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Abstract

Let M be a matroid on m elements and let r be its rank function. We show that any vector in the integer cone of the incidence vectors of bases of M can be written as nonnegative integer combination of at most $m + r(M) - 1$ incidence vectors of bases of M, and these can be determined in oracle-polynomial time.

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1 Introduction

Let H be a set of vectors in $\mathbf{R}^m$. We define

$$\begin{aligned} \text{cone}(H) &:= \Big\{ \sum_{h \in H} \lambda_h h \mid \lambda_h \in \mathbf{R}_+ \ (h \in H) \Big\}; \\ \text{lat}(H) &:= \Big\{ \sum_{h \in H} \lambda_h h \mid \lambda_h \in \mathbf{Z} \ (h \in H) \Big\}; \\ \text{int.cone}(H) &:= \Big\{ \sum_{h \in H} \lambda_h h \mid \lambda_h \in \mathbf{Z}_+ \ (h \in H) \Big\}. \end{aligned}$$

(Here $\mathbf{Z}_+$ is the set of nonnegative integers.) So, cone(H), lat(H) and int.cone(H) are the *cone*, the *lattice* and the *integer cone* generated by the vectors in H, respectively. We have the following containment:

$$\text{int.cone}(H) \subseteq \text{cone}(H) \cap \text{lat}(H). \tag{1}$$

The set H is called a *Hilbert base* if equality holds in (1). Thus, a set of vectors forms a Hilbert basis if any vector which can be written both as their nonnegative combination and their integer combination can also be written as their nonnegative integer combination.

Tutte [16] and, independently, Nash-Williams [12] established a necessary and sufficient conditions under which a given graph has k edge-disjoint spanning trees. Their theorem reads as follows.

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Theorem 1 (Tutte [16], Nash-Williams [12]) *A graph $G = (V, E)$ has k edge-disjoint spanning trees if and only if for every partition $(V_1, V_2, \ldots, V_t)$ of V, the number of edges joining vertices in distinct parts is at least $k(t-1)$.* ∎

It is a consequence of Theorem 1 that the incidence vectors of spanning trees of a graph form a Hilbert Base.

If M is a matroid, we denote its set of elements by $E(M)$, and the rank of a subset $S \subseteq E(M)$ by $r(S)$, or $r_M(S)$ in case of ambiguity. We sometimes write $r(M)$ for $r(E(M))$. It follows from Edmonds' generalization of Theorem 1, the so-called Matroid Partition Theorem, that incidence vectors of the bases of a Matroid form a Hilbert base.

Theorem 2 (Matroid Partition Theorem [4]) *A matroid M has k disjoint bases if and only if*

$$kr(S) + |E(M) - S| \geq kr(M)$$

for every subset S of $E(M)$. ∎

The problem we are going to study is related to a question posed by Cunningham [3]. Let M be a matroid. What is the least integer t such that any vector w in the integer cone of incidence vectors of bases of M can be written as a nonnegative integer combination of at most t incidence vector of bases of M? Cunningham [3] showed that his algorithm for testing membership in matroid polyhedra gives an upper bound of $m^4 + 1$ for t, where m is the number of elements in M.

Cook, Fonlupt and Schrijver [2] have proved the following integer analogue of Carathéodory's Theorem (for undefined terms see [14]).

Theorem 3 (Cook, Fonlupt, and Schrijver [2]) *Let C be a pointed cone, and let $H \subset \mathbf{Z}^m$ be its Hilbert base. If $w \in$ int.cone(H) then w is the positive linear combination of at most $2m - 1$ elements of H.* ∎

Sebő [15] has shown that the bound of $2m-1$ in Theorem 3 can be replaced by $2m-2$ and conjectured that the actual upper bound is m (this bound is the best possible). This conjecture is sometimes called the *Integral Carathéodory Conjecture* and remains open (for an updated discussion on this and others related conjectures see [7]). In particular, as incidence vectors of bases of a matroid form a Hilbert base of a pointed cone these results imply the upper bound of $2m-2$ for t.

