ON THE QUALITATIVE BEHAVIOR OF THE SOLUTIONS OF THE EQUATION OF THE TYPE $\ddot{x} + f_1(x)\dot{x} + f_2(x)\dot{x}^2 + g(x) = 0$

Hamilton Luiz Guidorizzi
Instituto de Matemática e Estatística
Universidade de São Paulo
Caixa Postal 20570 (Ag.Iguatemi) CEP 01498
São Paulo - SP - Brasil

Abstract

In this paper, working with the positive definite function

$$W_{\alpha_i,\beta}(x,y) = \int_0^{y/H_\beta(x)} \frac{s}{(1+\alpha_2)s^2 + \alpha_1 s + 1} ds + \ell n \beta^{-1/2} H_\beta(x)$$

where $H_\beta(x) = [2\int_0^x g(u)du + \beta]^{1/2}$, with $\beta > 0$, we study the qualitative behavior of the solutions of the equation $\ddot{x} + f_1(x)\dot{x} + f_2(x)\dot{x}^2 + g(x) = 0$ of the point of view of periodicity and oscillation.

1. Introduction

In [1], working with the positive definite function

$$V_{\alpha,\beta}(x,y) = \int_0^y \frac{s}{\alpha s^2 + \beta s + 1} ds + \int_0^x g(u)du$$

we establish sufficient conditions for the equation

$$\ddot{x} + f_1(x)\dot{x} + f_2(x)\dot{x}^2 + g(x) = 0 \tag{1.1}$$

to admit periodic solutions, as well as sufficient conditions for the solutions of (1.1) to be oscillating. Now, using the positive definite function

$$W_{\alpha_i,\beta}(x,y) = \int_0^{y/H_\beta(x)} \frac{s}{(1+\alpha_2)s^2 + \alpha_1 s + 1} ds + \ell n \beta^{-1/2} H_\beta(x), \tag{1.2}$$

where $H_\beta(x) = [2\int_0^x g(u)du + \beta]^{1/2}$, with $\beta > 0$, other of such conditions will be established.

2. The Positive Definite Function $W_{\alpha_i,\beta}$. Auxilliary Lemmas.

Throughout this work we suppose f_1, f_2 and g are functions from $\mathbf{R}$ into $\mathbf{R}$ satisfying the following conditions:

a) f_1, f_2 and g are of class C^1;

b) $xg(x) > 0$ for $x \neq 0$;

c) $\int_0^{+\infty} g(x)dx = +\infty = \int_0^{-\infty} g(x)dx$.

The equation (1.1) is equivalent to the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f_1(x)y - f_2(x)y^2 - g(x). \end{cases} \tag{2.1}$$

The condition a) ensures existence and uniqueness of solutions for (2.1) and b) ensures that $(0,0)$ is the only equilibrium point for (2.1). It can be immediately verified that the solutions of the equation

$$\frac{dy}{dx} = -f_1(x) - f_2(x)y - \frac{g(x)}{y}$$

do not admit vertical asymptotes, consequently the solutions of (2.1) do not admit them either.

Let $\alpha_1, \alpha_2, \beta \in \mathbf{R}$ be with $\beta > 0$. We indicate by $\Omega_{\alpha_i,\beta}$ the following open sets.
If $\alpha_1^2 - 4\alpha_2 - 4 < 0$ and $\alpha_2 > -1$ or $\alpha_1 = 0$ and $\alpha_2 = -1$ (2.2)

$$\Omega_{\alpha_i,\beta} = \mathbf{R}^2.$$

If $\alpha_1^2 - 4\alpha_2 - 4 \geq 0$, $\alpha_1 > 0$ and $\alpha_2 \geq -1$ (2.3)

$$\Omega_{\alpha_i,\beta} = \{(x,y) \in \mathbf{R}^2 \mid y > \rho_1 H_\beta(x),\ x \in \mathbf{R}\},$$

where

$$\rho_1 = \frac{-\alpha_1 + \sqrt{\alpha_1^2 - 4\alpha_2 - 4}}{2(\alpha_2 + 1)} \quad \text{if} \quad \alpha_2 > -1$$

or

$$\rho_1 = -\frac{1}{\alpha_1} \quad \text{if} \quad \alpha_2 = -1.$$

If $\alpha_1^2 - 4\alpha_2 - 4 \geq 0$, $\alpha_1 < 0$ and $\alpha_2 \geq -1$ (2.4)

