A NEW CHARACTERIZATION OF THE JACOBIAN CONJECTURE IN THE REAL PLANE AND SOME CONSEQUENCES

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Abstract. In this work we use tools of dynamical systems and algebraic geometry to give a new characterization of the Jacobian conjecture for polynomial maps in the the real plane in terms of the non-existence of real branches emerging from a singularity of a 1-parameter family of algebraic curves explicitly given. Using the Newton-Puiseux algorithm we state new sufficient conditions in order that the Jacobian conjecture in the real plane holds. We also provide an arbitrary degree family of polynomial maps satisfying the Jacobian conjecture.

1. Introduction

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map, i.e., $F(x, y) = (f(x, y), g(x, y))$ with $f, g \in \mathbb{R}[x, y]$. If we assume that its Jacobian $\det DF$ is nowhere zero, clearly F is a local diffeomorphism, but it is not always injective. Indeed there is a counterexample given in [22]. There are very general well-known additional sufficient conditions to ensure that F is a global diffeomorphism, see for instance [11, 16, 23].

The *Jacobian conjecture in the real plane* is an open problem that consists in to determine if any polynomial map F with non-zero constant $\det DF$ is injective. The conjecture was posed by Keller [20] in 1939 and since then it has been studied intensively and various partial results have been obtained. The reader can consult for example [4, 15] to obtain more information about the history of this well-known conjecture. We also mention that a formulation for polynomial maps in $\mathbb{C}^k$ of the Jacobian conjecture appears in the 16th problem of Smale's list [25] of open mathematical problems.

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Jacobian conjecture in $\mathbb{R}^2$: *Let $F = (f, g)$ be a real polynomial map with non-zero constant value of* $\det DF$. *Then F is a global diffeomorphism of the plane onto itself.*

In this work we give a new characterization of the Jacobian conjecture in $\mathbb{R}^2$ using tools from algebraic geometry and the qualitative theory of dynamical systems, see [18] as a seminal work.

Let $n = \deg(F) = \max\{\deg(f), \deg(g)\}$ and consider the real polynomial map

$$F(x,y) = (f(x,y), g(x,y)) = \left(x + \sum_{2 \le i \le n} f_i(x,y), y + \sum_{2 \le i \le n} g_i(x,y) \right), \tag{1}$$

with $\det DF \equiv 1$ and where f_i and g_i are homogeneous polynomials of degree i. Then $f_n \not\equiv 0$ and there is $k \in \mathbb{R}$ such that $g_n = k f_n$, see Proposition 10. We associate to F the 1-parameter family of algebraic curves $\mathcal{F}^{-1}(0)$ with polynomial

$$\mathcal{F}(u,v) = \mathcal{N}(u,v) - 2c(u^2+v^2)^{2n} = (1+k^2) f_n^2(u,v) + \cdots, \tag{2}$$

with arbitrary parameter $c \in \mathbb{R}$ and where $\mathcal{N}$ is defined as

$$H \circ \phi(u,v) = \frac{\mathcal{N}(u,v)}{2(u^2+v^2)^{2n}}, \tag{3}$$

with Bendixson map $\phi(u,v) = (u/(u^2+v^2), v/(u^2+v^2))$ and Hamiltonian $H = (f^2+g^2)/2$.

Theorem 1. *The map* (1) *with* $\det DF \equiv 1$ *is a global diffeomorphism of* $\mathbb{R}^2$ *if and only if its associated 1-parameter family of algebraic curves* $\mathcal{F}^{-1}(0)$ *has a singularity at the origin without real branches.*

We recall that, if $\mathcal{F}(u,v) = \sum_{i,j} f_{ij} u^i v^j$ with coefficients $f_{ij} \in \mathbb{R}$ and support $\mathrm{supp}(\mathcal{F}) = \{(j,i) \in \mathbb{N}^2 : f_{ij} \neq 0\}$, the Newton diagram $\mathbf{N}(\mathcal{F})$ of $\mathcal{F}$ is the boundary of the convex hull of the set $\bigcup_{(j,i)\in \mathrm{supp}(\mathcal{F})} \{(j,i) + \mathbb{R}^2_+\}$ where $\mathbb{R}^2_+$ is the positive quadrant. Looking at the local zero-set of $\mathcal{F}$ near the origin one is only interested in the different edges of $\mathbf{N}(\mathcal{F})$ with rational negative slopes $-k_1/k_2$, being k_1 and k_2 coprimes. We define $W(\mathbf{N}(\mathcal{F}))$ as the set whose elements are the former weights (k_2, k_1).

