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A countably compact free Abelian group of size continuum that admits a non-trivial convergent sequence ☆

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ABSTRACT

We show that it is consistent with *ZFC* that the free Abelian group of cardinality $\mathfrak{c}$ admits a topological group topology that makes it countably compact with a non-trivial convergent sequence.

1. Introduction

Recall that a topological space X is *countably compact* if every infinite subset of X has an accumulation point. Fuchs [6] (see also [10] and [9]) showed that free Abelian groups do not admit a compact group topology. In 1990, Tkachenko [12] constructed, under the Continuum Hypothesis, a countably compact group topology on the free Abelian group of cardinality $\mathfrak{c}$. His construction modifies the construction of Hájnal and Juhász [8] of a hereditarily finally dense (*HFD*) subgroup of $\{0,1\}^{\mathfrak{c}}$ with the property that all small projections are full. As a consequence, the resulting group does not have non-trivial convergent sequences. In 2007, Madariaga-Garcia and Tomita [11] showed that the free Abelian group of cardinality $\mathfrak{c}$ admits a countably compact group topology from the existence of $\mathfrak{c}$ incomparable selective ultrafilters, but this group also does not have any non-trivial convergent sequence. It is still unknown whether or not there exists such a topological group in *ZFC*.

The improvement of the technique of HFD's to other Abelian groups led to the characterization of the small Abelian groups that admit a countably compact group topology under Martin's Axiom (see [5]) or using forcing (see [3]). Thus, under some additional hypothesis, small Abelian groups that admit a countably compact group topology also admit a countably compact group topology without non-trivial convergent sequences. The following question is due to Dikranjan and Shakhmatov ([3, Questions 14.16 and 14.17] and [4, Question 24]):

Question 1.1. Let G be an infinite group admitting a countably compact (respectively, a pseudocompact) group topology. Does G have a countably compact (respectively, pseudocompact) group topology that contains a non-trivial convergent sequence?

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For pseudocompact Abelian groups, this question was solved positively by Galindo, Garcia-Ferreira and Tomita [7] who also noted that it is easy to show in *ZFC* that a torsion Abelian group that admits a countably compact group topology also admits a countably compact group topology with a non-trivial convergent sequence. Recently, it was shown by Tkachenko [13] that every infinite Abelian group satisfying $|G|^\omega = |G|$ and $|G| = r_0(G)$[1] admits a Hausdorff topological group topology making it a pseudocompact Fréchet–Urysohn[2] group, but a similar result cannot be obtained for countable compactness and free Abelian groups. Indeed, Tomita [14, Theorem 7] showed that a countably compact free Abelian group cannot be sequential.[3]

In this article, we show, assuming $\mathfrak{p} = \mathfrak{c}$, that the free Abelian group of cardinality $\mathfrak{c}$ admits a countably compact group topology with a non-trivial convergent sequence. This gives a partial answer to Question 1.1. This result can be also achieved by using selective ultrafilters as we shall describe at the end of the paper.

2. The topology on $\mathbb{Z}^{(\mathfrak{c})}$

We start this section with some basic notations and useful notions.

Let Λ be a set of ordinals. Given $g \in \mathbb{Z}^\Lambda$, the *support* of g is the set $\operatorname{supp} g = \{\mu \in \Lambda\colon\ g(\mu) \neq 0\}$. Thus, the direct sum $\mathbb{Z}^{(\Lambda)}$ is the set of all functions $g : \Lambda \to \mathbb{Z}$ with finite support. Given $g \in \mathbb{Z}^{(\mathfrak{c})}$, define $\|g\| = \sum |g(\mu)|$. We denote by $\mathbb{T}$ the unitary circle group identified with the metric group $(\mathbb{R}/\mathbb{Z}, +, \delta)$, where $\delta(x + \mathbb{Z}, y + \mathbb{Z}) = \min\{|x - y + a|\colon\ a \in \mathbb{Z}\}$.[4] In this context, an *open arc* of $\mathbb{T}$ with center x and diameter r is the set $\{y \in \mathbb{T}\colon\ \delta(x, y) < r\}$. For a subset A of $\mathbb{T}$, $\delta(A)$ will denote the diameter of A according to the metric δ. In what follows, $\mathcal{B}$ will denote the set of all non-empty open arcs of $\mathbb{T}$.

If $(x_n)_{n\in\omega}$ is a sequence in a topological space X, $x \in X$ and $A \in [\omega]^\omega$, then we write $x_n \to_{n\in A} x$ provided every open neighborhood of x contains all but finitely many elements of the set $\{x_n\colon\ n \in A\}$.

A *pseudointersection* of a family $\mathcal{G}$ of infinite sets is an infinite set X such that $X \subseteq^* G$ for every $G \in \mathcal{G}$. We say that a family $\mathcal{G}$ of infinite sets has the *strong finite intersection property* (SFIP, for short) if every finite subfamily of $\mathcal{G}$ has infinite intersection. The *pseudointersection number* $\mathfrak{p}$ is the smallest cardinality of $\mathcal{G} \in [\omega]^\omega$ with SFIP but with no pseudointersection.

It is known that if $\{\phi_i : G \to (H, \tau_i)\colon\ i \in I\}$ is a family of homomorphisms from a group G to a topological group (H, τ_i), then the initial[5] topology τ on G is a topological group topology (see, for instance, Proposition 3.1 of [2]). Following this idea, we shall define for each $g \in \mathbb{Z}^{(\mathfrak{c})} \setminus \{0\}$, a suitable homomorphism $\phi_g : \mathbb{Z}^{(\mathfrak{c})} \to \mathbb{T}$ and equip $\mathbb{Z}^{(\mathfrak{c})}$ with the initial topology induced by the family $\{\phi_g\colon\ g \in \mathbb{Z}^{(\mathfrak{c})} \setminus \{0\}\}$. By the properties of homomorphisms ϕ_g, this topology will be countably compact and will contain a non-trivial convergent sequence.