$$\Omega_{\alpha_i,\beta} = \{(x,y) \in \mathbf{R}^2 \mid y < \rho_2 H_\beta(x),\ x \in \mathbf{R}\}$$

where

$$\rho_2 = \frac{-\alpha_1 - \sqrt{\alpha_1^2 - 4\alpha_2 - 4}}{2(\alpha_2+1)} \quad \text{if} \quad \alpha_2 > -1$$

or

$$\rho_2 = -\frac{1}{\alpha_1} \quad \text{if} \quad \alpha_2 = -1.$$

If $\alpha_2 < -1$ (2.5)

$$\Omega_{\alpha_i,\beta} = \{(x,y) \in \mathbf{R}^2 \mid \rho_1 H_\beta(x) < y < \rho_2 H_\beta(x),\ x \in \mathbf{R}\}$$

where

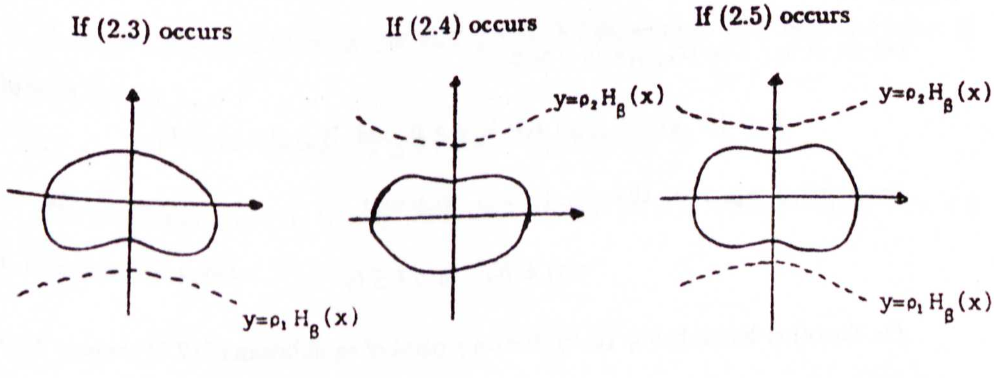
$$(\alpha_2+1)\rho_i^2 + \alpha_1\rho_i + 1 = 0 \qquad (i = 1,2)$$

with $\rho_1 < 0$ and $\rho_2 > 0$.

Let the positive definite function $W_{\alpha_i,\beta} : \Omega_{\alpha_i,\beta} \to \mathbf{R}$ be given by (1.2). It can be immediately verified if (2.2) occurs, then, for each $(x_0, y_0) \in \mathbf{R}^2$, the level curve

$$W_{\alpha_i,\beta}(x,y) = W_{\alpha_i,\beta}(x_0,y_0) \tag{2.6}$$

is a closed curve. If (2.3), (2.4) or (2.5) occurs, for each $(x_0, y_0) \in \Omega_{\alpha_i,\beta}$, the level curve (2.6) is a closed curve and shows the following aspect:



Observe that $W_{\alpha_i,\beta}(x,0)$ is strictly increasing in $[0,+\infty[$. Observe also that the derivative of $W_{\alpha_i,\beta}$ relative to the system (2.1) is given by

$$\dot{W}_{\alpha_i,\beta}(x,y) = \frac{y^2[\alpha_1 g(x) H_\beta^{-1}(x) - f_1(x)] + y^3[\alpha_2 g(x) H_\beta^{-2}(x) - f_2(x)]}{(\alpha_2+1)y^2 + \alpha_1 y H_\beta(x) + H_\beta^2(x)} \,. \tag{2.7}$$

The sign of $\dot{W}_{\alpha_i,\beta}(x,y)$ is the same of the numerator, because, $(\alpha_2+1)y^2+\alpha_1 y H_\beta(x) + H_\beta^2(x) > 0$ for $(x,y) \in \Omega_{\alpha_i,\beta}$.

Lemma 1. Suppose that the conditions a), b) and c) are verified. Suppose, also, there are α_1, α_2, β, $b \in \mathbf{R}$, with $\beta > 0$, $b > 0$ and

$$\alpha_1^2 - 4\alpha_2 - 4 < 0 \quad \text{and} \quad \alpha_2 > -1$$

or

$$\alpha_1 = 0 \quad \text{and} \quad \alpha_2 = -1$$

such that, for all $x \geq b$,

$$f_1(x) \geq \alpha_1 g(x) H_\beta^{-1}(x)$$

and

$$f_2(x) \geq \alpha_2 g(x) H_\beta^{-2}(x).$$

Then, for every solution $\gamma(t) = (x(t), y(t))$ of (2.1), with $\gamma(t_0) = (b, y_0)$, $y_0 > 0$, there is $t_1 > t_0$ such that $\gamma(t_1) = (x_1, 0)$, $x_1 > b$.