A related result has been proved by Chaourar [1]. A set $H \subset \mathbf{Z}^m$ of vectors is said a *strong Hilbert base* if for every face F of cone(H) there exists an element $h_F \in F \cap H$ with the following property: if w is a vector in the integer cone generated by H and F is the minimal face of cone(H) containing w, then $w - h_F \in$ cone(H). It is easy to see that for strong Hilbert bases the Integral Carathéodory Conjecture does hold. Chaourar [1] has shown that incidence vectors of bases of matroids without certain minors (the graphic matroid of the complete graph on four vertices is one of them) do form a strong Hilbert base. So, for those matroids m is an upper bound for t.

The following theorem constitutes our main result.

Theorem 4 *Let M be a matroid on a set E with m elements and let w be a vector in the integer cone generated by the incidence vectors of the bases of M. Then w can be written as a nonnegative integer combination of at most $m + r(M) - 1$ incidence vectors of bases of M, and these can be determined in oracle-polynomial time.*

Theorem 4 specializes to graphic matroids as follows:

Corollary 5 *Let G be a connected graph with n vertices and m edges and let w be a vector in the integer cone generated by the incidence vectors of.the spanning trees of G. Then w can be written as a nonnegative integer combination of at most $m + n - 2$ incidence vectors of the spanning trees of G, and these can be determined in polynomial time.*

This paper is structured as follows. In Section 2, we define all necessary terminology and we describe some auxiliary results. The proof of our main result is given in Section 3. Finally, in Section 4 we make a few last remarks.

2 Definitions and preliminaries

We assume knowledge of elementary matroid theory; for an introduction and undefined terms, see [13] and [17].

Let M be a matroid on a set E. If X and Y are disjoint subsets of E then $M/X \setminus Y$ denotes the matroid obtained from M by contracting X and deleting Y. By $M(X)$ we denote the restriction of M to X or the deletion of $E \setminus X$ from M.

Let w be a vector in $\mathbf{R}^{E(M)}$, for $S \subseteq E(M)$ we denote $\sum_{e \in S} w_e$ by $w(S)$. We denote by cone(M), lat(M) and int.cone(M) the cone, the lattice and the integer cone generated by the incidence vector of bases of M. For a subset S of E, the *incidence vector* χ^S of S is defined by the rule that $\chi^S_e = 1$ if $e \in S$, and $\chi^S_e = 0$ otherwise.

It follows from the Edmonds' description of the matroid polytope by linear inequalities (cf. [5, 6]) that a vector $x \in \mathbf{R}^{E(M)}$ belongs to cone(M) if and only if

$$\begin{aligned} x_e &\geq 0 && \text{for all } e \in E(M), \\ x(S) &\leq (x(E)/r(M))r(S) && \text{for all } S \text{ subset of } E(M). \end{aligned} \tag{2}$$

For each base B of M and $w \in$ cone(M) let

$$\mu_{w,B} := \max\{\mu \in \mathbf{R}_+ \mid w - \mu\chi^B \in \text{cone}(M)\}.$$

If $w \in \mathbf{R}^{E(M)}$ we call a subset S of $E(M)$ w-tight if $w(S) = (w(E(M))/r(M))r(S)$. (So, if $w \in$ cone(M) and $w_e > 0$ for every $e \in E(M)$ then the w-tight sets are in one to one correspondence with the faces of cone(M) containing w.)

Let cl be the closure operator of M, that is cl is the function from $2^{E(M)}$ to $2^{E(M)}$ defined, for every $S \subseteq E(M)$, by $\text{cl}(S) := \{e \in E(M) \mid r(S \cup \{e\}) = r(S)\}$.