We define the first determining polynomial $\mathcal{P}(\eta) = f_n^2(1,\eta)$. Assuming that $W(\mathbf{N}(\mathcal{F})) = \{(1,1)\}$, we associate to each non-zero real root α_0 of $\mathcal{P}$ the polynomial $\hat{\mathcal{F}}(u,w) = \mathcal{F}(u, \alpha_0 u + w)$. To each weight $(p_i, q_i) \in W(\mathbf{N}(\hat{\mathcal{F}}))$ with $q_i/p_i > 1$ for $i = 1, \ldots, \ell$, we define the second determining polynomials $\mathcal{P}_i(\eta)$ as

$$\hat{\mathcal{F}}(u^{p_i}, u^{q_i}\eta) = u^{m_i}\eta^{n_i}\left(\mathcal{P}_i(\eta) + \mathcal{O}(u)\right), \tag{4}$$

for some multiplicities $m_i, n_i \in \mathbb{N}$.

Using Theorem 1 and the Newton-Puiseux algorithm (see [10] for details) we prove the following result.

Corollary 2. *The map* (1) *with* $\det DF \equiv 1$ *and* $W(\mathbf{N}(\mathcal{F})) = \{(1,1)\}$ *is a global diffeomorphism of* $\mathbb{R}^2$ *in the following cases:*

(i) The first determining polynomial $\mathcal{P}(\eta)$ *has no non-zero real roots;*

(ii) All the second determining polynomials $\mathcal{P}_i(\eta)$*,* $i = 1, \ldots, \ell$*, have no non-zero real roots.*

Remark 3. It is worth to recall that $\mathbf{N}(\mathcal{F})$ is not coordinate free, that is, in general $\mathbf{N}(\mathcal{F}) \neq \mathbf{N}(\mathcal{F} \circ \Psi)$ for some analytic diffeomorphism Ψ around the origin of $\mathbb{R}^2$. In the easiest case we can take a linear change $\Psi(u,v) = (au + bv, cu + dv)$ with $ad - bc \neq 0$ so that, using (2), $\mathcal{F} \circ \Psi(u,v) = (1 + k^2)\tilde{f}_n^2(u,v) + \cdots$, where $\tilde{f}_n(u,v) = f_n \circ \Psi(u,v) = \sum_{i+j=n} a_{ij}u^i v^j$ and generically with $a_{0n}a_{n0} \neq 0$. In summary, recalling that if $\mathcal{F}$ has no real branches at the origin then a conjugation of $\mathcal{F}$ doesn't have them either, after a linear change of coordinates Ψ, we get $W(\mathbf{N}(\mathcal{F} \circ \Psi)) = \{(1,1)\}$ generically. Hence we can perform the Newton-Puiseux algorithm to $\mathcal{F} \circ \Psi$ instead of $\mathcal{F}$ to obtain a similar result as Corollary 2. Another option in the non-generic situation $W(\mathbf{N}(\mathcal{F})) \neq \{(1,1)\}$ is to use the Newton-Puiseux algorithm to each edge in $\mathbf{N}(\mathcal{F})$ to describe a procedure that generalizes Corollary 2. Although, for the sake of simplicity, we do not explicitly write here this generalization we will use it in the example of section 6.

Remark 4. We emphasize that the sufficient condition provided in [5] for the Jacobian conjecture in $\mathbb{R}^2$ to be true under the assumptions $\det DF \equiv 1$ and $W(\mathbf{N}(\mathcal{F})) = \{(1,1)\}$ is equivalent to statement (i) in Corollary 2. More specifically, in Theorem 1 of [5] it is proved that, if the homogeneous polynomials $f_n\, \partial_x f_n$ and $f_n\, \partial_y f_n$ do not have common real linear factors and $\det DF$ is nowhere zero, then F is injective. We notice that the former condition about the non-existence of common real linear factors is equivalent to the non-existence of real linear factors of just f_n (hence the non-existence of real roots of $\mathcal{P}(\eta)$). This is true because, by the Euler relation $x\partial_x f_n + y\partial_y f_n = n f_n$, the common factors of $\partial_x f_n$ and $\partial_y f_n$ must be also factors of f_n.

Now we present one of the results one can obtain as application of Theorem 1.

