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NONLINEAR ELLIPTIC EQUATIONS WITH CONCENTRATING REACTION TERMS AT AN OSCILLATORY BOUNDARY

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Dedicated to Peter Kloeden on his 70th aniversary.

Abstract. In this paper we analyze the asymptotic behavior of a family of solutions of a semilinear elliptic equation, with homogeneous Neumann boundary condition, posed in a two-dimensional oscillating region with reaction terms concentrated in a neighborhood of the oscillatory boundary $\theta_\varepsilon \subset \Omega_\varepsilon \subset \mathbb{R}^2$ when a small parameter $\varepsilon > 0$ goes to zero. Our main result is concerned with the upper and lower semicontinuity of the set of solutions in H^1. We show that the solutions of our perturbed equation can be approximated with one defined in a fixed limit domain, which also captures the effects of reaction terms that take place in the original problem as a flux condition on the boundary of the limit domain.

1. Introduction.

In this paper we analyze the asymptotic behavior of the family of solutions of the following semilinear elliptic equation with homogeneous Neumann boundary conditions:

$$\begin{cases} -\Delta u^\varepsilon + u^\varepsilon = \Phi(u^\varepsilon) + \dfrac{1}{\varepsilon}\chi^{\theta_\varepsilon} f(u^\varepsilon) & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u^\varepsilon}{\partial \nu^\varepsilon} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{1}$$

where $\Omega_\varepsilon \subset \mathbb{R}^2$ is an oscillating domain, Φ represents a reaction term acting in the whole domain and $\frac{1}{\varepsilon}\chi^{\theta_\varepsilon} f(u^\varepsilon)$ represents a reaction term concentrated in an extremely thin region θ_ε close to the border $\partial\Omega_\varepsilon$ which can also present oscillatory

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structure. See Figure 1 to visualize the oscillating domain Ω_ε, as well as the narrow oscillating neighborhood θ_ε.

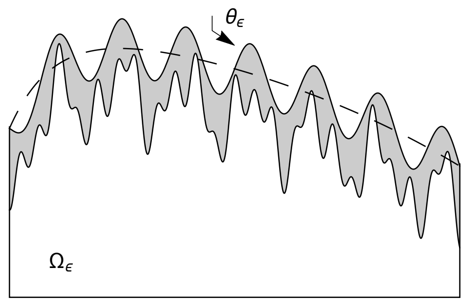

FIGURE 1. The oscillatory domain Ω_ε and strip θ_ε where reactions take place.

Under our assumptions, the two-dimensional family of oscillating regions Ω_ε approaches a bounded domain $\Omega \subset \mathbb{R}^2$, and the narrow strip θ_ε, that may also have an oscillatory behavior, degenerates into a fixed set $\Gamma \subset \partial\Omega$ as the positive parameter ε goes to zero.

We will show that the solutions of (1) converge in certain sense to be specified later to the solutions of the following problem posed in the fixed domain Ω:

$$\begin{cases} -\Delta u + u = \Phi(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \hat{\mu} f(u) & \text{on } \Gamma, \end{cases} \tag{2}$$

where $\hat{\mu}$ is a parameter related to the geometry of the oscillations of Ω_ε and θ_ε. Observe that the reaction term concentrated in θ_ε transforms as $\varepsilon \to 0$ into a boundary reaction term, in accordance to some results in the works [11, 20, 21].

We show that the family of solutions of (1) is upper semicontinuous at $\varepsilon = 0$, and under the additional condition on hyperbolicity of the solutions of the limit problem (2), we also obtain the lower semicontinuity. Moreover, we show that the perturbed equation (1) has one and only one solution nearby a solution of the limit equation for ε small enough.

As we will see, in order to show our results, we need to estimate and analyze the asymptotic behavior of concentrating integrals such as

$$\frac{1}{\varepsilon} \int_{\theta_\varepsilon} |u(x)|^q dx \tag{3}$$

for different values of $q \geq 1$ and open sets $\theta_\varepsilon \subset \Omega_\varepsilon \subset \mathbb{R}^2$. Notice the factor $1/\varepsilon$ in (3). The arrangement of this term with the narrow strip can be thought as a model to measure the concentration of u on θ_ε at $\varepsilon = 0$. In fact, a suitable control of this integral is useful to analyze models set in regions of $\mathbb{R}^2$ which present singular behavior. For instance, we mention our recent work [12], where an oscillating thin domain is studied.

Here, we are in agreement with the notation from papers [11, 20, 21] calling (3) as concentrating or concentrated integral. Indeed, this kind of problem was initially proposal in [11], where linear elliptic equations were considered with reaction and potential terms concentrated on the boundary. There, the neighborhood θ_ε has been set as a strip without oscillatory behavior in a fixed domain Ω. Later, the dynamical system given by a semilinear parabolic problem in the same situation was analyzed in [20, 21] where the upper semicontinuity of attractors at $\varepsilon = 0$ has been shown. In [3, 4] the results of [11, 20] were extended to a reaction-diffusion problem with delay. In these works, the boundary of the domain is always assumed to be smooth.

Subsequently some results of [11] were adapted in [5] to be considered in a semilinear elliptic problem posed on a Lipschitz fixed domain Ω with the ε-neighborhood presenting highly oscillatory behavior. The upper and lower semicontinuity of the attractor to the associated parabolic problem in smooth fixed domains were shown in [6].

Recently, some results from [11, 5] have been adapted in [1, 2] to a class of narrow strips θ_ε and bounded oscillatory domains Ω_ε. Under the restricted assumption $\Omega \subset \Omega_\varepsilon$ and $\theta_\varepsilon \subset \Omega_\varepsilon \setminus \Omega$ for all $\varepsilon > 0$, the authors have been able to estimate concentrating integrals and analyze the asymptotic behavior of semilinear elliptic equations as $\Omega_\varepsilon \to \Omega$ and $\partial\Omega_\varepsilon \to \partial\Omega$ when $\varepsilon \to 0$ in the sense of Hausdorff.

This paper is organized as follows: in Section 2 we introduce the assumptions, notations and the main result. In Section 3, we show some technical results concerning extension operators, Lebesgue-Bochner and Sobolev-Bochner generalized spaces needed to get our estimates. Following by Section 4, we prove some properties about concentrating integrals which are used in Section 5 to study the nonlinearities of our problem. Finally, in Section 6, we pass to the limit in a semilinear elliptic problem getting the upper semicontinuity of the solutions. Moreover, assuming hyperbolicity to the solutions of the limit equation, we also obtain the lower semicontinuity at $\varepsilon = 0$, and we will exclude the possibility that, near an equilibrium point of the limiting equation, may exist several different equilibrium points of the perturbed problem, and therefore, we will also prove some sort of uniqueness of the equilibrium points.

2. **Assumptions, notations and main result.**

To fix the problem, notation and main hypotheses, let us start considering problem (1) where

$$\begin{aligned} \Omega_\varepsilon &= \{(x_1, x_2) \in \mathbb{R}^2; x_1 \in (0,1),\ 0 < x_2 < G_\varepsilon(x_1)\} \quad \text{and} \\ \theta_\varepsilon &= \{(x_1, x_2) \in \mathbb{R}^2; x_1 \in (0,1),\ G_\varepsilon(x_1) - \varepsilon H_\varepsilon(x_1) < x_2 < G_\varepsilon(x_1)\} \end{aligned} \tag{4}$$

are set by functions G_ε, $H_\varepsilon : (0,1) \to \mathbb{R}$ satisfying conditions:

H(i) $G_\varepsilon(x_1) = m(x_1) + \varepsilon g(x_1/\varepsilon^\alpha)$ with $0 < \alpha \le 1$, where
(a) $m : (0,1) \to \mathbb{R}$ is C^1, bounded, with bounded derivative,
(b) $g : (0,1) \to \mathbb{R}$ is a C^1 bounded function, L_g-periodic with bounded derivative.
(c) $G_\varepsilon \to m$ as $\varepsilon \to 0$ uniformly in $(0,1)$.
(d) there are constants $G_0, G_1 > 0$ such that $G_0 \le G_\varepsilon(x) \le G_1$ for all $x \in (0,1)$.

H(ii) $H_\varepsilon(x_1) = h(x_1/\varepsilon^\beta)$, $\beta > 0$, where the function h is bounded, ie there are $H_0, H_1 \ge 0$ such that $H_0 \le H_\varepsilon(x) \le H_1$ for all $x \in (0,1)$, and L_h-periodic.

The vector $\nu^\varepsilon = (\nu_1^\varepsilon, \nu_2^\varepsilon)$ is the unit outward normal vector to the boundary $\partial\Omega_\varepsilon$, $\partial/\partial\nu^\varepsilon$ is the derivative in the direction of ν^ε, and $\chi^{\theta_\varepsilon}$ is the characteristic function of the neighborhood θ_ε. The nonlinearities $\Phi : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are bounded functions of class C^2 with bounded derivatives.

Under assumptions **H**, it is not difficult to associate (4) with the following limit sets

$$\begin{aligned} \Omega &= \{(x_1, x_2) \in \mathbb{R}^2; x_1 \in (0,1),\ 0 < x_2 < m(x_1)\} \text{ and} \\ \Gamma &= \{(x_1, x_2) \in \mathbb{R}^2; x_1 \in (0,1),\ x_2 = m(x_1)\}. \end{aligned} \tag{5}$$

As we mentioned in the introduction, passing to the limit in (1) we obtain equation (2) where $\hat{\mu} \in L^\infty(\Gamma)$ is given by

$$\hat{\mu} = \frac{\mu_h}{\sqrt{1 + m'^2}} \in L^\infty(\Gamma), \tag{6}$$

where $\mu_h \in L^\infty(\Gamma)$ is the weak* limit of H_ε. In fact, due to **H**(ii), it follows from [17, Teorema 2.6] that

$$H_\varepsilon \rightharpoonup \mu_h = \frac{1}{L_h} \int_0^{L_h} h(s) ds.$$

The coefficient $\hat{\mu}$ captures the influence of the small neighborhood θ_ε, as well as the geometry of the limit domain Ω. It also suggests with nonlinearity f a flux condition on the boundary, giving a qualitative idea on the effect of the concentrating reaction terms on the original problem.

Notice that, to obtain the convergence results, we have to compare functions defined in different functional spaces as $\varepsilon \to 0$. In order to do that, we consider the following family of operators

$$E_\varepsilon : H^1(\Omega) \to H^1(\Omega_\varepsilon) : u \mapsto E_\varepsilon u := R_\varepsilon P u \tag{7}$$

where $R_\varepsilon : H^1(\mathbb{R}^2) \to H^1(\Omega_\varepsilon)$ is the restriction operator to the open set Ω_ε and $P : H^1(\Omega) \to H^1(\mathbb{R}^2)$ is a continuous extension operator from functions defined in Ω to the whole plane $\mathbb{R}^2$. The existence of P is guaranteed by [19, Theorem 1.4.3.1].

From [7], we have

$$\|E_\varepsilon u\|_{H^1(\Omega_\varepsilon)} \to \|u\|_{H^1(\Omega)}, \quad \text{as } \varepsilon \to 0,$$

and then, we can compare solutions from (1) and (2) using the notion of E-convergence as in [14].

In general, consider a family of Banach spaces H_ε and a limit Banach space H_0. Besides, let $E_\varepsilon : H_0 \to H_\varepsilon$ a family of operators such that $\|E_\varepsilon u\|_{H_\varepsilon} \to \|u\|_{H_0}$ when $\varepsilon \to 0$.

Definition 2.1. We say that a sequence of $u^\varepsilon \in H_\varepsilon$ E-converges to $u_0 \in H_0$, if $\|u^\varepsilon - E_\varepsilon u\|_{H_\varepsilon} \to 0$ as $\varepsilon \to 0$. We denote this convergence by $u_\varepsilon \xrightarrow{E} u$.

If H_ε and H_0 are Hilbert spaces, we can define a weak E-convergence.

Definition 2.2. A sequence of $\{u^\varepsilon\}$, with $u^\varepsilon \in H_\varepsilon$, E-converges weakly to $u \in H_0$ if for any sequence E-convergent to w we have $(w^\varepsilon, u^\varepsilon)_{H_\varepsilon} \to (u, w)_{H_0}$ when $\varepsilon \to 0$. We may denote such convergence by $u^\varepsilon \overset{E}{\rightharpoonup} u$.

We also need a notion of compactness for sequences, and convergence for operators which are defined in different spaces. We recall the exposition from [14]. See also [7] and [12].

Definition 2.3. A sequence $\{u_n\}$, $u_n \in H_{\varepsilon_n}$ with $\varepsilon_n \to 0$, is E-precompact if for all subsequence $\{u_{n'}\}$ there are a subsequence $\{u_{n''}\}$ and an element $u \in H_0$ such that $u_{n''} \xrightarrow{E} u$. A family is said to be E-precompact if all sequence $\{u_n\}$, $u_n \in H_{\varepsilon_n}$ with $\varepsilon_n \to 0$, is E-precompact.

Definition 2.4. We say that a family of operators $\{T_\varepsilon\}$, with $T_\varepsilon : H_\varepsilon \to H_\varepsilon$, E-converges to $T : H_0 \to H_0$ when $\varepsilon \to 0$ if $T_\varepsilon u^\varepsilon \xrightarrow{E} Tu$ for any $u^\varepsilon \xrightarrow{E} u$. We denote this convergence by $T_\varepsilon \xrightarrow{EE} T$.

Furthermore we may define a notion of compact convergence for operators.

Definition 2.5. A family of compact operators $\{T_\varepsilon\}$, with $T_\varepsilon : H_\varepsilon \to H_\varepsilon$, converges compactly to $T : H_0 \to H_0$ when $\varepsilon \to 0$ if, for any family $\{u^\varepsilon\}$ with $\|u^\varepsilon\|_{H_\varepsilon}$ uniformly bounded, we have that $\{T_\varepsilon u^\varepsilon\}$ is E-precompact and $T_\varepsilon \xrightarrow{EE} T$. We denote this compact convergence by $T_\varepsilon \xrightarrow{CC} T$.

This notion of convergence can be extended to sets in the following manner: let J_ε be a family of sets in some Banach spaces Z_ε. We say that J_ε is

(i) upper semicontinuous at $\varepsilon = 0$ if $\mathrm{dist}_H(J_\varepsilon, E_\varepsilon J_0) \xrightarrow{\varepsilon\to 0} 0$;
(ii) lower semicontinuous at $\varepsilon = 0$ if $\mathrm{dist}_H(E_\varepsilon J_0, J_\varepsilon) \xrightarrow{\varepsilon\to 0} 0$.

Here, $\mathrm{dist}_H(A, B)$ denotes the Hausdorff semi-distance given by

$$\mathrm{dist}_H(A, B) = \sup_{x\in A} \inf_{y\in B} \|x - y\|_{Z_\varepsilon}.$$

Remark 1. In order to show the upper or lower semicontinuity of sets, the following characterizations are useful:

(i) The family $\{J_\varepsilon\}$ is upper semicontinuous at $\varepsilon = 0$ if every sequence $\{u_\varepsilon\}$, with $u_\varepsilon \in J_\varepsilon$ and $\varepsilon \to 0$, has a subsequence E-convergent to an element of J_0;
(ii) The family $\{J_\varepsilon\}$ is lower semicontinuous at $\varepsilon = 0$ if J_0 is compact and for all $u \in J_0$ exists a sequence $\{u_\varepsilon\}$, with $u_\varepsilon \in J_\varepsilon$ and $\varepsilon \to 0$, such that $u_\varepsilon \xrightarrow{E} u$.

Finally, for $\varepsilon > 0$, let us consider

$$\mathcal{E}_\varepsilon = \{u^\varepsilon \in H^1(\Omega_\varepsilon);\ u^\varepsilon \text{ is a solution of } (1)\}$$

and

$$\mathcal{E}_0 = \{u \in H^1(\Omega);\ u \text{ is a solution of } (2)\}.$$

The main goal of this work is to prove the upper and lower semicontinuity of the set $\mathcal{E}_\varepsilon$ at $\varepsilon = 0$:

Theorem 2.6. *If we consider the semilinear elliptic problem (1) then:*

(i) for any sequence $u^\varepsilon \in \mathcal{E}_\varepsilon$, with $\varepsilon \to 0$, there is a subsequence (also denoted by u^ε) and $u_0 \in \mathcal{E}_0$ such that $u^\varepsilon \xrightarrow{E} u_0$.
(ii) for any hyperbolic equilibrium point $u^* \in \mathcal{E}_0$, there is sequence $u^\varepsilon \in \mathcal{E}_\varepsilon$ such that $u^\varepsilon \xrightarrow{E} u^*$ when $\varepsilon \to 0$. Moreover, there are $\eta > 0$ and $\varepsilon_0 > 0$ such that exists an unique $u^\varepsilon \in \mathcal{E}_\varepsilon$ which satisfies

$$\|u^\varepsilon - E_\varepsilon u^*\|_{H^1(\Omega_\varepsilon)} \le \eta, \quad \textit{for all } \ 0 < \varepsilon < \varepsilon_0.$$

Remark 2. Recall that u^* is a hyperbolic equilibrium point of (2), if $\lambda = 0$ is not an eigenvalue of the linearized problem of (2) around u^*. For instance, if u^* is solution of (2) and is hyperbolic, then $\lambda = 0$ is not an eigenvalue of

$$\begin{cases} -\Delta v + v = \Phi'(u^*)\, v + \lambda\, v & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = \hat{\mu} f'(u^*)\, v & \text{on } \Gamma. \end{cases}$$

Furthermore, we notice that item (ii) of the Theorem 2.6 also give us a kind of uniqueness result to the solutions near a hyperbolic equilibrium point of the limit equation for sufficiently small ε.