For each ordinal $\xi < \mathfrak{c}$, the function $x_\xi : \mathfrak{c} \to \mathbb{Z}$ is defined by $x_\xi(\xi) = 1$ and $x_\xi(\mu) = 0$, for each $\mu \in \mathfrak{c} \setminus \{\xi\}$. If $\Lambda \subset \mathfrak{c}$ is a set of ordinals, then it is clear that $\{x_\xi\colon\ \xi \in \Lambda\}$ is an independent set that generates $\mathbb{Z}^{(\Lambda)}$. By using these generators we can define a homomorphism $\phi : \mathbb{Z}^{(\Lambda)} \to \mathbb{T}$ by the formula

$$\phi(g) = \phi\Big(\sum g(\mu) \cdot x_\mu\Big) = \sum g(\mu) \cdot \phi(x_\mu),$$

for each $g \in \mathbb{Z}^{(\Lambda)}$. So, in order to construct a certain homomorphism $\phi : \mathbb{Z}^{(\Lambda)} \to \mathbb{T}$, it is enough to define $\phi(x_\xi)$ for each $\xi \in \Lambda$.

Consider

$$\mathcal{F}_1 = \big\{ f \in \big(\mathbb{Z}^{(\mathfrak{c})}\big)^\omega\colon\ \forall n \in \omega\ \big(\big\| f(n)\big\| > n\big)\big\}$$

and

$$\mathcal{F}_2 = \left\{ f \in \big(\mathbb{Z}^{(\mathfrak{c})}\big)^\omega\colon\ \forall n \in \omega \left[\operatorname{supp} f(n) \cap \omega = \emptyset \wedge \operatorname{supp} f(n) \setminus \bigcup_{m<n} \operatorname{supp} f(m) \neq \emptyset\right]\right\}.$$

Put $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ and enumerate $\mathcal{F}$ as $\{f_\xi\colon\ \omega \leqslant \xi < \mathfrak{c}\}$ so that $\bigcup_{n\in\omega} \operatorname{supp} f_\xi(n) \subset \xi$, for each infinite $\xi < \mathfrak{c}$.

Lemma 2.1. *Let $g \in \mathbb{Z}^{(\mathfrak{c})} \setminus \{0\}$ and $E \in [\mathfrak{c}]^\omega$ be such that $\operatorname{supp} g \cup \omega \subset E$ and $\bigcup_{n\in\omega} \operatorname{supp} f_\xi(n) \subset E$, for all $\xi \in E \setminus \omega$. For each $\xi \in E \setminus \omega$, fix $R_\xi \in [\omega]^\omega$. Let $\{\theta_n\colon\ n \in \omega\}$ be an enumeration of $E \setminus \omega$ so that $|\{n \in \omega\colon\ \theta = \theta_n\}| = \omega$, for every $\theta \in E \setminus \omega$. Then, for each $m \in \omega$, there exist a function $\psi_m : E \to \mathcal{B}$, a finite set $G_m \subset E$, $b_{m-1} \in R_{\theta_{m-1}}$ (if $m > 0$) and $r_m > 0$ such that*:

(1) $0 \notin \sum g(\xi) \cdot \psi_0(\xi)$;
(2) $b_m > b_{m-1}$, *if* $m > 0$;

[1] For any Abelian group G, $r_0(G)$ denotes the torsion-free rank of G.
[2] A space X is called *Fréchet–Urysohn* if for every $x \in \operatorname{cl}_X(A)$, there exists a sequence $(a_n)_{n\in\omega}$ in A such that $a_n \to x$.
[3] A space X is called *sequential* if whenever $A \subset X$ is not closed, there exists a sequence $(a_n)_{n\in\omega}$ in A converging to a point outside of A.
[4] In what follows, an element $x + \mathbb{Z}$ of $\mathbb{T}$ will be denoted simply by x.
[5] That is, the coarsest topology on G that makes each homomorphism continuous.

(3) $G_m = G_{m-1} \cup \{\theta_m\} \cup \operatorname{supp} f_{\theta_{m-1}}(b_{m-1})$, *if* $m > 0$;
(4) $r_m = \frac{r_{m-1}}{2 \cdot \| f_{\theta_{m-1}}(b_{m-1}) \|}$, *if* $m > 0$;
(5) $\overline{\psi_m(\xi)} \subset \psi_{m-1}(\xi)$, *for each* $\xi \in E$ *and* $m > 0$;
(6) *If* $\xi \in G_m$, *then* $\delta(\psi_m(\xi)) = r_m$ *and if* $\xi \in E \setminus G_m$, *then* $\psi_m(\xi) = \mathbb{T}$;
(7) $\psi_{m-1}(\theta_{m-1}) \cap (\sum f_{\theta_{m-1}}(b_{m-1})(\mu) \cdot \psi_m(\mu)) \neq \emptyset$, *if* $m > 0$;
(8) *If* $m > 0$ *and* $\xi \in (G_m \setminus G_{m-1}) \cap \omega$, *then* $\delta(x, 0) < r_{m-1}$, *for each* $x \in \psi_m(\xi)$.

Proof. Set $b_{-1} = 0$, $r_0 = \frac{1}{4 \cdot \|g\|}$ and $G_0 = \operatorname{supp} g \cup \{\theta_0\}$. For each $\xi \in G_0$, choose $y_\xi \in \mathbb{R}$ such that

$$\sum g(\xi) \cdot y_\xi = \frac{1}{2}$$

and define $\psi_0(\xi)$ as the open arc of $\mathbb{T}$ centered at y_ξ with diameter r_0. If $\xi \in E \setminus G_0$, then let $\psi_0(\xi) = \mathbb{T}$. Then (6) holds and

$$\delta\Big(\sum g(\xi) \cdot \psi_0(\xi)\Big) \leqslant \|g\| \cdot r_0 = \frac{1}{4}.$$

Since $\frac{1}{2} \in \sum g(\xi) \cdot \psi_0(\xi)$, (1) must hold. Now, we start the successor stage. Fix $m \in \omega$ and suppose we have already defined $\psi_m : E \to \mathcal{B}$, $G_m \in [E]^{<\omega}$, $b_{m-1} \in R_{\theta_{m-1}}$ (if $m > 0$) and $r_m > 0$. Define G_{m+1} and r_{m+1} according to (3) and (4), respectively.