Proof:

Let $y_1 > y_0$, $L = W_{\alpha_i,\beta}(b, y_1)$ and

$$K = \{(x,y) \in \Omega_{\alpha_i,\beta} \mid x \geq b,\ y \geq 0 \text{ and } W_{\alpha_i,\beta}(x,y) \leq L\}.$$

From $\dot{x}(t_0) = y_0 > 0$, there is $t_2 > t_0$ such that

$$\gamma(t) \in K, \quad t_0 \leq t \leq t_2.$$

On the other hand, being $(0,0)$ the only point of equilibrium of (2.1) there is $t_3 > t_2$ such that

$$\gamma(t_3) \notin K.$$

From $\dot{W}_{\alpha_i,\beta}(x,y) \le 0$ for $x \ge b$ and $y \ge 0$, it follows that $\gamma(t)$ does not leave K through the arc

$$x \ge b, \quad y \ge 0 \quad \text{and} \quad W_{\alpha_i,\beta}(x,y) = L.$$

Therefore, there is $t_1 > t_0$ such that $\gamma(t_1) = (x_1, 0)$, $x_1 > b$. ■

Lemma 2. Suppose that the conditions a), b) and c) are verified. Suppose also there are α_1, α_2, β, $b \in \mathbf{R}$, with $b > 0$, $\beta > 0$, $\alpha_1 > 0$, $\alpha_2 \ge -1$ and $\alpha_1^2 - 4\alpha_2 - 4 \ge 0$, such that, for all $x \ge b$,

$$f_1(x) \ge \alpha_1 g(x) H_\beta^{-1}(x)$$

and

$$f_2(x) \le \alpha_2 g(x) H_\beta^{-2}(x).$$

Then, for every solution $\gamma(t) = (x(t), y(t))$ of (2.1) with $\gamma(t_0) = (x_0, 0)$, $x_0 > b$, there is $t_1 > t_0$ such that $\gamma(t_1) = (b, y_1)$, with $\rho_1 H_\beta(b) < y_1 < 0$ where $\rho_1 = \dfrac{-\alpha_1 + \sqrt{\alpha_1^2 - 4\alpha_2 - 4}}{2(\alpha_2 + 1)}$ if $\alpha_2 > -1$ or $\rho_1 = -\frac{1}{\alpha_1}$ if $\alpha_2 = -1$.

Proof:

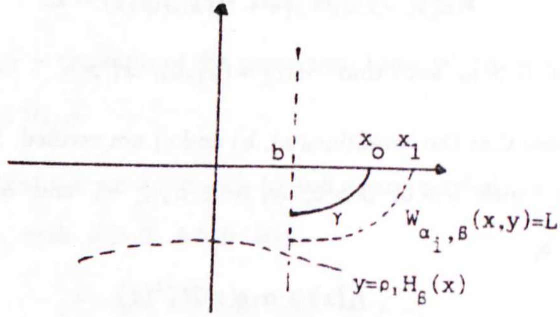
Let $W_{\alpha_i,\beta}(x_1, 0) = L$, with $x_1 > x_0$, and

$$K = \{(x,y) \in \Omega_{\alpha_i,\beta} \mid x \ge b, \ y \le 0 \ \text{and} \ W_{\alpha_i,\beta}(x,y) \le L\}.$$

From $\dot{W}_{\alpha_i,\beta}(x,y) \le 0$ for $x \ge b$ and $y \le 0$, it follows that $\gamma(t)$ does not leave K through the arc

$$x \ge b, \quad y \le 0 \quad \text{and} \quad W_{\alpha_i,\beta}(x,y) = L.$$

The conclusion there is $t_1 > t_0$ such that $\gamma(t_1) = (b, y_1)$, with $\rho_1 H_\beta(b) < y_1 < 0$ is, therefore, immediate.



■

In a similar way, we can prove the following lemma.