3. **Functional spaces and technical results.** In this section, we introduce the main functional spaces used throughout this paper and work with some of their properties. Then we set some technical results that will be useful in next sections. First, we define fractional Sobolev spaces.

Definition 3.1. For $s > 0$, $1 \leq p < \infty$ and $O \subset \mathbb{R}^n$, we denote by $W^{s,p}(O)$ and call fractional Sobolev space, the functional set given by the space of distributions defined in O such that

(i) $\partial^\alpha u \in L^p(O)$, for $|\alpha| \leq m$, when $s = m \in \mathbb{N}$;

(ii) $u \in W^{m,p}(O)$ and
$$\iint_{O\times O} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{n+\sigma p}} dxdy < \infty,$$
for $|\alpha| = m$, when $s = m + \sigma$ with $\sigma \in (0,1)$.

The norm in $W^{s,p}(O)$, that makes it Banach, is:
$$\|u\|^p_{W^{m,p}(O)} = \sum_{|\alpha|\leq m} \int_O |\partial^\alpha u(x)|^p dx \quad \text{in the case } (i)$$
and
$$\|u\|^p_{W^{s,p}(O)} = \|u\|^p_{W^{m,p}(O)} + \sum_{|\alpha|=m} \iint_{O\times O} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{n+\sigma p}} dxdy \quad \text{in the case } (ii).$$

Besides if $p = 2$ we denote it by $H^s(O)$, which is a Hilbert space.

Now let us introduce the Lebesgue and Sobolev-Bochner generalized spaces. Here, they are given in a similar way to [23], as a natural generalization to the Lebesgue and Sobolev spaces using Bochner integrals. The usual Lebesgue and Sobolev-Bochner spaces may be found, for instance, in [15, 17].

Definition 3.2. Let us consider a function $G : (0,1) \to \mathbb{R}$ satisfying $0 < G_0 \leq G(x) \leq G_1$, $\forall x \in (0,1)$, for some constants $0 < G_0 \leq G_1$. Let $1 \leq p \leq \infty$ e $1 \leq q < \infty$. The Lebesgue-Bochner generalized spaces, denoted by $L^p(0,1;L^q(0,G(x_1)))$, are defined by
$$L^p(0,1;L^q(0,G(x_1))) := \{u : \Omega_\varepsilon \to \mathbb{R} \text{ measurable}; u(x_1,\cdot) \in L^q(0,G(x_1)) \\ \text{for almost every } x_1 \in (0,1)\}.$$
They are Banach spaces with the norm
$$\|u\|_{L^p(0,1;L^q(0,G(x_1)))} = \begin{cases} \left(\int_0^1 \|u(x_1,\cdot)\|^p_{L^q(0,G(x_1))} dx_1 \right)^{1/p}, & p < \infty, \\ \operatorname*{ess\,sup}_{x\in(0,1)} \|u(x_1,\cdot)\|_{L^q(0,G(x_1))}, & p = \infty. \end{cases}$$

When $p = q = 2$ such space is Hilbert with the inner product
$$(u,v)_{L^2(0,1;L^2(0,G(x_1)))} = \int_0^1 (u(x_1,\cdot), v(x_1,\cdot))_{L^2(0,G(x_1))} dx_1.$$

Remark 3. Since $q < \infty$, the function $x_1 \mapsto \|u(x_1,\cdot)\|_{L^q(0,G(x_1))}$ is measurable by Fubini's Theorem. Then the space $L^p(0,1;L^q(0,G(x_1)))$ is well defined.

Analogously, we have that the Sobolev-Bochner generalized spaces, denoted by $L^p(0,1;W^{s,q}(0,G(x_1)))$ for $s>0$, are defined by

$$L^p(0,1;W^{s,q}(0,G(x_1))) := \{u \in L^p(0,1;L^q(0,G(x_1))); u(x_1,\cdot) \in W^{s,q}(0,G(x_1))\}.$$

Such spaces are Banach with the norm

$$\|u\|_{L^p(0,1;W^{s,q}(0,G(x_1)))} = \begin{cases} \left(\int_0^1 \|u(x_1,\cdot)\|^p_{W^{s,q}(0,G(x_1))} dx_1\right)^{1/p}, & p<\infty, \\ \operatorname*{ess\,sup}_{x_1\in(0,1)} \|u(x_1,\cdot)\|_{W^{s,q}(0,G(x_1))}, & p=\infty, \end{cases}$$

and, again, they are Hilbert spaces if $p=q=2$.

In general, it follows from [17, Proposition 3.59] that, if H is a Hilbert space and $1 \le p < \infty$, then the dual space of $L^p(0,1;H)$ is given by

$$[L^p(0,1;H)]' = L^q(0,1;H'),$$

where H' is the dual space of H and p,q are conjugates.

In our case we will consider the family of Lebesgue and Sobolev-Bochner generalized spaces for the function $G_\varepsilon(x_1) = m(x_1) + \varepsilon g(x_1/\varepsilon^\alpha)$ defined in hypothesis **H**(i) from (4).

Now we set important and nontrivial results that will help us to work with different definitions of Sobolev fractional spaces making their norms equivalent. The proofs are analogous to [12, Propositions 3.4, 3.5 and 3.6].

Lemma 3.3. *Fixed $\varepsilon>0$ and $x_1 \in (0,1)$, if we call $I_\varepsilon = (0,G_\varepsilon(x_1))$, there is a continuous linear extension operator $P: L^2(I_\varepsilon) \to L^2(\mathbb{R})$ such that $Pu=u$ in I_ε, with $\|Pu\|_{L^2(\mathbb{R})} \le \lambda_0\|u\|_{L^2(I_\varepsilon)}$, $\|Pu\|_{H^s(\mathbb{R})} \le \lambda_s\|u\|_{H^s(I_\varepsilon)}$ and $\|Pu\|_{H^1(\mathbb{R})} \le \lambda_1\|u\|_{H^1(I_\varepsilon)}$, for $0<s<1$, where the constants $\lambda_0, \lambda_s, \lambda_1 \ge 1$ are independent of $\varepsilon>0$ and $x_1 \in (0,1)$.*

Theorem 3.4. *Let $I_\varepsilon = (0,G_\varepsilon(x_1))$, with $\varepsilon>0$, $x_1\in(0,1)$ and $0<s<1$ fixed. Then there are $C_1, C_2 > 0$ independent of ε such that*

$$C_1\|u\|_{H^s(I_\varepsilon)} \le \|u\|_{H^s_{[]}(I_\varepsilon)} \le C_2\|u\|_{H^s(I_\varepsilon)}, \ \forall u \in H^s(I_\varepsilon),$$

where $H^s_{[]}(I_\varepsilon)$ is the complex interpolation space

$$H^s_{[]}(I_\varepsilon) = [L^2(I_\varepsilon), H^1(I_\varepsilon)]_s, \quad \text{for } 0<s<1.$$

Proposition 1. *For each $\varepsilon>0$, we have $H^1(\Omega_\varepsilon) \subseteq L^2(0,1;H^s(0,G_\varepsilon(x_1)))$ for all $0 \le s \le 1$, with constant of inclusion independent of ε. Besides $H^1(\Omega_\varepsilon) \subseteq L^2(0,1;H^s(0,G_\varepsilon(x_1)))$ with compact immersion if $0<s<1$.*

According to the properties of our domains Ω_ε defined in (4), we also have the important result.

Proposition 2. *The family of sets Ω_ε admits a continuous extension operator $P_\varepsilon: L^2(\Omega_\varepsilon) \to L^2(U)$, where the open set $U = U_1 \times U_2 \subset \mathbb{R}^2$ is such that the closure of Ω_ε is contained in U for all $\varepsilon>0$, and*

$$\|P_\varepsilon u^\varepsilon\|_{H^1(U)} \le C_0\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)},$$
$$\|P_\varepsilon u^\varepsilon\|_{L^2(U_1;H^s(U_2))} \le C_s\|u^\varepsilon\|_{L^2(0,1;H^s(0,G_\varepsilon(x)))},$$
$$\|P_\varepsilon u^\varepsilon\|_{L^2(U)} \le C_1\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)},$$

where the constants $C_0, C_s, C_1 > 0$ are independent of $\varepsilon>0$ and $0 \le s \le 1$.

Proof. By hypothesis $\mathbf{H}$(i), we have $|G'_\varepsilon(x)| \le C$ for all $x \in (0,1)$, with $C > 0$ independent of $\varepsilon > 0$. Thus, the proof follows from the extension operator defined in [10, Lemma 3.1]. □

Our next step is to prove some inclusions involving Sobolev fractional spaces and Sobolev-Bochner generalized spaces that will be useful in the further analysis of concentrating integrals.

Proposition 3. *For $\varepsilon > 0$ and considering the domains defined in (4), the following inclusions hold with immersion constants independent of ε.*

(a) $H^1(\Omega_\varepsilon) \subset L^\infty(0,1; L^2(0, G_\varepsilon(x)))$;
(b) if $q \ge 2$ *then* $H^1(\Omega_\varepsilon) \subset L^q(0,1; H^s(0, G_\varepsilon(x)))$, *where* $s = 2/q$;
(c) $H^1(\Omega_\varepsilon) \subset L^q(\Omega_\varepsilon)$, *for* $2 \le q \le 6$.

Proof. (a) For each $x_1 \in (0,1)$, we can use the extension operator given by Proposition 2 to get

$$\|u(x_1,\cdot)\|_{L^2(0,G_\varepsilon(x_1))} \le \|P_\varepsilon u(x_1,\cdot)\|_{L^2(0,G_1)}.$$

Hence,

$$\|u\|_{L^\infty(0,1;L^2(0,G_\varepsilon(x_1)))} \le \|P_\varepsilon u\|_{L^\infty(0,1;L^2(0,G_1))}. \tag{8}$$

From [15, Corollary 1.4.36] follows that

$$\|P_\varepsilon u\|_{L^\infty(0,1;L^2(0,G_1))} \le C\|P_\varepsilon u\|_{H^1(0,1;L^2(0,G_1))} \le C\|P_\varepsilon u\|_{H^1(U)}. \tag{9}$$

Thus using (9) in (8) and the continuity of P_ε with constant $C_1 = \|P_\varepsilon\|_{\mathcal{L}(H^1(\Omega_\varepsilon),H^1(U))}$ uniformly bounded for each ε, we have

$$\|u\|_{L^\infty(0,1;L^2(0,G_\varepsilon(x_1)))} \le C\|P_\varepsilon u\|_{H^1(U)} \le CC_1\|u\|_{H^1(\Omega_\varepsilon)},$$

which concludes the proof.

(b) First of all, let $q \ge 2$ and define $s = 2/q$, $0 < s \le 1$. For each $x_1 \in (0,1)$ fixed, we have by Theorem 3.4 and properties of interpolation spaces that there exists $C > 0$, independent of ε and x_1, such that

$$\begin{aligned}\|u(x_1,\cdot)\|_{H^s(0,G_\varepsilon(x_1))} &\le C\|u(x_1,\cdot)\|_{H^s_{[]}(0,G_\varepsilon(x_1))} \\ &\le C\|u(x_1,\cdot)\|^{1-s}_{L^2(0,G_\varepsilon(x_1))}\|u(x_1,\cdot)\|^s_{H^1(0,G_\varepsilon(x_1))}.\end{aligned}$$

Since by the item (a) $H^1(\Omega_\varepsilon) \subset L^\infty(0,1;L^2(0,G_\varepsilon(x_1)))$, we have that

$$\|u(x_1,\cdot)\|_{H^s(0,G_\varepsilon(x_1))} \le C\|u\|^{1-s}_{H^1(\Omega_\varepsilon)}\|u(x_1,\cdot)\|^s_{H^1(0,G_\varepsilon(x_1))}.$$

On the other hand, Proposition 1 implies $H^1(\Omega_\varepsilon) \subset L^2(0,1;H^1(0,G_\varepsilon(x_1)))$, and then,

$$\begin{aligned}\|u\|^q_{L^q(0,1;H^s(0,G_\varepsilon(x_1)))} &= \int_0^1 \|u(x_1,\cdot)\|^{2/s}_{H^s(0,G_\varepsilon(x_1))}dx_1 \\ &\le \int_0^1 \left(C\|u\|^{1-s}_{H^1(\Omega_\varepsilon)}\|u(x_1,\cdot)\|^s_{H^1(0,G_\varepsilon(x_1))}\right)^{2/s} dx_1 \\ &\le C^{2/s}\|u\|^{2(1-s)/s}_{H^1(\Omega_\varepsilon)}\int_0^1 \|u(x_1,\cdot)\|^2_{H^1(0,G_\varepsilon(x_1))}dx_1 \\ &\le C^{2/s}\|u\|^{2(1-s)/s}_{H^1(\Omega_\varepsilon)}\|u\|^2_{H^1(\Omega_\varepsilon)} = C^{2/s}\|u\|^{2/s}_{H^1(\Omega_\varepsilon)} = C^q\|u\|^q_{H^1(\Omega_\varepsilon)}.\end{aligned}$$

Thus, $H^1(\Omega_\varepsilon) \subseteq L^q(0,1;H^{2/q}(0,G_\varepsilon(x_1)))$ for $q \ge 2$.

(c) Since $L^q(\Omega_\varepsilon) = L^q(0,1; L^q(0, G_\varepsilon(x_1)))$ isometrically, we conclude the proof by item (b) if we show

$$H^{2/q}(0, G_\varepsilon(x_1)) \subseteq L^q(0, G_\varepsilon(x_1))$$

with constant of inclusion independent of $x_1 \in (0,1)$ and $\varepsilon > 0$.

If $q = 2$, it follows from the definition of the spaces. If $2 < q \le 4$, then $1/2 \le 2/q < 1$. Hence, by [26, Theorem 1.36] we get

$$H^{2/q}(\mathbb{R}) \subseteq H^{\frac{1}{2}}(\mathbb{R}) \subseteq L^r(\mathbb{R}),\ \forall r \ge 2.$$

In particular, it holds for $r = q$ with $2 \le q \le 4$. Besides, by the operator $P : H^s(0, G_\varepsilon(x_1)) \to H^s(\mathbb{R})$ from Lemma 3.3, whose norm is independent of $\varepsilon > 0$ and $x_1 \in (0,1)$ for any $1/2 < s < 1$, we have

$$\begin{aligned}\|u(x_1,\cdot)\|_{L^q(0,G_\varepsilon(x_1))} &\le \|Pu(x_1,\cdot)\|_{L^q(\mathbb{R})} \le C\|Pu(x_1,\cdot)\|_{H^{\frac{1}{2}}(\mathbb{R})} \\ &\le C\|Pu(x_1,\cdot)\|_{H^{2/q}(\mathbb{R})} \le C\|P\|\|u(x_1,\cdot)\|_{H^{2/q}(0,G_\varepsilon(x_1))}.\end{aligned}$$

Finally, if $4 < q \le 6$, then $1/3 \le 2/q < 1/2$. Again by [26, Theorem 1.36], we get

$$H^{2/q}(\mathbb{R}) \subseteq L^r(\mathbb{R}),\ \forall 2 \le r \le \frac{2}{1-2s}.$$

In particular, since $q = \dfrac{2}{s} \le \dfrac{2}{1-2s}$, we obtain that

$$H^{2/q}(\mathbb{R}) \subseteq L^q(\mathbb{R}).$$

Hence, we conclude the proof arguing as in the previous case $2 \le q \le 4$. □

4. **Concentrating integrals and its behavior at the limit.**

Our first results are about concentrating integrals. Notice that some estimates are given in different functional spaces. Under the conditions of Proposition 3, we may improve [12, Theorem 3.7] estimating the concentrated integrals with the $H^1(\Omega_\varepsilon)$ norm.