If $\xi \in (G_{m+1} \setminus G_m) \cap \omega$, then let $\tilde{\psi}_m(\xi)$ be the open arc of $\mathbb{T}$ centered at 0 with diameter r_m; if $\xi \in (G_{m+1} \setminus G_m) \setminus \omega$, then let $\tilde{\psi}_m(\xi)$ be any open arc of $\mathbb{T}$ with diameter r_m; finally, if $\xi \in G_m$, then let $\tilde{\psi}_m(\xi) = \psi_m(\xi)$.

In order to define $b_m \in R_{\theta_m}$ and ψ_{m+1}, we shall consider two cases:

Case I. $f_{\theta_m} \in \mathcal{F}_1$.

Choose $a_m \in \omega$ so that $\| f_{\theta_m}(n) \| \cdot r_m > 2$, for each $n > a_m$. Then fix $b_m \in R_{\theta_m}$ such that $b_m > a_m$ and $b_m > b_{m-1}$ (so, condition (2) holds). Consider the function $\tilde{\tilde{\psi}}_m : \operatorname{supp} f_{\theta_m}(b_m) \to \mathcal{B}$ which assigns to each point $\mu \in \operatorname{supp} f_{\theta_m}(b_m)$ the open arc $\tilde{\tilde{\psi}}_m(\mu)$ of $\mathbb{T}$ centered at the middle point of $\tilde{\psi}_m(\mu)$ with diameter $r_m/4$. For each $\xi \in E \setminus G_{m+1}$, define $\psi_{m+1}(\xi) = \mathbb{T}$. If $\xi \in G_{m+1} \setminus \operatorname{supp} f_{\theta_m}(b_m)$, then let $\psi_{m+1}(\xi)$ be the open arc of $\mathbb{T}$ centered at the middle point of $\tilde{\psi}_m(\xi)$ with diameter r_{m+1}. Next, we shall define $\psi_{m+1}(\xi)$ for each $\xi \in \operatorname{supp} f_{\theta_m}(b_m)$. Since $\| f_{\theta_m}(b_m) \| \cdot r_m > 2$, we must have that

$$\sum f_{\theta_m}(b_m)(\mu) \cdot \tilde{\tilde{\psi}}_m(\mu) = \mathbb{T}.$$

Hence, for each $\mu \in \operatorname{supp} f_{\theta_m}(b_m)$, there exists $x_\mu^m \in \tilde{\tilde{\psi}}_m(\mu)$ such that

$$\sum f_{\theta_m}(b_m)(\mu) \cdot x_\mu^m \in \psi_m(\theta_m).$$

We define $\psi_{m+1}(\mu)$ as the open arc of $\mathbb{T}$ centered at x_μ^m with diameter r_{m+1}. Thus, conditions (5)–(7) are verified. If $\xi \in (G_{m+1} \setminus G_m) \cap \omega$, then $\tilde{\tilde{\psi}}_m(\xi)$ is the open arc of $\mathbb{T}$ centered at 0 with diameter $r_m/4$. Since $x_\xi^m \in \tilde{\tilde{\psi}}_m(\xi)$ and $r_{m+1} \leqslant r_m/2$, it follows that $\delta(x, 0) < r_m$, for every $x \in \psi_{m+1}(\xi)$. So, condition (8) holds.

Case II. $f_{\theta_m} \in \mathcal{F}_2$.

Choose $a_m \in \omega$ so that $\operatorname{supp} f_{\theta_m}(n) \setminus G_m \neq \emptyset$, for each $n > a_m$. Then fix $b_m \in R_{\theta_m}$ such that $b_m > a_m$ and $b_m > b_{m-1}$ (so, condition (2) holds). For each $\xi \in E \setminus G_{m+1}$, define $\psi_{m+1}(\xi) = \mathbb{T}$. If $\xi \in G_{m+1} \setminus \operatorname{supp} f_{\theta_m}(b_m)$, then let $\psi_{m+1}(\xi)$ be the open arc of $\mathbb{T}$ centered at the middle point of $\tilde{\psi}_m(\xi)$ with diameter r_{m+1}. Since $f_{\theta_m} \in \mathcal{F}_2$, then $(G_{m+1} \setminus G_m) \cap \omega$ is contained in $G_{m+1} \setminus \operatorname{supp} f_{\theta_m}(b_m)$. So, condition (8) is verified. Finally, we shall define $\psi_{m+1}(\xi)$ for $\xi \in \operatorname{supp} f_{\theta_m}(b_m)$. Fix $\alpha \in \operatorname{supp} f_{\theta_m}(b_m) \setminus G_m$. For each $\xi \in \operatorname{supp} f_{\theta_m}(b_m) \setminus \{\alpha\}$, denote by z_ξ the middle point of $\tilde{\psi}_m(\xi)$. Since $\psi_m(\alpha) = \mathbb{T}$, we can find $z_\alpha \in \psi_m(\alpha)$ for which

$$\sum_{\mu \in \operatorname{supp} f_{\theta_m}(b_m) \setminus \{\alpha\}} f_{\theta_m}(b_m)(\mu) \cdot z_\mu + f_{\theta_m}(b_m)(\alpha) \cdot z_\alpha \in \psi_m(\theta_m).$$

Hence, for each $\xi \in \operatorname{supp} f_{\theta_m}(b_m)$, we define $\psi_{m+1}(\xi)$ as the open arc of $\mathbb{T}$ centered at z_ξ with diameter r_{m+1}. Thus, conditions (5)–(7) are verified. $\square$

Lemma 2.2. *Let* $g \in \mathbb{Z}^{(\mathfrak{c})} \setminus \{0\}$ *and* $E \in [\mathfrak{c}]^\omega$ *be such that* $\operatorname{supp} g \cup \omega \subset E$ *and* $\bigcup_{n \in \omega} \operatorname{supp} f_\xi(n) \subset E$, *for all* $\xi \in E \setminus \omega$. *For each* $\xi \in E \setminus \omega$, *fix* $R_\xi \in [\omega]^\omega$. *Then there exists a homomorphism* $\phi \colon \mathbb{Z}^{(E)} \to \mathbb{T}$ *such that*:

(1) $\phi(g) \neq 0$;
(2) *for each* $\xi \in E \setminus \omega$, *there exists* $S_\xi \in [R_\xi]^\omega$ *such that* $\phi(f_\xi(n)) \to_{n \in S_\xi} \phi(x_\xi)$; *and*
(3) $\phi(x_n) \to 0$.