Lemma 3. Suppose that the conditions a), b) and c) are verified. Suppose also there are α_1, α_2, β, $a \in \mathbb{R}$ with $\beta > 0$, $a < 0$ and

$$\alpha_1^2 - 4\alpha_2 - 4 < 0 \quad \text{and} \quad \alpha_2 > -1$$

or

$$\alpha_1 = 0 \quad \text{and} \quad \alpha_2 = -1$$

such that, for all $x \leq a$,

$$f_1(x) \geq \alpha_1 g(x) H_\beta^{-1}(x)$$

and

$$f_2(x) \leq \alpha_2 g(x) H^{-2}\beta(x).$$

Then, for every solution $\gamma(t)$ of (2.1), with $\gamma(t_0) = (a, y_0)$, $y_0 < 0$, there is $t_1 > t_0$ such that $\gamma(t_1) = (x_1, 0)$, $x_1 < a$.

Lemma 4. Suppose that the conditions a), b) and c) are verified and assume also that

1) there are α_{11}, α_{12}, β, $r_1 \in \mathbf{R}$, with $\beta > 0$ and $r_1 > 0$, such that, for $0 < x < r_1$,

$$f_1(x) \leq \alpha_1 g(x) H_\beta^{-1}(x)$$

and

$$f_2(x) \geq \alpha_2 g(x) H_\beta^{-2}(x);$$

2) there are α_{21}, α_{22}, β_1, $r_2 \in \mathbf{R}$, with $\beta_1 > 0$ and $r_2 < 0$, such that, for $r_2 < x < 0$,

$$f_1(x) \leq \alpha_{21} g(x) H_{\beta_1}^{-1}(x)$$

and

$$f_2(x) \leq \alpha_{22} g(x) H_{\beta_1}^{-2}(x).$$

Then, every solution of (2.1) starting at $(x_0, 0)$, with $x_0 > 0$, crosses the $y < 0$ half-axis and every solution starting at $(x_1, 0)$, with $x_1 < 0$, crosses the $y > 0$ half-axis.

Proof:

It is enough to observe that for $L > 0$ sufficiently small

$$W_{\alpha_{1i},\beta}(x, y) = L \quad \text{and} \quad W_{\alpha_{2i},\beta_1}(x, y) = L$$

are closed curves and that

$$\dot{W}_{\alpha_{1i},\beta}(x, y) \geq 0, \quad \text{for} \quad 0 < x < r_1 \quad \text{and} \quad y \leq 0,$$

and

$$\dot{W}_{\alpha_{2i},\beta_1}(x, y) \geq 0, \quad \text{for} \quad r_2 < x < 0 \quad \text{and} \quad y \geq 0. \qquad \blacksquare$$

3. Phase Portrait of the Equation (1.1) where $f_1(x) = \alpha_1 g(x) H_\beta^{-1}(x)$ and $f_2(x) = \alpha_2 g(x) H_\beta^{-2}(x)$

Theorem 1. Consider the equation

$$\ddot{x} + \alpha_1 g(x) H_\beta^{-1}(x)\dot{x} + \alpha_2 g(x) H_\beta^{-2}(x)\dot{x}^2 + g(x) = 0 \tag{3.1}$$

where α_1, α_2, $\beta \in \mathbf{R}$, with $\beta > 0$, and g satisfies the conditions a), b) and c) of the previous section. Then

1) If $\alpha_1^2 - 4\alpha_2 - 4 < 0$ and $\alpha_2 > -1$ or $\alpha_1 = 0$ and $\alpha_2 = -1$, then any non-trivial solution of (3.1) is periodic.

2) If $\alpha_1^2 - 4\alpha_2 - 4 \geq 0$, $\alpha_1 > 0$ and $\alpha_2 \geq -1$, then any non-trivial solution of (3.1) starting at (x_0, y_0), with $y_0 > \rho_1 H_\beta(x_0)$ being ρ_1 according to (2.3), is periodic. Besides, any solution starting at (x_0, y_0), with $y_0 \leq \rho_1 H_\beta(x_0)$, is not periodic.

3) If $\alpha_1^2 - 4\alpha_2 - 4 \geq 0$, $\alpha_1 < 0$ and $\alpha_2 \geq -1$, then any non-trivial solution of (3.1) starting at (x_0, y_0), with $y_0 < \rho_2 H_\beta(x_0)$ being ρ_2 according to (2.4), is periodic. Besides, any solution starting at (x_0, y_0), with $y_0 \geq \rho_2 H_\beta(x_0)$, is not periodic.