Theorem 4.1. *For $\varepsilon_0 > 0$ sufficiently small, there is a constant $C > 0$, independent of $\varepsilon \in (0, \varepsilon_0)$ and $u^\varepsilon \in H^1(\Omega_\varepsilon)$, such that, for all $\frac{1}{2} < s \le 1$, $0 < \varepsilon < \varepsilon_0$, we have*

$$\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |u^\varepsilon|^q \le C\|u^\varepsilon\|^q_{L^q(0,1;H^s(0,G_\varepsilon(x_1)))},\ \forall q \ge 1, \tag{10}$$

and

$$\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |u^\varepsilon|^2 \le C\left(\|u^\varepsilon\|^2_{H^s(\Omega_\varepsilon)} + \left\|\frac{\partial u^\varepsilon}{\partial x_2}\right\|^2_{L^2(\Omega_\varepsilon)}\right). \tag{11}$$

In particular,

$$\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |u^\varepsilon|^q \le C\|u^\varepsilon\|^q_{H^1(\Omega_\varepsilon)},\ 2 \le q < 4. \tag{12}$$

Proof. Take $u \in H^1(\Omega_\varepsilon)$. In a.e. $x_1 \in (0,1)$, we have $u(x_1,\cdot) \in H^1(0, G_\varepsilon(x_1))$. Define

$$z^* := G_0 - \varepsilon_0 H_1 \text{ and } z^\varepsilon := G_\varepsilon(x_1) - \varepsilon H_\varepsilon(x_1)$$

for $\varepsilon_0 > 0$ sufficiently small in such way that, for all $\varepsilon < \varepsilon_0$, we have

$$[z^\varepsilon - z^*, z^\varepsilon] \subset [0, G_\varepsilon(x_1)].$$

See Figure 2 for a representation.

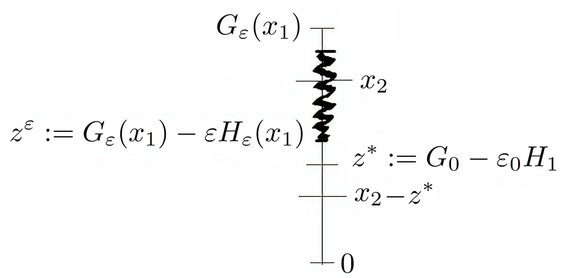

FIGURE 2. Fixed $x_1 \in (0,1)$ and $\varepsilon > 0$, we get a fiber of the oscillatory domain for $\varepsilon < \varepsilon_0$.

Since $(G_\varepsilon(x_1) - \varepsilon H_\varepsilon(x_1)) < x_2 < G_\varepsilon(x_1)$ and $\frac{1}{2} < s \le 1$, it follows from [19, Theorem 1.5.1.3] for $n = 1$ that there exists $K > 0$ independent of $\varepsilon > 0$ such that

$$|u(x_1, x_2)| \le K \|u(x_1, \cdot)\|_{H^s(x_2 - z^*, x_2)} \le K \|u(x_1, \cdot)\|_{H^s(0, G_\varepsilon(x_1))}.$$

Indeed, the interval where we are applying the result is fixed and independent of the parameters $\varepsilon > 0$ and $x_1 \in (0,1)$.

Hence,

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |u|^q &= \int_0^1 \frac{1}{\varepsilon}\int_{G_\varepsilon(x_1)-\varepsilon H_\varepsilon(x_1)}^{G_\varepsilon(x_1)} |u(x_1,x_2)|^q dx_2 dx_1 \\ &\le \int_0^1 \frac{1}{\varepsilon}\int_{G_\varepsilon(x_1)-\varepsilon H_\varepsilon(x_1)}^{G_\varepsilon(x_1)} K^q \|u(x_1,\cdot)\|^q_{H^s(0,G_\varepsilon(x_1))} dx_2 dx_1 \\ &\le K^q H_1 \int_0^1 \|u(x_1,\cdot)\|^q_{H^s(0,G_\varepsilon(x_1))} dx_1 = C_1 \|u\|^q_{L^q(0,1;H^s(0,G_\varepsilon(x_1)))},\end{aligned}$$

where C_2 is independent of ε, proving (10).

Consequently, taking $q = 2/s$, since by Proposition 3(b) we have $H^1(\Omega_\varepsilon) \subset L^q(0,1;H^s(0,G_\varepsilon(x_1)))$ for $\frac{1}{2} < s \le 1$ with constant independent of ε, it follows that

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |u|^q \le K^q \int_0^1 \|u(x_1,\cdot)\|^q_{H^s(0,G_\varepsilon(x_1))} dx_1 \\ = K^q \|u\|^q_{L^q(0,1;H^s(0,G_\varepsilon(x_1)))} \le C\|u\|^q_{H^1(\Omega_\varepsilon)},\end{aligned}$$

proving (12).

Now, let us prove (11). Here we use that $C^\infty(\Omega_\varepsilon)$ is dense in $H^1(\Omega_\varepsilon)$ (see [19, Theorem 1.4.2.2]). Let $u \in C^\infty(\Omega_\varepsilon)$ and fixed $x_1 \in (0,1)$. By Fundamental Theorem of Calculus, we have

$$u(x_1,x_2) = u(x_1,0) + \int_0^{x_2} \frac{\partial u}{\partial x_2}(x_1,s) ds.$$

Then

$$\begin{aligned}|u(x_1,x_2)|^2 &\le 2|u(x_1,0)|^2 + 2\left[\left(\int_0^{x_2}\left|\frac{\partial u}{\partial x_2}(x_1,s)\right|^2 ds\right)^{\frac{1}{2}}\left(\int_0^{x_2} 1^2 ds\right)^{\frac{1}{2}}\right]^2 \\ &\le 2|u(x_1,0)|^2 + 2G_\varepsilon(x_1)\int_0^{x_2}\left|\frac{\partial u}{\partial x_2}(x_1,s)\right|^2 ds.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_{G_\varepsilon(x_1)-\varepsilon H_\varepsilon(x_1)}^{G_\varepsilon(x_1)} |u(x_1,x_2)|^2 dx_2 &\le 2\int_{G_\varepsilon(x_1)-\varepsilon H_\varepsilon(x_1)}^{G_\varepsilon(x_1)} |u(x_1,0)|^2 dx_2 \\ &\quad + 2G_\varepsilon(x_1)\int_{G_\varepsilon(x_1)-\varepsilon H_\varepsilon(x_1)}^{G_\varepsilon(x_1)} \left(\int_0^{x_2}\left|\frac{\partial u}{\partial x_2}(x_1,s)\right|^2 ds\right) dx_2 \\ &\le 2\varepsilon H_1 |u(x_1,0)|^2 + 2G_1\varepsilon H_1 \int_0^{G_\varepsilon(x_1)} \left|\frac{\partial u}{\partial x_2}(x_1,x_2)\right|^2 dx_2.\end{aligned}$$

Hence, if $\gamma(u)$ is the trace of u given by [19, Theorem 1.5.1.3], we get

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\theta_\varepsilon}|u|^2 &= \frac{1}{\varepsilon}\int_0^1\int_{G_\varepsilon(x_1)-\varepsilon H_\varepsilon(x_1)}^{G_\varepsilon(x_1)} |u(x_1,x_2)|^2 dx_2 dx_1 \\ &\le 2H_1\int_0^1 |u(x_1,0)|^2 dx_1 + 2G_1H_1\int_0^1\int_0^{G_\varepsilon(x_1)} \left|\frac{\partial u}{\partial x_2}(x_1,x_2)\right|^2 dx_2 dx_1 \\ &\le 2H_1\|\gamma(u)\|^2_{L^2(0,1)} + 2H_1G_1\left\|\frac{\partial u}{\partial x_2}\right\|^2_{L^2(\Omega_\varepsilon)}.\end{aligned}$$

On the other hand, if $\Omega_0 = (0,1)\times(0,G_0)$, we have $\Omega_0\subset\Omega_\varepsilon$, and there exists a constant $c>0$ such that $\|\gamma(u)\|_{L^2(0,1)} \le c\|u\|_{H^s(\Omega_0)}$ for all $\frac{1}{2}<s\le 1$. Then, due to the previous inequality with $k=2H_1$,

$$\frac{1}{\varepsilon}\int_{\theta_\varepsilon}|u|^2 \le kc\|u\|^2_{H^s(\Omega_0)} + kG_1\left\|\frac{\partial u}{\partial x_2}\right\|^2_{L^2(\Omega_\varepsilon)} \le C_1\left(\|u\|^2_{H^s(\Omega_\varepsilon)} + \left\|\frac{\partial u}{\partial x_2}\right\|^2_{L^2(\Omega_\varepsilon)}\right)$$

with C_1 independent of ε. □

Notice that the above theorem is important because give us a better range of estimates with the $H^1(\Omega_\varepsilon)$ norm. However, the space may still varies with respect to the parameter ε.

Now we may study the behavior of the integrals which set the problem. We start analyzing the terms without concentration.

Proposition 4. *Let $U\subset\mathbb{R}^2$ an open set such that $\Omega_\varepsilon\subset U$ for all $\varepsilon>0$. If $u,\varphi\in H^1(U)$ then*

$$\int_{\Omega_\varepsilon} u(x_1,x_2)\varphi(x_1,x_2)dx_2dx_1 \longrightarrow \int_\Omega u(x_1,x_2)\varphi(x_1,x_2)dx_2dx_1, \qquad \text{as } \varepsilon\to 0.$$

Proof. Using [19, Theorem 1.4.2.1], we know that

$$C_c^\infty(\bar U) := \{u\in C^\infty(U);\ u = v_{|_U},\ \text{com } v\in C_c^\infty(\mathbb{R}^2)\}$$

is dense in $H^1(U)$. Hence, we can assume $u,\varphi\in C_c^\infty(\bar U)$. Then, since $G_\varepsilon(x_1) = m(x_1)+\varepsilon g_\varepsilon(x_1)$, where $g_\varepsilon(x_1) = g(x_1/\varepsilon^\alpha)$ with $0<\alpha\le 1$, performing the change of variables

$$y_1 = x_1, \quad y_2 = \frac{x_2 - m(x_1)}{\varepsilon g_\varepsilon(x_1)},$$

we get

$$\begin{aligned}\int_{\Omega_\varepsilon} u(x_1,x_2)\varphi(x_1,x_2)dx_2dx_1 &= \int_0^1 \int_0^{m(x_1)+\varepsilon g_\varepsilon(x_1)} u(x_1,x_2)\varphi(x_1,x_2)dx_2dx_1 \\ &= \int_0^1 \int_0^{m(x_1)} u(x_1,x_2)\varphi(x_1,x_2)dx_2dx_1 \\ &\quad + \int_0^1 \int_{m(x_1)}^{m(x_1)+\varepsilon g_\varepsilon(x_1)} u(x_1,x_2)\varphi(x_1,x_2)dx_2dx_1 \\ &= \int_\Omega u(x_1,x_2)\varphi(x_1,x_2)dx_2dx_1 \\ &\quad + \varepsilon \int_0^1 \int_0^1 u(y_1, m(y_1)+y_2\varepsilon g_\varepsilon(y_1))\varphi(y_1, m(y_1)+y_2\varepsilon g_\varepsilon(y_1))g_\varepsilon(y_1)dy_2dy_1 \\ &\to \int_\Omega u(x_1,x_2)\varphi(x_1,x_2)dx_2dx_1,\end{aligned}$$

since $u, \varphi \in C_c^\infty(\bar{U})$ and $g_\varepsilon(x_1)$ is bounded by Hypothesis $\mathbf{H}$(i) from the domain (4). Thus the result is valid through density properties. □

We can also prove results concerning to the behavior of the trace operator at $\varepsilon = 0$. Notice that, at the limit, a coefficient term appears capturing the geometry of the oscillating domain Ω_ε and the oscillatory strip θ_ε.

Proposition 5. *Let $U \subset \mathbb{R}^2$ an open set such that $\Omega_\varepsilon \subset U$ for all $\varepsilon > 0$. If $u, \varphi \in H^1(U)$ then*

$$\frac{1}{\varepsilon}\int_{\theta_\varepsilon} u(x_1,x_2)\varphi(x_1,x_2)dx_2dx_1 \longrightarrow \int_\Gamma \hat{\mu}\gamma(u)\gamma(\varphi)dS, \qquad \text{as } \varepsilon \to 0,$$

where γ is the trace operator given by [19, Theorem 1.5.1.3] *and $\hat{\mu}$ given by (6).*

Proof. Again, due to [19, Theorem 1.4.2.1], we know that $C_c^\infty(\bar{U}) := \{u \in C^\infty(U);\ u = v_{|U},\ \text{com } v \in C_c^\infty(\mathbb{R}^2)\}$ is dense in $H^1(U)$ and we can assume $u, \varphi \in C_c^\infty(\bar{U})$. Then, performing the change of variables

$$y_1 = x_1, \quad y_2 = \frac{x_2 - G_\varepsilon(x_1) + \varepsilon H_\varepsilon(x_1)}{\varepsilon H_\varepsilon(x_1)},$$

we get

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\theta_\varepsilon} u(x_1,x_2)\varphi(x_1,x_2)dx_2dx_1 &= \frac{1}{\varepsilon}\int_0^1 \int_{G_\varepsilon(x_1)-\varepsilon H_\varepsilon(x_1)}^{G_\varepsilon(x_1)} u(x_1,x_2)\varphi(x_1,x_2)dx_2dx_1 \\ = \int_0^1\int_0^1 & u(y_1, G_\varepsilon(y_1) - \varepsilon H_\varepsilon(y_1)(1-y_2))\varphi(y_1, G_\varepsilon(y_1) - \varepsilon H_\varepsilon(y_1)(1-y_2))H_\varepsilon(y_1)dy_2dy_1 \\ = \int_0^1\int_0^1 & (u(y_1, G_\varepsilon(y_1) - \varepsilon H_\varepsilon(y_1)(1-y_2)) - u(y_1, m(y_1))) \\ & \qquad \varphi(y_1, G_\varepsilon(y_1) - \varepsilon H_\varepsilon(y_1)(1-y_2))H_\varepsilon(y_1)dy_2dy_1 \\ + \int_0^1\int_0^1 & u(y_1, m(y_1))(\varphi(y_1, G_\varepsilon(y_1) - \varepsilon H_\varepsilon(y_1)(1-y_2)) - \varphi(y_1, m(y_1)))H_\varepsilon(y_1)dy_2dy_1 \\ + \int_0^1\int_0^1 & u(y_1, m(y_1))\varphi(y_1, m(y_1))(H_\varepsilon(y_1) - \mu_h)dy_2dy_1 \\ + \mu_h \int_0^1 & u(y_1, m(y_1))\varphi(y_1, m(y_1))dy_1 \to \mu_h \int_0^1 u(y_1, m(y_1))\varphi(y_1, m(y_1))\mu_h dy_1,\end{aligned}$$

as $\varepsilon \to 0$, since $G_\varepsilon \to m$ by Hypothesis **H**(i) from the domain (4). Finally, we obtain

$$\int_0^1 \mu_h u(y_1, m(y_1))\varphi(x_1, m(x_1))dy_1 = \int_\Gamma \hat{\mu}\gamma(u)\gamma(\varphi)dS$$

changing variables on the line integral, where $\hat{\mu}$ is given by (6), proving the result using density and trace operator properties. ☐

Remark 4. The function $\hat{\mu}$ given by (6) is independent of the parametrization chosen in Γ and, therefore, is unique.

We also have similar results to nonlinearities Φ, f.

Corollary 1. *Let $U \subset \mathbb{R}^2$ an open set such that $\Omega_\varepsilon \subset U$ for all $\varepsilon > 0$. If $u, \varphi \in H^1(U)$ and $\Phi, f : \mathbb{R} \to \mathbb{R}$ bounded functions of class C^1, then*

$$\int_{\Omega_\varepsilon} \Phi(u(x_1, x_2))\varphi(x_1, x_2)dx_2dx_1 \to \int_\Omega \Phi(u(x_1, x_2))\varphi(x_1, x_2)dx_2dx_1$$

and

$$\frac{1}{\varepsilon}\int_{\theta_\varepsilon} f(u(x_1, x_2))\varphi(x_1, x_2)dx_2dx_1 \to \int_\Gamma \hat{\mu}\gamma(f(u))\gamma(\varphi)dS,$$

as $\varepsilon \to 0$, where γ is the trace operator given by [19, Theorem 1.5.1.3] *and $\hat{\mu} \in L^\infty(\Gamma)$ is the coefficient given by (6).*

Proof. Arguing as in the proof of Propositions 4 and 5, we can assume $u, \varphi \in C_c^\infty(\bar{U})$. Then, using the same change of variables as before and noting that f is C^1, we have, for instance,

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\theta_\varepsilon} f(u)\varphi dx_2dx_1 &= \frac{1}{\varepsilon}\int_0^1 \int_{G_\varepsilon(x_1)-\varepsilon H_\varepsilon(x_1)}^{G_\varepsilon(x_1)} f(u(x_1, x_2))\varphi(x_1, x_2)dx_2dx_1\\ &= \int_0^1\int_0^1 f(u(y_1, G_\varepsilon(y_1) - \varepsilon H_\varepsilon(y_1)(1 - y_2)))\\ &\qquad \varphi(y_1, G_\varepsilon(y_1) - \varepsilon H_\varepsilon(y_1)(1 - y_2))H_\varepsilon(y_1)dy_2dy_1\\ &\to \int_\Gamma \hat{\mu}\gamma(f(u))\gamma(\varphi)dS, \quad \text{as } \varepsilon \to 0.\end{aligned}$$

The other convergence is analogous. ☐

The following corollaries possess similar proofs.

