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GLOBAL BIFURCATION RESULTS FOR A DELAY DIFFERENTIAL SYSTEM REPRESENTING A CHEMOSTAT MODEL.

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Abstract

We present a global bifurcation result for a first order system of delay differential equations depending on a real parameter. The system represent a chemostat model. The approach is topological and is based of a topological degree theory for nonlinear Fredholm maps between Banach spaces.

1 Introduction

In the paper we prove a global bifurcation result for periodic solutions to the following delayed first order system, depending on a real parameter $\lambda \geq 0$,

$$\begin{cases} s'(t) = Ds^0(t) - Ds(t) - \dfrac{\lambda}{\gamma}\,\mu(s(t))x(t) & t \geq 0 \\ x'(t) = x(t)\big[\lambda\mu(s(t-\tau)) - D\big] & t \geq 0 \\ s(\theta) = \phi(\theta) \quad \text{and} \quad x(0) = x_0 & \text{if } \ \theta \in [-\tau, 0], \end{cases} \tag{1}$$

in which the following conditions hold:

(a) $s^0 : \mathbb{R} \to \mathbb{R}$ is continuous, positive and ω-periodic, where $\omega > 0$ is given,

(b) $\mu : [0, +\infty) \to [0, +\infty)$ is C^1 and verifies $\mu(0) = 0$ and $\mu'(s) > 0$, for any $s \in [0, +\infty)$,

(c) D, γ and the delay τ are positive constants,

(d) $\phi : [-\tau, 0] \to \mathbb{R}$ is continuous.

For a given λ, a *solution* of (1) is defined as a pair (s, x) of maps $s : [-\tau, +\infty) \to \mathbb{R}$ and $x : [0, +\infty) \to \mathbb{R}$, such that s is continuous, its restriction to $[0, +\infty)$ is C^1, x is C^1 and both satisfy the equations in (1) as well as the boundary conditions.

System (1) has been studied in [1] and it represents a chemostat model, with a delay. The chemostat is a continuous bioreactor with a constant volume, in which one or more microbial species are cultivated in a liquid medium containing a set of resources with, in particular, a specific nutrient. The maps $s(t)$ and $x(t)$ are, respectively, the densities of the nutrient and of the microbial species at time t. The device receives continuously an input of liquid volume, described by $s^0(t)$, containing a variable concentration of the specific nutrient. It expulses continuously towards the exterior an output of liquid volume containing a mixing of microbial biomass and nutrient. The model described by the system (1) assumes that the consumption of the nutrient has no immediate effects on the microbial growth, but we have a time interval $[t - \tau, t]$ in which the microbial species metabolize(s) the nutrient.

If (s, x) is any solution of (1) such that x vanishes at some t_0, then x turns out to be identically zero. Thus, the first equation in system (1) becomes linear and appears in the form

$$v'(t) = Ds^0(t) - Dv(t). \tag{2}$$

Among the infinite solutions of (2), one and only one is ω-periodic, it is positive and can be written as

$$v^*(t) = \int_{-\infty}^{t} e^{-D(t-r)} D s^0(r)\, dr$$

For a sake of simplicity, assume that

$$\frac{1}{\omega}\int_0^\omega \mu(v^*(t))\, dt = D.$$

In general, the relation between the average of $\lambda(\mu \circ v^*)$ and D is crucial. In [1], the authors prove that

(a) *if $\lambda < 1$ (resp. $\lambda > 1$) and (s, x) is an ω-periodic solution, different from $(v^*, 0)$, then, $x(t) < 0$ (resp. $x(t) > 0$) for all $t \in \mathbb{R}$;*

(b) *if $\lambda = 1$, no ω-periodic solution is different from $(v^*, 0)$.*

Solutions with $x(t) < 0$ are not interesting form a biological point of view, as x is the densitiy of the microbial species at time t. Observing items (a) and (b) above, it is quite natural to ask if $(v^*, 0)$ is a bifurcation point for ω-periodic solutions of (1) with x positive, as well as it is important to investigate the global behaviour of the connected components of such solutions whose closures (in a suitable topology) contain $(v^*, 0)$, analogously to the classical bifurcation results of Rabinowitz in [3].

2 Main Results

We need some notation. If (s, x) is an ω-periodic solution of (1) for a given λ, the triple (λ, s, x) will be called *ω-triple*. An ω-triple (λ, s, x) such that $\lambda \neq 1$ and $(s, x) \neq (v^*, 0)$ will be called *nontrivial*. We consider an ω-triple as an element of $E := \mathbb{R} \times C^1_\omega \times C^1_\omega$, where

$$C^1_\omega = \{u \in C^1([0, \omega], \mathbb{R}) : u(0) = u(\omega) \text{ and } u'(0) = u'(\omega)\},$$

is a Banach space with the usual norm. Our main result is the following global bifurcation theorem.

Theorem 2.1. *There exist in E exactly two connected components $\mathcal{C}_+$ and $\mathcal{C}_-$ of nontrivial ω-triples, which are unbounded, contain $(1, v^*, 0)$ in their closure and are such that every $(\lambda, s, x) \in \mathcal{C}_+$ verifies $\lambda > 1$, $0 < s < v^*$ and $x > 0$, while every $(\lambda, s, x) \in \mathcal{C}_-$ verifies $\lambda < 1$, $s > v^*$ and $x < 0$.*

The proof is given by a topological approach based on a concept of degree introduced in [2] for Fredholm maps of index zero between Banach spaces or smooth Banach manifolds. This degree is based on a notion of topological orientation for nonlinear Fredholm maps of index zero. The degree coincides with the Brouwer degree for C^1 maps between finite dimensional oriented manifolds of Euclidean spaces; in the infinite dimensional case and for C^1 compact vector fields, it coincides with the Leray–Schauder degree.

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