4) If $\alpha_2 < -1$, then any non-trivial solution of (3.1) starting at (x_0, y_0), with

$$\rho_1 H_\beta(x_0) < y_0 < \rho_2 H_\beta(x_0),$$

being ρ_1 and ρ_2 according to (2.5), is periodic. Besides, any solution starting at (x_0, y_0), with $y_0 \geq \rho_2 H_\beta(x_0)$ or $y_0 \leq \rho_1 H_\beta(x_0)$, is not periodic.

Proof:

It is enough to observe that

$$\dot{W}_{\alpha_i,\beta}(x,y) = 0 \quad \text{for} \quad (x,y) \in \Omega_{\alpha_i,\beta}$$

and, for each $(x_0, y_0) \in \Omega_{\alpha_i,\beta}$, the level curve $W_{\alpha_i,\beta}(x,y) = W_{\alpha_i,\beta}(x_0,y_0)$ is a closed curve. ∎

4. Sufficient Conditions for Oscillating Solutions

A solution $x = x(t)$ of (1.1) is *oscilllating* if there is a sequence $(t_n)_{n\geq 1}$ tending monotonically to $+\infty$ such that $x(t_n) = 0$ for $n \geq 1$.

Theorem 2. Suppose that the conditions a), b) and c) of the section 2 are verified. Suppose also that

1) there are α_{11}, α_{12}, β, $b \in \mathbb{R}$, with $\beta > 0$, $b > 0$ and

$$\alpha_{11}^2 - 4\alpha_{12} - 4 < 0 \quad \text{and} \quad \alpha_{12} > -1$$

or

$$\alpha_{11} = 0 \quad \text{and} \quad \alpha_{12} = -1$$

such that, for all $x \geq b$,

$$f_1(x) \geq \alpha_{11} g(x) H_\beta^{-1}(x)$$

and

$$f_2(x) \geq \alpha_{12} g(x) H_\beta^{-2}(x);$$

2) there are α_{21}, α_{22}, β_1, $a \in \mathbb{R}$, with $\beta_1 > 0$, $a < 0$ and

$$\alpha_{21}^2 - 4\alpha_{22} - 4 < 0 \quad \text{and} \quad \alpha_{22} > -1$$

or

$$\alpha_{21} = 0 \quad \text{and} \quad \alpha_{22} = -1$$

such that, for all $x \leq a$,

$$f_1(x) \geq \alpha_{21} g(x) H_{\beta_1}^{-1}(x)$$

and

$$f_2(x) \leq \alpha_{22} g(x) H_{\beta_1}^{-2}(x).$$

3) there are τ_{11}, τ_{12}, β_2, $r \in \mathbb{R}$, with $\beta_2 > 0$ and $r > 0$, such that, for $0 < x < r$,

$$f_1(x) \leq \tau_{11} g(x) H_{\beta_2}^{-1}(x)$$

and

$$f_2(x) \geq \tau_{12} g(x) H_{\beta_2}^{-2}(x);$$

4) there are τ_{21}, τ_{22}, β_3, $r_1 \in \mathbf{R}$, with $\beta_3 > 0$ and $r_1 < 0$, such that, for $r_1 < x < 0$,

$$f_1(x) \leq \tau_{21} g(x) H_{\beta_3}^{-1}(x)$$

and

$$f_2(x) \leq \tau_{22} g(x) H_{\beta_3}^{-2}(x).$$

Then, any non-trivial solution of (1.1) is oscilating.

Proof:

From hypotheses 2-3 and lemmas 3-4 and by the fact that the solutions of (2.1) do not admit vertical asymptotes, it follows that every solution starting at $(x_0, 0)$, $x_0 > 0$, crosses the $x < 0$ half-axis. From hypotheses 1-4 and lemmas 1-4, it follows that every solution starting at $(x_1, 0)$, $x_1 < 0$, crosses the $x > 0$ half-axis. Therefore, any non-trivial solution is oscillating. ■

Remark 1. Combining the hypotheses of this theorem 2 with ones of theorem 1 in [1], other sufficient conditions for the solutions to be oscillating can be established.