Corollary 2. *Let $U \subset \mathbb{R}^2$ an open set such that $\Omega_\varepsilon \subset U$ for all $\varepsilon > 0$. If $u, \varphi, \psi \in H^1(U)$, then*

$$\int_{\Omega_\varepsilon} u(x_1, x_2)\varphi(x_1, x_2)\psi(x_1, x_2)dx_2dx_1 \to \int_\Omega u(x_1, x_2)\varphi(x_1, x_2)\psi(x_1, x_2)dx_2dx_1$$

and

$$\frac{1}{\varepsilon}\int_{\theta_\varepsilon} u(x_1, x_2)\varphi(x_1, x_2)\psi(x_1, x_2)dx_2dx_1 \to \int_\Gamma \hat{\mu}\gamma(u)\gamma(\varphi)\gamma(\psi)dS,$$

as $\varepsilon \to 0$, where γ is the trace operator given by [19, Theorem 1.5.1.3] *and $\hat{\mu} \in L^\infty(\Gamma)$ is the coefficient given by (6).*

Corollary 3. *Let $U \subset \mathbb{R}^2$ an open set such that $\Omega_\varepsilon \subset U$ for all $\varepsilon > 0$. If $u, \varphi, \psi \in H^1(U)$ and $f, \Phi : \mathbb{R} \to \mathbb{R}$ bounded functions of class C^1, then*

$$\int_{\Omega_\varepsilon} \Phi(u(x_1,x_2))\varphi(x_1,x_2)\psi(x_1,x_2)dx_2dx_1 \to \int_{\Omega} \Phi(u(x_1,x_2))\varphi(x_1,x_2)\psi(x_1,x_2)dx_2dx_1$$

and

$$\frac{1}{\varepsilon}\int_{\theta_\varepsilon} f(u(x_1,x_2))\varphi(x_1,x_2)\psi(x_1,x_2)dx_2dx_1 \to \int_\Gamma \hat{\mu}\gamma(f(u))\gamma(\varphi)\gamma(\psi)dS,$$

as $\varepsilon \to 0$, where γ is the trace operator given by [19, Theorem 1.5.1.3] *and $\hat{\mu} \in L^\infty(\Gamma)$ is the coefficient given by (6).*

5. **Nonlinear maps.** In this section we discuss the main properties of the maps used to describe the reaction terms on the nonlinearities of the elliptic problems (1) and (2). For $\frac{1}{2} < s < 1$, consider the Sobolev-Bochner spaces

$$X_\varepsilon = L^2(0,1; H^s(0, G_\varepsilon(x_1))) \text{ and } X'_\varepsilon = L^2(0,1; \{H^s(0, G_\varepsilon(x_1))\}').$$

Then define

$$\begin{aligned} F_\varepsilon : H^1(\Omega_\varepsilon) &\to X'_\varepsilon \\ u &\mapsto F_\varepsilon(u) : X_\varepsilon \to \mathbb{R} \\ &\qquad v \mapsto \langle F_\varepsilon(u), v\rangle = \int_{\Omega_\varepsilon} \Phi(u)v + \frac{1}{\varepsilon}\int_{\theta_\varepsilon} f(u)v, \end{aligned} \tag{13}$$

where $\Phi, f \in C^2(\mathbb{R})$ are bounded functions with bounded derivatives.

Remark 5. Notice that the assumption $\Phi, f \in C^2(\mathbb{R})$ bounded with bounded derivatives it is not a big restriction since we are interested in analyze $f(u)$ when u is uniformly bounded in $L^\infty(\Omega_\varepsilon)$. More details can be found in [9, Remark 2.2] or [7, Remark 2.2].

Remark 6. Notice that $L^2(\Omega_\varepsilon) \subset X_\varepsilon$ with constant independent of ε. Indeed, it follows from [24, Proposition 2.1] that, if $u \in X_\varepsilon$,

$$\begin{aligned} \|u\|^2_{L^2(\Omega_\varepsilon)} &= \int_0^1 \int_0^{G_\varepsilon(x_1)} |u(x_1,x_2)|^2 dx_2dx_1 = \int_0^1 \|u(x_1,\cdot)\|^2_{L^2(0,G_\varepsilon(x_1))}dx_1 \\ &\le \int_0^1 C\|u(x_1,\cdot)\|^2_{H^s(0,G_\varepsilon(x_1))}dx_1 = C\|u\|^2_{X_\varepsilon}, \end{aligned}$$

where $C > 0$ is independent of the domain and, furthermore, of ε.

Now we prove an analogous result to [6, Lemma 3.6] and [25, Lemma 3.1].

Proposition 6. *The function F_ε defined in (13) satisfies for constants independent of ε:*

(a) there exists $K > 0$ such that

$$\sup_{u^\varepsilon \in H^1(\Omega_\varepsilon)} \|F_\varepsilon(u^\varepsilon)\|_{X'_\varepsilon} \le K;$$

(b) F_ε is globally Lipschitz continuous, that is, there exists $L > 0$ such that

$$\|F_\varepsilon(u_1^\varepsilon) - F_\varepsilon(u_2^\varepsilon)\|_{X'_\varepsilon} \le L\|u_1^\varepsilon - u_2^\varepsilon\|_{H^1(\Omega_\varepsilon)}, \ \forall u_1^\varepsilon, u_2^\varepsilon \in H^1(\Omega_\varepsilon).$$

(c) F_ε is Frechet differentiable, with

$$\begin{aligned}F'_\varepsilon : H^1(\Omega_\varepsilon) &\to \mathcal{L}(H^1(\Omega_\varepsilon), X'_\varepsilon)\\ u^\varepsilon &\mapsto F'_\varepsilon(u^\varepsilon) : H^1(\Omega_\varepsilon) \to X'_\varepsilon\\ w^\varepsilon &\mapsto F'_\varepsilon(u^\varepsilon)(w^\varepsilon) : X_\varepsilon \to \mathbb{R}\\ v^\varepsilon &\mapsto \langle F'_\varepsilon(u^\varepsilon)(w^\varepsilon), v^\varepsilon\rangle = \int_{\Omega_\varepsilon} \Phi'(u^\varepsilon) w^\varepsilon v^\varepsilon + \frac{1}{\varepsilon}\int_{\theta_\varepsilon} f'(u^\varepsilon) w^\varepsilon v^\varepsilon;\end{aligned}$$

(d) fixed $u^\varepsilon \in H^1(\Omega_\varepsilon)$, there is $\bar{C} > 0$ such that

$$\|F'_\varepsilon(u^\varepsilon)(w_2^\varepsilon - w_1^\varepsilon)\|_{X'_\varepsilon} \le \bar{C}\|w_2^\varepsilon - w_1^\varepsilon\|_{H^1(\Omega_\varepsilon)},\ \forall w_1^\varepsilon, w_2^\varepsilon \in H^1(\Omega_\varepsilon);$$

(e) there are $\vartheta \in (0,1)$ and $M > 0$ such that

$$\|F'_\varepsilon(u^\varepsilon) - F'_\varepsilon(v^\varepsilon)\|_{\mathcal{L}(H^1(\Omega_\varepsilon), X'_\varepsilon)} \le M\|u^\varepsilon - v^\varepsilon\|_{X_\varepsilon}^{\vartheta},\ \forall u^\varepsilon, v^\varepsilon \in X_\varepsilon;$$

(f) there is $k > 0$ such that

$$\|F_\varepsilon(u^\varepsilon + v^\varepsilon) - F_\varepsilon(u^\varepsilon) - F'_\varepsilon(u^\varepsilon)v^\varepsilon\|_{X'_\varepsilon} \le k\|v^\varepsilon\|_{H^1(\Omega_\varepsilon)}^{1+\delta},\ \forall \delta \in (0,1),\ \forall u^\varepsilon, v^\varepsilon \in H^1(\Omega_\varepsilon).$$

Proof. (a) For $u^\varepsilon \in H^1(\Omega_\varepsilon)$,

$$\|F_\varepsilon(u^\varepsilon)\|_{X'_\varepsilon} = \sup_{\|v^\varepsilon\|_{X_\varepsilon}=1} |\langle F_\varepsilon(u^\varepsilon), v^\varepsilon\rangle|.$$

Hence, if $v^\varepsilon \in X_\varepsilon$, it follows from Theorem 4.1 and Remark 6 that

$$\begin{aligned}|\langle F_\varepsilon(u^\varepsilon), v^\varepsilon\rangle| &\le \int_{\Omega_\varepsilon} |\Phi(u^\varepsilon)v^\varepsilon| + \frac{1}{\varepsilon}\int_{\theta_\varepsilon} |f(u^\varepsilon)v^\varepsilon|\\ &\le \left(\int_{\Omega_\varepsilon} |\Phi(u^\varepsilon)|^2\right)^{\frac12}\left(\int_{\Omega_\varepsilon} |v^\varepsilon|^2\right)^{\frac12} + \left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |f(u^\varepsilon)|^2\right)^{\frac12}\left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |v^\varepsilon|^2\right)^{\frac12}\\ &\le \left(\sup_{x\in\mathbb{R}} |\Phi(x)|\right) G_1^{\frac12}\|v^\varepsilon\|_{L^2(\Omega_\varepsilon)} + \left(\sup_{x\in\mathbb{R}} |f(x)|\right) H_1^{\frac12} C\|v^\varepsilon\|_{X_\varepsilon} \le K\|v^\varepsilon\|_{X_\varepsilon}\end{aligned}$$

Therefore

$$\sup_{u^\varepsilon \in H^1(\Omega_\varepsilon)} \|F_\varepsilon(u^\varepsilon)\|_{X'_\varepsilon} \le K.$$

(b) Indeed, for any $u_1^\varepsilon, u_2^\varepsilon \in H^1(\Omega_\varepsilon)$, we have

$$\|F_\varepsilon(u_1^\varepsilon) - F_\varepsilon(u_2^\varepsilon)\|_{X'_\varepsilon} = \sup_{\|v^\varepsilon\|_{X_\varepsilon}=1} |\langle F_\varepsilon(u_1^\varepsilon), v^\varepsilon\rangle - \langle F_\varepsilon(u_2^\varepsilon), v^\varepsilon\rangle|.$$

Using Mean Value Theorem, with Theorem 4.1 and Remark 6 again, we get

$$\begin{aligned}|\langle F_\varepsilon(u_1^\varepsilon), v^\varepsilon\rangle - \langle F_\varepsilon(u_2^\varepsilon), v^\varepsilon\rangle| &= |\langle F_\varepsilon(u_1^\varepsilon) - F_\varepsilon(u_2^\varepsilon), v^\varepsilon\rangle|\\ &\le \int_{\Omega_\varepsilon} |(\Phi(u_1^\varepsilon) - \Phi(u_2^\varepsilon))v^\varepsilon| + \frac{1}{\varepsilon}\int_{\theta_\varepsilon} |(f(u_1^\varepsilon) - f(u_2^\varepsilon))v^\varepsilon|\\ &\le \left(\int_{\Omega_\varepsilon} |\Phi(u_1^\varepsilon) - \Phi(u_2^\varepsilon)|^2\right)^{\frac12}\left(\int_{\Omega_\varepsilon} |v^\varepsilon|^2\right)^{\frac12}\end{aligned}$$

$$
\begin{aligned}
&+ \left(\frac{1}{\varepsilon} \int_{\theta_\varepsilon} |f(u_1^\varepsilon) - f(u_2^\varepsilon)|^2 \right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon} \int_{\theta_\varepsilon} |v^\varepsilon|^2 \right)^{\frac{1}{2}} \\
\leq& \left(\sup_{x\in\mathbb{R}} |\Phi'(x)| \right) \|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Omega_\varepsilon)} \|v^\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&+ \left(\sup_{x\in\mathbb{R}} |f'(x)| \right) C^2 \|u_1^\varepsilon - u_2^\varepsilon\|_{H^1(\Omega_\varepsilon)} \|v^\varepsilon\|_{X_\varepsilon} \\
\leq& L\|u_1^\varepsilon - u_2^\varepsilon\|_{H^1(\Omega_\varepsilon)} \|v^\varepsilon\|_{X_\varepsilon}.
\end{aligned}
$$

Thus

$$
\|F_\varepsilon(u_1^\varepsilon) - F_\varepsilon(u_2^\varepsilon)\|_{X_\varepsilon'} \leq L\|u_1^\varepsilon - u_2^\varepsilon\|_{H^1(\Omega_\varepsilon)}
$$

and, therefore, F_ε is globally Lipschitz with constant independent of ε.

(c) In fact, if $u^\varepsilon, h^\varepsilon \in H^1(\Omega_\varepsilon)$ and $v^\varepsilon \in X_\varepsilon$, applying Mean Value Theorem,

$$
\begin{aligned}
&|\langle F_\varepsilon(u^\varepsilon + h^\varepsilon) - F_\varepsilon(u^\varepsilon) - F_\varepsilon'(u^\varepsilon)h^\varepsilon, v^\varepsilon\rangle| \leq \\
&\quad \leq \int_{\Omega_\varepsilon} |\Phi(u^\varepsilon + h^\varepsilon) - \Phi(u^\varepsilon) - \Phi'(u^\varepsilon)h^\varepsilon||v^\varepsilon| \\
&\quad\quad + \frac{1}{\varepsilon} \int_{\theta_\varepsilon} |f(u^\varepsilon + h^\varepsilon) - f(u^\varepsilon) - f'(u^\varepsilon)h^\varepsilon||v^\varepsilon| \\
&\quad \leq \left(\int_{\Omega_\varepsilon} |\Phi(u^\varepsilon + h^\varepsilon) - \Phi(u^\varepsilon) - \Phi'(u^\varepsilon)h^\varepsilon|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega_\varepsilon} |v^\varepsilon|^2 \right)^{\frac{1}{2}} \\
&\quad\quad + \left(\frac{1}{\varepsilon} \int_{\theta_\varepsilon} |f(u^\varepsilon + h^\varepsilon) - f(u^\varepsilon) - f'(u^\varepsilon)h^\varepsilon|^2 \right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon} \int_{\theta_\varepsilon} |v^\varepsilon|^2 \right)^{\frac{1}{2}}
\end{aligned}
$$

where $u^\varepsilon(x) \leq \xi^\varepsilon(x), \zeta^\varepsilon(x) \leq (u^\varepsilon + h^\varepsilon)(x)$.

Thus it follows that

$$
\begin{aligned}
|\langle F_\varepsilon(u^\varepsilon + h^\varepsilon) - F_\varepsilon(u^\varepsilon) - F_\varepsilon'(u^\varepsilon)h^\varepsilon, v^\varepsilon\rangle| &\leq \left(\int_{\Omega_\varepsilon} |(\Phi'(\zeta^\varepsilon) - \Phi'(u^\varepsilon))h^\varepsilon|^2 \right)^{\frac{1}{2}} \|v^\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&+ C \left(\frac{1}{\varepsilon} \int_{\theta_\varepsilon} |(f'(\xi^\varepsilon) - f'(u^\varepsilon))h^\varepsilon|^2 \right)^{\frac{1}{2}} \|v^\varepsilon\|_{X_\varepsilon}
\end{aligned} \tag{14}
$$

We will analyze the second part of (14). Notice that, applying Mean Value Theorem again, we get

$$
|(f'(\xi^\varepsilon) - f'(u^\varepsilon))h^\varepsilon|^2 \leq |f''(\eta^\varepsilon)|^2 |\xi^\varepsilon - u^\varepsilon|^2 |h^\varepsilon|^2 \leq \left(\sup_{x\in\mathbb{R}} |f''(x)| \right) |h^\varepsilon|^4 \tag{15}
$$

for $\xi^\varepsilon(x) \leq \eta^\varepsilon(x) \leq u^\varepsilon(x)$, for all $x \in \Omega_\varepsilon$.