Proof. Let $\{\psi_m\colon m \in \omega\}$, $\{G_m\colon m \in \omega\}$, $\{b_m\colon m \in \omega\}$ and $\{r_m\colon m \in \omega\}$ be as in Lemma 2.1. Since $\mathbb{T}$ is a compact metric space and the sequence $(r_n)_{n\in\omega}$ converges to 0, it follows that the set $\bigcap_{n\in\omega} \psi_n(\xi) = \bigcap_{n\in\omega} \overline{\psi_n(\xi)}$ is a singleton, for every $\xi \in E$. For each $\xi \in E$, let $\phi(x_\xi)$ be the unique element of the intersection $\bigcap_{n\in\omega} \psi_n(\xi)$. Since $\{x_\xi\colon \xi \in E\}$ is an independent set that generates the group $\mathbb{Z}^{(E)}$, we extend ϕ to a homomorphism from $\mathbb{Z}^{(E)}$ into $\mathbb{T}$ by defining

$$\phi(h) = \sum h(\mu) \cdot \phi(x_\mu)$$

for each $h \in \mathbb{Z}^{(E)}$. By construction, we have that $\phi(h) \in \sum h(\mu) \cdot \psi_0(\mu)$, for each $h \in \mathbb{Z}^{(E)}$. Also, by the first condition of Lemma 2.1, we have that $\phi(g) \neq 0$.

For each $\xi \in E \setminus \omega$, define $I_\xi = \{m \in \omega\colon \xi = \theta_m\}$ and $S_\xi = \{b_m\colon m \in I_\xi\}$. It is evident that $S_\xi \in [R_\xi]^\omega$. Furthermore, if $m, n \in I_\xi$ and $m < n$, then $b_m < b_n$. We claim that $\phi(f_\xi(b_m)) \to_{m \in I_\xi} \phi(x_\xi)$. Indeed, we know that

$$\phi\big(f_{\theta_m}(b_m)\big) = \sum f_{\theta_m}(b_m)(\mu) \cdot \phi(x_\mu) \in \sum f_{\theta_m}(b_m)(\mu) \cdot \psi_{m+1}(\mu)$$

and $\phi(x_{\theta_m}) \in \psi_m(\theta_m)$. On the other hand, we have that

$$\begin{aligned}\delta\big(\phi\big(f_{\theta_m}(b_m)\big), \phi(x_{\theta_m})\big) &\leqslant \delta\Big(\sum f_{\theta_m}(b_m)(\mu) \cdot \psi_{m+1}(\mu)\Big) + \delta\big(\psi_m(\theta_m)\big)\\ &\leqslant \big\| f_{\theta_m}(b_m)\big\| \cdot 2 \cdot r_{m+1} + r_m < 2 \cdot r_m.\end{aligned}$$

Since $r_m \to 0$, it follows that $\phi(f_\xi(b_m)) \to_{m \in I_\xi} \phi(x_\xi)$. In other words, $\phi(f_\xi(n)) \to_{n \in S_\xi} \phi(x_\xi)$.

It remains to show that the sequence $(\phi(x_n))_{n\in\omega}$ converges to 0. In fact, this follows directly from condition (8) of Lemma 2.1 which guarantees that the set $\{n \in \omega\colon \delta(\phi(x_n), 0) \geqslant r_m\}$ is finite for all $m \in \omega$. □

Our next aim is to extend the homomorphism obtained in Lemma 2.2 to $\mathbb{Z}^{(\mathfrak{c})}$ in such a way that condition (2) holds for every element of $\mathcal{F}$.

Lemma 2.3. *Let* g *be an element of* $\mathbb{Z}^{(\mathfrak{c})} \setminus \{0\}$ *and* $\{R_\xi\colon \xi \in \mathfrak{c} \setminus \omega\} \subset [\omega]^\omega$. *Then there exists a homomorphism* $\phi : \mathbb{Z}^{(\mathfrak{c})} \to \mathbb{T}$ *such that*:

(1) $\phi(g) \neq 0$;
(2) *for each* $\xi \in \mathfrak{c} \setminus \omega$, *there exists* $S_\xi \in [R_\xi]^\omega$ *such that* $\phi(f_\xi(n)) \to_{n \in S_\xi} \phi(x_\xi)$; *and*
(3) $\phi(x_n) \to 0$.

Proof. Given $g \in \mathbb{Z}^{(\mathfrak{c})} \setminus \{0\}$, fix $E \in [\mathfrak{c}]^\omega$ such that $\operatorname{supp} g \cup \omega \subset E$ and $\bigcup_{n\in\omega} \operatorname{supp} f_\xi(n) \subset E$, whenever $\xi \in E \setminus \omega$.[6] Let $\phi : \mathbb{Z}^{(E)} \to \mathbb{T}$ be a homomorphism satisfying the conclusion of Lemma 2.2.

Let $\{\alpha_\xi\colon \xi < \mathfrak{c}\}$ be a strictly increasing enumeration of $\mathfrak{c} \setminus E$. Since $\mathbb{T}$ is sequentially compact, there exists $S_{\alpha_0} \in [R_{\alpha_0}]^\omega$ so that the sequence $\phi(f_{\alpha_0}(n))_{n \in S_{\alpha_0}}$ is convergent. Define $\phi(x_{\alpha_0})$ as the limit point of this sequence and extend ϕ to a homomorphism from $\mathbb{Z}^{(E\cup\{\alpha_0\})}$ into $\mathbb{T}$.