Proposition 6 *Let M be a matroid on a set E and let $w \in$ cone(M) with $w_e > 0$ for every $e \in E$. If S is a w-tight subset of E then S is closed (a flat).*

Proof. Suppose that S is not closed. Then there exists a $e \in \text{cl}(S) \setminus S$. Thus,

$$\begin{aligned} w(S \cup \{e\}) &= w(S) + w(e) \\ &= (w(E)/r(M))r(S) + w_e \quad \text{(as } S \text{ is } w\text{-tight)} \\ &= (w(E)/r(M))r(S \cup \{e\}) + w_e \quad \text{(as } e \in \text{cl}(S)) \\ &> (w(E)/r(M))r(S \cup \{e\}) \quad \text{(as } w_e > 0). \end{aligned}$$

This contradicts the hypothesis that $w \in \text{cone}(M)$. ∎

Let f be a set-function on a set E, i.e., a function defined on the class of all subsets of E. The function f is called *submodular* if

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y), \tag{3}$$

for all subsets X and Y of E. Similarly, f is called *supermodular* if $-f$ is submodular, i.e. if f satisfies (3) with the opposite sign. f is modular if f is both submodular and supermodular, i.e. if f satisfies (3) with equality. The following proposition is a well-know consequence of the submodularity of the rank function of a matroid.

Proposition 7 *Let M be a matroid on a set E and let $w \in \text{cone}(M)$. If S_1 and S_2 are w-tight subsets of $E(M)$ then $S_1 \cap S_2$ and $S_1 \cup S_2$ are w-tight sets.* ∎

Proposition 8 *Let M be a matroid on a set E and let $w \in \text{cone}(M)$ with $w_e > 0$ for every $e \in E$. If B is a base of M such that $\chi^B(S) = r(S)$ for every w-tight subset S of E then $\mu_{w,B} > 0$.*

Proof. Suppose that $\mu_{w,B} = 0$. By (2), as $w_e > 0$ for every $e \in B$, we have that there exists a w-tight set S such that

$$w(S) - \mu\chi^B(S) \quad > \quad (w(E)/r(M) - \mu)r(S) \quad \text{for all } \mu > 0.$$

This implies that $\chi^B(S) < r(S)$ contradicting the hypothesis. ∎

We shall make use of the Discrete Separation Theorem of Frank [8, 9] in order to prove the main result in the next section.

Theorem 9 (Discrete Separation Theoreom [8, 9]) *Let p and b be super- and submodular set-function on a set E, respectively. There is a finite modular set-function m for which $p(S) \leq m(S) \leq b(S)$ for all $S \subseteq E$ if and only if $p(S) \leq b(S)$ for all $S \subseteq E$. If p and b are integer-valued, m can be chosen to be integer-valued. Moreover, if such a m exists, it can be found in polynomial time, provided that a polynomial-time subroutine is available for minimizing a submodular set-function.* ∎

We say that $S \subseteq E$ minimizes a set-function f if $f(S)$ is as small as possible. We assume that a submodular function f is given by an oracle returning $f(S)$ for each query $S \subseteq E$. Minimizing a submodular set-function can be done in oracle-polynomial time via the ellipsoid algorithm, as shown by Grötschel, Lovász and Schrijver [10] (see also [11]).

3 Proof of Theorem 4

In this section, we prove Theorem 4. We first establish a key lemma from which our main result follows easily. If M is a matroid on E and w is a vector in $\mathbf{R}^E$ then by $d(w)$ we denote the minimum dimension of a face of cone(M) containing w. Roughly, this key lemma states that if w is a non-null vector in int.cone(M) and B_1 is an arbitrary base of M then either $d(w - \mu_{w,B_1}\chi^{B_1}) < d(w)$ (i.e. μ_{w,B_1} is an integer) or there exists a base B_2 of M such that $d(w - \mu_{w,B_1}\chi^{B_1} - \chi^{B_2}) < d(w)$. So, in order to 'bring' a vector of int.cone(M) to a facet of cone(M) one needs at most two incidence vectors of bases of M.

Lemma 10 *Let M be a matroid on a set E and let $w \in$ int.cone(M) with $w_e > 0$ for all $e \in E$. Let X be a closed set of M such that*

(i) $w(X) - (w(E)/r_M(E) - 1)r_M(X) \geq 0$;

(ii) $w(S) - (w(E)/r_M(E) - 1)r_M(S) \leq w(X) - (w(E)/r_M(E) - 1)r_M(X)$ *for all* $S \subseteq X$.

