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A primary obstruction to topological embeddings and its applications

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Abstract

Let $f : M \to N$ be a proper continuous map between topological manifolds with $m = \dim M < \dim N = m + k$. A primary obstruction $\theta(f) \in H^c_{m-k}(M; \mathbf{Z}_2)$ for f to be homotopic to a topological embedding has been defined and studied from a differentiable topological viewpoint by the first and the third authors in [5], where H^c_* denotes the singular homology group based on infinite chains. In this paper, using algebraic topological methods, we show that $\bar{f}_*\theta(f) \in \check{H}^c_{m-k}(f(M); \mathbf{Z}_2)$ always vanishes, where $\bar{f} = f : M \to f(M)$ and $\check{H}^c_*$ denotes the Čech homology group based on infinite chains. This enables us to obtain various results without assuming the existence of differentiable structures on the manifolds. For example, we show the vanishing of the i-th Stiefel-Whitney class $w_i(f) \in H^i(M; \mathbf{Z}_2)$ of the stable normal bundle of an arbitrary proper topological embedding $f : M \to N$ for all $i > k$. We also give various new characterizations of differentiable embeddings among generic differentiable maps as refinements of the results obtained in [5], [7]. Furthermore, we give a result concerning the number of connected components of the complement of a codimension-1 continuous map with a normal crossing point, which generalizes the results obtained in [3], [1], [2] and [5]. We also study the R-bordism invariance of the homology class $\theta(f)$.

1 Introduction

This is a continuation of the studies by the first and the third authors [4], [5], [6] and [7].

Let M and N be topological manifolds of dimensions m and n respectively and suppose $k = n - m > 0$. For a proper continuous map $f : M \to N$, a homology class $\theta(f) \in H^c_{m-k}(M; \mathbf{Z}_2)$ has been defined in [5, Definition 2.5], where H^c_* denotes the (singular)

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homology of the compatible family with respect to compact subsets (see [22, Chapter 6, §3]) or equivalently the (singular) homology based on infinite chains (see [18, §5 and §65]). This is a homotopy invariant of f and has the property that, when M and N are smooth* manifolds, if f is homotopic† to a proper smooth embedding, then $\theta(f)$ vanishes. Furthermore, when M is compact and M and N are smooth manifolds, if f is homotopic to a topological embedding‡, then $\theta(f)$ vanishes (see [5]). For all these results, the differentiable structures on both M and N have played an important role, since the technique of generic differentiable maps and the results due to Ronga [19] about such generic maps have been extensively used.

In this paper, we study the homology class $\theta(f)$ of a proper continuous map $f : M \to N$ from an algebraic topological viewpoint, so that we need no differentiability hypothesis on M or N. Our main result is Corollary 3.4, which states that $\bar{f}_*\theta(f) \in \check{H}^c_{m-k}(f(M); \mathbf{Z}_2)$ always vanishes, where $\bar{f} = f : M \to f(M)$ and $\check{H}^c_*$ denotes the Čech homology group based on infinite chains. This result implies, as a direct corollary, that if f is a proper (not necessarily locally flat) topological embedding, then $\theta(f) \in H^c_{m-k}(M; \mathbf{Z}_2)$ vanishes (Corollary 3.7). This means that the homology class $\theta(f)$ can always be regarded as a primary obstruction to the existence of a homotopy between a given map and a topological embedding.

Using the above mentioned result, we give various related results as applications. First, as Corollary 3.9, we show that the top Stiefel-Whitney class of the stable normal bundle of a proper topological embedding, defined via the Stiefel-Whitney classes of the manifolds involved, coincides with the modulo 2 normal Euler class of the embedding (see [9, Chapter VIII, §11]). Furthermore, we show that the i-th Stiefel-Whitney class $w_i(f) \in H^i(M; \mathbf{Z}_2)$ of the stable normal bundle of a codimension-k proper topological embedding $f : M \to N$ vanishes for all $i > k$ (Corollary 3.10). Such results have been well-known for differentiable embeddings, but, to the authors' knowledge, the results for topological embeddings have not appeared in the literature until now. Then we give an application concerning continuous maps of the real projective plane into 3-dimensional manifolds (Proposition 4.1), generalizing results of [6]. Furthermore, we give various new characterizations of differentiable embeddings among generic differentiable maps, which are refinements of results obtained in [5], [7]. We also give a result concerning the number of connected components of the complement of a codimension-1 map with a normal crossing point, which generalizes results of [3], [1], [2].

*In this paper, a manifold or a map is *smooth* if it is of class C^∞.

†In this paper, we say that two proper maps f and $g : M \to N$ are *homotopic* if there exists a homotopy $F : M \times [0,1] \to N$ between f and g such that F is a proper map.

‡In this paper, a continuous map is said to be a *topological embedding* if it is a homeomorphism onto its image. Thus the topological embeddings in this paper may not necessarily be locally flat.

We also give various new results about the homology class $\theta(f)$ itself. For example, we show that if the source manifold M is compact and the closure of the self-intersection set of f has topological dimension strictly less than $m-k$, then $\theta(f) \in H_{m-k}(M; \mathbf{Z}_2)$ vanishes (Corollary 3.17). We also show that the homology class $\theta(f)$ is invariant under R-bordism (see [11]) in an appropriate certain sense.

The paper is organized as follows. In §2, we recall the definition of the homology class $\theta(f)$. In §3, we show our key theorem (Theorem 3.1) and give various corollaries. In §4, we give an application to maps of the real projective plane into 3-dimensional manifolds. In §5, we give various new characterizations of differentiable embeddings among generic differentiable maps when the source manifold is compact. For example, we show that a generic map of class C^2 is a differentiable embedding if and only if the $(m-k+1)$-th reduced Čech homology group of the mapping cone of $\bar{f} : M \to f(M)$ vanishes (Theorem 5.2). Furthermore, we show that such a generic map is a differentiable embedding if and only if $\bar{f}_* : H_*(M; \mathbf{Z}_2) \to \check{H}_*(f(M); \mathbf{Z}_2)$ is an isomorphism (Corollary 5.4). We will also give an example of a generic immersion $f : M \to N$ such that $H_*(M; R)$ is isomorphic to $H_*(f(M); R)$ for every commutative ring R with unit but that f is not an embedding (Example 5.6). In §6, we study continuous maps with a differentiable normal crossing point of multiplicity two. For such a continuous map $f : M \to N$, we will find a nonzero element $\mu \in H_{m-k}(A_1; \mathbf{Z}_2)$ such that A_1 is a compact ANR (absolute neighborhood retract) containing the self-intersection set of f, $j_{1*}\mu = \theta(f)$ with $j_1 : A_1 \to M$ the inclusion map, and $(f|A_1)_*\mu = 0$ in $\check{H}_{m-k}(f(A_1); \mathbf{Z}_2)$ (Proposition 6.2). Using this result, we will show that the number of connected components of the complement of such a map is greater than or equal to three, provided that $H_1(N; \mathbf{Z}_2) = 0$ (see Corollary 6.4), generalizing results obtained in [3], [1], [2] concerning immersions with normal crossings. This is also a refinement of a result obtained in [5]. Note that a more general result has been obtained in [20]; however, our method is totally different from that used in [20] and our argument can also be applied to maps with codimension not necessarily equal to one (see Corollary 6.3). In §7, we will define the notion of R-bordism between two continuous maps $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ (see [11]). In that case, $H_*(M_1; \mathbf{Z}_2)$ and $H_*(M_2; \mathbf{Z}_2)$ are naturally isomorphic to each other. We will show that if f_1 and f_2 are R-bordant, then $\theta(f_1)$ and $\theta(f_2)$ correspond to each other under this natural isomorphism (Theorem 7.4). This result is closely related to the result obtained in [6] concerning the bordism invariance of $\theta(f)$. We also give some examples of continuous maps f such that $\theta(f) = 0$, but that f is not homotopic (or R-bordant) to a topological embedding.

We note that throughout the sections §§4–7, the source manifold M will always be compact, since we will use the exactness of Čech homology for compact pairs (see [14]). Thus, in these sections, the homology class $\theta(f)$ will always be an element of the usual homology group $H_{m-k}(M;\mathbf{Z}_2)$, which is isomorphic to $H^c_{m-k}(M;\mathbf{Z}_2)$. We also note that $\check{H}^c_* \cong \check{H}_*$ for compact spaces, where $\check{H}_*$ denotes the (usual) Čech homology group.

Throughout the paper, all manifolds are paracompact and have no boundary. The symbol "$\cong$" denotes an appropriate isomorphism between algebraic objects.