On the other side,

$$
|(f'(\xi^\varepsilon) - f'(u^\varepsilon))h^\varepsilon|^2 = |f'(\xi^\varepsilon) - f'(u^\varepsilon)|^2 |h^\varepsilon|^2 \leq 2 \left(\sup_{x\in\mathbb{R}} |f'(x)| \right)^2 |h^\varepsilon|^2. \tag{16}
$$

Then putting (15) and (16) together, we have

$$
|(f'(\xi^\varepsilon) - f'(u^\varepsilon))h^\varepsilon|^2 \leq K \min\{|h^\varepsilon|^2, 1\} |h^\varepsilon|^2. \tag{17}
$$

However, for all $\delta \in [0, 1]$,

$$
\min\{|h^\varepsilon|^2, 1\} = \min\{|h^\varepsilon|^2, 1\}^\delta \min\{|h^\varepsilon|^2, 1\}^{1-\delta} \leq |h^\varepsilon|^{2\delta}
$$

and, thus, (17) became

$$|(f'(\xi^\varepsilon) - f'(u^\varepsilon))h^\varepsilon|^2 \le K_2 |h^\varepsilon|^{2(1+\delta)}, \ \forall \delta \in [0,1].$$

Analogously, using the properties of Φ we may say that, for the first part of (14),

$$|(\Phi'(\zeta^\varepsilon) - \Phi'(u^\varepsilon))h^\varepsilon|^2 \le K_1 |h^\varepsilon|^{2(1+\delta)}, \ \forall \delta \in [0,1].$$

Then it follows from (14) and using Remark 6 that

$$|\langle F_\varepsilon(u^\varepsilon + h^\varepsilon) - F_\varepsilon(u^\varepsilon) - F'_\varepsilon(u^\varepsilon)h^\varepsilon, v^\varepsilon\rangle| \le K_1 \left(\int_{\Omega_\varepsilon} |h^\varepsilon|^{2(1+\delta)} \right)^{\frac{1}{2}} \|v^\varepsilon\|_{X_\varepsilon} + K_2 \left(\frac{1}{\varepsilon} \int_{\theta_\varepsilon} |h^\varepsilon|^{2(1+\delta)} \right)^{\frac{1}{2}} \|v^\varepsilon\|_{X_\varepsilon}.$$

Furthermore, if $\delta \in (0,1)$, we can use Theorem 4.1 to get

$$\begin{aligned} &\|F_\varepsilon(u^\varepsilon + h^\varepsilon) - F_\varepsilon(u^\varepsilon) - F'_\varepsilon(u^\varepsilon)h^\varepsilon\|_{X'_\varepsilon} \\ &\le K_1 \left(\int_{\Omega_\varepsilon} |h^\varepsilon|^{2(1+\delta)} \right)^{\frac{1}{2}} + K_2 \left(\frac{1}{\varepsilon} \int_{\theta_\varepsilon} |h^\varepsilon|^{2(1+\delta)} \right)^{\frac{1}{2}} \le C \|h^\varepsilon\|^{1+\delta}_{H^1(\Omega_\varepsilon)}. \end{aligned}$$

Consequently,

$$\frac{\|F_\varepsilon(u^\varepsilon + h^\varepsilon) - F_\varepsilon(u^\varepsilon) - F'_\varepsilon(u^\varepsilon)h^\varepsilon\|_{X'_\varepsilon}}{\|h^\varepsilon\|_{H^1(\Omega_\varepsilon)}} \le C^2 \|h^\varepsilon\|^{\delta}_{H^1(\Omega_\varepsilon)} \to 0$$

when $\|h^\varepsilon\|_{H^1(\Omega_\varepsilon)} \to 0$ and, thus, F_ε is Frechet differentiable.

(d) Indeed, since $\Phi, \Phi', \Phi'', f, f', f''$ are bounded, if $w_1^\varepsilon, w_2^\varepsilon \in H^1(\Omega_\varepsilon)$ and $v^\varepsilon \in X_\varepsilon$, we have by Theorem 4.1

$$\begin{aligned} &|\langle F'_\varepsilon(u^\varepsilon)w_2^\varepsilon - F'_\varepsilon(u^\varepsilon)w_1^\varepsilon, v^\varepsilon\rangle| \\ &\le \int_{\Omega_\varepsilon} |(\Phi'(u^\varepsilon)w_2^\varepsilon - \Phi'(u^\varepsilon)w_1^\varepsilon)v^\varepsilon| + \frac{1}{\varepsilon}\int_{\theta_\varepsilon} |(f'(u^\varepsilon)w_2^\varepsilon - f'(u^\varepsilon)w_1^\varepsilon)v^\varepsilon| \\ &\le \left(\sup_{x\in\mathbb{R}} |\Phi'(x)| \right) \left(\int_{\Omega_\varepsilon} |w_2^\varepsilon - w_1^\varepsilon|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega_\varepsilon} |v^\varepsilon|^2 \right)^{\frac{1}{2}} \\ &\quad + \left(\sup_{x\in\mathbb{R}} |f'(x)| \right) \left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |w_2^\varepsilon - w_1^\varepsilon|^2 \right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |v^\varepsilon|^2 \right)^{\frac{1}{2}} \\ &\le C \|w_2^\varepsilon - w_1^\varepsilon\|_{H^1(\Omega_\varepsilon)} \|v\|_{X_\varepsilon}. \end{aligned}$$

It follows that

$$\|F'_\varepsilon(u^\varepsilon)(w_2^\varepsilon - w_1^\varepsilon)\|_{X'_\varepsilon} \le C \|w_2^\varepsilon - w_1^\varepsilon\|_{H^1(\Omega_\varepsilon)},$$

proving the result.

(e) If $u^\varepsilon, v^\varepsilon \in H^1(\Omega_\varepsilon)$ and $w^\varepsilon \in X_\varepsilon$,

$$\|F'_\varepsilon(u^\varepsilon) - F'_\varepsilon(v^\varepsilon)\|_{\mathcal{L}(H^1(\Omega_\varepsilon), X'_\varepsilon)} = \sup_{\|w^\varepsilon\|_{H^1(\Omega_\varepsilon)}=1} \ \sup_{\|z^\varepsilon\|_{X_\varepsilon}=1} \langle (F'_\varepsilon(u^\varepsilon) - F'_\varepsilon(v^\varepsilon))w^\varepsilon, z^\varepsilon\rangle.$$

Hence, if $w^\varepsilon \in H^1(\Omega_\varepsilon)$ and $z^\varepsilon \in X_\varepsilon$, it follows from Theorem 4.1 and Hölder's Inequality Generalized with $3 < q < 4$ and $4 < p < 6$ (since $1/p + 1/q = \frac{1}{2}$) that

$$\begin{aligned}
&|\langle (F'_\varepsilon(u^\varepsilon) - F'_\varepsilon(v^\varepsilon))w^\varepsilon, z^\varepsilon\rangle| \\
&\leq \int_{\Omega_\varepsilon} |(\Phi'(u^\varepsilon) - \Phi'(v^\varepsilon))w^\varepsilon z^\varepsilon| + \frac{1}{\varepsilon}\int_{\theta_\varepsilon} |(f'(u^\varepsilon) - f'(v^\varepsilon))w^\varepsilon z^\varepsilon| \\
&\leq \left(\int_{\Omega_\varepsilon} |\Phi'(u^\varepsilon) - \Phi'(v^\varepsilon)|^p\right)^{\frac{1}{p}} \left(\int_{\Omega_\varepsilon} |w^\varepsilon|^q\right)^{\frac{1}{q}} \left(\int_{\Omega_\varepsilon} |z^\varepsilon|^2\right)^{\frac{1}{2}} \\
&+ \left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |f'(u^\varepsilon) - f'(v^\varepsilon)|^p\right)^{\frac{1}{p}} \left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |w^\varepsilon|^q\right)^{\frac{1}{q}} \left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |z^\varepsilon|^2\right)^{\frac{1}{2}} \\
&\leq C\left[\left(\int_{\Omega_\varepsilon} |\Phi'(u^\varepsilon) - \Phi'(v^\varepsilon)|^p\right)^{\frac{1}{p}} + \left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |f'(u^\varepsilon) - f'(v^\varepsilon)|^p\right)^{\frac{1}{p}}\right] \|w^\varepsilon\|_{H^1(\Omega_\varepsilon)} \|z^\varepsilon\|_{X_\varepsilon}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|F'_\varepsilon(u^\varepsilon) - F'_\varepsilon(v^\varepsilon)\|_{\mathcal{L}(H^1(\Omega_\varepsilon), X'_\varepsilon)} &= \sup_{\|z^\varepsilon\|_{X_\varepsilon}=1} |\langle (F'_\varepsilon(u^\varepsilon) - F'_\varepsilon(v^\varepsilon))w^\varepsilon, z^\varepsilon\rangle| \\
&\leq \left[\left(\int_{\Omega_\varepsilon} |\Phi'(u^\varepsilon) - \Phi'(v^\varepsilon)|^p\right)^{\frac{1}{p}} + \left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |f'(u^\varepsilon) - f'(v^\varepsilon)|^p\right)^{\frac{1}{p}}\right].
\end{aligned}$$

Now, for all $x \in \Omega_\varepsilon$, we have

$$|f'(u^\varepsilon(x)) - f'(v^\varepsilon(x))| \leq 2\left(\sup_{x\in\mathbb{R}} |f'(x)|\right).$$

On the other hand, by Mean Value Theorem,

$$|f'(u^\varepsilon(x)) - f'(v^\varepsilon(x))| \leq \left(\sup_{x\in\mathbb{R}} |f''(x)|\right) |u^\varepsilon(x) - v^\varepsilon(x)|.$$

Thus, if $\vartheta \in (0,1)$,

$$\begin{aligned}
|f'(u^\varepsilon) - f'(v^\varepsilon)|^p &\leq K_1 \min\{1, |u^\varepsilon - v^\varepsilon|^p\} \\
&= K_1 \min\{1, |u^\varepsilon - v^\varepsilon|^p\}^\vartheta \min\{1, |u^\varepsilon - v^\varepsilon|^p\}^{1-\vartheta} \\
&\leq K_1 |u^\varepsilon - v^\varepsilon|^{\vartheta p}.
\end{aligned}$$

Taking ϑ such that $\vartheta p = 2$ (ie, for some $1/3 < \vartheta < \frac{1}{2}$), it follows that

$$\begin{aligned}
\left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |f'(u^\varepsilon) - f'(v^\varepsilon)|^p\right)^{1/p} &\leq \left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} K_1 |u^\varepsilon - v^\varepsilon|^2\right)^{1/p} \\
&\leq M_1 \|u^\varepsilon - v^\varepsilon\|^\vartheta_{H^1(\Omega_\varepsilon)}.
\end{aligned}$$

In a similar way,

$$\begin{aligned}
\left(\int_{\Omega_\varepsilon} |\Phi'(u^\varepsilon) - \Phi'(v^\varepsilon)|^p\right)^{1/p} &\leq \left(\int_{\Omega_\varepsilon} \bar{K}_2 |u^\varepsilon - v^\varepsilon|^2\right)^{1/p} \\
&\leq \bar{M}_2 \|u^\varepsilon - v^\varepsilon\|^\vartheta_{H^1(\Omega_\varepsilon)}.
\end{aligned}$$

Furthermore, for some $\vartheta \in (0,1)$,

$$\|F'_\varepsilon(u^\varepsilon) - F'_\varepsilon(v^\varepsilon)\|_{\mathcal{L}(H^1(\Omega_\varepsilon), X'_\varepsilon)} \leq M \|u^\varepsilon - v^\varepsilon\|^\vartheta_{H^1(\Omega_\varepsilon)}$$

(f) If $u^\varepsilon, v^\varepsilon \in H^1(\Omega_\varepsilon)$ and $z^\varepsilon \in X_\varepsilon$,

$$\begin{aligned}
&\langle F_\varepsilon(u^\varepsilon + v^\varepsilon) - F_\varepsilon(u^\varepsilon) - F'_\varepsilon(u^\varepsilon)v^\varepsilon, z^\varepsilon\rangle = \\
&= \int_{\Omega_\varepsilon} (\Phi(u^\varepsilon + v^\varepsilon) - \Phi(u^\varepsilon) - \Phi'(u^\varepsilon)v^\varepsilon)z^\varepsilon + \frac{1}{\varepsilon}\int_{\theta_\varepsilon} (f(u^\varepsilon + v^\varepsilon) - f(u^\varepsilon) - f'(u^\varepsilon)v^\varepsilon)z^\varepsilon.
\end{aligned}$$

Hence, we can argue as in the proof of item (c) to obtain, for any $\delta \in (0,1)$, that

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\theta_\varepsilon} &|f(u^\varepsilon+v^\varepsilon)-f(u^\varepsilon)-f'(u^\varepsilon)v^\varepsilon||z^\varepsilon| \leq \\ &\leq \left(\int_{\Omega_\varepsilon}|\Phi(u^\varepsilon+v^\varepsilon)-\Phi(u^\varepsilon)-\Phi'(u^\varepsilon)v^\varepsilon|^2\right)^{\frac{1}{2}}\left(\int_{\Omega_\varepsilon}|z^\varepsilon|^2\right)^{\frac{1}{2}}+ \\ &\quad+\left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon}|f(u^\varepsilon+v^\varepsilon)-f(u^\varepsilon)-f'(u^\varepsilon)v^\varepsilon|^2\right)^{\frac{1}{2}}\left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon}|z^\varepsilon|^2\right)^{\frac{1}{2}} \\ &\leq C_1\left(\int_{\Omega_\varepsilon}|v^\varepsilon|^{2(1+\delta)}\right)^{\frac{1}{2}}\|z^\varepsilon\|_{X_\varepsilon}+C_2\left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon}|v^\varepsilon|^{2(1+\delta)}\right)^{\frac{1}{2}}\|z^\varepsilon\|_{X_\varepsilon} \\ &\leq k\|v^\varepsilon\|^{1+\delta}_{H^1(\Omega_\varepsilon)}\|z^\varepsilon\|_{X_\varepsilon}.\end{aligned}$$

Therefore,

$$\|F_\varepsilon(u^\varepsilon+v^\varepsilon)-F_\varepsilon(u^\varepsilon)-F'_\varepsilon(u^\varepsilon)v^\varepsilon\|_{X'_\varepsilon}\leq k\|v^\varepsilon\|^{1+\delta}_{H^1(\Omega_\varepsilon)},\ \forall\delta\in(0,1),$$

which concludes the proof. □

Remark 7. The results from Proposition 6 are also valid if

$$\begin{aligned}F_\varepsilon : H^1(\Omega_\varepsilon) &\to H^{-1}(\Omega_\varepsilon)\\ u &\mapsto F_\varepsilon(u) : H^1(\Omega_\varepsilon)\to\mathbb{R}\\ &\qquad v\mapsto\langle F_\varepsilon(u),v\rangle=\int_{\Omega_\varepsilon}\Phi(u)v+\frac{1}{\varepsilon}\int_{\theta_\varepsilon}f(u)v\end{aligned}$$

or

$$\begin{aligned}F_\varepsilon : X_\varepsilon &\to H^{-1}(\Omega_\varepsilon)\\ u &\mapsto F_\varepsilon(u) : H^1(\Omega_\varepsilon)\to\mathbb{R}\\ &\qquad v\mapsto\langle F_\varepsilon(u),v\rangle=\int_{\Omega_\varepsilon}\Phi(u)v+\frac{1}{\varepsilon}\int_{\theta_\varepsilon}f(u)v.\end{aligned}$$

This is a consequence of Proposition 1 and Theorem 4.1.

6. **Upper and lower semicontinuity of solutions.** Our main goal in this section is to prove Theorem 2.6, passing to the limit in problem (1). First of all, we write equations (1) and (2) in an abstract way. Next, we combine the results from the previous sections with those ones from [7, 9] concerned with compact convergence to obtain upper and lower semicontinuity to $\mathcal{E}_\varepsilon$ at $\varepsilon = 0$.

6.1. **Abstract setting and existence of solutions.** In order to write problem (1) in an abstract way, we consider the linear operator

$$\begin{aligned}A_\varepsilon : D(A_\varepsilon)\subset L^2(\Omega_\varepsilon)&\to L^2(\Omega_\varepsilon)\\ u^\varepsilon&\mapsto A_\varepsilon u^\varepsilon=-\Delta u^\varepsilon+u^\varepsilon\end{aligned}$$

with $D(A_\varepsilon)=\{u^\varepsilon\in H^2(\Omega_\varepsilon); \frac{\partial u^\varepsilon}{\partial\nu^\varepsilon}=0\}$.

Let $Z^0_\varepsilon = L^2(\Omega_\varepsilon)$, $Z^1_\varepsilon = D(A_\varepsilon)$ and consider the scale of Hilbert spaces constructed by complex interpolation between Z^0_ε and Z^1_ε. In our context, such spaces isometrically coincide with the fractional power space of the operator A_ε (see [26, Theorem 16.1]). Such scale can be extended to negative exponents such as

$Z_\varepsilon^{-\alpha} = (Z_\varepsilon^{\alpha})'$ for $\alpha > 0$. Notice that $Z_\varepsilon^{\frac{1}{2}} = H^1(\Omega_\varepsilon)$ and $Z_\varepsilon^{-\frac{1}{2}} = (H^1(\Omega_\varepsilon))'$. Hence, if we consider the realizations of A_ε in this scale, we have $A_{\varepsilon,-\frac{1}{2}} \in \mathcal{L}(Z_\varepsilon^{\frac{1}{2}}, Z_\varepsilon^{-\frac{1}{2}})$ with

$$\langle A_{\varepsilon,-\frac{1}{2}}\, u^\varepsilon, \varphi^\varepsilon\rangle = \int_{\Omega_\varepsilon} \nabla u^\varepsilon \nabla \varphi^\varepsilon + u^\varepsilon \varphi^\varepsilon,\ \forall \varphi^\varepsilon \in H^1(\Omega_\varepsilon).$$

With some abuse of notation we identify all different realizations of this operator writing as A_ε. Then the problem (1) can be rewrite as

$$A_\varepsilon u^\varepsilon = F_\varepsilon(u^\varepsilon), \tag{18}$$

where the map F_ε is given by

$$\begin{aligned} F_\varepsilon : H^1(\Omega_\varepsilon) &\to X_\varepsilon' \\ u^\varepsilon &\mapsto F_\varepsilon(u^\varepsilon) : L^2(0,1; H^s(0, G_\varepsilon(x_1))) \to \mathbb{R} \\ &\qquad v^\varepsilon \mapsto \langle F_\varepsilon(u^\varepsilon), v^\varepsilon\rangle = \int_{\Omega_\varepsilon} \Phi(u^\varepsilon) v^\varepsilon + \frac{1}{\varepsilon}\int_{\theta_\varepsilon} f(u^\varepsilon) v^\varepsilon, \end{aligned}$$

with $\frac{1}{2} < s < 1$.