Let $\xi < \mathfrak{c}$ be an ordinal and suppose that ϕ was extended to a homomorphism (which we will also denote by ϕ) from $\mathbb{Z}^{(E\cup\{\alpha_\mu\colon \mu<\xi\})}$ into $\mathbb{T}$. Since $\alpha_\xi = \min \mathfrak{c} \setminus (E \cup \{\alpha_\mu\colon \mu < \xi\})$ and $\bigcup_{n\in\omega} \operatorname{supp} f_{\alpha_\xi}(n) \subset \alpha_\xi$, then $\phi(f_{\alpha_\xi}(n))$ is defined for every $n \in \omega$. From the sequential compactness of $\mathbb{T}$, it follows that there exists $S_{\alpha_\xi} \in [R_{\alpha_\xi}]^\omega$ such that the sequence $\phi(f_{\alpha_\xi}(n))_{n \in S_{\alpha_\xi}}$ is convergent. Define $\phi(x_{\alpha_\xi})$ as the limit point of this sequence and extend ϕ to a homomorphism from $\mathbb{Z}^{(E\cup\{\alpha_\mu\colon \mu<\xi+1\})}$ into $\mathbb{T}$. By induction, we extend ϕ to a homomorphism from $\mathbb{Z}^{(\mathfrak{c})}$ into $\mathbb{T}$ satisfying (1)–(3). □

Lemma 2.4. $[\mathfrak{p} = \mathfrak{c}]$ *Let* $\{g_\alpha\colon \alpha < \mathfrak{c}\}$ *be an enumeration of* $\mathbb{Z}^{(\mathfrak{c})} \setminus \{0\}$. *Then, for each* $\alpha < \mathfrak{c}$, *there exist a family* $\{S_{\xi,\alpha}\colon \xi \in \mathfrak{c} \setminus \omega\}$ *of infinite subsets of* ω *and a homomorphism* $\phi_\alpha : \mathbb{Z}^{(\mathfrak{c})} \to \mathbb{T}$ *such that*:

(1) $S_{\xi,\beta} \subset^* S_{\xi,\alpha}$ *whenever* $\alpha < \beta < \mathfrak{c}$ *and* $\xi \in \mathfrak{c} \setminus \omega$;
(2) $\phi_\alpha(g_\alpha) \neq 0$;
(3) $\phi_\alpha(f_\xi(n)) \to_{n \in S_{\xi,\alpha}} \phi_\alpha(x_\xi)$; *and*
(4) $\phi_\alpha(x_n) \to 0$.

Proof. For each $\xi \in \mathfrak{c} \setminus \omega$, set $R_{\xi,0} = \omega$. Applying Lemma 2.3 to $g = g_0$ and $R_\xi = R_{\xi,0}$, we obtain a homomorphism $\phi_0 : \mathbb{Z}^{(\mathfrak{c})} \to \mathbb{T}$ and $S_{\xi,0} \in [R_{\xi,0}]^\omega$ for each $\xi \in \mathfrak{c} \setminus \omega$, such that:

[6] The existence of such an E is guaranteed by Lemma 3.5 of [11].

(1) $\phi_0(g_0) \neq 0$;
(2) $\phi_0(f_\xi(n)) \to_{n \in S_{\xi,0}} \phi_0(x_\xi)$; and
(3) $\phi_0(x_n) \to 0$.

Fix $\beta < \mathfrak{c}$ and suppose that, for each $\alpha < \beta$, we have constructed a family $\{S_{\xi,\alpha}\colon \xi \in \mathfrak{c} \setminus \omega\}$ of infinite subsets of ω and a homomorphism $\phi_\alpha : \mathbb{Z}^{(\mathfrak{c})} \to \mathbb{T}$ satisfying conditions (1)–(4) of Lemma 2.4, with $\mathfrak{c}$ replaced by β.

If $\beta = \alpha + 1$, then we set $R_{\xi,\beta} = S_{\xi,\alpha}$ for each $\xi \in \mathfrak{c} \setminus \omega$ and apply Lemma 2.3 to $g = g_\beta$ and $R_\xi = R_{\xi,\beta}$. If β is a limit ordinal, then for each $\xi \in \mathfrak{c} \setminus \omega$, consider the family $\{S_{\xi,\alpha}\colon \alpha < \beta\}$. By inductive hypothesis, this family has the SFIP and, from $\mathfrak{p} = \mathfrak{c}$, such family has a pseudointersection $R_{\xi,\beta}$. Apply Lemma 2.3 to $g = g_\beta$ and $R_\xi = R_{\xi,\beta}$. $\square$

Theorem 2.5. $[\mathfrak{p} = \mathfrak{c}]$ *The free Abelian group of cardinality $\mathfrak{c}$ admits a topological group topology that makes it countably compact with a non-trivial convergent sequence.*

Proof. Fix an enumeration $\{g_\alpha\colon \alpha < \mathfrak{c}\}$ of $\mathbb{Z}^{(\mathfrak{c})} \setminus \{0\}$. For each $\alpha < \mathfrak{c}$, let $\{S_{\xi,\alpha}\colon \xi \in \mathfrak{c} \setminus \omega\}$ be a family of infinite subsets of ω and let $\phi_\alpha : \mathbb{Z}^{(\mathfrak{c})} \to \mathbb{T}$ be a homomorphism satisfying conditions (1)–(4) of Lemma 2.4. We equip $\mathbb{Z}^{(\mathfrak{c})}$ with the initial topology τ induced by the family of homomorphisms $\{\phi_\alpha\colon \alpha < \mathfrak{c}\}$. We know that $(\mathbb{Z}^{(\mathfrak{c})}, \tau)$ is a topological group and that $\{\bigcap_{\alpha \in F} \phi_\alpha^{-1}(V)\colon F \in [\mathfrak{c}]^{<\omega} \wedge V$ is a neighborhood of $0\}$ is a local basis at 0 (see Proposition 3.1 of [2]). Since $\phi_\alpha(x_n) \to 0$ for each $\alpha < \mathfrak{c}$, we must have that $x_n \to 0$. Thus, $(\mathbb{Z}^{(\mathfrak{c})}, \tau)$ has a non-trivial convergent sequence. It is evident that the second condition of Lemma 2.4 guarantees that the topology τ is Hausdorff.