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2 Definition of $\theta(f)$

Let M and N be topological manifolds of dimensions m and n respectively such that $k = n - m > 0$ and $f : M \to N$ a proper continuous map.

Let $[M] \in H^c_m(M;\mathbf{Z}_2)$ denote the fundamental class of M, where H^c_* denotes the (singular) homology of the compatible family with respect to compact subsets (see [22, Chapter 6, §3]) or equivalently the (singular) homology based on infinite chains (see [18, §5 and §65]). Denote by $U_f \in H^k(N;\mathbf{Z}_2)$ the Poincaré dual of $f_*[M] \in H^c_m(N;\mathbf{Z}_2)$; in other words, $f_*[M] = U_f \frown [N]$, where $[N] \in H^c_n(N;\mathbf{Z}_2)$ is the fundamental class of N.

Let the total Stiefel-Whitney classes of M and N be denoted by $w(M) \in H^*(M;\mathbf{Z}_2)$ and $w(N) \in H^*(N;\mathbf{Z}_2)$ respectively and let $\bar{w}(M) \in H^*(M;\mathbf{Z}_2)$ denote the dual Stiefel-Whitney class of M; i.e., $\bar{w}(M) = w(M)^{-1}$. Define $w(f) = f^*w(N) \smile \bar{w}(M)$, which is called the *total Stiefel-Whitney class of the stable normal bundle of* f. We denote by $w_k(f) \in H^k(M;\mathbf{Z}_2)$ the degree k term of $w(f)$, which is the k-th Stiefel-Whitney class of the stable normal bundle of f.

Definition 2.1 We define

$$\theta(f) = (f^*U_f - w_k(f)) \frown [M] \in H^c_{m-k}(M;\mathbf{Z}_2).$$

Note that this is a homotopy invariant of f. We also note that when M is compact, $\theta(f)$ is an element of the usual homology group $H_{m-k}(M;\mathbf{Z}_2)$, since $H_{m-k}(M;\mathbf{Z}_2) \cong H^c_{m-k}(M;\mathbf{Z}_2)$.

The above homology class has been originally defined in [5] and denoted by $\theta_1(f)$. In this paper, we use the notation $\theta(f)$ instead of $\theta_1(f)$, which will cause no confusion.

Many important observations about f^*U_f and $w_k(f) \in H^k(M;\mathbf{Z}_2)$ have been given in [5, §2]. For example, it has been remarked that $\theta(f)$ depends only on the map $f : M \to V$, where V is an arbitrary neighborhood of $f(M)$ in N.

The reason why we use the homology class instead of the corresponding cohomology class is that when M and N are smooth manifolds, $\theta(f)$ coincides with the fundamental class carried by the closure of the self-intersection set of a generic map homotopic to f (see [19], [5]).

It will be shown in the next section that if f is homotopic to a topological embedding, then $\theta(f)$ vanishes. In other words, $\theta(f)$ can be regarded as a primary obstruction to the existence of such a homotopy.

3 Key theorem and corollaries

In this section, we first prove the following key theorem and give its important corollaries.

Theorem 3.1 *Let $f : M \to N$ be a proper continuous map of an m-dimensional topological manifold M into an $(m+k)$-dimensional topological manifold N with $k > 0$. Then $f_*\theta(f) \in H^c_{m-k}(N;\mathbf{Z}_2)$ always vanishes.*

Proof. First recall that

$$\begin{aligned} f_*\theta(f) &= f_*((f^*U_f - w_k(f)) \frown [M]) &(1)\\ &= f_*((f^*U_f) \frown [M]) - f_*(w_k(f) \frown [M]), &(2)\end{aligned}$$

where $w_k(f)$ is equal to the degree k term of $(f^*w(N)) \smile \bar{w}(M)$. As to the first term of the equation (2), we have

$$\begin{aligned} f_*((f^*U_f) \frown [M]) &= U_f \frown f_*[M] &(3)\\ &= U_f \frown (U_f \frown [N]) &(4)\\ &= (U_f \smile U_f) \frown [N]. &(5)\end{aligned}$$

On the other hand, as to the second term of the equation (2), we have

$$f_*(((f^*w(N)) \smile \bar{w}(M)) \frown [M]) = f_*((f^*w(N)) \frown (\bar{w}(M) \frown [M])). \quad (6)$$

Denoting by Sq_h the Steenrod squaring operation on the *homology* as defined in [17, Problem 11-F, p.136], we have

$$\bar{w}(M) \frown [M] = Sq_h[M]. \quad (7)$$

Thus, with the usual Steenrod squaring operation on the *cohomology* being denoted by Sq, the equation (6) is equal to

$$\begin{aligned} f_*((f^*w(N)) \frown (\bar{w}(M) \frown [M])) &= f_*((f^*w(N)) \frown Sq_h[M]) &(8)\\ &= w(N) \frown f_*Sq_h[M] &(9)\\ &= Sq(v(N)) \frown Sq_h(f_*[M]) &(10)\\ &= Sq_h(v(N) \frown f_*[M]) &(11)\\ &= Sq_h(v(N) \frown (U_f \frown [N])) &(12)\\ &= Sq_h((v(N) \smile U_f) \frown [N]), &(13)\end{aligned}$$

where $v(N)$ denotes the total Wu class of N and the equation (10) follows from the Wu formula (see [17, Theorem 11.14]). Thus, we have only to show that $(U_f \smile U_f) \frown [N]$ is equal to the degree $m-k$ term of $Sq_h((v(N) \smile U_f) \frown [N])$ in view of the equations (2), (5), (6) and (13). Let ξ be an arbitrary element of $H_c^{m-k}(N;\mathbf{Z}_2)$, where H_c^* denotes the cohomology group with compact support. By the universal coefficient theorem, we have only to show that

$$<\xi, (U_f \smile U_f) \frown [N]> = <\xi, Sq_h((v(N) \smile U_f) \frown [N])> . \tag{14}$$

As to the left hand side, we have

$$<\xi, (U_f \smile U_f) \frown [N]> = <\xi \smile (U_f \smile U_f), [N]> . \tag{15}$$

As to the right hand side of the equation (14), we have

$$\begin{aligned}
<\xi, Sq_h((v(N) \smile U_f) \frown [N])> &= <\overline{Sq}(\xi), (v(N) \smile U_f) \frown [N]> & (16)\\
&= <(\overline{Sq}(\xi) \smile U_f) \smile v(N), [N]> & (17)\\
&= <Sq(\overline{Sq}(\xi) \smile U_f), [N]> & (18)\\
&= <Sq(\overline{Sq}(\xi)) \smile Sq(U_f), [N]> & (19)\\
&= <\xi \smile Sq(U_f), [N]>, & (20)
\end{aligned}$$

where $\overline{Sq}$ denotes the inverse of the automorphism Sq on the cohomology with compact support (see [17, Problem 11-E, p.136]), the equation (16) follows from [17, Problem 11-F, p.136], the equation (18) follows from the property of the Wu class, and the equation (20) follows from the definition of $\overline{Sq}$. Since the degree $m+k$ term of $\xi \smile Sq(U_f)$ is equal to the cup product of ξ and the degree $2k$ term of $Sq(U_f)$, we see that

$$<\xi \smile Sq(U_f), [N]> = <\xi \smile (U_f \smile U_f), [N]>, \tag{21}$$

which is equal to the left hand side of the equation (14) by the equation (15), since $U_f \in H^k(N;\mathbf{Z}_2)$. This completes the proof. ||

Remark 3.2 It is well-known that for noncompact manifolds, we have the Poincaré duality isomorphism between the cohomology with compact support and the usual homology (for example, see [17, Appendix A]). However, in the above proof, we have implicitly used the duality isomorphisms $H_i^c(M;\mathbf{Z}_2) \cong H^{m-i}(M;\mathbf{Z}_2)$ and $H_j^c(N;\mathbf{Z}_2) \cong H^{n-j}(N;\mathbf{Z}_2)$ (see [18, §65]) and also the corresponding formulas about the Steenrod squaring operations. Such results are not explicitly written in the literature, but can be proved by standard arguments.