Then one can find in oracle-polynomial time a base B of M such that $(w - \chi^B) \in$ int.cone(M) and X is $(w - \chi^B)$-tight (i.e. $d(w - \chi^B) < d(w)$, or, equivalently, $\mu_{w,B} = 1$).

Proof. Let $k := w(E)/r(M) - 1$ and $Y := E \setminus X$. Let $w_X \in \mathbf{Z}^X$ be the vector such that $(w_X)_e := w_e$ for all $e \in X$ (i.e. w_X is the restriction of w to X). Analogously, we define $w_Y \in \mathbf{Z}^Y$ as the restriction of w to Y. In order to prove the present lemma we shall proceed as follows:

(a) Firstly, we show that one can find in oracle-polynomial time an independent set I_X of $M(X)$ such that $w_X - \chi^{I_X}$ can be written as a nonnegative integer combination of k incidence vectors of bases of $M(X)$ (i.e. $w_X - \chi^{I_X} \in$ int.cone($M(X)$));

(b) Secondly, we prove that one can find in polynomial time an independent set $I_Y \subseteq Y$ of M/I_X such that $w_Y - \chi^{I_Y}$ can be written as nonnegative integer combination of k incidence vectors of bases of M/X (i.e. $w_Y - \chi^{I_Y} \in$ int.cone(M/X)); and

(c) Finally, we verify that $B := I_X \cup I_Y$ is a base of M such that $w - \chi^B \in$ int.cone(M) and X is $(w - \chi^B)$-tight (implying that $\mu_{w,B} = 1$).

Proof of (a). Let $l := w(X) - kr_M(X)$. By hypothesis l is nonnegative. Define $b : 2^X \to \mathbf{Z}_+$ by $b(S) := \min\{l, r_M(S)\}$. The set-function b is the rank function of a matroid on the set E of rank l (this matroid is obtained from M by a sequence of 'truncations', cf. [13]), so, in particular, b is submodular. Now, define $p : 2^X \to \mathbf{Z}_+$ by $p(S) := w_X(S) - kr_M(S)$. The set-function p is supermodular.

It can be easily verified that $p(S) \leq b(S)$ for all $S \subseteq X$. Indeed, for all $S \subseteq X$ with $r_M(S) \geq l$ we have that

$$\begin{aligned} p(S) &= w(S) - kr_M(S) \\ &\leq w(X) - kr_M(X) \quad \text{(by (ii))} \\ &= l = b(S), \end{aligned}$$

and for all $S \subset X$ with $r_M(S) < l$ we obtain that

$$\begin{aligned} p(S) &= w(S) - kr_M(S) \\ &\leq (k+1)r_M(S) - r(S) \quad (\text{as } v \in \text{cone}(M)) \\ &= r_M(S) = b(S). \end{aligned}$$

So we conclude that $p(S) \leq b(S)$ for all $S \subseteq E$ and by the Discrete Separation Theorem there exists an integer-valued modular set-function m such that

$$p(S) \leq m(S) \leq b(S) \text{ for all } S \subseteq X. \tag{4}$$

The above inequality implies that $p(\emptyset) = m(\emptyset) = b(\emptyset) = 0$ and thus

$$m(S) = \sum_{e \in S} m(\{e\}) \text{ for all } S \subseteq X.$$

Inequality (4) also implies that $p(X) = m(X) = b(X) = l$.

We claim that $m(\{e\}) \geq 0$ for all $e \in E$. Suppose not. Let $X_- := \{e \in X \mid m(\{e\}) < 0\} \neq \emptyset$ and let $X_+ := \{e \in X \mid m(\{e\}) > 0\}$. Then

$$\begin{aligned} b(X_+) &\geq m(X_+) \\ &= m(X) - m(X_-) \\ &= b(X) - m(X_-) \\ &> b(X), \end{aligned}$$

which contradicts the fact that b is the rank function of a matroid. So, as m is integer-valued and $m(\{e\}) \leq b(\{e\}) = 1$ we conclude that $m(\{e\}) \in \{0, 1\}$ for all $e \in X$.