Theorem 5. *Let $0 \neq k \in \mathbb{R}$ be such that $g_n = kf_n$ and assume that $f_{n-1}^2(1,k)+g_{n-1}^2(1,k) \neq 0$ and $k(-kf_{n-1}^2(1,k)+2f_{n-1}(1,k)g_{n-1}(1,k)+kg_{n-1}^2(1,k)) < 0$. Assume moreover that $\sum_{\substack{i+j=n+1\\ i,j\neq n}} J_{ij} \equiv 0$ where J_{ij} is the Jacobian determinant of the polynomial map (f_i, g_j). Then the Jacobian conjecture in $\mathbb{R}^2$ holds for family* (1) *with* $\det DF \equiv 1$.

Remark 6. Along the proof of Theorem 5 we see that the strange condition $\sum_{\substack{i+j=n+1\\ i,j\neq n}} J_{ij} \equiv 0$ is equivalent to impose that f_n only has a linear factor in its factorization over $\mathbb{C}[x,y]$ that has the form $f_n(x,y) = c_n(y-kx)^n$ with nonzero $c_n \in \mathbb{R}$.

2. Some background

2.1. **Global centers.** A singular point $(x_0, y_0) \in \mathbb{R}^2$ of a vector field in $\mathbb{R}^2$ is a *center* if it possesses a punctured neighborhood foliated by periodic orbits. The *period annulus* of a center is the maximal open punctured neighborhood with that property. The center is *global* if its period annulus is $\mathbb{R}^2 \backslash \{(x_0, y_0)\}$.

The next result due to Sabatini, see Theorem 2.3 of [24], will be used to prove Theorem 1.

Theorem 7 ([24])**.** *Let $F = (f, g)$ be a real polynomial map with nowhere zero* $\det DF$ *and $F(0,0) = (0,0)$. Then the following statements are equivalent.*

- *(a) The origin is a global center for the Hamiltonian vector field $\mathcal{X}_H = (-\partial_y H, \partial_x H)$ with $H = (f^2 + g^2)/2$.*
- *(b) F is a global diffeomorphism of the plane onto itself.*

2.2. **Bendixson's compactification: blowing-down the polycicle at infinity.** We state the following result which although is straightforward it is a key piece in the proof of Theorem 1. We recall here that a singularity of a planar vector field is called *monodromic* if the local orbits of it turn around the singularity. Examples of monodromic singularities are centers and foci.

Theorem 8. *Let the origin be the unique singularity of a real planar Hamiltonian polynomial vector field $\mathcal{X}$ of degree m. We consider the Bendixson compactification $\tilde{\mathcal{X}} = (u^2+v^2)^m \, \phi_*(\mathcal{X})$ of $\mathcal{X}$ where ϕ_* is the pull-back associated to $\phi(x,y) = (u(x,y), v(x,y)) = (x/(x^2+y^2), y/(x^2+y^2))$. If the origin is a center of $\mathcal{X}$ then it is a global center if and only if the origin of $\tilde{\mathcal{X}}$ is a monodromic singularity.*

Remark 9. To use Theorem 8 we must to study the monodromy of the origin of the associated vector field $\tilde{\mathcal{X}} = \tilde{P}(u,v)\partial_u + \tilde{Q}(u,v)\partial_v$. There are several monodromy algorithms for singularities of polynomial planar vector fields developed in the literature, see for instance [1,2,7,21]. It is worth to mention that all the necessary monodromic conditions stated in Theorem 3 of [1] holds for the origin of $\tilde{\mathcal{X}}$ but the sufficient monodromic condition (4) in Theorem 4 of [1] fails. When Theorem 4 in [1] does not work the monodromic algorithms involve blowing-up the singularity and its implementation with arbitrary degree is quite difficult, thus we follow another path.

The construction of the Bendixson compactification can be found in Chapt. 13 of [3] or Chapt. 5 of [14]. It follows that the monomials in the components of $\tilde{\mathcal{X}}$ have minimum degree $m+2$, see below. Hence $(0,0)$ is a singular point of $\tilde{\mathcal{X}}$. Notice that Bendixson compactification blows-down the polycycle Γ at infinity of $\mathcal{X}$ into the singularity at the origin of $\tilde{\mathcal{X}}$ preserving their monodromic nature. This argument proves Theorem 8 taking into account that the existence of a global center of $\mathcal{X}$ is equivalent to the one-side monodromy of Γ, that is, there is a well defined Poincaré map in a transversal section to Γ with one endpoint on Γ.