We also note that the universal coefficient theorem works for cohomology with compact support and homology based on infinite chains. In fact, for a noncompact manifold X, the cup product is a bilinear map

$$H^i_c(X;\mathbf{Z}_2) \times H^j(X;\mathbf{Z}_2) \to H^{i+j}_c(X;\mathbf{Z}_2)$$

or

$$H^i(X;\mathbf{Z}_2) \times H^j(X;\mathbf{Z}_2) \to H^{i+j}(X;\mathbf{Z}_2),$$

the cap product is a bilinear map

$$H^i(X;\mathbf{Z}_2) \times H^c_j(X;\mathbf{Z}_2) \to H^c_{j-i}(X;\mathbf{Z}_2),$$

and the Cronecker delta is a bilinear map

$$H^i_c(X;\mathbf{Z}_2) \times H^c_i(X;\mathbf{Z}_2) \to \mathbf{Z}_2.$$

Furthermore, Sq_h is defined on the homology based on infinite chains and Sq is defined both on the usual cohomology and on the cohomology with compact support.

Remark 3.3 When the manifolds M and N are differentiable, we can also prove the above theorem by using a generic differentiable map (see §5) homotopic to f and by using [7, Lemma 3.1].

Corollary 3.4 *Let $f: M \to N$ be a proper continuous map of an m-dimensional topological manifold M into an $(m+k)$-dimensional topological manifold N with $k > 0$. Then $\bar{f}_*\theta(f) \in \check{H}^c_{m-k}(f(M);\mathbf{Z}_2)$ always vanishes, where $\bar{f} = f: M \to f(M)$.*

Remark 3.5 It is known that for ANR's, the Čech homology groups are naturally isomorphic to the singular homology groups (for example, see [22]). Thus, for ANR's, we always identify the two homology groups, especially for manifolds. In particular, we have $\theta(f) \in \check{H}^c_{m-k}(M;\mathbf{Z}_2) = H^c_{m-k}(M;\mathbf{Z}_2)$.

Proof of Corollary 3.4. Take an arbitrary open neighborhood V of $f(M)$ in N. Since V is a topological manifold, by applying Theorem 3.1 to $f_V = f: M \to V$, we see that $(f_V)_*\theta(f_V) = 0$ in $H^c_{m-k}(V;\mathbf{Z}_2)$. Since $\theta(f) = \theta(f_V)$ in $H^c_{m-k}(M;\mathbf{Z}_2)$, we see that $(f_V)_*\theta(f) = 0$ in $H^c_{m-k}(V;\mathbf{Z}_2)$.

Let $i_V: f(M) \to V$ denote the inclusion map. Then, by the above argument, we have $(i_V)_*(\bar{f}_*\theta(f)) = (f_V)_*\theta(f) = 0$ in $H^c_{m-k}(V;\mathbf{Z}_2)$ for all V. Since $\check{H}^c_{m-k}(f(M);\mathbf{Z}_2)$ is identified with the inverse limit

$$\varprojlim H^c_{m-k}(V;\mathbf{Z}_2),$$

we see that $\bar{f}_*\theta(f) = 0$ in $\check{H}^c_{m-k}(f(M);\mathbf{Z}_2)$. This completes the proof. ||

We have the following immediate corollaries.

Corollary 3.6 *Let $f : M \to N$ be a proper continuous map of an m-dimensional topological manifold M into an $(m+k)$-dimensional topological manifold N with $k > 0$. If $\bar{f}_* : H^c_{m-k}(M; \mathbf{Z}_2) \to \check{H}^c_{m-k}(f(M); \mathbf{Z}_2)$ is a monomorphism, then $\theta(f) = 0$ in $H^c_{m-k}(M; \mathbf{Z}_2)$.*

Corollary 3.7 *Let $f : M \to N$ be a proper topological embedding of an m-dimensional topological manifold M into an $(m+k)$-dimensional topological manifold N with $k > 0$. Then $\theta(f) \in H^c_{m-k}(M; \mathbf{Z}_2)$ always vanishes.*

The above corollary shows that if a proper continuous map $f : M \to N$ between topological manifolds ($\dim M < \dim N$) is homotopic to a (not necessarily locally flat) topological embedding, then $\theta(f)$ vanishes. In other words, $\theta(f)$ can be regarded as a primary obstruction to the existence of such a homotopy. This fact has already been shown in [5] when the manifolds M and N admit differentiable structures and M is compact.

Remark 3.8 A topological embedding $f : M \to N$ between topological manifolds is proper if and only if $f(M)$ is a closed subset of N.

For a proper topological embedding $f : M \to N$, denote by $\tau(f) \in H^k(N, N - f(M); \mathbf{Z}_2)$ the Thom class and set $\chi(f) = f^*\tau(f) \in H^k(M; \mathbf{Z}_2)$, where $f^* : H^k(N, N - f(M); \mathbf{Z}_2) \to H^k(M; \mathbf{Z}_2)$ is the homomorphism induced by the composition of $\bar{f} : M \to f(M)$ and the inclusions $f(M) \to N \to (N, N - f(M))$. The cohomology class $\chi(f)$ is called the *normal Euler class* of f (see [9, Chapter VIII, §11]).

Corollary 3.9 *Let $f : M \to N$ be a proper topological embedding of an m-dimensional topological manifold M into an $(m+k)$-dimensional topological manifold N with $k > 0$. Then we always have $w_k(f) = \chi(f) = f^*U_f \in H^k(M; \mathbf{Z}_2)$.*

Proof. By [9, Chapter VIII, 11.25 Proposition], we see that the element $\chi(f) \frown [M] \in H^c_{m-k}(M; \mathbf{Z}_2)$ is Poincaré dual to $f^*(U_f) \in H^k(M; \mathbf{Z}_2)$ (see also [9, Chapter VIII, §11.27]). Hence $\chi(f) = f^*U_f$. On the other hand, by Corollary 3.7 and the definition of $\theta(f)$ (see Definition 2.1), we have $w_k(f) = f^*U_f$. This completes the proof. ||

Corollary 3.10 *Let $f : M \to N$ be a proper topological embedding of an m-dimensional topological manifold M into an $(m+k)$-dimensional topological manifold N with $k > 0$. Then the i-th Stiefel-Whitney class $w_i(f) \in H^i(M; \mathbf{Z}_2)$ of the stable normal bundle of f vanishes for all $i > k$.*

Proof. Consider the composition

$$\tilde{f} : M \xrightarrow{f} N = N \times \{0\} \xrightarrow{\eta} N \times \mathbf{R}^{i-k},$$

where η is the inclusion map. It is not difficult to see that $H^c_m(N \times \mathbf{R}^{i-k}; \mathbf{Z}_2)$ is naturally isomorphic to $H^c_{m-(i-k)}(N; \mathbf{Z}_2) \otimes H^c_{i-k}(\mathbf{R}^{i-k}; \mathbf{Z}_2) \cong H^c_{m-(i-k)}(N; \mathbf{Z}_2)$. Thus we see that $U_{\tilde{f}} = 0 \in H^i(N \times \mathbf{R}^{i-k}; \mathbf{Z}_2)$ and hence that $\theta(\tilde{f})$ coincides with the Poincaré dual of $w_i(\tilde{f})$. On the other hand, since $\tilde{f}$ is a proper topological embedding, $\theta(\tilde{f})$ vanishes by Corollary 3.7. Thus we have $w_i(\tilde{f}) = 0 \in H^i(M; \mathbf{Z}_2)$. Thus we have only to show the following.

Lemma 3.11 *We have $w_i(\tilde{f}) = w_i(f) \in H^i(M; \mathbf{Z}_2)$ for all i.*

Proof. We have only to show that $w(N) \in H^*(N; \mathbf{Z}_2)$ corresponds to $w(N \times \mathbf{R}) \in H^*(N \times \mathbf{R}; \mathbf{Z}_2)$ by the natural isomorphism $H^*(N; \mathbf{Z}_2) \cong H^*(N \times \mathbf{R}; \mathbf{Z}_2)$. Furthermore, by the Wu formula, this reduces to showing that $v(N)$ corresponds to $v(N \times \mathbf{R})$, where v denotes the total Wu class. For this, we have only to show that, for an arbitrary $x \in H^{n+1-j}_c(N \times \mathbf{R}; \mathbf{Z}_2)$, we have

$$< x \smile v'_j(N), [N \times \mathbf{R}] > = < Sq^j(x), [N \times \mathbf{R}] >,$$

where $v'_j(N) \in H^j(N \times \mathbf{R}; \mathbf{Z}_2)$ is the element which corresponds to $v_j(N) \in H^j(N; \mathbf{Z}_2)$. Since $H^{n+1-j}_c(N \times \mathbf{R}; \mathbf{Z}_2) \cong H^{n-j}_c(N; \mathbf{Z}_2) \otimes H^1_c(\mathbf{R}; \mathbf{Z}_2)$, we have $x = x' \times \zeta$ for some $x' \in H^{n-j}_c(N; \mathbf{Z}_2)$, where $\zeta \in H^1_c(\mathbf{R}; \mathbf{Z}_2) \cong \mathbf{Z}_2$ is the generator. Then, with the generator of $H^0(\mathbf{R}; \mathbf{Z}_2) \cong \mathbf{Z}_2$ being denoted by 1, we have

$$\begin{aligned} < x \smile v'_j(N), [N \times \mathbf{R}] > &= < (x' \times \zeta) \smile (v_j(N) \times 1), [N] \times [\mathbf{R}] > \\ &= < x' \smile v_j(N), [N] > \\ &= < Sq^j(x'), [N] > \\ &= < Sq^j(x') \times \zeta, [N] \times [\mathbf{R}] > \\ &= < Sq^j(x), [N \times \mathbf{R}] > . \end{aligned}$$

This completes the proof of Lemma 3.11 and hence Corollary 3.10. ||

Remark 3.12 Probably, it would also be possible to prove the above result by using the stable normal microbundles as in [16], [12], [15, Essay IV, Appendix A]. Here we have given a proof which does not depend on the existence of such bundle structures.

