdots, u_m)u_k < 0, \text{ for } |u_k| > \xi_i^k, \tag{3.30}$$

$k = 1, \cdots, m$, $i = 1,2$, will be also assumed from now on. Under the just stated assumptions the problem (3.28) has a global attractor $\mathcal{A}_k$, [8], and with the same proof as for the problem (3.13) we get global in time and asymptotically independent on the initial data (u_0, v_0) estimates of the solutions $(u(t), v(t))$ of (3.28) in $L^\infty(\Omega, \mathbb{R}^m)$:

$$\sup_t(\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)}) \le \max\left\{\|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)}, \sum_{k=1}^{m}(\xi_1^k + \xi_2^k)\right\}, \tag{3.31}$$

and

$$\limsup_{t\to\infty}(\|u(t)\|_{L^\infty(\Omega,\mathbb{R}^m)} + \|v(t)\|_{L^\infty(\Omega,\mathbb{R}^m)}) \le \sum_{k=1}^{m}(\xi_1^k + \xi_2^k). \tag{3.32}$$

Therefore we can assume, without loss of generality, that f_1 and f_2 are globally Lipschitz and globally bounded functions.

If we introduce the linear change of variables:

$$w = \frac{u-v}{\sqrt{2}},\ z = \frac{u+v}{\sqrt{2}}, \tag{3.33}$$

then the problem (3.28) will be rewritten in the form

$$\begin{aligned} w_t &= Lw - 2kw + \tfrac{1}{\sqrt{2}}f_1(\tfrac{w+z}{\sqrt{2}}) - \tfrac{1}{\sqrt{2}}f_2(\tfrac{z-w}{\sqrt{2}}), \\ z_t &= Lz + \tfrac{1}{\sqrt{2}}f_1(\tfrac{w+z}{\sqrt{2}}) + \tfrac{1}{\sqrt{2}}f_2(\tfrac{z-w}{\sqrt{2}}), \end{aligned} \tag{3.34}$$

suitable to apply Theorem 2. This system has a global attractor $\tilde{A}_k$ and we have:

Theorem 3.8 *Under the assumption (3.30) the family $\{\tilde{A}_k, k \le \infty\}$ of global attractors corresponding to the semigroups $T_k(t)$ is upper semicontinuous at infinity, where $\tilde{A}_\infty = \{(0,z) : z \in \mathcal{A}\}$, and $\mathcal{A}$ denotes the global attractor of the limiting problem*

$$z_t = Lz + \frac{1}{\sqrt{2}}f_1(\frac{z}{\sqrt{2}}) + \frac{1}{\sqrt{2}}f_2(\frac{z}{\sqrt{2}}), \tag{3.35}$$

with homogeneous Dirichlet boundary condition.

Remark 1. We need to point out here, that the linear part of the right hand side of the system (3.34) is quasi-monotone increasing in functional arguments (cf. [26]). Recall, that the function $F(x,z) = (F_1,(x,z),\cdots,F_\nu(x,z))$ defined in $D \subset \mathbb{R}^k \times \mathbb{R}^\nu$ is quasi-monotone increasing in z if, for every $z = (z_1,\cdots,z_\nu)$, $\bar{z} = (\bar{z}_1,\cdots,\bar{z}_\nu)$,

$$F_i(x,z) \le F_i(x,\bar{z}),\ z_i = \bar{z}_i,,\ z_j \le \bar{z}_j,\ j = 1,\cdots,\nu. \tag{3.36}$$

This property allows for successful using of the a priori estimates technique for systems of ordinary and partial differential equations (see [26]).

Remark 2. The associated with (3.34) system of ordinary differential equations

$$\begin{aligned} \dot{y}(t) &= -k(y-z), \\ \dot{z}(t) &= -k(z-y), \end{aligned} \tag{3.37}$$

$y = (y_1,\cdots,y_m)$, $z = (z_1,\cdots,z_m)$ is a special example of the so-called Chapman-Kolmogorov systems (cf. [13]). Asymptotic behavior of solutions of such systems when time goes to infinity was studied in details in the theory of stochastic processes.

Changing back the variables to u and v we have the following result:

Theorem 3.9 *The family of attractors $\{\mathcal{A}_k, k \le \infty\}$ is upper semicontinuous at infinity where $\mathcal{A}_\infty = \{(u,u) : u \in \mathcal{A}\}$ and $\mathcal{A}$ is the global attractor of the problem*

$$u_t = \Delta u + \tfrac{1}{2}f_1(u) + \tfrac{1}{2}f_2(u). \tag{3.38}$$

This theorem and the definition of upper semicontinuity of attractors immediately imply the following result.

Corollary 3.5 *Given $\epsilon > 0$ there is a $k_0 > 0$ such that, for $k \geq k_0$ and for any $(u_0, v_0) \in X^\alpha \times X^\alpha$ the solution $(u(t), v(t))$ of (3.13), with $(u(0), v(0)) = (u_0, v_0)$, satisfy*

$$\limsup_{t\to\infty} \|u(t) - v(t)\|_{X^\alpha} \leq \epsilon.$$

3.2.3 Coupled parabolic equations on a 1-d lattice

As a final example consider a system in which each single equation (except the first and the last one) is linearly-coupled with its two neighbors