Thus, $u^\varepsilon \in H^1(\Omega_\varepsilon)$ is a solution of (18) if, and only if, $u^\varepsilon = A_\varepsilon^{-1} F_\varepsilon(u^\varepsilon)$. Then, $u^\varepsilon \in H^1(\Omega_\varepsilon)$ must be a fixed point of $A_\varepsilon^{-1}F_\varepsilon : H^1(\Omega_\varepsilon) \to H^1(\Omega_\varepsilon)$. The existence of such solutions follows from Schaefer Fixed Point Theorem [18, Section 9.2.2, Theorem 4].

Indeed, as we will see in Proposition 11, we have that the operator $A_\varepsilon^{-1}F_\varepsilon$ is compact. Hence, to conclude the existence, we just need to prove that

$$O_\varepsilon = \{\varphi^\varepsilon \in H^1(\Omega_\varepsilon);\ \varphi^\varepsilon = A_\varepsilon^{-1} F_\varepsilon(\varphi^\varepsilon)\}$$

is a bounded set. Now, it is a direct consequence from Hölder's Inequality and Theorem 4.1 since

$$\begin{aligned} \|\varphi^\varepsilon\|^2_{H^1(\Omega_\varepsilon)} &\le \int_{\Omega_\varepsilon} |\Phi(\varphi^\varepsilon)\varphi^\varepsilon| + \frac{1}{\varepsilon}\int_{\theta_\varepsilon} |f(\varphi^\varepsilon)\varphi^\varepsilon| \\ &\le \left(\sup_{x\in\mathbb{R}} |\Phi(x)|\right) G_1^{\frac{1}{2}} \left(\int_{\Omega_\varepsilon} |\varphi^\varepsilon|^2\right)^{\frac{1}{2}} + \left(\sup_{x\in\mathbb{R}} |f(x)|\right) H_1^{\frac{1}{2}} \left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |\varphi^\varepsilon|^2\right)^{\frac{1}{2}} \\ &\le C\|\varphi^\varepsilon\|_{H^1(\Omega_\varepsilon)}, \end{aligned}$$

for any $\varphi^\varepsilon \in O_\varepsilon$.

In a similar way, we can analyze the limit problem given in (2). We first consider the linear operator $A_0 \in \mathcal{L}(H^1(\Omega), H^1(\Omega)')$ with

$$\langle A_0\, u, \varphi\rangle = \int_\Omega \nabla u \nabla\varphi + u\varphi,\ \forall \varphi \in H^1(\Omega),$$

and then, we set the nonlinearity

$$\begin{aligned} F_0 : H^1(\Omega) &\to L^2(0,1; \{H^s(0, m(x_1))\}') \\ u &\mapsto F_0(u) : L^2(0,1; H^s(0, m(x_1))) \to \mathbb{R} \\ &\qquad v \mapsto \langle F_0(u), v\rangle = \int_\Omega \Phi(u) v + \int_\Gamma \hat{\mu}\gamma(f(u))\gamma(v) dS. \end{aligned}$$

Then the limit problem (2) can be rewritten as

$$A_0 u = F_0(u) \tag{19}$$

and, with this notation, $u \in H^1(\Omega)$ is a solution of (19) if, and only if, $u = A_0^{-1}F_0(u)$. In other words, $u \in H^1(\Omega)$ is a fixed point of $A_0^{-1}F_0 : H^1(\Omega) \to H^1(\Omega)$. Again, the existence of a solution follows from Schauder's Fixed Point Theorem.

6.2. **Extension operator.** A particular continuous linear extension operator is useful here. For the proof see [7, Proposition 4.1].

Proposition 7. *Let Ω_ε be the family of domains defined in (4). Then, for each $1 \le p \le \infty$, there are $\varepsilon_0 > 0$ and a continuous extension operator $P_{\Omega_\varepsilon} : L^1(\Omega_\varepsilon) \to L^1(\mathbb{R}^2)$ such that, with the notation $X(V) = L^p(V)$ or $W^{1,p}(V)$ for an open set $V \subset \mathbb{R}^2$, P_{Ω_ε} transforms functions of $X(\Omega_\varepsilon)$ in $X(\mathbb{R}^2)$ with*

$$\|P_{\Omega_\varepsilon}\|_{\mathcal{L}(X(\Omega_\varepsilon),X(\mathbb{R}^2))} \le K, \ \text{for } 0 < \varepsilon < \varepsilon_0,$$

for some $K > 0$ independent of ε.

Moreover, P_{Ω_ε} is constructed in such way that $P_{\Omega_\varepsilon} u \equiv 0$ outside an open set U, where U contain the closure of Ω_ε for all $\varepsilon > 0$.

Remark 8. The construction of operators P_{Ω_ε} allows us to introduce a new family of operator $P_{\Omega_\varepsilon,V} : X(\Omega_\varepsilon) \to X(V)$ given by $P_{\Omega_\varepsilon,V} = R_V P_{\Omega_\varepsilon}$, where R_V is the restriction to the open set V. Using this notation, $P_{\Omega_\varepsilon} = P_{\Omega_\varepsilon,\mathbb{R}^2}$. We also have $\|P_{\Omega_\varepsilon,V}\|_{\mathcal{L}(X(\Omega_\varepsilon),X(V))} \le C$ independent of ε (see [7, Remark 4.2]).

The next lemma is convenient to get E-convergence results in Ω_ε (see [7, Lemma 4.3]).

Lemma 6.1. *Let $\{u^\varepsilon\}$ be a family in $H^1(\Omega_\varepsilon)$ with $\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)} \le M$. Then*

(i) there is a subsequence of u^ε, denoted by u^{ε_k}, and $u_0 \in H^1(\Omega)$ such that $u^{\varepsilon_k} \overset{E}{\rightharpoonup} u_0$;
(ii) there is a subsequence of u^ε, denoted by u^{ε_n}, and $u \in H^1(U)$ such that $P_{\Omega_{\varepsilon_n},U} u^{\varepsilon_n} \rightharpoonup u$ in $H^1(U)$ and $u^{\varepsilon_n} \overset{E}{\rightharpoonup} u|_\Omega$.

6.3. **Continuity of the equilibria set.** We first show that the solutions are uniformly bounded in $L^\infty(\Omega_\varepsilon)$.

Proposition 8. *If $u^\varepsilon \in H^1(\Omega_\varepsilon)$ is a solution of (18), then there is $C > 0$ independent of $\varepsilon > 0$ such that $\|u^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \le C$.*

Proof. If $u^\varepsilon \in H^1(\Omega_\varepsilon)$ is solution of (18), we have for all $\varphi^\varepsilon \in H^1(\Omega_\varepsilon)$ that

$$\int_{\Omega_\varepsilon} \frac{\partial u^\varepsilon}{\partial x_1}\frac{\partial \varphi^\varepsilon}{\partial x_1} + \int_{\Omega_\varepsilon} \frac{\partial u^\varepsilon}{\partial x_2}\frac{\partial \varphi^\varepsilon}{\partial x_2} + \int_{\Omega_\varepsilon} u^\varepsilon \varphi^\varepsilon = \int_{\Omega_\varepsilon} \Phi(u^\varepsilon)\varphi^\varepsilon + \frac{1}{\varepsilon}\int_{\theta_\varepsilon} f(u^\varepsilon)\varphi^\varepsilon.$$

Now, if $A_{\varepsilon,k} := \{(x_1,x_2) \in \Omega_\varepsilon;\ u^\varepsilon(x_1,x_2) > k\}$ for $k > 0$, take $\varphi^\varepsilon = (u^\varepsilon - k)^+ \in H^1(\Omega_\varepsilon)$, where

$$(u^\varepsilon - k)^+(x_1,x_2) = \begin{cases} u^\varepsilon(x_1,x_2) - k, & \text{if } (x_1,x_2) \in A_{\varepsilon,k}, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\int_{\Omega_\varepsilon} \frac{\partial u^\varepsilon}{\partial x_1}\frac{\partial (u^\varepsilon - k)^+}{\partial x_1} + \int_{\Omega_\varepsilon} \frac{\partial u^\varepsilon}{\partial x_2}\frac{\partial (u^\varepsilon - k)^+}{\partial x_2} + \int_{\Omega_\varepsilon} u^\varepsilon (u^\varepsilon - k)^+$$
$$= \int_{\Omega_\varepsilon} \Phi(u^\varepsilon)(u^\varepsilon - k)^+ + \frac{1}{\varepsilon}\int_{\theta_\varepsilon} f(u^\varepsilon)(u^\varepsilon - k)^+.$$

Thus using Hölder's Inequality, Theorem 4.1 and the definition of $A_{\varepsilon,k}$, we get

$$\|(u^\varepsilon - k)^+\|^2_{H^1(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon\cap A_{\varepsilon,k}} \Phi(u^\varepsilon)(u^\varepsilon - k)^+ + \frac{1}{\varepsilon}\int_{\theta_\varepsilon\cap A_{\varepsilon,k}} f(u^\varepsilon)(u^\varepsilon - k)^+ - \int_{\Omega_\varepsilon\cap A_{\varepsilon,k}} k(u^\varepsilon - k)^+$$

$$\le \left(\int_{\Omega_\varepsilon\cap A_{\varepsilon,k}} |\Phi(u^\varepsilon)|^2\right)^{\frac12}\left(\int_{\Omega_\varepsilon\cap A_{\varepsilon,k}} |u^\varepsilon - k|^2\right)^{\frac12} + \left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon\cap A_{\varepsilon,k}} |f(u^\varepsilon)|^2\right)^{\frac12}\left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon\cap A_{\varepsilon,k}} |u^\varepsilon - k|^2\right)^{\frac12}$$

$$\le \left(\sup_{x\in\mathbb{R}}|\Phi(x)|\right)|A_{\varepsilon,k}|^{\frac12}\|u^\varepsilon - k\|_{H^1(\Omega_\varepsilon)} + \left(\sup_{x\in\mathbb{R}}|f(x)|\right)\frac{|\theta_\varepsilon|^{\frac12}|A_{\varepsilon,k}|^{\frac12}}{\varepsilon^{\frac12}}\|u^\varepsilon - k\|_{H^1(\Omega_\varepsilon)}.$$

Since the set θ_ε has order ε, we obtain that

$$\|u^\varepsilon - k\|_{H^1(\Omega_\varepsilon)} \le C_1|A_{\varepsilon,k}|^{\frac12} \tag{20}$$

where $C_1 > 0$ is independent of $\varepsilon > 0$.

Otherwise, notice that for p, q conjugates (in other words, $1/p + 1/q = 1$) we have

$$\|(u^\varepsilon - k)^+\|_{L^1(A_{\varepsilon,k})} = \int_{A_{\varepsilon,k}} (u^\varepsilon - k) \le \left(\int_{A_{\varepsilon,k}} 1^p\right)^{1/p}\left(\int_{A_{\varepsilon,k}} (u^\varepsilon - k)^q\right)^{1/q}$$
$$\le |A_{\varepsilon,k}|^{1/p}\|(u^\varepsilon - k)\|_{L^q(\Omega_\varepsilon)}. \tag{21}$$

From Proposition 3(c), we have that $H^1(\Omega_\varepsilon) \subseteq L^q(\Omega_\varepsilon)$ for $2 \le q \le 4$. Thus, taking $2 < q < 4$ and its conjugate $1 < p < 2$, we obtain from (20) in (21) that

$$\|(u^\varepsilon - k)^+\|_{L^1(A_{\varepsilon,k})} \le C_2|A_{\varepsilon,k}|^{1/p}\|(u^\varepsilon - k)\|_{H^1(\Omega_\varepsilon)} \le K|A_{\varepsilon,k}|^{\frac12 + 1/p} = K|A_{\varepsilon,k}|^{1+\delta}$$

for some $\delta > 0$ since $\frac12 < 1/p < 1$.

Therefore, applying [22, Lemma 5.1] we obtain $\|u^\varepsilon\|_{L^\infty(\Omega_\varepsilon)}$ uniformly bounded, proving the result. □

We also need the following lemma.

Lemma 6.2. *Let $u^\varepsilon, w^\varepsilon \in H^1(\Omega_\varepsilon)$ given by $w^\varepsilon = A_\varepsilon^{-1}F_\varepsilon(u^\varepsilon)$. Then $\|w^\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C$ for some $C > 0$ independent of ε.*

Proof. Since $w^\varepsilon = A_\varepsilon^{-1}F_\varepsilon u^\varepsilon$, it follows that, for any $\varphi^\varepsilon \in H^1(\Omega_\varepsilon)$,

$$\int_{\Omega_\varepsilon} \frac{\partial w^\varepsilon}{\partial x_1}\frac{\partial \varphi^\varepsilon}{\partial x_1} + \int_{\Omega_\varepsilon} \frac{\partial w^\varepsilon}{\partial x_2}\frac{\partial \varphi^\varepsilon}{\partial x_2} + \int_{\Omega_\varepsilon} w^\varepsilon\varphi^\varepsilon = \int_{\Omega_\varepsilon} \Phi(u^\varepsilon)\varphi^\varepsilon + \frac{1}{\varepsilon}\int_{\theta_\varepsilon} f(u^\varepsilon)\varphi^\varepsilon.$$

Therefore, taking $\varphi^\varepsilon = w^\varepsilon$, we have from Hölder's Inequality, the limitation of Φ, f and Theorem 4.1 that

$$\|w^\varepsilon\|^2_{H^1(\Omega_\varepsilon)} \le \left(\int_{\Omega_\varepsilon} |\Phi(u^\varepsilon)|^2\right)^{\frac12}\left(\int_{\Omega_\varepsilon} |w^\varepsilon|^2\right)^{\frac12} + \left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |f(u^\varepsilon)|^2\right)^{\frac12}\left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |w^\varepsilon|^2\right)^{\frac12}$$
$$\le \left(\sup_{x\in\mathbb{R}}|\Phi(x)|\right)G_1^{\frac12}\|w^\varepsilon\|_{H^1(\Omega_\varepsilon)} + \left(\sup_{x\in\mathbb{R}}|f(x)|\right)H_1^{\frac12}\|w^\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C\|w^\varepsilon\|_{H^1(\Omega_\varepsilon)},$$

which shows the result. □

Next, we analyze the asymptotic behavior of the nonlinear terms of the problem.

Proposition 9. *Let $w^\varepsilon, u^\varepsilon \in H^1(\Omega_\varepsilon)$ and $w, u \in H^1(U)$ such that $P_{\Omega_\varepsilon,U}(u^\varepsilon) \rightharpoonup u$ and $P_{\Omega_\varepsilon,U}(w^\varepsilon) \rightharpoonup w$ in $H^1(U)$, where $P_{\Omega_\varepsilon,U}$ is the extension operator given by Proposition 7. Then*

$$\int_{\Omega_\varepsilon} \Phi(u^\varepsilon)w^\varepsilon \to \int_\Omega \Phi(u)w \quad \text{and} \quad \frac{1}{\varepsilon}\int_{\theta_\varepsilon} f(u^\varepsilon)w^\varepsilon \to \int_\Gamma \hat{\mu}\gamma(f(u))\gamma(w)dS,$$

where $\hat{\mu}$ is given by (6).