Remark 3.13 We do not know if a result similar to Corollary 3.10 holds for topological immersions (i.e., locally injective continuous maps) as well.

Corollaries 3.7 and 3.10 imply, for example, that if the dual Stiefel-Whitney class $\bar{w}_i(M)$ of an m-dimensional topological manifold M does not vanish, then for all $j \leq i$, M cannot be topologically embedded in $\mathbf{R}^{m+j}$ as a closed subset. This is a very strong result, since we are considering topological embeddings which are not necessarily locally flat.

Corollary 3.14 *Let $f : M \to N$ be a proper continuous map of a connected m-dimensional topological manifold M into an $2m$-dimensional topological manifold N. Then $\theta(f) \in H_0(M; \mathbf{Z}_2)$ always vanishes.*

Proof. Since $\bar{f}_* : H_0^c(M; \mathbf{Z}_2) \to \check{H}_0^c(f(M); \mathbf{Z}_2)$ is always a monomorphism, we have the result by Corollary 3.6. ||

For a continuous map $f : M \to N$ between topological manifolds, we set

$$M(f) = \{x \in M : f^{-1}(f(x)) \neq \{x\}\},$$

which is called the *self-intersection set* of f (see [19], [5]).

In the following results, the source manifold M will be compact. The following is a refinement of [5, Theorem 6.1].

Corollary 3.15 *Let $f : M \to N$ be a continuous map of an m-dimensional closed topological manifold M into an $(m+k)$-dimensional topological manifold N with $k > 0$. Set $A = \overline{M(f)}$ and $B = f(A)$. Then there exists an element $\mu \in \check{H}_{m-k}(A; \mathbf{Z}_2)$ such that*

$$j_*\mu = \theta(f) \in \check{H}_{m-k}(M; \mathbf{Z}_2) = H_{m-k}(M; \mathbf{Z}_2) \quad \text{and} \quad (f|A)_*\mu = 0 \in \check{H}_{m-k}(B; \mathbf{Z}_2),$$

where $j : A \to M$ is the inclusion map (when $A = \emptyset$, we regard $\check{H}_{m-k}(A; \mathbf{Z}_2) = 0 = \check{H}_{m-k}(B; \mathbf{Z}_2)$).

Proof. We may assume that $A \neq \emptyset$ by Corollary 3.7. By an argument as in [7, §3], we have the following exact sequence:

$$\check{H}_{m-k}(A; \mathbf{Z}_2) \xrightarrow{\alpha} \check{H}_{m-k}(M; \mathbf{Z}_2) \oplus \check{H}_{m-k}(B; \mathbf{Z}_2) \xrightarrow{\psi} \check{H}_{m-k}(f(M); \mathbf{Z}_2),$$

where $\alpha = (j_*, (f|A)_*)$ and $\psi = \bar{f}_* + j'_*$ with $j' : B \to f(M)$ the inclusion map. Since $\psi(\theta(f), 0) = 0$ by Theorem 3.1, there exists an element $\mu \in \check{H}_{m-k}(A; \mathbf{Z}_2)$ such that $\alpha(\mu) = (j_*\mu, (f|A)_*\mu) = (\theta(f), 0)$. This completes the proof. ||

Corollary 3.15 shows that the "support" of the homology class $\theta(f) \in H_{m-k}(M; \mathbf{Z}_2)$ is contained in the closure A of the self-intersection set of f. We do not know if $\mu \neq 0 \in \check{H}_{m-k}(A; \mathbf{Z}_2)$ (see Proposition 6.2 in §6).

Remark 3.16 By the same argument, we can show that if $v \in \check{H}_{m-k}(B; \mathbf{Z}_2)$ satisfies $j'_*v = 0$ in $\check{H}_{m-k}(f(M); \mathbf{Z}_2)$, then there exists an element $\mu_v \in \check{H}_{m-k}(A; \mathbf{Z}_2)$ such that $j_*\mu_v = \theta(f)$ and $(f|A)_*\mu_v = v$.

Corollary 3.17 *Let $f : M \to N$ be a continuous map of an m-dimensional closed topological manifold M into an $(m+k)$-dimensional topological manifold N with $k > 0$. Set $A = \overline{M(f)}$. If the topological dimension of A is strictly less than $m - k$, then $\theta(f) \in H_{m-k}(M; \mathbf{Z}_2)$ vanishes.*

For the definition and the properties of the topological dimension, see [13].

Proof. Since the topological dimension of A is strictly less than $m - k$, we have $\check{H}_{m-k}(A; \mathbf{Z}_2) = 0$ by [13, Theorem VIII 4 (p.152)]. Thus, by the exact sequence as in the proof of Corollary 3.15, we see that $\bar{f}_* : \check{H}_{m-k}(M; \mathbf{Z}_2) \to \check{H}_{m-k}(f(M); \mathbf{Z}_2)$ is a monomorphism. Then the result follows from Corollary 3.6. ||

4 Application

In this section, we give an application of the results obtained in the previous section to maps of the real projective plane into 3-dimensional manifolds.

Proposition 4.1 *Let $f : \mathbf{R}P^2 \to N$ be a continuous map of the real projective plane into a 3-dimensional topological manifold N. If $f_*[\mathbf{R}P^2] = 0$ in $H_2(N; \mathbf{Z}_2)$, then $\bar{f}_* : H_1(\mathbf{R}P^2; \mathbf{Z}_2) \to \check{H}_1(f(\mathbf{R}P^2); \mathbf{Z}_2)$ is the zero map.*

Proof. By an argument similar to that in [6, §4], f is bordant to a constant map (see [8]) and hence we have

$$\theta(f) = \bar{w}_1(\mathbf{R}P^2) \frown [\mathbf{R}P^2],$$

which is the generator of $H_1(\mathbf{R}P^2; \mathbf{Z}_2) \cong \mathbf{Z}_2$. Then the result follows from Corollary 3.4. This completes the proof. ||

As a direct corollary to the above proposition, we obtain the following.

Corollary 4.2 ([6, Corollary 4.2]) *Let $f : \mathbf{R}P^2 \to N$ be a continuous map of the real projective plane into a 3-dimensional topological manifold N. If $f_*[\mathbf{R}P^2] = 0$ in $H_2(N; \mathbf{Z}_2)$, then f is not a topological embedding.*

5 Characterizations of differentiable embeddings

In this section, using results of §3, we give various new characterizations of differentiable embeddings among generic differentiable maps.

Definition 5.1 Let $f : M \to N$ be a map of class C^2 between smooth manifolds with $\dim M < \dim N$. We say that f is *generic for the double points* if it is so in the sense of Ronga [19]. See also [7, Definition 2.1].

Note that the set of all C^2 maps which are generic for the double points is a dense subset of the mapping space $C^2(M, N)$ endowed with the Whitney C^2-topology.

In the following, $\widetilde{\check{H}}_*$ denotes the reduced Čech homology group. The following is a refinement of [5, Corollary 4.10].