Theorem 3. Suppose that the conditions a), b) and c) of the section 2 are verified. Assume also that

1) there are α_2, $\beta \in \mathbf{R}$, with $\beta > 0$, such that

$$f_2(x) = \alpha_2 g(x) H_\beta^{-2}(x), \quad x \in \mathbf{R};$$

2) there is $\alpha_1 \in \mathbf{R}$ such that

$$f_1(x) > \alpha_1 g(x) H_\beta^{-1}(x), \quad \text{for } x \neq 0;$$

3) there are τ_1, $r \in \mathbf{R}$, with $\tau_1 > 0$ and $r > 0$, such that, for $0 < |x| < r$,

$$f_1(x) \leq \tau_1 |g(x)| H_\beta^{-1}(x).$$

Then, for any non-trivial solution $x = x(t)$ of (1.1), with $(x(0), \dot{x}(0)) \in \Omega_{\alpha_i,\beta}$, is oscillating and $(x(t), \dot{x}(t))$ approaches to the origin as $t \to +\infty$.

Proof: Immediate. ■

Remark 2. If the hypothesis 3), in theorem 3, is not satisfied, it can only assert that any solution $\gamma(t)$ of (2.1), with $\gamma(0) \in \Omega_{\alpha_i,\beta}$, approaches to the origin as $t \to +\infty$.

5. Sufficient Conditions for Existence of Periodic Solutions

The next theorem establishes a sufficient condition for any non-trivial solution of

$$\ddot{x} + f_2(x)\dot{x}^2 + g(x) = 0 \tag{4.1}$$

to be periodic.

Theorem 4. Suppose that the conditions a), b) and c) of the section 2 are verified. Assume also that

1) there are β, $b \in \mathbf{R}$, with $\beta > 0$ and $b > 0$, such that, for all $x \geq b$,

$$f_2(x) \geq -g(x)H_\beta^{-2}(x);$$

2) there are β_1, $a \in \mathbf{R}$, with $\beta_1 > 0$ and $a < 0$, such that, for all $x \leq a$,

$$f_2(x) \leq -g(x)H_{\beta_1}^{-2}(x).$$

Then, any non-trivial solution of (4.1) is periodic.

Proof:

The origin $(0,0)$ is a local center (see [2]). Then, according to theorem 2, any non-trivial solution of (4.1) is oscillating. Therefore, any non-trivial solution is periodic. (Observe if $y = y(x)$ is a solution of $\frac{dy}{dx} = -f_2(x)y - \frac{g(x)}{y}$. The same occurs with $y = -y(x)$.) ■

The next theorem establishes sufficient conditions for any non-trivial solution of (1.1) to be periodic, where f_1, f_2 and g are supposed odd functions.

Theorem 5. Suppose that the conditions a), b) and c) of the section 2 are verified.

Assume also that

1) f_1, f_2 and g are odd functions,

2) there are α_{11}, α_{12}, α_{22}, β, $b \in \mathbf{R}$, with $\beta > 0$, $b > 0$, $\alpha_{22} > -1$, $\alpha_{11}^2 - 4\alpha_{22} - 4 < 0$ and $\alpha_{12}^2 - 4\alpha_{22} - 4 < 0$, such that, for all $x \geq b$,

$$\alpha_{11} g(x) H_\beta^{-1}(x) \leq f_1(x) \leq \alpha_{12} g(x) H_\beta^{-1}(x)$$

and

$$f_2(x) \geq \alpha_{22} g(x) H_\beta^{-2}(x);$$

3) there are τ_1, τ_2, β_1, $r \in \mathbf{R}$, with $\beta_1 > 0$ and $r > 0$, such that, for $0 < x < r$,

$$f_1(x) \leq \tau_1 g(x) H_{\beta_1}^{-1}(x)$$

and

$$f_2(x) \geq \tau_2 g(x) H_{\beta_1}^{-2}(x).$$

Then, any non-trivial solution of (2.1) is periodic.

Proof:

The conditions

$$f_1(x) \geq \alpha_{11} g(x) H_\beta^{-1}(x)$$

and

$$f_2(x) \geq \alpha_{22} g(x) H_\beta^{-2}(x)$$

for $x \geq b$, and the hypothesis 3 ensures that every solution starting at $(0, y_0)$, $y_0 > 0$, crosses the $y < 0$ half-axis. The conditions

$$f_1(x) \leq \alpha_{12} g(x) H_\beta^{-1}(x)$$

and

$$f_2(x) \geq \alpha_{22} g(x) H_\beta^{-2}(x)$$

for $x \geq b$, ensure that, for each $y_1 < 0$, there is $x_0 > 0$ such that the solution starting at $(x_0, 0)$ crosses the $y < 0$ half-axis at $(0, y_1)$. The hypothesis 1 ensures that every solution $\gamma(t)$ of (2.1), with $\gamma(t_1) = (0, y_0)$, $y_0 > 0$, and $\gamma(t_2) = (0, y_1)$, $y_1 < 0$, for some t_1 and t_2, is periodic. Therefore, any non-trivial solution of (2.1) is periodic. ■

Remark 3. The hypothesis 2, in theorem 5, can be replaced by: there are $b > 0$, $\beta_1 > 0$, $\beta_2 < 0$ and $\alpha > 0$, with $\alpha^2 - 4\beta_1 < 0$ and $\alpha^2 - 4\beta_2 < 0$, such that, for all $x \geq b$,

$$\beta_2 g(x) \leq f_1(x) \leq \beta_1 g(x)$$

and

$$f_2(x) \geq \alpha g(x)$$

(see [1]).