Proof. To prove the first convergence, notice that using the Main Value Theorem we obtain

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon} \Phi(u^\varepsilon)w^\varepsilon - \int_\Omega \Phi(u)w\right| &\le \left|\int_{\Omega_\varepsilon} \Phi(u^\varepsilon)(w^\varepsilon - w)\right| + \left|\int_{\Omega_\varepsilon} (\Phi(u^\varepsilon) - \Phi(u))w\right| \\
&\quad + \left|\int_{\Omega_\varepsilon} \Phi(u)w - \int_\Omega \Phi(u)w\right| \\
&\le \left(\int_{\Omega_\varepsilon} |\Phi(u^\varepsilon)|^2\right)^{\frac{1}{2}} \left(\int_{\Omega_\varepsilon} |w^\varepsilon - w|^2\right)^{\frac{1}{2}} + \left(\int_{\Omega_\varepsilon} |\Phi(u^\varepsilon) - \Phi(u)|^2\right)^{\frac{1}{2}} \left(\int_{\Omega_\varepsilon} |w|^2\right)^{\frac{1}{2}} \\
&\quad + \left|\int_{\Omega_\varepsilon} \Phi(u)w - \int_\Omega \Phi(u)w\right| \\
&\le \left(\sup_{x\in\mathbb{R}} |\Phi(x)|\right) G_1^{\frac{1}{2}} \|w^\varepsilon - w\|_{L^2(\Omega_\varepsilon)} + \left(\sup_{x\in\mathbb{R}} |\Phi'(x)|\right) \|u^\varepsilon - u\|_{X_\varepsilon} \|w\|_{L^2(\Omega_\varepsilon)} \\
&\quad + \left|\int_{\Omega_\varepsilon} \Phi(u)w - \int_\Omega \Phi(u)w\right| = i + ii + iii
\end{aligned}$$

Since $P_{\Omega_\varepsilon,U}(u^\varepsilon) \rightharpoonup u$ and $P_{\Omega_\varepsilon,U}(w^\varepsilon) \rightharpoonup w$ in $H^1(U)$, we have that $P_{\Omega_\varepsilon,U}(u^\varepsilon) \to u$ and $P_{\Omega_\varepsilon,U}(w^\varepsilon) \to w$ in $L^2(U)$. Using that Φ and Φ' are uniformly bounded and properties from the extension operator given by Proposition 7, we obtain

$$i = \left(\sup_{x\in\mathbb{R}} |\Phi(x)|\right) G_1^{\frac{1}{2}} \|w^\varepsilon - w\|_{L^2(\Omega_\varepsilon)} \le \left(\sup_{x\in\mathbb{R}} |f(x)|\right) H_1^{\frac{1}{2}} \|P_{\Omega_\varepsilon,U} w^\varepsilon - w\|_{L^2(U)} \to 0$$

and

$$\begin{aligned}
ii &= \left(\sup_{x\in\mathbb{R}} |\Phi'(x)|\right) \|u^\varepsilon - u\|_{L^2(\Omega_\varepsilon)} \|w\|_{L^2(\Omega_\varepsilon)} \\
&\le \left(\sup_{x\in\mathbb{R}} |f'(x)|\right) \|P_{\Omega_\varepsilon,U} u^\varepsilon - u\|_{L^2(U)}\| \|w\|_{L^2(U)} \to 0.
\end{aligned}$$

Since $iii \to 0$ by Corollary 1, we obtain the first result.

On the other side, to prove the second convergence we have

$$\begin{aligned}
&\left|\frac{1}{\varepsilon}\int_{\theta_\varepsilon} f(u^\varepsilon)w^\varepsilon - \int_\Gamma \hat{\mu}\gamma(f(u))\gamma(w)dS\right| \le \left|\frac{1}{\varepsilon}\int_{\theta_\varepsilon} f(u^\varepsilon)(w^\varepsilon - w)\right| \\
&\quad + \left|\frac{1}{\varepsilon}\int_{\theta_\varepsilon} (f(u^\varepsilon) - f(u))w\right| + \left|\frac{1}{\varepsilon}\int_{\theta_\varepsilon} f(u)w - \int_\Gamma \hat{\mu}\gamma(f(u))\gamma(w)dS\right| \\
&\le \left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |f(u^\varepsilon)|^2\right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |w^\varepsilon - w|^2\right)^{\frac{1}{2}} + \left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |f(u^\varepsilon) - f(u)|^2\right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon} |w|^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$+\left|\frac{1}{\varepsilon}\int_{\theta_\varepsilon} f(u)w - \int_\Gamma \hat{\mu}\gamma(f(u))\gamma(w)dS\right|$$
$$\leq \left(\sup_{x\in\mathbb{R}}|f(x)|\right) H_1^{\frac{1}{2}}\|w^\varepsilon - w\|_{X_\varepsilon} + \left(\sup_{x\in\mathbb{R}}|f'(x)|\right)\|u^\varepsilon - u\|_{X_\varepsilon}\|w\|_{H^1(\Omega_\varepsilon)}$$
$$+\left|\frac{1}{\varepsilon}\int_{\theta_\varepsilon} f(u)w - \int_\Gamma \hat{\mu}\gamma(f(u))\gamma(w)dS\right| = I + II + III,$$

with $X_\varepsilon = L^2(0,1;H^s(0,G_\varepsilon(x_1)))$ for $\frac{1}{2} < s < 1$.

Notice that, since we are working on $\mathbb{R}^2$, $U \subset U_1 \times U_2$, with $U_1, U_2 \subset \mathbb{R}$ open sets, $(0,1) \subset U_1$ and $(0, G_\varepsilon(x_1)) \subset U_2$ for all $x_1 \in (0,1)$ and $0 < \varepsilon < \varepsilon_0$. Therefore $H^1(U) \subset H^1(U_1 \times U_2) \subset L^2(U_1; H^s(U_2)) =: X_U$, where the last inclusion is compact by Proposition 1. Thus, for some $k_1, k_2 > 0$,

$$I = \left(\sup_{x\in\mathbb{R}}|f(x)|\right) H_1^{\frac{1}{2}}\|w^\varepsilon - w\|_{X_\varepsilon} \leq k_1 H_1^{\frac{1}{2}}\|P_{\Omega_\varepsilon,U}w^\varepsilon - w\|_{X_U} \to 0$$

and

$$II = \left(\sup_{x\in\mathbb{R}}|f'(x)|\right)\|u^\varepsilon - u\|_{X_\varepsilon}\|w\|_{H^1(\Omega_\varepsilon)} \leq k_2\|P_{\Omega_\varepsilon,U}u^\varepsilon - u\|_{X_U}\|w\|_{H^1(U)} \to 0.$$

Finally $III \to 0$ again by Corollary 1 and we conclude the proof. □

Proposition 10. *Let $u^\varepsilon, v^\varepsilon \in H^1(\Omega_\varepsilon)$ and $u, v \in H^1(U)$ such that $P_{\Omega_\varepsilon,U}(u^\varepsilon) \rightharpoonup u$ and $P_{\Omega_\varepsilon,U}(v^\varepsilon) \rightharpoonup v$ in $H^1(U)$, where $P_{\Omega_\varepsilon,U}$ is the extension operator given by Proposition 7. Then, for all $\varphi \in H^1(U)$,*

$$\int_{\Omega_\varepsilon} \Phi'(u^\varepsilon)v^\varepsilon\varphi \to \int_\Omega \Phi'(u)v\varphi \quad \text{and} \quad \frac{1}{\varepsilon}\int_{\theta_\varepsilon} f'(u^\varepsilon)v^\varepsilon\varphi \to \int_\Gamma \hat{\mu}\gamma(f'(u))\gamma(v)\gamma(\varphi)dS,$$

where $\hat{\mu}$ is given by (6).

Proof. Indeed, to prove the first result we have

$$\left|\int_{\Omega_\varepsilon} \Phi'(u^\varepsilon)v^\varepsilon\varphi - \int_\Omega \Phi'(u)v\varphi\right| \leq \left|\int_{\Omega_\varepsilon} \Phi'(u^\varepsilon)(v^\varepsilon - v)\varphi\right| + \left|\int_{\Omega_\varepsilon} (\Phi'(u^\varepsilon) - \Phi'(u))v\varphi\right|$$
$$+\left|\int_{\Omega_\varepsilon} \Phi'(u)v\varphi - \int_\Omega \Phi'(u)v\varphi\right| = i + ii + iii$$

Remembering that Φ, Φ' are uniformly bounded and that $P_{\Omega_\varepsilon,U}(u^\varepsilon) \rightharpoonup u$ and $P_{\Omega_\varepsilon,U}(v^\varepsilon) \rightharpoonup v$ in $H^1(U)$ implies $P_{\Omega_\varepsilon,U}(u^\varepsilon) \to u$ and $P_{\Omega_\varepsilon,U}(v^\varepsilon) \to v$ in $L^2(U)$, we can analyze each term on the right:

$$i = \left|\int_{\Omega_\varepsilon} \Phi'(u^\varepsilon)(v^\varepsilon - v)\varphi\right| \leq \left(\sup_{x\in\mathbb{R}}|\Phi'(x)|\right)\left(\int_{\Omega_\varepsilon}|v^\varepsilon - v|^2\right)^{\frac{1}{2}}\left(\int_{\Omega_\varepsilon}|\varphi|^2\right)^{\frac{1}{2}}$$
$$\leq \left(\sup_{x\in\mathbb{R}}|\Phi'(x)|\right)\|v^\varepsilon - v\|_{L^2(\Omega_\varepsilon)}\|\varphi\|_{L^2(\Omega_\varepsilon)}$$
$$\leq \left(\sup_{x\in\mathbb{R}}|\Phi'(x)|\right)\|P_{\Omega_\varepsilon,U}v^\varepsilon - v\|_{L^2(U)}\|\varphi\|_{L^2(U)} \to 0$$

and using the Sobolev inclusion [26, Theorem 1.36] we have, for some $C > 0$ independent of ε that

$$\begin{aligned}
ii &= \left|\int_{\Omega_\varepsilon}(\Phi'(u^\varepsilon)-\Phi'(u))v\varphi\right| \le \int_{\Omega_\varepsilon}|(\Phi'(u^\varepsilon)-\Phi'(u))v\varphi| \\
&\le \left(\sup_{x\in\mathbb{R}}|\Phi''(x)|\right)\left(\int_{\Omega_\varepsilon}|u^\varepsilon-u|^2\right)^{\frac{1}{2}}\left(\int_{\Omega_\varepsilon}|v|^4\right)^{1/4}\left(\int_{\Omega_\varepsilon}|\varphi|^4\right)^{1/4} \\
&\le \|u^\varepsilon-u\|_{L^2(\Omega_\varepsilon)}\|v\|_{L^4(\Omega_\varepsilon)}\|\varphi\|_{L^4(\Omega_\varepsilon)} \le \|P_{\Omega_\varepsilon,U}u^\varepsilon-u\|_{L^2(U)}\|v\|_{L^4(U)}\|\varphi\|_{L^4(U)} \\
&\le C\|P_{\Omega_\varepsilon,U}u^\varepsilon-u\|_{L^2(U)}\|v\|_{H^1(U)}\|\varphi\|_{H^1(U)} \to 0.
\end{aligned}$$

For iii, using Corollary 3,

$$iii = \left|\int_{\Omega_\varepsilon}\Phi'(u)v\varphi - \int_\Omega \Phi'(u)v\varphi\right| \to 0,$$

proving the first result.

To prove the second convergence, we have

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{\theta_\varepsilon} f'(u^\varepsilon)v^\varepsilon\varphi &= \frac{1}{\varepsilon}\int_{\theta_\varepsilon} f'(u^\varepsilon)(v^\varepsilon-v)\varphi + \frac{1}{\varepsilon}\int_{\theta_\varepsilon}(f'(u^\varepsilon)-f'(u))v\varphi + \\
&+ \frac{1}{\varepsilon}\int_{\theta_\varepsilon} f'(u)v\varphi = I + II + III.
\end{aligned}$$

Analyzing each term separately and using the definition of X_U given in the proof of Proposition 9:

$$\begin{aligned}
I &= \frac{1}{\varepsilon}\int_{\theta_\varepsilon} f'(u^\varepsilon)(v^\varepsilon-v)\varphi \le \left(\sup_{x\in\mathbb{R}}|f'(x)|\right)\left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon}|v^\varepsilon-v|^2\right)^{\frac{1}{2}}\left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon}|\varphi|^2\right)^{\frac{1}{2}} \\
&\le C\|v^\varepsilon-v\|_{X_\varepsilon}\|\varphi\|_{H^1(\Omega_\varepsilon)} \le C\|P_{\Omega_\varepsilon,U}v^\varepsilon-v\|_{X_U}\|\varphi\|_{H^1(U)} \to 0.
\end{aligned}$$

Since f' is C^1, applying Corollary 3, we get

$$III = \frac{1}{\varepsilon}\int_{\theta_\varepsilon} f'(u)v\varphi \to \int_\Gamma \hat{\mu}\gamma(f'(u))\gamma(\varphi)\gamma(\psi)dS.$$

Finally, notice that we can rewrite II as

$$\begin{aligned}
\Psi_\varepsilon : H^1(U) &\to \mathbb{R} \\
\varphi &\mapsto \frac{1}{\varepsilon}\int_{\theta_\varepsilon}(f'(u^\varepsilon)-f'(u))v\varphi.
\end{aligned}$$

It follows that Ψ is a bounded linear operator in $H^1(U)$ since, using Theorem 4.1,

$$|\Psi_\varepsilon(\varphi)| \le 2\left(\sup_{x\in\mathbb{R}}|f'(x)|\right)\left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon}|v|^2\right)^{\frac{1}{2}}\left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon}|\varphi|^2\right)^{\frac{1}{2}} \le C\|v\|_{H^1(U)}\|\varphi\|_{H^1(U)}.$$

Besides, for all $\varphi \in C_c^\infty(\bar{U})$,

$$\begin{aligned}
\Psi_\varepsilon(\varphi) &= \frac{1}{\varepsilon}\int_{\theta_\varepsilon}(f'(u^\varepsilon)-f'(u))v\varphi \\
&\le \left(\sup_{x\in\mathbb{R}}|f''(x)|\right)\left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon}|u^\varepsilon-u|^2\right)^{\frac{1}{2}}\left(\frac{1}{\varepsilon}\int_{\theta_\varepsilon}|v|^2\right)^{\frac{1}{2}}\|\varphi\|_\infty \\
&\le K\|P_{\Omega_\varepsilon,U}u^\varepsilon-u\|_{X_U}\|v\|_{H^1(U)}\|\varphi\|_{H^1(U)} \to 0
\end{aligned}$$

and then, by density, we have $II = \Psi_\varepsilon(\varphi) \to 0$, for all $\varphi \in H^1(U)$. This concludes the proof. □

For now on, consider the spaces $H_\varepsilon = H^1(\Omega_\varepsilon)$ and $H_0 = H^1(\Omega)$ in the context of Definition 2.1. We prove the result which guarantee the upper and lower semicontinuity of the set of solutions from (18) at $\varepsilon = 0$.

Proposition 11. *Using the notations from (18) and (19), we have that* $A_\varepsilon^{-1}F_\varepsilon \xrightarrow{CC} A_0^{-1}F_0$.

Proof. To prove the compact convergence, we verify separately each item.

(a) $A_\varepsilon^{-1}F_\varepsilon$ is a compact operator, for each $\varepsilon > 0$.

Since by Proposition 1 $H^1(\Omega_\varepsilon) \hookrightarrow X_\varepsilon$ with compact immersion, we have $X'_\varepsilon \hookrightarrow H^{-1}(\Omega_\varepsilon)$ compactly. Also, F_ε is a Lipschitz function by Proposition 6(b). Thus, we get the result from

$$H^1(\Omega_\varepsilon) \xrightarrow{F_\varepsilon} X'_\varepsilon \xrightarrow{i} H^{-1}(\Omega_\varepsilon) \xrightarrow{A_\varepsilon^{-1}} H^1(\Omega_\varepsilon).$$

(b) If $\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)} \le K$, then $\{A_\varepsilon^{-1}F_\varepsilon(u^\varepsilon)\}$ is E-precompact.

Let $\{u^\varepsilon\}$ such that $\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)} \le K$. By Lemma 6.1 we obtain a subsequence, that we still call u^ε, such that $P_{\Omega_\varepsilon,U}u^\varepsilon \rightharpoonup u$ in $H^1(U)$ and $u^\varepsilon \overset{E}{\rightharpoonup} u|_\Omega$ for some $u \in H^1(U)$. Consider $w^\varepsilon = A_\varepsilon^{-1}F_\varepsilon(u^\varepsilon)$. By Lemma 6.2, $\|w^\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C$ and, thus, again by Lemma 6.1, there exists a subsequence, also called w^ε, and $w \in H^1(U)$ such that $P_{\Omega_\varepsilon,U}w^\varepsilon \rightharpoonup w$ in $H^1(U)$ and $w^\varepsilon \overset{E}{\rightharpoonup} w|_\Omega$.