Theorem 5.2 *Let $f : M \to N$ be a map of class C^2 between smooth manifolds which is generic for the double points with $m = \dim M < \dim N = m + k$ and M closed. Then f is a differentiable embedding if and only if $\widetilde{\check{H}}_{m-k+1}(C_{\bar{f}}; \mathbf{Z}_2) = 0$, where $C_{\bar{f}}$ is the mapping cone of the map $\bar{f} = f : M \to f(M)$.*

Proof. Let $Z_{\bar{f}}$ be the mapping cylinder of $\bar{f}$ and we regard M to be naturally embedded in $Z_{\bar{f}}$. Denote by $i : M \to Z_{\bar{f}}$ the inclusion map. Since the spaces M and $Z_{\bar{f}}$ are compact, we have the following exact sequence of the Čech homology (see [14], [10]):

$$\begin{array}{rlll} & \check{H}_{m-k+1}(M; \mathbf{Z}_2) & \xrightarrow{i_*} \check{H}_{m-k+1}(Z_{\bar{f}}; \mathbf{Z}_2) \longrightarrow \check{H}_{m+k-1}(Z_{\bar{f}}, M; \mathbf{Z}_2) \\ \longrightarrow & \check{H}_{m-k}(M; \mathbf{Z}_2) & \xrightarrow{i_*} \check{H}_{m-k}(Z_{\bar{f}}; \mathbf{Z}_2). \end{array}$$

Note that the map

$$i_* : \check{H}_*(M; \mathbf{Z}_2) \to \check{H}_*(Z_{\bar{f}}; \mathbf{Z}_2)$$

is equivalent to

$$\bar{f}_* : \check{H}_*(M; \mathbf{Z}_2) \to \check{H}_*(f(M); \mathbf{Z}_2).$$

By an argument similar to that of [7, p.76], we see that

$$i_* : \check{H}_{m-k+1}(M; \mathbf{Z}_2) \to \check{H}_{m-k+1}(Z_{\bar{f}}; \mathbf{Z}_2)$$

is always a monomorphism. Furthermore, $\check{H}_{m-k+1}(Z_{\bar{f}}, M; \mathbf{Z}_2)$ is always isomorphic to the reduced Čech homology group $\widetilde{\check{H}}_{m-k+1}(C_{\bar{f}}; \mathbf{Z}_2)$. Thus, if $\widetilde{\check{H}}_{m-k+1}(C_{\bar{f}}; \mathbf{Z}_2) = 0$, then

$$\bar{f}_* : \check{H}_{m-k+1}(M; \mathbf{Z}_2) \to \check{H}_{m-k+1}(f(M); \mathbf{Z}_2)$$

is an isomorphism and

$$\bar{f}_* : \check{H}_{m-k}(M; \mathbf{Z}_2) \to \check{H}_{m-k}(f(M); \mathbf{Z}_2)$$

is a monomorphism. Thus, by Corollary 3.6, $\theta(f) \in H_{m-k}(M; \mathbf{Z}_2)$ vanishes. Then by [7, Theorem 2.2], we see that f is a differentiable embedding.

Conversely, if f is a differentiable embedding, it is easy to show that $\widetilde{\check{H}}_{m-k+1}(C_{\bar{f}}; \mathbf{Z}_2) = 0$ since $C_{\bar{f}}$ is contractible. This completes the proof. ||

As the above argument shows, we also have the following.

Proposition 5.3 *Let $f : M \to N$ be a map of class C^2 between smooth manifolds which is generic for the double points with $m = \dim M < \dim N = m + k$ and M closed. Then f is a differentiable embedding if and only if $\bar{f}_* : \check{H}_{m-k+1}(M; \mathbf{Z}_2) \to \check{H}_{m-k+1}(f(M); \mathbf{Z}_2)$ is an epimorphism and $\bar{f}_* : \check{H}_{m-k}(M; \mathbf{Z}_2) \to \check{H}_{m-k}(f(M); \mathbf{Z}_2)$ is a monomorphism.*

Corollary 5.4 *Let $f : M \to N$ be a map of class C^2 between smooth manifolds which is generic for the double points with $m = \dim M < \dim N = m + k$ and M closed. Then f is a differentiable embedding if and only if $\bar{f}_* : \check{H}_*(M; \mathbf{Z}_2) \to \check{H}_*(f(M); \mathbf{Z}_2)$ is an isomorphism.*

In the following, for a topological space X, we set

$$\begin{aligned}
\beta_j(X; \mathbf{Z}_2) &= \dim_{\mathbf{Z}_2} H_j(X; \mathbf{Z}_2), \\
\beta_j(X; \mathbf{Z}) &= \operatorname{rank} H_j(X; \mathbf{Z}), \\
\check{\beta}_j(X; \mathbf{Z}_2) &= \dim_{\mathbf{Z}_2} \check{H}_j(X; \mathbf{Z}_2), \\
\check{\beta}_j(X; \mathbf{Z}) &= \operatorname{rank} \check{H}_j(X; \mathbf{Z}).
\end{aligned}$$

As a related result, we have the following.

Proposition 5.5 *Let $f : M \to N$ be a map of class C^2 between smooth manifolds which is generic for the double points with $\dim N = \dim M + 1$ and M closed. Suppose that $H_1(N; \mathbf{Z}_2) = 0$. Then f is a differentiable embedding if and only if $\check{\beta}_m(M; \mathbf{Z}_2) = \check{\beta}_m(f(M); \mathbf{Z}_2)$ and $\check{\beta}_m(M; \mathbf{Z}) = \check{\beta}_m(f(M); \mathbf{Z})$.*

Proof. For $\mathbf{Z}_2$ and $\mathbf{Z}$ coefficients, we have the exact sequence

$$0 \to H^0(N) \to H^0(N - f(M)) \to H^1(N, N - f(M)) \to H^1(N).$$

By our assumption, we have $H^1(N) = 0$ and by Alexander duality, $H^1(N, N - f(M))$ is isomorphic to $\check{H}_m(f(M))$, since N is orientable (see [22]). Thus we have

$$\begin{aligned}
\beta_0(N; \mathbf{Z}_2) + \check{\beta}_m(f(M); \mathbf{Z}_2) &= \beta_0(N - f(M); \mathbf{Z}_2) \\
&= \beta_0(N - f(M); \mathbf{Z}) \\
&= \beta_0(N; \mathbf{Z}) + \check{\beta}_m(f(M); \mathbf{Z}).
\end{aligned}$$

Since we have

$$\beta_0(N; \mathbf{Z}_2) = \beta_0(N; \mathbf{Z}),$$

if

$$\check{\beta}_m(M; \mathbf{Z}_2) = \check{\beta}_m(f(M); \mathbf{Z}_2) \text{ and } \check{\beta}_m(M; \mathbf{Z}) = \check{\beta}_m(f(M); \mathbf{Z}),$$

then we have

$$\beta_m(M;\mathbf{Z}_2) = \beta_m(M;\mathbf{Z})$$

and hence M must be orientable. Then the result follows from [7, Corollary 2.6][§]. This completes the proof. ||

The above proposition suggests the following question: if a map $f : M \to N$ of class C^2 between smooth manifolds is generic for the double points and satisfies $\check{H}_*(M;R) \cong \check{H}_*(f(M);R)$ for every commutative ring R, then is f a differentiable embedding? The answer to this question is in fact negative as the following example shows.

Example 5.6 Let $\varphi : S^1 \to \mathbf{R}^2$ be a self-transverse immersion with exactly one normal crossing point at the origin as in Figure 1. Let $\tau : \mathbf{R}^2 \to \mathbf{R}^2$ be the reflection with respect to the second coordinate axis. We may assume that there exists an orientation preserving involution $\bar{\tau} : S^1 \to S^1$ such that $\varphi \circ \bar{\tau} = \tau \circ \varphi$. Set $M = S^1 \times [0,1]/(p,1) \sim (\bar{\tau}(p),0)$ and $N = \mathbf{R}^2 \times [0,1]/(q,1) \sim (\tau(q),0)$, which are diffeomorphic to the torus and the open solid Klein bottle respectively. Then define the smooth immersion $f : M \to N$ by $f(p,t) = (\varphi(p),t)$. We will show that $\check{H}_*(M;R)$ is isomorphic to $\check{H}_*(f(M);R)$ for any commutative ring R with unit.