Remark 4. If the hypothesis 2, in theorem 5, is replaced by: there are α_1, α_2, β, $b \in \mathrm{R}$, with $\beta > 0$ and $b > 0$, such that, for all $x \geq b$,

$$f_1(x) \geq \alpha_1 g(x) H_\beta^{-1}(x)$$

and

$$f_2(x) \geq \alpha_2 g(x) H_\beta^{-2}(x).$$

Then, we can only assert that every solution $\gamma(t)$ of (2.1), with $\gamma(0) = (x_0, y_0) \in \Omega_{\alpha_i, \beta}$, $x_0 \geq b$ and $y_0 \geq 0$, is periodic.

Theorem 6. Suppose that the conditions a), b) and c) of the section 2 are verified. Suppose also that

1) there are α_1, α_2, β, $b \in \mathrm{R}$, with $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta > 0$, $b > 0$ and $\alpha_1^2 - 4\alpha_2 - 4 \geq 0$, such that, for all $x \geq b$,

$$f_1(x) \geq \alpha_1 g(x) H_\beta^{-1}(x)$$

and

$$-g(x)H_\beta^{-2}(x) \le f_2(x) \le \alpha_2 g(x)H_\beta^{-2}(x);$$

2) there are α_{11}, α_{12}, β_1, $a \in \mathbb{R}$, with $\beta_1 > 0$, $a < 0$ and

$$\alpha_{11}^2 - 4\alpha_{12} - 4 < 0 \quad \text{and} \quad \alpha_{12} > -1$$

or

$$\alpha_{11} = 0 \quad \text{and} \quad \alpha_{12} = -1$$

such that, for all $x \le a$,

$$f_1(x) \ge \alpha_{11} g(x) H_{\beta_1}^{-1}(x)$$

and

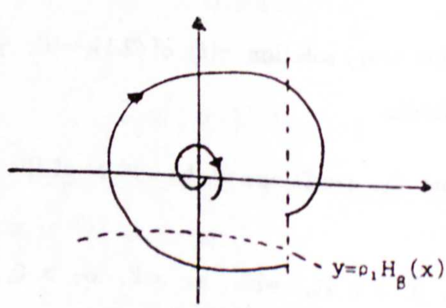
$$f_2(x) \le \alpha_{12} g(x) H_{\beta_1}^{-2}(x);$$

3) the origin is repulsive.

Then, the system (2.1) admits at least one non-trivial periodic solution.

Proof:

The hypotheses ensure that any non-trivial solution is oscillating. The hypothesis 1) ensures that every solution starting at (b, y_0), with $y_0 > 0$, crosses the straight line $x = b$ at (b, y_1), with $\rho_1 H_\beta(b) < y_1 < 0$, where $\rho_1 = \dfrac{-\alpha_1 + \sqrt{\alpha_1^2 - 4\alpha_2 - 4}}{2(\alpha_2 + 1)}$.



From theorem of Poincaré–Bendixson, there is at least one non-trivial periodic solution.

■

Remark 5. A sufficient condition for the origin to be repulsive is: there is $r > 0$ such that, for $0 < |x| < r$, $f_1(x) < 0$. It is enough to observe that the derivative of the positive definite function given by

$$V(x,y) = \frac{y^2}{2} e^{2\int_0^x f_2(s)ds} + \int_0^x e^{2\int_0^s f_2(u)du} g(s) ds$$

relative to the system (2.1) is

$$\dot{V}(x,y) = -f_1(x) y^2 e^{2\int_0^x f_2(s)ds}.$$

Remark 6. Combining the hypotheses of the theorem 6 with ones of the theorem 3 in [1], other sufficient conditions for existence of periodic solutions can be established.