Letting $\mathcal{X} = P(x,y)\partial_x + Q(x,y)\partial_y$ with degree $m = \deg(\mathcal{X}) := \max\{\deg P, \deg Q\}$, and using that the Bendixson map $\phi(x,y) = (u,v) = (x/(x^2+y^2), y/(x^2+y^2))$ has inverse $\phi^{-1}(u,v) = (x,y) = (u/(u^2+v^2), v/(u^2+v^2))$, we get the explicit expression of the associated vector field $\tilde{\mathcal{X}} = \tilde{P}(u,v)\partial_u + \tilde{Q}(u,v)\partial_v$ defined in Theorem 8 which is

$$\begin{aligned}
\tilde{P}(u,v) &= (u^2+v^2)^m \left((v^2-u^2)\,(P\circ\phi^{-1})(u,v) - 2uv\,(Q\circ\phi^{-1})(u,v)\right),\\
\tilde{Q}(u,v) &= (u^2+v^2)^m \left((v^2-u^2)\,(Q\circ\phi^{-1})(u,v) - 2uv\,(P\circ\phi^{-1})(u,v)\right).
\end{aligned}$$

We write $P(x,y) = \sum_{1\le i\le m} P_i(x,y)$ and $Q(x,y) = \sum_{1\le i\le m} Q_i(x,y)$ with P_i and Q_i homogeneous polynomials of degree i. Then we have

$$\begin{aligned}
\tilde{P}(u,v) &= (v^2-u^2)\sum_{1\le i\le m}(u^2+v^2)^{m-i}P_i(u,v)\\
&\quad -2uv\sum_{1\le i\le m}(u^2+v^2)^{m-i}Q_i(u,v),\\
\tilde{Q}(u,v) &= (u^2-v^2)\sum_{1\le i\le m}(u^2+v^2)^{m-i}Q_i(u,v)\\
&\quad -2uv\sum_{1\le i\le m}(u^2+v^2)^{m-i}P_i(u,v).
\end{aligned}$$

We therefore have the expansion $\tilde{P}(u,v) = \tilde{P}_{m+2}(u,v) + \cdots$, $\tilde{Q}(u,v) = \tilde{Q}_{m+2}(u,v) + \cdots$ where $\tilde{P}_{m+2}$ and $\tilde{Q}_{m+2}$ are homogeneous polynomials of degree $m+2$ and the dots denote higher order terms. More specifically

$$\begin{aligned}\tilde{P}_{m+2}(u,v) &= (v^2-u^2)P_m(u,v) - 2uvQ_m(u,v),\\ \tilde{Q}_{m+2}(u,v) &= (u^2-v^2)Q_m(u,v) - 2uvP_m(u,v).\end{aligned}$$

We remark that, when m is even, the origin of $\tilde{\mathcal{X}}$ always possesses characteristic directions because $u\tilde{Q}_{m+2}(u,v) - v\tilde{P}_{m+2}(u,v)$ is an homogeneous polynomial of odd degree. Since $\mathcal{X}$ has degree m, that is, $P_m^2(x,y) + Q_m^2(x,y) \not\equiv 0$, it follows that $\tilde{P}_{m+2}^2(u,v) + \tilde{Q}_{m+2}^2(u,v) \not\equiv 0$ too. Therefore, for the origin of $\tilde{\mathcal{X}}$ to be monodromic it is necessary that m be odd.

2.3. **Branches of analytic curves at singularities.** In this subsection we consider real analytic functions $\mathcal{F}(x,y)$ defined in a neighborhood of the origin of $\mathbb{R}^2$. We assume that $\mathcal{F}(0,0)=0$ and $\nabla\mathcal{F}(0,0) = (0,0)$, hence the origin is a singular point of $\mathcal{F}$. By Newton-Puiseux Theorem (see [6] for instance) there exists a local finite factorization

$$\mathcal{F}(x,y) = U(x,y)\, x^m \prod_i (y - y_i^*(x))^{m_i} \tag{5}$$

where U is a real analytic unit, that is, U is a real-valued analytic function defined in a neighborhood of the origin with $U(0,0) \neq 0$, $m \in \mathbb{N}\cup\{0\}$, $m_i \in \mathbb{N}$ and the $y_i^*(x)$ are complex-valued holomorphic functions of x^{1/n_i} with $y_i^*(0) = 0$ called *branches* of $\mathcal{F}$ at the origin. The exponents n_i are positive integers called the *indices* of the branches $y_i^*(x)$.