Figure 1 here

Set $M_1 = S^1 \times [0,1/2]$ and $M_2 = S^1 \times [1/2,1]$, which are considered to be subspaces of M. Then we have the following Mayer-Vietoris exact sequence of R-modules:

$$\begin{array}{rcl} & & H_2(f(M_1);R) \oplus H_2(f(M_2);R) \to H_2(f(M);R) \\ \to H_1(f(M_1)\cap f(M_2);R) & \xrightarrow{\eta} & H_1(f(M_1);R) \oplus H_1(f(M_2);R) \to H_1(f(M);R) \\ \to \tilde{H}_0(f(M_1)\cap f(M_2);R) & \longrightarrow & \tilde{H}_0(f(M_1);R) \oplus \tilde{H}_0(f(M_2);R). \end{array}$$

Thus we see that $H_2(f(M);R)$ is isomorphic to the kernel of η and $H_1(f(M);R)$ is isomorphic to the direct sum of the cokernel of η and R. With respect to appropriate bases for $H_1(f(M_1)\cap f(M_2);R)$ and $H_1(f(M_1);R) \oplus H_1(f(M_2);R)$, η is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

[§]In [7], the manifold N is assumed to be connected. However, this assumption is redundant, if we replace the condition $\beta_0(N - f(M)) = \beta_0(M) + 1$ by the condition $\beta_0(N - f(M)) = \beta_0(M) + \beta_0(N)$ in [7, Corollary 2.6].

This matrix is easily seen to be equivalent to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we see that $H_1(f(M); R)$ and $H_2(f(M); R)$ are isomorphic to $R \oplus R$ and R respectively. On the other hand, since M is diffeomorphic to the torus $S^1 \times S^1$, $H_1(M; R)$ and $H_2(M; R)$ are also isomorphic to $R \oplus R$ and R respectively. Hence $H_*(M; R) = \check{H}_*(M; R)$ is isomorphic to $H_*(f(M); R) = \check{H}_*(f(M); R)$ for every commutative ring R with unit. Nevertheless, f is not an embedding.

The above example shows that the condition $H_1(N; \mathbf{Z}_2) = 0$ is essential in Proposition 5.5. Note also that, in the above example, $\bar{f}_* : H_1(M; R) \to H_1(f(M); R)$ is not an isomorphism for $R = \mathbf{Z}_2, \mathbf{Z}$.

6 Maps with a normal crossing point

In this section, we study continuous maps which have normal crossing points of multiplicity two.

Definition 6.1 Let $f : M \to N$ be a continuous map between smooth manifolds. A point $q \in N$ is said to be a *differentiable normal crossing point of multiplicity* r, if there exists an open ball neighborhood U of q in N such that $f^{-1}(U)$ is a disjoint union of open disks $V_1, \ldots, V_r$ in M, that $f|V_i : V_i \to U$ are differentiable embeddings of class C^1 ($i = 1, \ldots, r$), and that, for a C^1 diffeomorphism $\varphi : U \to \mathbf{R}^n$ with $\varphi(q) = 0$ ($n = \dim N$), $\varphi \circ f(V_i)$ ($i = 1, \ldots, r$) are linear subspaces of $\mathbf{R}^n$ in general position. Compare this definition with that of [20].

The following is a refinement of Corollary 3.15 for maps with a differentiable normal crossing point.

Proposition 6.2 *Let $f : M \to N$ be a continuous map of an m-dimensional closed smooth manifold M into an $(m+k)$-dimensional smooth manifold N with $k > 0$. Set $A = \overline{M(f)}$. If f has a differentiable normal crossing point of multiplicity two, then there exist a compact ANR A_1 in M containing A and a nonzero element $\mu \in H_{m-k}(A_1; \mathbf{Z}_2)$ such that*

$$j_{1*}\mu = \theta(f) \in H_{m-k}(M; \mathbf{Z}_2) \text{ and } (f|A_1)_*\mu = 0 \in \check{H}_{m-k}(B_1; \mathbf{Z}_2),$$

where $j_1 : A_1 \to M$ is the inclusion map and $B_1 = f(A_1)$.

Proof. Let $q \in f(M)$ be a differentiable normal crossing point of multiplicity two and let U, V_1, V_2 and $\varphi : U \to \mathbf{R}^n$ be as in Definition 6.1. Let U' be the open ball neighborhood of q in U which corresponds to the open unit disk in $\mathbf{R}^n$ by φ. Set $V_i' = V_i \cap f^{-1}(U')$ $(i = 1, 2)$ and $A_1 = (M - (V_1' \cup V_2')) \cup f^{-1}(Z)$, where Z is a sufficiently small closed tubular neighborhood of $f(V_1) \cap f(V_2)$ in U (see Figure 2). Note that A_1 contains A and is a compact ANR.

Figure 2 here

Let $\{W_n\}_{n=1}^{\infty}$ be a sequence of open neighborhoods of $B_1 = f(A_1)$ in N such that $B_1 = \cap_{n=1}^{\infty} W_n$, $\overline{W}_{n+1} \subset W_n$ for all $n = 1, 2, 3, \ldots$, and that $\overline{W}_n$ are all compact. Then, for each n, there exists a C^2 map $g_n : M \to N$ which is generic for the double points and approximates f such that g_n is homotopic to f, that $g_n|(g_n^{-1}(U'))$ is an immersion with normal crossings without triple points with the set Y_n of normal crossing points of multiplicity two being an open disk of dimension $m - k$ properly embedded in U', that $Z \cap U'$ is a closed tubular neighborhood of Y_n in U', $\overline{M(g_n)} \subset A_1$, $g_n(A_1) \subset W_n$, and that $g_n|A_1$ and $f|A_1 : A_1 \to W_n$ are homotopic.

Set $\mu_n = i_{n*}[\overline{M(g_n)}] \in H_{m-k}(A_1; \mathbf{Z}_2)$, where $i_n : \overline{M(g_n)} \to A_1$ is the inclusion map and $[\overline{M(g_n)}] \in H_{m-k}(\overline{M(g_n)}; \mathbf{Z}_2)$ is the fundamental class of $\overline{M(g_n)}$ in the sense of Ronga [19]. Let T_n be a $2k$-dimensional open disk properly embedded in U' which intersects Y_n transversely at one point such that $T_n \cap Z$ is a $2k$-dimensional closed disk. Since the image of μ_n in $H_{m-k}(A_1, A_1 - g_n^{-1}(T_n); \mathbf{Z}_2) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is nonzero, we see that $\mu_n \in H_{m-k}(A_1; \mathbf{Z}_2)$ is nonzero.

Since A_1 is a compact ANR, $H_{m-k}(A_1; \mathbf{Z}_2)$ is finitely generated and hence is a finite group. Thus there exist an infinite subsequence $\{\mu_{n_l}\}$ of $\{\mu_n\}$ and an element $\mu \in H_{m-k}(A_1; \mathbf{Z}_2)$ such that $\mu_{n_l} = \mu \neq 0$ for all l. Then we have $j_{1*}\mu = \theta(g_{n_l}) = \theta(f)$ by [19], since g_{n_l} and f are homotopic, and

$$(f|A_1)_*\mu = (g_{n_l}|A_1)_*\mu_{n_l} = 0 \in H_{m-k}(W_{n_l}; \mathbf{Z}_2)$$

by [7, Lemma 3.1], since g_{n_l} is generic for the double points. Since the inverse limit

$$\varprojlim H_{m-k}(W_{n_l}; \mathbf{Z}_2)$$

is identified with $\check{H}_{m-k}(B_1; \mathbf{Z}_2)$, we see that $(f|A_1)_*\mu = 0$ in $\check{H}_{m-k}(B_1; \mathbf{Z}_2)$. This completes the proof. ||

The following corollary follows from Proposition 6.2 together with an argument similar to that in the proof of [5, Lemma 6.3].

Corollary 6.3 *Let $f : M \to N$ be a continuous map of an m-dimensional closed smooth manifold M into an $(m+k)$-dimensional smooth manifold N with $k > 0$. If f has a differentiable normal crossing point of multiplicity two and $\theta(f) = 0$ in $H_{m-k}(M; \mathbf{Z}_2)$, then $\bar{f}_* : H_{m-k+1}(M; \mathbf{Z}_2) \to \check{H}_{m-k+1}(f(M); \mathbf{Z}_2)$ is not an epimorphism.*

The following is a refinement of [5, Proposition 6.4].