Remark 7. The hypotheses 1 and 2 in theorem 6 can be replaced by:

1') there are $\alpha_1 < 0$, $\alpha_2 > 0$, $\beta > 0$ and $a < 0$, with $\alpha_1^2 - 4\alpha_2 - 4 \geq 0$, such that, for all $x \leq a$,

$$f_1(x) \geq \alpha_1 g(x) H_\beta^{-1}(x)$$

and

$$\alpha_2 g(x) H_\beta^{-2}(x) \leq f_2(x) \leq -g(x) H_\beta^{-2}(x);$$

2') there are α_{11}, α_{12}, β_1, $b \in \mathbf{R}$, with $\beta_1 > 0$, $b > 0$ and

$$\alpha_{11}^2 - 4\alpha_{12} - 4 < 0 \quad \text{and} \quad \alpha_{12} > -1$$

or

$$\alpha_{11} = 0 \quad \text{and} \quad \alpha_{12} = -1$$

such that, for all $x \geq b$,

$$f_1(x) \geq \alpha_{11} g(x) H_{\beta_1}^{-1}(x)$$

and

$$f_2(x) \geq \alpha_{12} g(x) H_{\beta_1}^{-2}(x).$$

Corollary. Suppose that the conditions a), b) and c) of the section 2 are verified. Suppose also that

1) there are $\alpha \geq -1$ and $\beta > 0$ such that

$$f_2(x) = \alpha g(x) H_\beta^{-2}(x);$$

2) there are $\alpha_1 > 0$ and $b > 0$, with $\alpha_1^2 - 4\alpha_2 - 4 \geq 0$, such that, for all $x \geq b$,

$$f_1(x) \geq \alpha_1 g(x) H_\beta^{-1}(x);$$

3) there are α_{11}, $a \in \mathbf{R}$, with $a < 0$ and

$$\alpha_{11}^2 - 4\alpha_2 - 4 < 0 \quad \text{if} \quad \alpha_2 > -1$$

or

$$\alpha_{11} = 0 \quad \text{if} \quad \alpha_2 = -1$$

such that, for all $x \leq a$,

$$f_1(x) \geq \alpha_{11} g(x) H_\beta^{-1}(x);$$

4) the origin is repulsive.

Then, the system (2.1) admits at least one non-trivial periodic solution.

Remark 8. In this corollary, a sufficient condition for the origin to be repulsive is: there are r_1, $r \in \mathbf{R}$, with $r > 0$, such that, for $0 < |x| < r$, $f_1(x) < r_1 g(x) H_\beta^{-1}(x)$.

To close, we observe that the positive definite function

$$\overline{W}_{\alpha_i,\beta}(x,y) = \int_0^{y/H_\beta(x)} \frac{s^{2n-1}}{s^{2n} + \sum_{i=1}^k \alpha_i s^i + 1} ds + \ell n[\beta^{-(2n)^{-1}} H_\beta(x)]$$

where $H_\beta(x) = \left[2n \int_0^x g(u)du + \beta\right]^{(2n)^{-1}}$, with $\beta > 0$, can be utilized for studying the qualitative behavior of the solutions of the system

$$\begin{cases} \dot{x} = y^{2n-1} \\ \dot{y} = -\sum_{i=1}^{k} f_i(x) y^i - g(x). \end{cases} \tag{4.2}$$

When $k \leq 2n$, the results obtained in this work can be extended for the system (4.2).

Systems of the type

$$\begin{cases} \dot{x} = y^{2n-1} \\ \dot{y} = -\sum_{i=1}^{n} f_i(x) y^{2i-1} - g(x) \end{cases}$$

are studied in [3] and [4].

References

[1] Guidorizzi, H.L. *Oscillating and periodic solutions of equations of the type* $\ddot{x} + f_1(x)\dot{x} + f_2(x)\dot{x}^2 + g(x) = 0$. To appear in J. Math. anal. and. appl.

[2] Utz, W.R. *Periodic solutions of a non-linear second order differential equation.* SIAM J. Appl. Math. 19 (1970), 56-59.

[3] Guidorizzi, H.L. *Oscillating and periodic solutions of equations of the type* $\ddot{x} + \dot{x}\sum_{i=1}^{n} f_i(x)|\dot{x}|^{\delta_i} + g(x) = 0$. To appear in J. Math anal. and appl.

[4] Guidorizzi, H.L. *On periodic solutions of systems of the type* $\dot{x} = H(y)$, $\dot{y} = -\sum_{i=1}^{n} f_i(x)H_i(y) - g(x)$. To appear.