Using branching theory we know that any branch y_i^* of $\mathcal{F}$ emerging from $(x,y) = (0,0)$ can be locally expressed as convergent Puiseux series determined by the descending sections of the Newton diagram $\mathbf{N}(\mathcal{F})$ of $\mathcal{F}$ with rational negative slopes $-k_1/k_2$, that is with $(k_2,k_1) \in W(\mathbf{N}(\mathcal{F}))$ being k_1 and k_2 coprimes. Thus any branch $y_i^*(x)$ in (5) has the form

$$y_i^*(x) = \alpha_0 x^{k_1/k_2} + o(x^{k_1/k_2}) \tag{6}$$

with $\alpha_0 \in \mathbb{C}\backslash\{0\}$ and $0 < k_1/k_2 \in \mathbb{Q}$. The leading coefficients α_0 are the nonzero roots of a *determining polynomial* $\mathcal{P}(\eta)$ associated to each descending segment of $\mathbf{N}(F)$, and defined by the first term of the expansion

$$\mathcal{F}(x^{k_2}, x^{k_1}\eta) = x^{s_1}\,\eta^{s_2}\big[\mathcal{P}(\eta) + \mathcal{O}(x)\big], \tag{7}$$

for some $s_i \in \mathbb{N}\cup\{0\}$. The Newton-Puiseux algorithm can be used to determine the higher order terms of the branch (6) appearing in its convergent Puiseux series

$$y_i^*(x) = \sum_{j\geq 0} \alpha_j x^{\frac{k_1}{k_2}+\frac{j}{n_i}}.$$

All the branches $y_i^*(x)$ are analytic functions of x^{1/n_i}. A branch $y_i^*(x)$ is called a *simple branch* if α_0 is a simple root of $\mathcal{P}$. For simple branches $n_i = k_2$ and moreover $\alpha_j \in \mathbb{R}$ for all j provided that $\alpha_0 \in \mathbb{R}$, see a proof in [26].

3. Proof of Theorem 1

Proof. We consider the Hamiltonian vector field $\mathcal{X}_H = (-\partial_y H, \partial_x H)$ with Hamiltonian $H = (f^2+g^2)/2$. By Theorem 7, $F = (f, g)$ is a global diffeomorphism in $\mathbb{R}^2$ if and only if the origin is a global center of $\mathcal{X}_H$.

Using the Bendixson's compactification $\tilde{\mathcal{X}} = (u^2+v^2)^{2n-1}\,\phi_*(\mathcal{X})$ of $\mathcal{X}_H$ as it was described in Theorem 8, we know that the origin is a global center of $\mathcal{X}_H$ if and only if the origin of $\tilde{\mathcal{X}}$ is a monodromic singularity. We claim that the above turns out into the equivalent condition that the associated 1-parameter family of algebraic curves $\mathcal{F}^{-1}(0)$ defined in (2) has a singularity at the origin without real branches finishing the proof.

To prove the claim we just notice that $\tilde{H} = H \circ \phi(u,v)$ with $\phi(u,v) = (u/(u^2+v^2), v/(u^2+v^2))$ is a rational first integral of the polynomial vector field $\tilde{\mathcal{X}}$ because H is a first integral of $\mathcal{X}_H$. Since all the orbits of $\tilde{\mathcal{X}}$ are contained in the level sets $\{\tilde{H} = c\}$ of $\tilde{H}$, and taking into account (3), it follows that any orbit of $\tilde{\mathcal{X}}$ must lie in one element of the 1-parameter family of algebraic curves $\mathcal{F}^{-1}(0)$ for certain value $c \in \mathbb{R}$. The monodromy of the origin of $\tilde{\mathcal{X}}$ is therefore equivalent to the non-existence of real branches of $\mathcal{F}^{-1}(0)$. □

4. Proof of Corollary 2

We stated in the introduction the contains of the following well-known proposition, see for example Lemma 10.2.4 in [15]. Anyway we include it here with very simple proof for the sake of completeness.

Proposition 10. *If* $\det DF \equiv 1$ *then there is* $k \in \mathbb{R}$ *such that* $g_n = k f_n$.

Proof. Using the Euler identity for homogeneous functions $\mathcal{X}_E(f_n) = n f_n$ and $\mathcal{X}_E(g_n) = n g_n$ where $\mathcal{X}_E = u\partial_u + v\partial_v$ we know that there exist functions $\tilde{\mathrm{f}}$ and $\tilde{\mathrm{g}}$ such that and $f_n = x^n \tilde{\mathrm{f}}(\sigma)$ and $g_n = x^n \tilde{\mathrm{g}}(\sigma)$ where $\sigma = y/x$. Since $\det DF \equiv 1$, the Jacobian of the homogeneous map

(f_n, g_n) is identically zero, which implies that $\tilde{\mathrm{f}}(\sigma)\tilde{\mathrm{g}}'(\sigma) - \tilde{\mathrm{g}}(\sigma)\tilde{\mathrm{f}}'(\sigma) = 0$ and therefore $\tilde{\mathrm{g}}(\sigma) = k\tilde{\mathrm{f}}(\sigma)$ finishing the proof.