Claim. *For every $\xi \in \mathfrak{c} \setminus \omega$, x_ξ is an accumulation point of the sequence $(f_\xi(n))_{n \in \omega}$.*

Proof of the claim. Fix $\xi \in \mathfrak{c} \setminus \omega$. According to Lemma 2.3, we know that

$$\phi_\alpha\big(f_\xi(n)\big) \to_{n \in S_{\xi,\alpha}} \phi_\alpha(x_\xi)$$

for every $\alpha < \mathfrak{c}$. Let $V = \bigcap_{\alpha \in F} \phi_\alpha^{-1}(V_\alpha)$, where $F \in [\mathfrak{c}]^{<\omega}$ and V_α is a neighborhood of $\phi_\alpha(x_\xi)$, for each $\alpha \in F$. If $\gamma = \max F$, then $S_{\xi,\gamma} \subseteq^* S_{\xi,\alpha}$ for each $\alpha \in F$. Choose $m \in \omega$ so that $\phi_\alpha(f_\xi(n)) \in V_\alpha$ for each $n \in S_{\xi,\alpha}$ with $n \geqslant m$. In other words, $f_\xi(n) \in \bigcap_{\alpha \in F} \phi_\alpha^{-1}(V_\alpha) = V$ for each $n \in S_{\xi,\alpha}$ with $n \geqslant m$. Therefore, x_ξ is an accumulation point of the sequence $(f_\xi(n))_{n \in \omega}$. $\square$

Next we shall prove that $(\mathbb{Z}^{(\mathfrak{c})}, \tau)$ is countably compact. To do that consider an arbitrary function $f : \omega \to \mathbb{Z}^{(\mathfrak{c})}$. If $\{\|f(n)\|\colon n \in \omega\}$ is unbounded, then there exists a strictly increasing function $j : \omega \to \omega$ such that $f \circ j \in \mathcal{F}_1$. Thus, $f = f_\xi$, for some $\xi \in \mathfrak{c} \setminus \omega$. Then, by the claim, x_ξ is an accumulation point of the sequence $(f(n))_{n \in \omega}$. Assume that $\{\|f(n)\|\colon n \in \omega\}$ is bounded. Define $f_1, f_2 : \omega \to \mathbb{Z}^{(\mathfrak{c})}$ by

$$f_1(n) = \sum_{\xi \in \omega} f(n)(\xi) \cdot x_\xi \quad \text{and} \quad f_2(n) = \sum_{\xi \in \mathfrak{c} \setminus \omega} f(n)(\xi) \cdot x_\xi$$

for every $n \in \omega$. Observe that the set $\{\|f_1(n)\|\colon n \in \omega\}$ is also bounded. Then it is possible to find a strictly increasing function $j : \omega \to \omega$ such that the sequence $(\|f_1 \circ j(n)\|)_{n \in \omega}$ is constant and either $f_2 \circ j \in \mathcal{F}_2$ or $f_2 \circ j$ is constant. Notice that, in both cases, the sequence $(f_2(j(n)))_{n \in \omega}$ has an accumulation point and observe that the sequence $(f_1(j(n)))_{n \in \omega}$ converges to some χ_F where $F \subseteq \omega$ is finite. Suppose that $f_2 \circ j = f_\xi$ for some $\xi \in \mathfrak{c} \setminus \omega$. Then $\chi + \chi_F$ is an accumulation point of the sequence $(f_1(j(n)) + f_\xi(n))_{n \in \omega}$. As $f = f_1 + f_2$, we deduce that the sequence $(f(n))_{n \in \omega}$ also has an accumulation point in $\mathbb{Z}^{(\mathfrak{c})}$. Therefore, $\mathbb{Z}^{(\mathfrak{c})}$ is countably compact. $\square$

3. An example from selective ultrafilters

The set of all free ultrafilters over ω will be denoted by ω^*. Bernstein [1] defined the following concept, which is an important tool for the study of countable compactness.

Definition 3.1. Let $p \in \omega^*$ and $\{x_n\colon n \in \omega\}$ be a sequence in a topological space X. We say that $x \in X$ is a *p-limit point* of $\{x_n\colon n \in \omega\}$ if, for every neighborhood U of x, the set $\{n \in \omega\colon x_n \in U\}$ is an element of p. In this case, we write $x = p\text{-}\lim\{x_n\colon n \in \omega\}$.

It is not difficult to prove that a topological space X is countably compact iff each sequence in X has a p-limit point, for some $p \in \omega^*$.

Definition 3.2. We say that $p \in \omega^*$ is *selective* if, for each partition $\{A_n\colon n \in \omega\}$ of ω into non-empty sets, either $A_n \in p$, for some $n \in \omega$ or, for each $n \in \omega$, there exists $a_n \in A_n$ such that $\{a_n\colon n \in \omega\} \in p$.

Two selective ultrafilters p and q are said to be *incomparable* if there exists no bijection $f:\omega\to\omega$ such that $\beta f(p)=q$, where βf is the Stone–Čech extension of f.

It is possible to modify the previous example by using a construction via selective ultrafilters and some results and ideas from the papers [11] and [15]. The frame of the construction is as before: we choose adequate sequences in $\mathbb{Z}^{(\mathfrak{c})}$ to work with (namely, the ones in $\mathcal{F}$), construct $\mathfrak{c}$ many homomorphisms from $\mathbb{Z}^{(\mathfrak{c})}$ into $\mathbb{T}$ satisfying suitable conditions and consider the initial topology on $\mathbb{Z}^{(\mathfrak{c})}$ given by these homomorphisms.

The next lemma is proved in [11, Lemma 3.2] and it will be used to carry on the construction of the homomorphisms.