Let $I_X := \{e \in X \mid m(\{e\}) = 1\}$. We claim that I_X is an independent set of $M(X)$. Indeed,

$$\begin{aligned} \chi_e^{I_X} = m(\{e\}) &\geq 0 && \text{for all } e \in X, \\ \chi^{I_X}(S) = m(S) &\leq r_M(S) && \text{for all } S \subseteq E \end{aligned}$$

implying that χ^{I_X} is an integer vector in the matroid polytope of $M(X)$ and therefore I_X is an independent set of $M(X)$ (cf. [5, 6]) with $|I_X| = m(X) = l$.

It remains to be shown that $v_X - \chi^{I_X} \in \text{int.cone}(M(X))$. Let us first observe that $w(X) - \chi^{I_X}(X) = w(X) - l = kr_M(X)$. For all $S \subseteq X$ we have that

$$\begin{aligned} kr(S) + w_X(X \setminus S) - \chi^{I_X}(X \setminus S) &= kr_M(S) + w(X) - w(S) - \chi^{I_X}(X) + \chi^{I_X}(S) \\ &= w(X) - \chi^{I_X}(X) + \chi^{I_X}(S) - (w(S) - kr_M(S)) \\ &= kr_M(X) + \chi^{I_X}(S) - (w(S) - kr_M(S)) \\ &\geq kr_M(X) + b(S) - p(S) \\ &\geq kr_M(X). \end{aligned}$$

Thus, by the Matroid Partition Theorem we conclude that $w_X - \chi^{I_X}$ is the sum of k incidence vectors of bases of M. This completes the proof of (a). ∎

Proof of (b). This proof is almost identical to the proof of (a). Define $b : 2^Y \to \mathbf{Z}_+$ by $b(S) := r_{M/I_x}(S)$ and define $p : 2^Y \to \mathbf{Z}_+$ by $p(S) := w_Y(S) - kr_{M/X}(S)$. The set-functions p

and b are super- and submodular on Y, respectively. We claim that $p(S) \le b(S)$ for all $S \subseteq Y$. Indeed,

$$\begin{aligned} p(S) &= w_Y(S) - kr_{M/X}(S) \\ &= w(X \cup Y) - w(X) - kr_M(X \cup S) + kr_M(X) \\ &= w(X \cup Y) - kr_M(X \cup S) - (w(X) - kr_M(X)) \\ &= w(X \cup Y) - kr(X \cup S) - |I_X| \\ &\le r(X \cup S) - |I_X| \\ &= r_{M/I_X}(S), \end{aligned}$$

for all $S \subseteq Y$. By the Discrete Separation Theorem there exists an integer-valued modular set-function m such that $p(S) \le m(S) \le b(S)$ for all $S \subseteq Y$. Analogously to the proof of (a) one can verify that $p(Y) = m(Y) = b(Y) = r_M(E) - |I_X|$ and $m(\{e\}) \in \{0,1\}$.

Let $I_Y := \{e \in Y \mid m(\{e\}) = 1\}$. We have that

$$\begin{aligned} \chi_e^{I_Y} = m(\{e\}) &\ge 0 \qquad \text{for all } e \in Y, \\ \chi^{I_Y}(S) = m(\{S\}) &\le r_{M/I_X}(S) \quad \text{for all } S \subseteq Y. \end{aligned}$$

Hence, χ^{I_Y} is an integer vector in the matroid polytope of M/I_X and therefore I_Y is an independent set of M/I_X (cf. [5, 6]).