□

Proof of Corollary 2. By (2) we have

$$\mathcal{F}(u, v) = (1 + k^2) f_n^2(u, v) + \cdots ,$$

and despite this expression, the function $\mathcal{F}$ is not necessary positive defined in a punctured neighborhood of the origin because the homogeneous polynomial f_n can have a linear factor in $\mathbb{R}[u, v]$.

The algebraic curve $\mathcal{F}^{-1}(0)$ has a singularity at $(u, v) = (0, 0)$ and its eventual branches emanating from it can be expressed as convergent Puiseux series $v^*(u) = \alpha_0 u + o(u)$ since $W(\mathbf{N}(\mathcal{F})) = \{(1, 1)\}$ by assumption. The leading coefficient α_0 is a non-zero root of the determining polynomial $\mathcal{P}(\eta) = (1 + k^2) f_n^2(1, \eta)$, see (7).

If $\alpha_0 \in \mathbb{R}$, we take $f_n(u, v) = \sum_{i+j=n} f_{ij} u^i v^j$ and we assume that $f_{0n} f_{n0} \neq 0$ so that $f_n^2(u, v) = f_{n0}^2 u^{2n} + \cdots + f_{0n}^2 v^{2n}$ which is clearly a necessary and sufficient condition for having $W(\mathbf{N}(\mathcal{F})) = \{(1, 1)\}$, since $f_n \not\equiv 0$.

In order to compute the next term in the Puiseux expansion of the branch $v^*(u)$ we perform the change $v \mapsto w$ with $v = \alpha_0 u + w$. Then $\mathcal{F}$ is transformed into $\hat{\mathcal{F}}(u, w) = \mathcal{F}(u, \alpha_0 u + w) = (1 + k^2) f_n^2(u, \alpha_0 u + w) + \cdots$ with $f_n^2(u, \alpha_0 u + w) = \hat{f}_{n0}^2 u^{2n} + \cdots + \hat{f}_{0n}^2 w^{2n}$. Indeed $\hat{f}_{n0} = 0$ since $f_n(u, \alpha_0 u) \equiv 0$ and consequently there is at least one edge in $\mathbf{N}(\hat{\mathcal{F}})$ with weights different from $(1, 1)$.

By the Newton-Puiseux algorithm we know that the eventual branches must be of the form $v^*(u) = \alpha_0 u + \alpha_1 u^{q_i/p_i} + o(u^{q_i/p_i})$ with $(p_i, q_i) \in W(\mathbf{N}(\hat{\mathcal{F}}))$ satisfying $q_i/p_i > 1$ and α_1 a non-zero real root of the second determining polynomial $\mathcal{P}_i(\eta)$ defined as

$$\hat{\mathcal{F}}(u^{p_i}, u^{q_i}\eta) = u^{m_i} \eta^{n_i} \left(\mathcal{P}_i(\eta) + \mathcal{O}(u) \right).$$

The proof of the statements (i) and (ii) finishes recalling Theorem 1 and imposing that all the possible branches $v^*(u)$ be complex-valued either because either $\alpha_0 \in \mathbb{C} \backslash \mathbb{R}$ or $\alpha_1 \in \mathbb{C} \backslash \mathbb{R}$, respectively. □

5. Proof of Theorem 5

Proof. Let $F = (f, g) = (x + \cdots + f_n, y + \cdots + k f_n)$ and consider the terms of degree $n - 1$ in the polynomial $\det DF$, that is, we look at $\mathcal{J}_{n-1} = \sum_{i+j=n+1} J_{ij}$ where J_{ij} denotes the Jacobian determinant of the polynomial map (f_i, g_j). Imposing the condition $\sum_{\substack{i+j=n+1 \\ i,j \neq n}} J_{ij} \equiv 0$, and taking into account that $\mathcal{J}_{n-1} \equiv 0$ since $\det DF = 1$, we obtain that f_n

must satisfy the linear partial differential equation $\partial_x f_n + k\partial_y f_n \equiv 0$. The general solution of this equation is an arbitrary function of $y - kx$, hence $f_n(x,y) = c_n(y-kx)^n$ with nonzero $c_n \in \mathbb{R}$. Then $W(\mathbf{N}(\mathcal{F})) = \{(1,1)\}$ thanks to (2) provided that $k \neq 0$, and the determining polynomial $\mathcal{P}(\eta)$ is, up to a multiplicative constant, given by $(\eta - k)^{2n}$.