Lemma 3.3. *Let $g\in\mathbb{Z}^{(\mathfrak{c})}\setminus\{0\}$ and $E\in[\mathfrak{c}]^{\omega}$ be such that $\omega\subset E$, $\operatorname{supp} g\subset E$ and $\bigcup_{n\in\omega}\operatorname{supp} f_\xi(n)\subset E$, whenever $\xi\in E\setminus\omega$. Also, let $\{p_\xi:\ \xi\in E\setminus\omega\}$ be a family of incomparable selective ultrafilters. There exist a family $\{E_k:\ k\in\omega\}$ of finite subsets of E, a strictly increasing sequence $\{b_k:\ k\in\omega\}$ of natural numbers, a sequence $\{r_k:\ k\in\omega\}$ of positive real numbers and a function $i:\omega\to E\setminus\omega$ such that*:

(1) $\operatorname{supp} g\subset E_0$;
(2) $E=\bigcup_{k\in\omega}E_k$;
(3) $E_{k+1}\supset E_k\cup\bigcup\{\operatorname{supp} f_{i(m)}(b_m):\ m\leqslant k\}$, *for each* $k\in\omega$;
(4) $i(k)\in E_k$, *for each* $k\in\omega$;
(5) $\{b_k:\ k\in i^{-1}(\{\xi\})\}\in p_\xi$, *for each* $\xi\in E\setminus\omega$;
(6) *If* $f_{i(k)}\in\mathcal{F}_1$, *then* $\|f_{i(k)}(b_k)\|\cdot r_k>2$, *for each* $k\in\omega$;
(7) *If* $f_{i(k)}\in\mathcal{F}_2$, *then* $\operatorname{supp} f_{i(k)}(b_k)\setminus E_k\neq\emptyset$, *for each* $k\in\omega$;
(8) $r_0=\frac{1}{4\cdot\|g\|}$;
(9) $r_{k+1}=\frac{r_k}{2\cdot\|f_{i(k)}(b_k)\|}$, *for each* $k\in\omega$.

The careful choice of the sequence $\{b_k:\ k\in\omega\}$ allows us to obtain the following result, whose proof is quite similar to the proof of Lemma 2.2.

Lemma 3.4. *Let $g\in\mathbb{Z}^{(\mathfrak{c})}\setminus\{0\}$ and $E\in[\mathfrak{c}]^{\omega}$ be such that $\omega\subset E$, $\operatorname{supp} g\subset E$ and $\bigcup_{n\in\omega}\operatorname{supp} f_\xi(n)\subset E$, whenever $\xi\in E\setminus\omega$. Also, let $\{p_\xi:\ \xi\in E\setminus\omega\}$ be a family of incomparable selective ultrafilters. There exists a homomorphism $\phi_g:\mathbb{Z}^{(E)}\to\mathbb{T}$ such that*:

(i) $\phi_g(g)\neq 0$;
(ii) $\phi_g(x_\xi)=p_\xi\text{-}\lim\{\phi_g(f_\xi(n)):\ n\in\omega\}$, *for each* $\xi\in E\setminus\omega$;
(iii) $\phi_g(x_n)\to 0$.

Proof. Let $\{E_k:\ k\in\omega\}$, $\{b_k:\ k\in\omega\}$, $\{r_k:\ k\in\omega\}$ and $i:\omega\to E\setminus\omega$ be as in Lemma 3.3.

For each $\xi\in E_0$, let $y_\xi\in\mathbb{R}$ be such that

$$\sum g(\xi)\cdot y_\xi=\frac{1}{2}$$

and define $\psi_0(\xi)$ as the open arc centered at y_ξ with diameter r_0. It follows that

$$\delta\Bigl(\sum g(\xi)\cdot\psi_0(\xi)\Bigr)\leqslant\sum\bigl|g(\xi)\bigr|\cdot\delta\bigl(\psi_0(\xi)\bigr)=\|g\|\cdot r_0=\frac{1}{4}.$$

Since

$$\frac{1}{2}\in\sum g(\xi)\cdot\psi_0(\xi)$$

we must have that

$$0\notin\sum g(\xi)\cdot\psi_0(\xi).$$

Finally, if $\xi\in E\setminus E_0$, then let $\psi_0(\xi)=\mathbb{T}$.

Fix $m\in\omega$ and suppose that we have already defined $\psi_m:E\to\mathcal{B}$. We shall construct $\psi_{m+1}:E\to\mathcal{B}$ with the following properties:

(1) $\overline{\psi_{m+1}(\xi)}\subset\psi_m(\xi)$, for every $\xi\in E$;
(2) If $\xi\in E\setminus E_{m+1}$, then $\psi_{m+1}(\xi)=\mathbb{T}$; If $\xi\in E_{m+1}$, then $\delta(\psi_{m+1}(\xi))=r_{m+1}$;
(3) $\psi_m(i(m))\cap\sum f_{i(m)}(b_m)(\mu)\cdot\psi_{m+1}(\mu)\neq\emptyset$;
(4) If $\xi\in(E_{m+1}\setminus E_m)\cap\omega$, then $\delta(x,0)<r_m$, for every $x\in\psi_{m+1}(\xi)$.

If $\xi \in (E_{m+1} \setminus E_m) \cap \omega$, then let $\tilde{\psi}_m(\xi)$ be the open arc of $\mathbb{T}$ centered at 0 with diameter r_m; if $\xi \in (E_{m+1} \setminus E_m) \setminus \omega$, then let $\tilde{\psi}_m(\xi)$ be any open arc of $\mathbb{T}$ with diameter r_m; finally, if $\xi \in E_m$, then let $\tilde{\psi}_m(\xi) = \psi_m(\xi)$.

If $\xi \in E \setminus E_{m+1}$, then let $\psi_{m+1}(\xi) = \mathbb{T}$. If $\xi \in E_{m+1} \setminus \operatorname{supp} f_{i(m)}(b_m)$, then let $\psi_{m+1}(\xi)$ be the open arc of $\mathbb{T}$ centered at the middle point of $\tilde{\psi}_m(\xi)$ with diameter r_{m+1}. It is evident that (1), (2) and (4) are satisfied for $\xi \in E_{m+1} \setminus \operatorname{supp} f_{i(m)}(b_m)$. Let us now define $\psi_{m+1}(\xi)$ for $\xi \in \operatorname{supp} f_{i(m)}(b_m)$.