Let us observe that $w_Y(Y) - \chi^{I_Y}(Y) = w_Y(Y) - |I_Y| = kr_{M/X}(Y)$. For all $S \subseteq Y$ we have that

$$\begin{aligned} kr_{M/X}(S) + w_Y(Y \setminus S) - \chi^{I_Y}(Y \setminus S) &= kr_{M/X}(S) + v_Y(Y) - w_Y(S) - \chi^{I_Y}(Y) + \chi^{I_Y}(S) \\ &= w_Y(Y) - \chi^{I_Y}(Y) + \chi^{I_Y}(S) - (w_Y(S) - kr_{M/X}(S)) \\ &= kr_{M/X}(Y) + \chi^{I_Y}(S) - (w_Y(S) - kr_{M/X}(S)) \\ &\ge kr_{M/X}(Y) + b(S) - p(S) \\ &\ge kr_{M/X}(Y). \end{aligned}$$

Thus, by the the Matroid Partition Theorem it follows that $w_Y - \chi^{I_Y}$ is the sum of k incidence vectors of bases of M. This completes the proof of (b). ∎

Proof of (c). It follows from (a) and (b) that: I_X is an independent set of $M(X)$ of cardinality l; I_Y is an independent set of M/I_X; and $|I_X \cup I_Y| = r_M(E)$. Hence, $B := I_X \cup I_Y$ is a base of M.

From (a) we know that there exist bases $B_{X,1}, B_{X,2}, \ldots, B_{X,k}$ of $M(X)$ such that

$$w_X - \chi^{I_X} = \chi^{B_{X,1}} + \cdots + \chi^{B_{X,k}}, \tag{5}$$

and from (b) we have that there exist bases $B_{Y,1}, B_{Y,2}, \ldots, B_{Y,k}$ of M/X such that

$$w_Y - \chi^{I_Y} = \chi^{B_{Y,1}} + \cdots + \chi^{B_{Y,k}}, \tag{6}$$

Hence, $B_{X,1} \cup B_{Y,1}, B_{X,2} \cup B_{Y,2}, \ldots, B_{X,k} \cup B_{Y,k}$ are bases of M such that

$$\begin{aligned} w - \chi^B &= w_X + w_Y - \chi^{I_X} - \chi^{I_Y} \\ &= \chi^{B_{X,1}} + \cdots + \chi^{B_{X,k}} + \chi^{B_{Y,1}} + \cdots + \chi^{B_{Y,k}} \quad \text{(by (5) and (6))} \\ &= \chi^{B_{X,1} \cup B_{Y,1}} + \cdots + \chi^{B_{X,k} \cup B_{Y,k}}. \end{aligned}$$

Moreover,

$$\begin{aligned} w(X) - \chi^B(X) &= w(X) - |I_X| \\ &= kr_M(X). \end{aligned}$$

Thus, X is a $(w - \chi^B)$-tight set. This completes the proof of Lemma 10. ∎

Lemma 11 *Let M be a matroid on a set E and let $w \in int.cone(M)$ with $w_e > 0$ for every $e \in E$ and such that E contains no w-tight proper subset.*

(*i*) *Suppose B_1 is a base of M such that $w - \chi^{B_1} \notin cone(M)$. Then one can find in oracle-polynomial time a base B_2 of M such that $w - \chi^{B_2} \in cone(M)$ and E contains a $w - \chi^{B_2}$-tight proper subset (in particular, $\mu_{w,B_2} = 1$).*

(*ii*) *Suppose B_1 is a base of M such that μ_{w,B_1} is not an integer and let $v := w - \lfloor \mu_{w,B_1} \rfloor \chi^{B_1}$. Then one can find in oracle-polynomial time a base B_2 of M such that $v - \chi^{B_2} \in cone(M)$ and E contains a $(v - \chi^{B_2})$-tight proper subset (in particular, $\mu_{v,B_2} = 1$).*

Proof. One can easily derive (*ii*) from (*i*). So, we shall prove only (*i*).