To continue we need first to compute higher order terms in the expression of $\mathcal{F}$ given in (2). It is straightforward to see that $\mathcal{F}(u,w) = \sum_{i\geq 2n} \mathcal{F}_i(u,w)$ being $\mathcal{F}_i$ homogeneous polynomials of degree i and

$$
\begin{aligned}
\mathcal{F}_{2n}(u,v) &= (1+k^2) f_n^2(u,v),\\
\mathcal{F}_{2n+1}(u,v) &= 2(u^2+v^2) f_n(u,v)(f_{n-1}(u,v) + k g_{n-1}(u,v)),\\
\mathcal{F}_{2n+2}(u,v) &= (u^2+v^2)^2 (f_{n-1}^2(u,v) + 2 f_{n-2}(u,v) f_n(u,v) +\\
&\quad 2k f_n(u,v) g_{n-2}(u,v) + g_{n-1}^2(u,v)).
\end{aligned}
$$

Now the eventual branches of $\mathcal{F}$ are $v^*(u) = ku + o(u)$. We perform the linear change of variables $v \mapsto w$ defined by $v = ku + w$, such that $\mathcal{F}(u,v)$ is transformed into $\hat{\mathcal{F}}(u,w) = \mathcal{F}(u, ku+w) = \sum_{i\geq 2n} \hat{\mathcal{F}}_i(u,w)$ being $\hat{\mathcal{F}}_i(u,w) = \mathcal{F}_i(u, ku+w)$ homogeneous polynomials of degree i. Thus we obtain

$$
\begin{aligned}
\hat{\mathcal{F}}_{2n}(u,w) &= (1+k^2) c_n^2 w^{2n},\\
\hat{\mathcal{F}}_{2n+1}(u,w) &= 2c_n w^n ((1+k^2)u^2 + 2kuw + w^2) \times\\
&\quad (f_{n-1}(u, ku+w) + k g_{n-1}(u, ku+w)),\\
\hat{\mathcal{F}}_{2n+2}(u,w) &= ((1+k^2)u^2 + 2kuw + w^2)^2 \times\\
&\quad (f_{n-1}^2(u, ku+w) + g_{n-1}^2(u, ku+w) +\\
&\quad 2c_n w^n (f_{n-2}(u, ku+w) + k g_{n-2}(u, ku+w))).
\end{aligned}
$$

Due to $\hat{\mathcal{F}}_{2n}$ the point $(2n, 0)$ is a vertex of $\mathbf{N}(\hat{\mathcal{F}})$. Moreover, the point $(0, 2n+1) \notin \mathbf{N}(\hat{\mathcal{F}})$ because $\hat{\mathcal{F}}_{2n+1}(u, 0) = 0$. We impose that $(0, 2n+2) \in \mathbf{N}(\hat{\mathcal{F}})$, that is, the homogeneous polynomial $f_{n-1}^2(u, ku+w) + g_{n-1}^2(u, ku+w)$ contains the monomial u^{2n-2}; equivalently $f_{n-1}^2(1,k) + g_{n-1}^2(1,k) \neq 0$. In this situation we claim that $\mathbf{N}(\hat{\mathcal{F}})$ has only one edge the one connecting the points $(2n, 0)$ and $(0, 2n+2)$. The claim follows just observing first that the points with integer coordinates in $\mathrm{supp}(\hat{\mathcal{F}})$ coming from $\hat{\mathcal{F}}_{2n+1}$, that is those on the line $x + y = 2n+1$, have abscissas greater or equal than n since w^n divides $\hat{\mathcal{F}}_{2n+1}$ and secondly because the point $(n, n+1)$ is the intersection between the former line and the edge of $\mathbf{N}(\hat{\mathcal{F}})$.