Case I. $f_{i(m)} \in \mathcal{F}_1$.

We have that

$$\sum f_{i(m)}(b_m)(\mu) \cdot \tilde{\tilde{\psi}}_m(\mu) = \mathbb{T}$$

where $\tilde{\tilde{\psi}}_m(\mu)$ is the open arc of $\mathbb{T}$ centered at the middle point of $\tilde{\psi}_m(\mu)$ with diameter $r_m/4$. Therefore, for each $\mu \in \operatorname{supp} f_{i(m)}(b_m)$, there exists $x_\mu^m \in \tilde{\tilde{\psi}}_m(\mu)$ such that

$$\sum f_{i(m)}(b_m)(\mu) \cdot x_\mu^m \in \psi_m\big(i(m)\big).$$

Define $\psi_{m+1}(\mu)$ as the open arc of $\mathbb{T}$ centered at x_μ^m with diameter r_{m+1}. Thus, conditions (1)–(4) are verified.

Case II. $f_{i(m)} \in \mathcal{F}_2$.

Fix $\alpha \in \operatorname{supp} f_{i(m)}(b_m) \setminus E_m$. For each $\xi \in \operatorname{supp} f_{i(m)}(b_m) \setminus \{\alpha\}$, denote by z_ξ the middle point of $\tilde{\psi}_m(\xi)$. Since $\psi_m(\alpha) = \mathbb{T}$, there exists $z_\alpha \in \psi_m(\alpha)$ such that

$$\sum_{\mu \in \operatorname{supp} f_{i(m)}(b_m) \setminus \{\alpha\}} f_{i(m)}(b_m)(\mu) \cdot z_\mu + f_{i(m)}(b_m)(\alpha) \cdot z_\alpha \in \psi_m\big(i(m)\big).$$

For each $\xi \in \operatorname{supp} f_{i(m)}(b_m)$, define $\psi_{m+1}(\xi)$ as the open arc of $\mathbb{T}$ centered at z_ξ with diameter r_{m+1}. The conditions (1)–(4) are verified, since $\operatorname{supp} f_{i(m)}(b_m) \cap \omega = \emptyset$.

Since $\mathbb{T}$ is a complete metric space and $(r_k)_{k\in\omega}$ is a sequence of positive real numbers that converges to 0, we conclude that if $\xi \in E$, then $\bigcap_{k\in\omega} \psi_k(\xi) = \bigcap_{k\in\omega} \overline{\psi_k(\xi)}$ is a singleton. We shall denote by $\phi(x_\xi)$ the unique element of the intersection $\bigcap_{k\in\omega} \psi_k(\xi)$. Since $\{x_\xi\colon \xi \in E\}$ is an independent set that generates $\mathbb{Z}^{(E)}$, it is possible to extend ϕ to a homomorphism $\phi_g : \mathbb{Z}^{(E)} \to \mathbb{T}$.

We have that

$$\phi_g(g) = \sum g(\mu) \cdot \phi_g(x_\mu) \in \sum g(\mu) \cdot \psi_0(\mu)$$

and, therefore, $\phi_g(g) \neq 0$.

Fix $\xi \in E \setminus \omega$. For each $k \in i^{-1}(\{\xi\})$, we have that

$$\phi_g\big(f_{i(k)}(b_k)\big) \in \sum f_{i(k)}(b_k)(\mu) \cdot \psi_{k+1}(\mu)$$

and

$$\phi_g(x_{i(k)}) \in \psi_k\big(i(k)\big).$$

Thus,

$$\begin{aligned} \delta\big(\phi_g\big(f_{i(k)}(b_k)\big), \phi_g\big(x_{i(k)}\big)\big) &\leqslant \delta\Big(\sum f_{i(k)}(b_k)(\mu) \cdot \psi_{k+1}(\mu)\Big) + \delta\big(\psi_k\big(i(k)\big)\big) \\ &< 2 \cdot r_k. \end{aligned}$$

Since $r_k \to 0$, the sequence $\{\phi_g(f_\xi(b_k))\colon k \in i^{-1}(\xi)\}$ converges to $\phi_g(x_\xi)$. Therefore,

$$\phi_g(x_\xi) = p_\xi\text{-}\lim\big\{\phi_g\big(f_\xi(n)\big)\colon n \in \omega\big\}.$$

It remains to show that the sequence $\{\phi_g(x_n)\colon n \in \omega\}$ converges to 0. It is sufficient to note that if $k \in \omega$ is fixed, then the set $\{n \in \omega\colon \delta(\phi_g(x_n), 0) \geqslant r_k\}$ is finite (since it is contained in E_k). Thus, (iii) is verified. □

If $p \in \omega^*$, then every sequence in $\mathbb{T}$ admits a p-limit point. So, minor modifications in the proof of Lemma 2.3 guarantee that it is possible to extend each homomorphism $\phi_g : \mathbb{Z}^{(E)} \to \mathbb{T}$ obtained from Lemma 3.4 to $\mathbb{Z}^{(\mathfrak{c})}$ in the following sense:

Lemma 3.5. *For each $g \in \mathbb{Z}^{(\mathfrak{c})} \setminus \{0\}$, there exists a homomorphism $\phi_g : \mathbb{Z}^{(\mathfrak{c})} \to \mathbb{T}$ such that*:

(i) $\phi_g(g) \neq 0$;
(ii) $\phi_g(x_\xi) = p_\xi\text{-}\lim\{\phi_g(f_\xi(n)) \colon n \in \omega\}$, *for every* $\xi \in \mathfrak{c} \setminus \omega$;
(iii) $\phi_g(x_n) \to 0$.

We are ready to state the main result of this section.

Theorem 3.6. *Assuming the existence of $\mathfrak{c}$ incomparable selective ultrafilters, the free Abelian group of cardinality $\mathfrak{c}$ admits a countably compact group topology with a non-trivial convergent sequence.*

The proof of Theorem 3.6 is analogous to the proof of Theorem 2.5.
We finish the paper with the following question.

Question 3.7. Is it consistent with *ZFC* that the additive group $\mathbb{R}$ admits a sequentially compact group topology?

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