By the hypothesis and Proposition 8 we have that $0 < \mu_{w,B_1} < 1$. As μ_{w,B_1} is not an integer then by (2) there exists a $(w - \mu_{w,B_1}\chi^{B_1})$-tight proper subset X of E, i.e.,

$$w(X) - \mu_{w,B_1}\chi^{B_1}(X) = (w(E)/r_M(M) - \mu_{w,B_1})r_M(X). \tag{7}$$

By Proposition 6 we know that X is a closed set. (Thus, M/X is a simple matroid.) Let $k := w(E)/r_M(E) - 1$. For all $S \subsetneq X$ we have that

$$\begin{aligned} w(S) - kr_M(S) &= (1 - \mu_{w,B_1})r_M(S) + \mu_{v,B_1}\chi^{B_1}(S) \text{ (by definition of } \mu_{w,B_1}) \\ &\leq (1 - \mu_{w,B_1})r(X) + \mu_{w,B_1}\chi^{B_1}(X) \text{ (as } 0 < \mu_{v,B_1} < 1) \\ &= v(X) - kr_M(X) \text{ (by (7)).} \end{aligned}$$

Then by Lemma 10 one can find in oracle-polynomial time a base B_2 of M such that $(w - \chi^{B_2}) \in$ int.cone(M) and X is $(w - \chi^{B_2})$-tight. ∎

We are now ready to prove Theorem 4.

Proof of Theorem 4. We may assume that M is simple and that $w_e > 0$ for all $e \in E$. We proceed by induction on $m + r(M)$. For $r(M) = 1$ we have that w can be written as a nonnegative integer combination of m incidence vectors of bases of M. So, we assume $r(M) > 1$.

Let B_1 an arbitrary base of M and let $v := w - \lfloor \mu_{w,B_1} \rfloor \chi^{B_1}$. We have to consider two possibilities.

Case 1. μ_{w,B_1} is an integer.

From the description (2) of cone(M) by linear inequalities there exists some $e \in E$ such that $v(e) = 0$ or there exists a proper subset X of E which is v-tight.

Let us first suppose that $E_0 := \{e \in E \mid v_e = 0\}$ is nonempty. Let $E' := E \setminus E_0$ and let v' be the restriction of v to E'. By induction hypothesis v' can be written as a nonnegative integer combination of at most $m - |E_0| + r_M(E') - 1 \leq m + r(M) - 2$ incidence vectors of bases of $M(E')$. Hence, w can be written as a nonnegative combination of at most $m + r(M) - 1$ incidence vectors of bases of M.

So, we may assume that $E_0 = \emptyset$. Let X a maximal proper v-tight set of E. From Proposition 6 we know that X is closed (so M/X is a simple matroid). Let $M(X_1), \ldots, M(X_p)$ be the connected components of $M(X)$ (i.e. $M(X)$ is the direct sum of $M(X_1), \ldots, M(X_p)$.) Let $k := w(E)/r(M) - \lfloor \mu_{w,B_1} \rfloor$, $Y := E \setminus X$, let v_Y be the restriction of v to Y and let v_{X_i} be the restriction of v to X_i ($i = 1, \ldots, p$). One can verify that:

(a) v_Y is the sum of k incidence vectors of bases of M/X ($v_Y \in \text{int.cone}(M/X)$);

(b) v_{X_i} is the sum of k incidence vectors of bases of $M(X_i)$ ($i = 1, \ldots, p$) ($v_{X_i} \in \text{int.cone}(M(X_i))$ ($i = 1, \ldots, p$));

(c) $r(M/X) < r(M)$ and $r(M(X_i)) < r(M)$ ($i = 1, \ldots, p$).

Thus, by the induction hypothesis it follows that v_Y can be written as a nonnegative integer combination of at most $|Y| + r(M/X) - 1$ incidence vectors of bases of M/X and v_{X_i} can be written as a nonnegative integer combination of at most $|X_i| + r(M(X_i)) - 1$ incidence vectors of bases of $M(X_i)$ ($i = 1, \ldots, p$). These combinations can be glued together to form a nonnegative integer combination of v of at most

$$|Y| + \sum_{i=1}^{p} |X_i| + r(M/X) + \sum_{i=1}^{p} r(M(X_i)) - (p+1) - p \quad = \quad m + r(M) - 2p - 1 \qquad (8)$$

incidence vector of bases of M. This glueing can be done as follows. Let $B_{Y,1}, \ldots, B_{Y,k}$ be bases of M/X such that:

- $v_Y = \chi^{B_{Y,1}} + \cdots + \chi^{B_{Y,k}}$;
- there exist indexes $1 = l_{Y,0} < l_{Y,1} < \ldots < l_{Y,k_Y} = k$ with $k_Y < |Y| + r(M/X) - 1$ such that
$$B_{Y,l_{Y,j}} = B_{Y,l_{Y,j}+1} = \ldots = B_{Y,l_{Y,j+1}} \quad \text{for } j = 0, \ldots, t_Y - 1.$$

For $i = 1, \ldots, p$ let $B_{X_i,1}, \ldots, B_{X_i,k}$ be bases of $M(X_i)$ such that:

- $v_{X_i} = \chi^{B_{X_i,1}} + \cdots + \chi^{B_{X_i,k}}$;
- there exist indexes $1 = l_{X_i,0} < l_{X_i,1} < \ldots < l_{X_i,k_{X_i}} = k$ with $k_{X_i} < |X_i| + r(M(X_i)) - 1$ such that
$$B_{X_i,l_{X_i,j}} = B_{X_i,l_{X_i,j}+1} = \ldots = B_{X_i,l_{X_i,j+1}} \quad \text{for } j = 0, \ldots, t_{X_i} - 1.$$

Then, by (a) and (b) above, $B_i := B_{Y,i} \cup B_{X_1,i} \cup \ldots \cup B_{X_p,i}$ is a base of M ($i = 1, \ldots, k$) and the sum

$$\chi^{B_1} + \cdots + \chi^{B_k}$$

shows that v can be written as a nonnegative integer combination of at most

$$k_Y + \sum_{i=1}^{p} k_{X_i} - p \;\; \leq \;\; |Y| + \sum_{i=1}^{p} |X_i| + r(M/X) + \sum_{i=1}^{p} r(M(X_i)) - (p+1) - p$$

bases of M (this explains the extra $-p$ term in (8)).

Therefore, w can be written as a nonnegative integer combination os at most $m+r(M)-2p \leq m + r(M) - 2$ incidence vectors of bases of M.

Case 2. μ_{w,B_1} is not an integer.

From Lemma 11 it follows that there exists a base B_2 of M so that $\mu_{v,B_2} = 1$ and there exists a proper $(v - \chi^{B_2})$-tight set X of E. Now, proceeding similarly as in the Case 1 simply replacing w by v and B_1 by B_2 one can show that v can be written as a nonnegative integer combination of at most

$$m + r(M) - |E_0| - 2p \;\; \leq \;\; m + r(M) - 2,$$

incidence vectors of bases of M, where $E_0 = \{e \in E \mid v_e - \chi_e^{B_2} = 0\}$ and p is the number of connected components of $M(X)$.

Thus, w can be written as a nonnegative integer combination of at most $m + r(M) - 1$ incidence vectors of bases of M. This completes the proof of Theorem 4. ∎

4 Conclusions

We would like to make a few remarks. Using methods of the present paper it might be possible to derive a sharper upper bound for the least number t needed in order to write any vector in the integer cone formed by the incidence vectors of bases of a matroid as a nonnegative integer combination of at most t of these incidence vectors. If one can show that for matroids with rank lesser than k we have $t \leq k$ then the bound od $m + r(M) - 1$ in Theorem 4 can be replaced by $m + r(M) - k$. In the proof of Theorem 4 we somehow stressed the fact that if the tight set X found has $p \geq 2$ connected components then one would need fewer distinct incidence vectors in the nonnegative integer combination derived for w. The 'bad' X's sets for the proof are precisely the connected hyperplanes of M. If one can show that: for any $w \in$ int.cone(M), there exists a base B of M such that the X set obtained in Theorem 4 from $(w - \chi^B)$ is not a connected hyperplane. Then one could proved (using the proof of Theorem 4) that m is a upper bound for t (this bound is best possible).

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