The associated weights of the edge of $\mathbf{N}(\hat{\mathcal{F}})$ are $(p_1, q_1) = (2n, 2n+2)$, hence $q_1/p_1 > 1$ and we need to compute the associated second determining polynomial $\mathcal{P}_1(\eta)$. Following (4) we see that

$$\hat{\mathcal{F}}(u^{2n}, u^{2n+2}\eta) = u^{4n(1+n)} \left(\mathcal{P}_1(\eta) + \mathcal{O}(u)\right),$$

where $\mathcal{P}_1(\eta) = (1+k^2)(c_n^2\eta^{2n} + (1+k^2)f_{n-1}^2(1,k) + 2c_n\eta^n(f_{n-1}(1,k) + kg_{n-1}(1,k)))$. Then η^n is a root of a quadratic polynomial with discriminant $\Delta = k(-kf_{n-1}^2(1,k) + 2f_{n-1}(1,k)g_{n-1}(1,k) + kg_{n-1}^2(1,k))$. We impose that $\Delta < 0$ that guarantee $\eta^n \in \mathbb{C}\backslash\mathbb{R}$ and so $\eta \in \mathbb{C}\backslash\mathbb{R}$. Therefore $\hat{\mathcal{F}}$ has no real branches at the origin and the same happens to $\mathcal{F}$. This finishes the proof applying Theorem 1. □

Remark 11. In the case $k \neq 0$, applying the theory of asymptotic solutions at the origin of the differential equation $\tilde{P}(u,v)dv - \tilde{Q}(u,v)du = 0$ associated to the vector field $\tilde{\mathcal{X}}$ explained in [13,17], according to [8,9]. This theory is based on the so-called Newton diagram $\mathbf{N}(\tilde{\mathcal{X}})$ of $\tilde{\mathcal{X}}$ that is also a polygonal with associated weights $W(\mathbf{N}(\tilde{\mathcal{X}})) \subset \mathbb{N}^2$, see for example [17] to a precise definition of it. The outcome of this analysis is that the unique balance ku has associated a formal Gâteaux derivative with polynomial $V(j) \equiv 0$ which is a degenerate case that does not provide us any information about the index N of the branch v^* having Puiseux series $v^*(u) = ku + \sum_{i\geq 1} \alpha_i u^{1+i/N}$. This case is commented in [19, page 298], see also page 5 in [12], and the proposed resolution of this degeneracy consists in to perform a change of variables to compute the next non-zero term in $v^*(u) = ku + o(u)$ that is, the term $\alpha_j u^{1+j/N}$ for some $j \geq 1$ such that $\alpha_i = 0$ for all $i < j$.

6. Example with $k = 0$

We present an example where $W(\mathbf{N}(\mathcal{F})) \neq \{(1,1)\}$ and moreover $W(\mathbf{N}(\mathcal{X})) \neq W(\mathbf{N}(\mathcal{F}))$.

We take the map $F(x,y) = (f(x,y), g(x,y)) = (x + ay^3 + by^4, y)$ with $(a,b) \in \mathbb{R}^2$ that has $\det DF \equiv 1$. One has $W(\mathbf{N}(\mathcal{X})) = \{(4,7),(1,1)\}$ and $W(\mathbf{N}(\mathcal{F})) = \{(4,7)\}$, hence we get that $W(\mathbf{N}(\mathcal{X})) \cap W(\mathbf{N}(\mathcal{F})) = (p,q) = (4,7)$ and the eventual real branches are of the form $v^*(u) = \alpha_0 u^{7/4} + o(u^{7/4})$ with α_0 non-zero real root of $\mathcal{P}(\eta) = (1+b\eta^4)^2$. So the real branch may exists only if $b < 0$. Going to the next level in the Newton-Puiseux algorithm we obtain, defining $b = -B^2$, and computing $\hat{\mathcal{F}}(u,w) = \mathcal{F}(u^4, \alpha_0 u^7 + w)$, we get $W(\mathbf{N}(\hat{\mathcal{F}})) = \{(p_1,q_1)\} = \{(1,7)\}$. Moreover, the second determining polynomials $\mathcal{P}_1(\eta)$ is computed by $\hat{\mathcal{F}}(u, u^7\eta) = -u^{56}\left(\mathcal{P}_1(\eta) + \mathcal{O}(u)\right)$, with $\mathcal{P}_1(\eta) = (-1 + B(\alpha_0 + \eta)^2)^2(1 + B(\alpha_0+\eta)^2)^2$. Now we have that the eventual real branches are $v^*(u) = \alpha_0 u^{7/4} + \alpha_1 u^7 + o(u^7)$ with α_1 non-zero real root of $\mathcal{P}_1(\eta)$. Going further in the Newton-Puiseux algorithm we compute $\mathcal{F}^\dagger(u,z) = \mathcal{F}(u, \alpha_1 w^7 + z)$, and we get $W(\mathbf{N}(\mathcal{F}^\dagger)) = \{(p_2,q_2)\} = \{(1,7)\}$. Since

$q_2/p_2 \leq q_1/p_1$ the branch $v^*(u)$ does not exist. This is a proof of the obvious fact that F is invertible.

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