If we call $u_0 = u|_\Omega$ and $w_0 = w|_\Omega$, we have that $w_0 = A_0^{-1}F_0(u_0)$. Indeed, $w^\varepsilon \overset{E}{\rightharpoonup} w_0$ implies for any $v \in H^1(U)$ that $(w^\varepsilon, v)_{H^1(\Omega_\varepsilon)} \to (w_0, v)_{H^1(\Omega)}$. On other hand, by Proposition 9 we have

$$(w^\varepsilon, v)_{H^1(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} \Phi(u^\varepsilon)v + \frac{1}{\varepsilon}\int_{\theta_\varepsilon} f(u^\varepsilon)v \to \int_\Omega \Phi(u_0)v + \int_\Gamma \hat{\mu}\gamma(f(u_0))\gamma(v)dS.$$

Thus, since the limit is unique, we get, for all $v \in H^1(U)$,

$$\langle A_0 w_0, v\rangle = (w_0, v)_{H^1(\Omega)} = \int_\Omega \Phi(u_0)v + \int_\Gamma \hat{\mu}\gamma(f(u_0))\gamma(v)dS = \langle F_0(u_0), v\rangle,$$

and, therefore, $w_0 = A_0^{-1}F_0(u_0)$. Now, let us prove $\|w^\varepsilon\|_{H^1(\Omega_\varepsilon)} \to \|w_0\|_{H^1(\Omega)}$, implying $w^\varepsilon \xrightarrow{E} w_0$ by [7, Proposition 3.2]. As a matter of fact, using Proposition 9 again, we have

$$\begin{aligned}
&\|w^\varepsilon\|^2_{H^1(\Omega_\varepsilon)} = (w^\varepsilon, w^\varepsilon)_{H^1(\Omega)} = (A_\varepsilon^{-1}F_\varepsilon(u^\varepsilon), w^\varepsilon)_{H^1(\Omega)} \\
&= \int_{\Omega_\varepsilon} \Phi(u^\varepsilon)w^\varepsilon + \frac{1}{\varepsilon}\int_{\theta_\varepsilon} f(u^\varepsilon)w^\varepsilon \to \int_\Omega \Phi(u_0)w_0 + \int_\Gamma \hat{\mu}\gamma(f(u_0))\gamma(w_0)dS \\
&= (A_0^{-1}F_0(u_0), w_0)_{H^1(\Omega)} = (w_0, w_0)_{H^1(\Omega)} = \|w_0\|^2_{H^1(\Omega)}.
\end{aligned}$$

(c) If $u^\varepsilon \xrightarrow{E} u$, then $A_\varepsilon^{-1}F_\varepsilon(u^\varepsilon) \xrightarrow{E} A_0^{-1}F_0(u)$.

Indeed, if we assume that $u^\varepsilon \xrightarrow{E} u$, we get $\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C$, for some $C > 0$ independent of ε. In particular, for any subsequence of u^ε, we can find another subsequence, denoting all by u^ε, such that, using the same argument of the previous item, we have $P_{\Omega_\varepsilon,U}(u^\varepsilon) \rightharpoonup u$, with $u_0 = u|_\Omega$ and, for this subsequence, $A_\varepsilon^{-1}F_\varepsilon(u^\varepsilon) \xrightarrow{E} A_0^{-1}F_0(u_0)$. As we can prove this for any subsequence, we obtain the E-convergence of all family, that is, $A_\varepsilon^{-1}F_\varepsilon(u^\varepsilon) \xrightarrow{E} A_0^{-1}F_0(u_0)$.

∎

Finally, we can conclude the upper and lower semicontinuity of the equilibrium set at $\varepsilon = 0$ proving Theorem 2.6. Indeed, from Proposition 11 and [9, Proposition 5.6], we have:

Proposition 12. *For any family $\{u^\varepsilon\}$, $u^\varepsilon \in H^1(\Omega_\varepsilon)$ solution of (18), there is $u_* \in H^1(\Omega)$ solution of (19) and a subsequence still denoted by u^ε, such that $u^\varepsilon \xrightarrow{E} u_*$.*

Moreover, with the assumption that the limit solution is hyperbolic, we can get lower semicontinuity of the equilibrium set. More precisely, from Proposition 11 and [9, Proposition 5.7] we have

Proposition 13. *If $u_* \in H^1(\Omega)$ solution of (19) is hyperbolic, then there is a sequence $\{u_*^\varepsilon\}$, $u_*^\varepsilon \in H^1(\Omega_\varepsilon)$ solution of (18), such that $u_*^\varepsilon \xrightarrow{E} u_*$.*

Remark 9. In the case when all equilibria points of the limit equation (19) are hyperbolic, we have that all of them are isolated and there is only a finite number of them (see [9, Corollary 5.4 or Proposition 5.5]).

Notice that the continuity above does not exclude the possibility that near an equilibrium point of the limiting equation may exist several different equilibrium points of the perturbed problem. We show that is possible to obtain some sort of uniqueness of the equilibrium points concluding the proof of Theorem 2.6.

First we will prove an important result about the compact convergence of the operators $A_\varepsilon^{-1}F'_\varepsilon(u_*^\varepsilon)$ if $u_*^\varepsilon \in H^1(\Omega_\varepsilon)$ is a sequence of solutions from (18) that is E-convergent.

Proposition 14. *If $\{u^\varepsilon\}$ is a sequence of solutions of (18), $u^\varepsilon \in H^1(\Omega_\varepsilon)$, and $u_0 \in H^1(\Omega)$ is solution of (19) then $A_\varepsilon^{-1}F'_\varepsilon(u^\varepsilon) \xrightarrow{CC} A_0^{-1}F'_0(u_0)$ whenever $u^\varepsilon \xrightarrow{E} u_0$.*

Proof. We prove by steps, as in Proposition 11.

(i) $A_\varepsilon^{-1}F'_\varepsilon(u^\varepsilon)$ is compact, for each $\varepsilon > 0$.

Since $H^1(\Omega_\varepsilon) \hookrightarrow X_\varepsilon$ with compact immersion by Proposition 1, we have

$$H^1(\Omega_\varepsilon) \xrightarrow{F'_\varepsilon(u^\varepsilon)} X'_\varepsilon \xrightarrow{i} H^{-1}(\Omega_\varepsilon) \xrightarrow{A_\varepsilon^{-1}} H^1(\Omega_\varepsilon),$$

where $F'_\varepsilon(u^\varepsilon)$ is continuous by Proposition 6(d), proving the affirmation.

(ii) $A_\varepsilon^{-1}F'_\varepsilon(u^\varepsilon)v^\varepsilon$ is E-precompact whenever $\|v^\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C$.

Let $\{v^\varepsilon\}$ family in $H^1(\Omega_\varepsilon)$ such that $\|v^\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C$ and define $w^\varepsilon = A_\varepsilon^{-1}F'_\varepsilon(u^\varepsilon)v^\varepsilon$. Then for any $\varphi^\varepsilon \in H^1(\Omega_\varepsilon)$,

$$\int_{\Omega_\varepsilon} \frac{\partial w^\varepsilon}{\partial x_1}\frac{\partial \varphi^\varepsilon}{\partial x_1} + \int_{\Omega_\varepsilon} \frac{\partial w^\varepsilon}{\partial x_2}\frac{\partial \varphi^\varepsilon}{\partial x_2} + \int_{\Omega_\varepsilon} w^\varepsilon \varphi^\varepsilon = \int_{\Omega_\varepsilon} \Phi'(u^\varepsilon)v^\varepsilon\varphi^\varepsilon + \frac{1}{\varepsilon}\int_{\theta_\varepsilon} f'(u^\varepsilon)v^\varepsilon\varphi^\varepsilon.$$

If $\varphi^\varepsilon = w^\varepsilon$ follows by Theorem 4.1

$$\begin{aligned}\|w^\varepsilon\|^2_{H^1(\Omega_\varepsilon)} &= \int_{\Omega_\varepsilon} \Phi'(u^\varepsilon)v^\varepsilon w^\varepsilon + \frac{1}{\varepsilon}\int_{\theta_\varepsilon} f'(u^\varepsilon)v^\varepsilon w^\varepsilon \\ &\le \left(\sup_{x\in\mathbb{R}}|\Phi'(x)|\right)\|v^\varepsilon\|_{H^1(\Omega_\varepsilon)}\|w^\varepsilon\|_{H^1(\Omega_\varepsilon)} + \left(\sup_{x\in\mathbb{R}}|f'(x)|\right)C^2\|v^\varepsilon\|_{H^1(\Omega_\varepsilon)}\|w^\varepsilon\|_{H^1(\Omega_\varepsilon)}\end{aligned}$$

and, thus, $\|w^\varepsilon\|_{H^1(\Omega_\varepsilon)} \le K$, for some $K > 0$ independent of ε. Therefore, by Lemma 6.1 we obtain subsequences, also denoted by v^ε, w^ε, and $v, w \in H^1(U)$ such that $P_{\Omega_\varepsilon,U}(v^\varepsilon) \rightharpoonup v$ and $P_{\Omega_\varepsilon,U}(w^\varepsilon) \rightharpoonup w$ both in $H^1(U)$, with $v^\varepsilon \overset{E}{\rightharpoonup} v|_\Omega$ and $w^\varepsilon \overset{E}{\rightharpoonup} w|_\Omega$.

Now if we call $v_0 = v|_\Omega$ and $w_0 = w|_\Omega$ we may prove that $w_0 = A_0^{-1} F_0'(u_0) v_0$. Indeed, for $\varphi \in H^1(U)$

$$(w^\varepsilon, \varphi)_{H^1(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} \Phi'(u^\varepsilon) v^\varepsilon \varphi + \frac{1}{\varepsilon} \int_{\theta_\varepsilon} f'(u^\varepsilon) v^\varepsilon \varphi. \tag{22}$$

On one hand, using Proposition 10, we have

$$\begin{aligned}\int_{\Omega_\varepsilon} \Phi'(u^\varepsilon) v^\varepsilon \varphi + \frac{1}{\varepsilon} \int_{\theta_\varepsilon} f'(u^\varepsilon) v^\varepsilon \varphi \to \int_\Omega \Phi'(u_0) v_0 \varphi + \int_\Gamma \hat{\mu} \gamma(f'(u_0)) \gamma(v_0) \gamma(\varphi) dS \\ = (A_0^{-1} F_0'(u_0) v_0, \varphi)_{H^1(\Omega)}.\end{aligned}$$

However, since $w^\varepsilon \xrightarrow{E} w|_\Omega$,

$$(w^\varepsilon, \varphi)_{H^1(\Omega_\varepsilon)} \to (w_0, \varphi)_{H^1(\Omega)}.$$

Thus $w_0 = A_0^{-1} F_0'(u_0) v_0$.

Finally, we show that $w^\varepsilon \xrightarrow{E} w_0$. By [7, Proposition 3.2], it is enough to prove $\|w^\varepsilon\|_{H^1(\Omega_\varepsilon)} \to \|w_0\|_{H^1(\Omega)}$. But, if we take $\varphi = w^\varepsilon$ in (22) we obtain arguing as in the proof of Proposition 11, the norm convergence.

(iii) $A_\varepsilon^{-1} F_\varepsilon'(u^\varepsilon) v^\varepsilon \xrightarrow{E} A_0^{-1} F_0'(u_0) v_0$ se $v^\varepsilon \xrightarrow{E} v_0$.

To prove that $w^\varepsilon \xrightarrow{E} w_0$ for the whole sequence it is enough to use an analogous proof of this step in Proposition 11.

□

The following lemma is the last one that we need to conclude the uniqueness of equilibrium points near a hyperbolic limit solution.

Lemma 6.3. *If $u_*^\varepsilon \in H^1(\Omega_\varepsilon)$ is a solution of (18) then there is $K > 0$ such that, for all $v^\varepsilon \in H^1(\Omega_\varepsilon)$ with $\|v^\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq 1$, we have*

$$\|A_\varepsilon^{-1}(F_\varepsilon(u_*^\varepsilon + v^\varepsilon) - F_\varepsilon(u_*^\varepsilon) - F_\varepsilon'(u_*^\varepsilon) v^\varepsilon)\|_{H^1(\Omega_\varepsilon)} \leq K \|v^\varepsilon\|^{1+\delta}_{H^1(\Omega_\varepsilon)}, \quad \textit{for some } \delta \in (0,1).$$

Proof. Let $w^\varepsilon = A_\varepsilon^{-1}(F_\varepsilon(u_*^\varepsilon + v^\varepsilon) - F_\varepsilon(u_*^\varepsilon) - F_\varepsilon'(u_*^\varepsilon) v^\varepsilon)$. This implies that, for all $\varphi^\varepsilon \in H^1(\Omega_\varepsilon)$,

$$\begin{aligned}\int_{\Omega_\varepsilon} \frac{\partial w^\varepsilon}{\partial x_1} \frac{\partial \varphi^\varepsilon}{\partial x_1} + \int_{\Omega_\varepsilon} \frac{\partial w^\varepsilon}{\partial x_2} \frac{\partial \varphi^\varepsilon}{\partial x_2} + \int_{\Omega_\varepsilon} w^\varepsilon \varphi^\varepsilon = \int_{\Omega_\varepsilon} (\Phi(u_*^\varepsilon + v^\varepsilon) - \Phi(u_*^\varepsilon) - \Phi'(u_*^\varepsilon) v^\varepsilon) \varphi^\varepsilon \\ + \frac{1}{\varepsilon} \int_{\theta_\varepsilon} (f(u_*^\varepsilon + v^\varepsilon) - f(u_*^\varepsilon) - f'(u_*^\varepsilon) v^\varepsilon) \varphi^\varepsilon.\end{aligned}$$

Taking $\varphi^\varepsilon = w^\varepsilon$, the left side of the equation becomes $\|w^\varepsilon\|^2_{H^1(\Omega_\varepsilon)}$. For the right side, with a fixed $1 < p < 2$ in a way that its conjugate q is $2 < q < 4$, follows by Theorem 4.1 that

$$\begin{aligned}&\int_{\Omega_\varepsilon} (\Phi(u_*^\varepsilon + v^\varepsilon) - \Phi(u_*^\varepsilon) - \Phi'(u_*^\varepsilon) v^\varepsilon) w^\varepsilon + \frac{1}{\varepsilon} \int_{\theta_\varepsilon} (f(u_*^\varepsilon + v^\varepsilon) - f(u_*^\varepsilon) - f'(u_*^\varepsilon) v^\varepsilon) w^\varepsilon \\ &\leq \left(\int_{\Omega_\varepsilon} |\Phi(u_*^\varepsilon + v^\varepsilon) - \Phi(u_*^\varepsilon) - \Phi'(u_*^\varepsilon) v^\varepsilon|^p \right)^{1/p} \left(\int_{\Omega_\varepsilon} |w^\varepsilon|^q \right)^{1/q} + \\ &\quad + \left(\frac{1}{\varepsilon} \int_{\theta_\varepsilon} |f(u_*^\varepsilon + v^\varepsilon) - f(u_*^\varepsilon) - f'(u_*^\varepsilon) v^\varepsilon|^p \right)^{1/p} \left(\frac{1}{\varepsilon} \int_{\theta_\varepsilon} |w^\varepsilon|^q \right)^{1/q}\end{aligned}$$

$$\leq \left(\int_{\Omega_\varepsilon} |\Phi(u_*^\varepsilon + v^\varepsilon) - \Phi(u_*^\varepsilon) - \Phi'(u_*^\varepsilon) v^\varepsilon|^p \right)^{1/p} \|w^\varepsilon\|_{H^1(\Omega_\varepsilon)} + \\ + \left(\frac{1}{\varepsilon} \int_{\theta_\varepsilon} |f(u_*^\varepsilon + v^\varepsilon) - f(u_*^\varepsilon) - f'(u_*^\varepsilon) v^\varepsilon|^p \right)^{1/p} \|w^\varepsilon\|_{H^1(\Omega_\varepsilon)}.$$

By Proposition 6(f) we obtain, for $\delta \in (0,1)$ such that $p(1+\delta) = 2$ or, in other words, $2/p = (1+\delta)$,

$$\|w^\varepsilon\|^2_{H^1(\Omega_\varepsilon)} \leq \left(\int_{\Omega_\varepsilon} |v^\varepsilon|^2 \right)^{\frac{1}{2}\frac{2}{p}} \|w^\varepsilon\|_{H^1(\Omega_\varepsilon)} + C \left(\frac{1}{\varepsilon} \int_{\theta_\varepsilon} |v^\varepsilon|^2 \right)^{\frac{1}{2}\frac{2}{p}} \|w^\varepsilon\|_{H^1(\Omega_\varepsilon)} \\ \leq C^2 \|v^\varepsilon\|^{2/p}_{H^1(\Omega_\varepsilon)} \|w^\varepsilon\|_{H^1(\Omega_\varepsilon)} = K \|v^\varepsilon\|^{1+\delta}_{H^1(\Omega_\varepsilon)} \|w^\varepsilon\|_{H^1(\Omega_\varepsilon)}$$

and, thus,

$$\|w^\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq K \|v^\varepsilon\|^{1+\delta}_{H^1(\Omega_\varepsilon)}$$

proving the result. □

Now we can conclude the uniqueness of the equilibrium as ε is close to zero.

Proposition 15. *If u_0^* is a hyperbolic equilibrium of (19), then there exist $\eta > 0$ and $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, there exists one, and only one, u_*^ε solution of (18) such that $\|u_*^\varepsilon - E_\varepsilon u_0^*\|_{H^1(\Omega_\varepsilon)} \leq \eta$. Furthermore $u_*^\varepsilon \xrightarrow{E} u_0^*$.*

Proof. This is a consequence of [7, Proposition 5.5] or [9, Theorem 5.8]. □

Finally, we can prove the main result of this section.

Proof of Theorem 2.6. The item (a) follows from Theorem 12. On the other hand, (b) follows from Theorem 13 and Proposition 15. □

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