$$\begin{aligned} u_{1t} &= \Delta u_1 + k(u_2 - u_1) + f_1(u_1), \\ u_{it} &= \Delta u_i + k(u_{i+1} - u_i) - k(u_i - u_{i-1}) + f_i(u_i), \ 2 \leq i \leq m, \\ u_{mt} &= \Delta u_m - k(u_m - u_m) + f_m(u_m), \end{aligned} \tag{3.39}$$

together with homogeneous Neumann boundary conditions. The corresponding linear system is again of the Chapman-Kolmogorov type with the right hand side quasi-monotone increasing. Under the dissipativeness condition:

$$f_i(u_i)u_i < 0, \quad \text{for } |u_i| > \xi, \ 1 \leq i \leq m,$$

the above system was studied in [7] with the use of the *method of invariant regions*. It was proved that (3.39) has a global attractor $\mathcal{A}_k$ on X^α, $\alpha > \frac{n}{4}$. It is also proved that the attractor $\mathcal{A}_k$, $k \geq 0$, satisfy

$$u(x) \in [-\xi, \xi]^m, \ \forall x \in \Omega \ \forall u \in \mathcal{A}_k,$$

This allow us to assume, without loss of generality, that the functions $f_i : \mathbb{R} \to \mathbb{R}$, $1 \leq i \leq m$, are globally Lipschitz and globally bounded.

To transform (3.39) into a form suitable to use Theorem 2 we need to change the variables. Let $-kK$ be a matrix of coefficients of the linear part of (3.39):

$$-K = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}_{m\times m} \tag{3.40}$$

and let $\xi_1 = 0 < \xi_2 \leq \cdots \leq \xi_m$ be the sequence of eigenvalues of $-K$ and $(z_1, \cdots, z_m)$ be a corresponding sequence of eigenvectors normalized in $\mathbb{R}^m$ with respect to the Euclidean norm. Setting $Z = (z_1, \cdots, z_m)$ we introduce the new unknown by the formula

$$v := Z^{-1}u, \tag{3.41}$$

so that the system (3.39) is transformed into

$$v_t = \Delta v - kZ^{-1}KZv + Z^{-1}f(Zv), \tag{3.42}$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$ is defined by $f(u) = (f_1(u_1), \cdots, f_m(v_m))^\top$ or componentwise,

$$(v_j)_t = \Delta v_j - k\xi_j v_j + \sum_{q=1}^{m} f_q \left(\sum_{i=1}^{m} z_i^q v_i \right) z_j^q,$$

with $\xi_1 = 0$ and $\xi_i > 0$ for $j = 2, \cdots, m$. This problem has a global attractor $\tilde{A}_k$ and can now be rewritten in the abstract form

$$\begin{aligned} \dot{x} &= A_k x + f^e(x, y), \\ \dot{y} &= Ay + g^e(x, y), \end{aligned} \tag{3.43}$$

where A is the Laplacian with homogeneous Neumann boundary condition, $A_k := \mathrm{diag}(A - k\xi_2, \cdots, A - k\xi_m)$, $f^e : [X^\alpha]^{m-1} \times X^\alpha \to [X]^m$ and $g^e : [X^\alpha]^{m-1} \times X^\alpha \to X$ are given by

$$f^e = \begin{pmatrix} f_2^e \\ \vdots \\ f_m^e \end{pmatrix},$$

$$f_j^e(\phi_1, \cdots, \phi_m)(x) = \sum_{q=1}^{m} f_q \left(\sum_{i=1}^{m} z_i^q \phi_i(x) \right) z_j^q, \ 2 \le j \le m,$$

and

$$g^e(\phi_1, \cdots, \phi_m)(x) = \sum_{q=1}^{m} f_q \left(\sum_{i=1}^{m} z_i^q \phi_i(x) \right) z_1^q.$$

Theorem 3.10 *The family of attractors $\{\tilde{\mathcal{A}}_k, k \le \infty\}$ is upper semicontinuous at infinity, where $\tilde{\mathcal{A}}_\infty = \{(v, 0, \cdots, 0) \in [X^\alpha]^{m-1} : v \in \mathcal{A}\}$ and $\mathcal{A}$ is the attractor for the problem*

$$v_t = \Delta v + \sum_{q=1}^{m} f_q \left(z_1^q v \right) z_1^q,$$

with homogeneous Neumann boundary condition.

The proof of this result follows from Theorem 2 and it is easily seen that f^e, g^e and P satisfy the required hypotheses.

Changing back the variables v_i to u_i we have the following result:

Theorem 3.11 *The family of attractors $\{\mathcal{A}_k, k \le \infty\}$ is upper semicontinuous at infinity where $\mathcal{A}_\infty = \{(u, \cdots, u) : u \in \mathcal{A}\}$ and $\mathcal{A}$ is the global attractor of the problem*

$$u_t = \Delta u + \sum_{q=1}^{m} f_q (u) z_1^q, \tag{3.44}$$

with homogeneous Neumann boundary condition.

This theorem and the definition of upper semicontinuity of attractors immediately imply the following result.

Corollary 3.6 *Given $\epsilon > 0$ there is a $k_0 > 0$ such that, for $k \ge k_0$ and for any $(u_{10}, \cdots, u_{m0}) \in [X^\alpha]^m$ the solution $(u_1(t), \cdots, u_m(t))$ of (3.39), with $(u_1(0), \cdots, u_m(0)) = (u_{10}, \cdots, u_{m0})$, satisfy*

$$\limsup_{t \to \infty} \sum_{q=2}^{m} \|u_{q-1}(t) - u_q(t)\|_{X^\alpha} \le \epsilon.$$

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