Corollary 6.4 *Let $f : M \to N$ be a continuous map of a connected orientable m-dimensional closed smooth manifold M into a connected $(m+1)$-dimensional smooth manifold N with $H_1(N; \mathbf{Z}_2) = 0$. If f has a differentiable normal crossing point of multiplicity two, then the number of connected components of $N - f(M)$ is greater than or equal to three.*

Proof. By an argument similar to that in the proof of Proposition 5.5, we see that the number of connected components of $N - f(M)$ is equal to $\dim_{\mathbf{Z}_2} \check{H}_m(f(M); \mathbf{Z}_2) + 1$. On the other hand, by an argument similar to that in the proof of [5, Lemma 6.3], we have the following exact sequence:

$$\check{H}_m(A_1; \mathbf{Z}_2) \to \check{H}_m(B_1; \mathbf{Z}_2) \oplus \check{H}_m(M; \mathbf{Z}_2) \to \check{H}_m(f(M); \mathbf{Z}_2),$$

where A_1 and B_1 are as in Proposition 6.2. Note that $\check{H}_m(A_1; \mathbf{Z}_2) = 0$, since $A_1 \neq M$ and M is connected. Thus $\bar{f}_* : \check{H}_m(M; \mathbf{Z}_2) \to \check{H}_m(f(M); \mathbf{Z}_2)$ is a monomorphism.

On the other hand, by our assumption, it is easy to see that $\theta(f) \in H_{m-1}(M; \mathbf{Z}_2)$ vanishes. Thus by Corollary 6.3, we have $\dim_{\mathbf{Z}_2} \check{H}_m(f(M); \mathbf{Z}_2) \geq \dim_{\mathbf{Z}_2} \check{H}_m(M; \mathbf{Z}_2) + 1 = 2$. Thus the number of connected components of $N - f(M)$ is greater than or equal to three. This completes the proof. ||

Corollary 6.4 generalizes the results in [3], [1], [2] about codimension-1 immersions with normal crossings. In fact, using Corollary 6.4, we can prove the converse of the Jordan-Brouwer theorem for codimension-1 C^2 maps which are generic for the double points (see [7, Corollary 2.6]).

Note that a more general result concerning maps with a normal crossing point of multiplicity $r \geq 2$ has been obtained in [20]. See also [21].

7 R-bordism invariance of $\theta(f)$

In this section, we show that the homology class $\theta(f)$ is invariant under a certain kind of bordism.

Let M_1, M_2 and N be topological manifolds, where M_1 and M_2 are closed and have the same dimension m.

Definition 7.1 Two continuous maps $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are said to be *R-bordant* if there exist a compact $(m+1)$-dimensional topological manifold with boundary, W, and a continuous map $F : W \to N$ such that
(1) W is a cobordism between M_1 and M_2; i.e, ∂W is identified with the disjoint union of M_1 and M_2,
(2) there exist retractions $r_1 : W \to M_1$ and $r_2 : W \to M_2$, and
(3) $F|M_1 = f_1$ and $F|M_2 = f_2$.

For example, if $M_1 = M_2$ and f_1 and f_2 are homotopic, then they are R-bordant.

Note that if f_1 and f_2 are R-bordant, then there exists a canonical isomorphism between $H_*(M_1; \mathbf{Z}_2)$ and $H_*(M_2; \mathbf{Z}_2)$ (see [11, Theorem 1.2]). If fact, $(r_2 \circ i_1)_* : H_*(M_1; \mathbf{Z}_2) \to H_*(M_2; \mathbf{Z}_2)$ and $(r_1 \circ i_2)_* : H_*(M_2; \mathbf{Z}_2) \to H_*(M_1; \mathbf{Z}_2)$ are isomorphisms and are inverse of each other, where $i_1 : M_1 \to W$ and $i_2 : M_2 \to W$ denote the inclusion maps. Furthermore, we have $f_{1*} = f_{2*} \circ (r_2 \circ i_1)_* : H_*(M_1; \mathbf{Z}_2) \to H_*(N; \mathbf{Z}_2)$ and $f_{2*} = f_{1*} \circ (r_1 \circ i_2)_* : H_*(M_2; \mathbf{Z}_2) \to H_*(N; \mathbf{Z}_2)$, since

$$
\begin{aligned}
f_{2*} \circ (r_2 \circ i_1)_* &= F_* \circ i_{2*} \circ r_{2*} \circ i_{1*} && (22)\\
&= F_* \circ i_{1*} \circ r_{1*} \circ i_{2*} \circ r_{2*} \circ i_{1*} && (23)\\
&= F_* \circ i_{1*} && (24)\\
&= f_{1*}, && (25)
\end{aligned}
$$

where the equation (23) follows from [11, Lemma 1.1].

When the manifolds M_1 and M_2 admit differentiable structures, we can characterize the above relation as follows.

Proposition 7.2 *Let M_1 and M_2 be closed smooth manifolds with the same dimension and N a topological manifold. Then two continuous maps $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are R-bordant if and only if there exist two continuous maps $g_1 : M_1 \to M_2$ and $g_2 : M_2 \to M_1$ such that $g_{1*} : H_*(M_1; \mathbf{Z}_2) \to H_*(M_2; \mathbf{Z}_2)$ and $g_{2*} : H_*(M_2; \mathbf{Z}_2) \to H_*(M_1; \mathbf{Z}_2)$ are inverse isomorphisms and that $f_{2*} \circ g_{1*} = f_{1*} : H_*(M_1; \mathbf{Z}_2) \to H_*(N; \mathbf{Z}_2)$ and $f_{1*} \circ g_{2*} = f_{2*} : H_*(M_2; \mathbf{Z}_2) \to H_*(N; \mathbf{Z}_2)$.*

Proof. If f_1 and f_2 are R-bordant, then the continuous maps $g_1 = r_2 \circ i_1 : M_1 \to M_2$ and $g_2 = r_1 \circ i_2 : M_2 \to M_1$ satisfy the required conditions as has been seen above.

Suppose the existence of g_1 and g_2. Then by the proof of [11, Theorem 1.7], we see that $[M_1, \mathrm{id} \times g_1] = [M_2, g_2 \times \mathrm{id}] \in \mathcal{N}_m(M_1 \times M_2)$, where $\mathcal{N}_m$ denotes the m-th bordism group with $m = \dim M_1 = \dim M_2$ (see [8]) and "id" denotes the identity map. Consider the images under the homomorphism

$$\mathcal{N}_m(M_1 \times M_2) \xrightarrow{(\mathrm{id} \times (f_1 \circ p_1))_*} \mathcal{N}_m((M_1 \times M_2) \times N),$$

where $p_1 : M_1 \times M_2 \to M_1$ denotes the projection to the first factor. Then we have $[M_1, \mathrm{id} \times g_1 \times f_1] = [M_2, g_2 \times \mathrm{id} \times (f_1 \circ g_2)] \in \mathcal{N}_m(M_1 \times M_2 \times N)$. Since $(f_1 \circ g_2)_* = f_{2*}$ by our assumption, by an argument using the method of Whitney numbers (see [8, (17.2) Theorem]), we see that $[M_2, g_2 \times \mathrm{id} \times (f_1 \circ g_2)] = [M_2, g_2 \times \mathrm{id} \times f_2] \in \mathcal{N}_m(M_1 \times M_2 \times N)$. Hence we have $[M_1, \mathrm{id} \times g_1 \times f_1] = [M_2, g_2 \times \mathrm{id} \times f_2]$. Thus there exists a (smooth) cobordism W between M_1 and M_2 and a continuous map $\tilde{F} : W \to M_1 \times M_2 \times N$ such that $\tilde{F}|M_1 = \mathrm{id} \times g_1 \times f_1$ and $\tilde{F}|M_2 = g_2 \times \mathrm{id} \times f_2$. Then $r_1 = \pi_1 \circ \tilde{F} : W \to M_1$ and $r_2 = \pi_2 \circ \tilde{F} : W \to M_2$ give retractions which restrict to g_2 and g_1 on M_2 and M_1 respectively, and $F = \pi_3 \circ \tilde{F} : W \to N$ gives a continuous map such that $F|M_1 = f_1$ and $F|M_2 = f_2$, where $\pi_1 : M_1 \times M_2 \times N \to M_1$, $\pi_2 : M_1 \times M_2 \times N \to M_2$ and $\pi_3 : M_1 \times M_2 \times N \to N$ denote the projections. Thus f_1 and f_2 are R-bordant. This completes the proof. ||

Corollary 7.3 *Let M be a closed smooth manifold and N a topological manifold. Then two continuous maps $f_1 : M \to N$ and $f_2 : M \to N$ are R-bordant if $f_{1*} = f_{2*} : H_*(M; \mathbf{Z}_2) \to H_*(N; \mathbf{Z}_2)$.*

The following is closely related to the main theorem of [6] concerning the bordism invariance of the homology class $\theta(f)$.

Theorem 7.4 *Let $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ be continuous maps of m-dimensional closed topological manifolds M_1 and M_2 into an $(m+k)$-dimensional topological manifold N with $k > 0$. If f_1 and f_2 are R-bordant, then $\theta(f_1) \in H_{m-k}(M_1; \mathbf{Z}_2)$ corresponds to $\theta(f_2) \in H_{m-k}(M_2; \mathbf{Z}_2)$ by the canonical isomorphism.*

Proof. Let ξ be an arbitrary element of $H^*(N; \mathbf{Z}_2)$. Then we have

$$\begin{aligned}
(r_2 \circ i_1)_*((f_1^*\xi) \frown [M_1]) &= r_{2*} \circ i_{1*}((i_1^*(F^*\xi)) \frown [M_1]) && (26)\\
&= r_{2*}((F^*\xi) \frown i_{1*}[M_1]) && (27)\\
&= r_{2*}((F^*\xi) \frown i_{2*}[M_2]) && (28)\\
&= r_{2*} \circ i_{2*}((i_2^*(F^*\xi)) \frown [M_2]) && (29)\\
&= (f_2^*\xi) \frown [M_2], && (30)
\end{aligned}$$

where the equation (28) follows from the fact that $i_{1*}[M_1] = i_{2*}[M_2]$ in $H_m(W; \mathbf{Z}_2)$. This together with the fact that $f_{1*}[M_1] = f_{2*}[M_2] \in H_m(N; \mathbf{Z}_2)$ implies that $f_1^*(U_{f_1}) \frown [M_1] \in H_{m-k}(M_1; \mathbf{Z}_2)$ corresponds to $f_2^*(U_{f_2}) \frown [M_2] \in H_{m-k}(M_2; \mathbf{Z}_2)$.

On the other hand, we have

$$(r_2 \circ i_1)_*(((f_1^* w(N)) \smile \bar{w}(M_1)) \frown [M_1]) \qquad (31)$$

$$
\begin{aligned}
&= r_{2*} \circ i_{1*}((i_1^*(F^*w(N))) \frown (\bar{w}(M_1) \frown [M_1])) && (32)\\
&= r_{2*}((F^*w(N)) \frown i_{1*}(\bar{w}(M_1) \frown [M_1])) && (33)\\
&= r_{2*}((F^*w(N)) \frown i_{1*}(((r_2 \circ i_1)^*\bar{w}(M_2)) \frown [M_1])) && (34)\\
&= r_{2*}((F^*w(N)) \frown i_{1*}((i_1^*(r_2^*\bar{w}(M_2))) \frown [M_1])) && (35)\\
&= r_{2*}((F^*w(N)) \frown ((r_2^*\bar{w}(M_2)) \frown i_{1*}[M_1])) && (36)\\
&= r_{2*}((F^*w(N)) \frown ((r_2^*\bar{w}(M_2)) \frown i_{2*}[M_2])) && (37)\\
&= r_{2*}(((F^*w(N)) \smile (r_2^*\bar{w}(M_2))) \frown i_{2*}[M_2]) && (38)\\
&= r_{2*} \circ i_{2*}((i_2^*((F^*w(N)) \smile (r_2^*\bar{w}(M_2)))) \frown [M_2]) && (39)\\
&= ((f_2^*w(N)) \smile \bar{w}(M_2)) \frown [M_2], && (40)
\end{aligned}
$$

where the equation (34) follows from [11, Lemma 1.4]. Thus $((f_1^*w(N)) \smile \bar{w}(M_1)) \frown [M_1] \in H_*(M_1; \mathbf{Z}_2)$ corresponds to $((f_2^*w(N)) \smile \bar{w}(M_2)) \frown [M_2] \in H_*(M_2; \mathbf{Z}_2)$.

Hence $\theta(f_1) = (f_1^*(U_{f_1}) - w_k(f_1)) \frown [M_1] \in H_{m-k}(M_1; \mathbf{Z}_2)$ corresponds to $\theta(f_2) = (f_2^*(U_{f_2}) - w_k(f_2)) \frown [M_2] \in H_{m-k}(M_2; \mathbf{Z}_2)$. This completes the proof. ||

Combining the above theorem with Corollary 7.3, we obtain the following.

Corollary 7.5 *Let M be an m-dimensional closed smooth manifold and N an $(m+k)$-dimensional topological manifold with $k > 0$. If two continuous maps $f_1 : M \to N$ and $f_2 : M \to N$ satisfies $f_{1*} = f_{2*} : H_*(M; \mathbf{Z}_2) \to H_*(N; \mathbf{Z}_2)$, then $\theta(f_1) = \theta(f_2) \in H_{m-k}(M; \mathbf{Z}_2)$.*

Combining Theorem 7.4 with Corollary 3.7, we have the following.

Corollary 7.6 *Let $f : M \to N$ be a continuous map of an m-dimensional closed topological manifold M into an $(m+k)$-dimensional topological manifold N with $k > 0$. If f is R-bordant to a topological embedding, then $\theta(f) = 0$ in $H_{m-k}(M; \mathbf{Z}_2)$.*

The converse of Corollary 7.6 does not hold as is seen in the following example.

Example 7.7 Consider a continuous map $f : \mathbf{C}P^2 \to \mathbf{R}^5$. Since $H_3(\mathbf{C}P^2; \mathbf{Z}_2) = 0$, $\theta(f)$ always vanishes. Suppose that f is R-bordant to a topological embedding $g : M \to \mathbf{R}^5$. Then $H^*(M; \mathbf{Z}_2) \cong H^*(\mathbf{C}P^2; \mathbf{Z}_2)$ and by [11, Lemma 1.4], $\bar{w}_2(M)$ corresponds to $\bar{w}_2(\mathbf{C}P^2)$ under the isomorphism. Since $\bar{w}_2(\mathbf{C}P^2)$ does not vanish, we see that $\bar{w}_2(M) = w_2(g)$ also does not vanish. Thus g cannot be a topological embedding by Corollary 3.10. Thus $f : \mathbf{C}P^2 \to \mathbf{R}^5$ is not R-bordant to a topological embedding, although $\theta(f) = 0$.

By an argument similar to the above example, we can show the following.

Proposition 7.8 *Let $f : M \to N$ be a continuous map of an m-dimensional closed topological manifold M into an $(m+k)$-dimensional topological manifold N with $k > 0$. If f is R-bordant to a topological embedding, then $w_i(f) = 0$ in $H^i(M; \mathbf{Z}_2)$ for all $i > k$.*

The above proposition suggests the following problem.

Problem 7.9 Let $f : M \to N$ be a continuous map of an m-dimensional closed topological manifold M into an $(m+k)$-dimensional topological manifold N with $k > 0$. If $\theta(f) = 0$ in $H_{m-k}(M; \mathbf{Z}_2)$ and $w_i(f) = 0$ in $H^i(M; \mathbf{Z}_2)$ for all $i > k$, then is f R-bordant to a topological embedding?

Example 7.10 Consider a continuous map $f : S^1 \to N$, where N is an arbitrary 2-dimensional topological manifold. Then by Corollary 3.14, $\theta(f) \in H_0(S^1; \mathbf{Z}_2)$ always vanishes. Furthermore, obviously we have $w_i(f) = 0$ for all $i > 1$. On the other hand, it is not difficult to show that every element of $H_1(N; \mathbf{Z}_2)$ can be represented by a topological embedding of S^1 into N. Thus, by Corollary 7.5, we see that every continuous map $f : S^1 \to N$ is R-bordant to a topological embedding. In fact, when N is smooth, every continuous map f is R-bordant to a smooth embedding.

In Problem 7.9, if we replace the condition "R-bordant" with "homotopic", then we have a counter example as follows.

Example 7.11 Consider the continuous map $f : S^1 \to \mathbf{C} - \{0\}$ defined by $f(z) = z^2$, where we identify S^1 with the unit circle in $\mathbf{C}$. Then, by the previous example, it is R-bordant to a smooth embedding of S^1 into $\mathbf{C} - \{0\}$. Suppose that f is homotopic to a topological embedding $g : S^1 \to \mathbf{C} - \{0\}$. By the Schoenflies theorem, $g(S^1)$ bounds a region U in $\mathbf{C}$ homeomorphic to the closed 2-dimensional disk. If $0 \notin U$, then g is null-homotopic in $\mathbf{C} - \{0\}$, which is a contradiction. If $0 \in U$, then g represents a generator of $H_1(\mathbf{C} - \{0\}; \mathbf{Z}_2)$. Thus g is not homotopic to f, which is again a contradiction. Thus, f is not *homotopic* to a topological embedding, although it is R-bordant to a topological embedding.

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