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by

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POISSONIAN TREE CONSTRUCTED FROM INDEPENDENT POISSON POINT PROCESSES

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ABSTRACT. In this work a connected graph without cycles, a *tree* is constructed, with a single *infinite self-avoiding path*, an end. The vertices of the tree are points of an infinite sequence of Independent Poisson Point Processes defined in $\mathbb{R}^d$, such that for every $k \geq 1$, the rate of k–th process X_k is λ_k. We name such a graph of *One-Ended Poissonian Tree.*

1. INTRODUCTION

In this work a connected graph without cycles, a *tree* is constructed, with a single *infinite self-avoiding path*, an end. The vertices of the tree are points of an infinite sequence of Independent Poisson Point Processes defined in $\mathbb{R}^d$, such that for every $k \geq 1$, the rate of k–th process X_k is λ_k. We name such a graph of *One-Ended Poissonian Tree.* (Ferrari et all., 2007) constructed a Poissonian Tree with a unique end for the points of a stationary Poisson process when it is defined in $S \subset \mathbb{R}^d$, for $d \leq 3$ and show that for $d \geq 4$ the graph has infinetely many components, a forest. (Holroyd et all., 2003) build a one-ended Poissonian Tree in a deterministic isometry-invariant way for any d-dimensional Poisson process. (Gangopadhyay et all., 2004) show that there is a one-ended tree from a sequence of independent process defined in $\mathbb{Z}^d$, $d \leq 3$, where each point of $\mathbb{Z}^d$ has distribution Bernoulli(p) and a forest for $d \geq 4$.

In section 1 the algorithm of construction of the Poissonian Tree is given, as well. The definition of its elements and a sufficient condition for the existence of a unique tree consisting of all the points of each one of the realizations of the infinite Poisson Point Process (independent). The One-Ended Poissonian Tree is constructed in section 2 for processes defined in $\mathbb{R}$ and whose sequence of rates is such that $\liminf \lambda_k = 0$. In subsection 2.1 we establish conditions on the sequence of the rates for the almost surely existence of a One-Ended Poissonian Tree and for the non-existence of this tree with positive probability. In section 3, teorema 5, we have such graph for processes in $\mathbb{R}^d$ and $\lambda_k = \alpha^k$, where $\alpha \in (0, 1)$ is the ratio decay among the rates of the processes.

For all $k \geq 1$, any point ξ_k belonging to X_k will be said a point of "generation k". We will also consider that the point ξ_k will define its position in X_k, indicated for X_{ξ_k}. ξ_{k+1} of

Key words and phrases. One-Ended Poissonian Tree, Poisson Point Process, Random Tree, Random Matching, Succession Line, Coalescing.

FIGURE 1. Succession Line of a Sequence of Unidimensional Independent Poisson Point Processes.

X_{k+1} will be said an *ancestor* of ξ_k, if this belongs to the *Voronoi Cell* of ξ_{k+1}, in this case ξ_k will be *successor* of ξ_{k+1}.

Consider the graph whose vertices are the points of the sequence formed by infinite Poisson Point Process (independent), such that the endvertices of each edge are determined by pairs of points $(\xi_k \xi_{k+1})$, that is, each point of the process X_k is adjacent to the nearest point in X_{k+1}. The distance between any two points ξ_k^1 and ξ_k^2 of the same "generation k", for all $k \geq 1$, will be denoted by $D(\xi_k^1, \xi_k^2) = |\xi_k^1 - \xi_k^2|$. We will represent by $D_k := |X_{b_k} - X_{a_k}| = |b_k - a_k|$ the distance between the k–th ancestors of a_1 and b_1, chosen arbitrarily in X_1.

It follows that for each point of X_k, there exists a unique ancestor in X_{k+1}, which guarantees the non-existence of *cycles*. To proof the existence of a unique infinite self-avoiding path, and therefore *connectedness*, it is sufficient that given any two points in X_1 (a_1 and b_1), there exists $k = k(a_1, b_1) > 1$ such that a_k and b_k *coalesce in probability* ($a_k \overset{\mathbf{P}}{=} b_k$), that is, the sufficient condition for the existence of the Poissonian Tree is given by

$$a_k \overset{\mathbf{P}}{=} b_k := \lim_{k\to\infty} \mathbf{P}(D_k \neq 0) = 0 \Leftrightarrow \lim_{k\to\infty} \mathbf{P}(a_k = b_k) = 1 \tag{1.1}$$

Remark 1. *The condition established in (1.1) tell us that chosen any two points in the first generation. (That is, a_1 and b_1 are selected in a deterministic way in X_1), the probability of the event of coalescing between its respective ancestors, a_k and b_k, for k sufficiently large, tends to one. We could consider in place of a_1 and b_1, any two points a_m and b_m in X_m, because the coalescing of these points, will imply the coalescing of any others of its successors in X_j, where $1 \leq j < m$.*

Definition 1. *We define that a_k and b_k coalescing almost surely, $a_k \overset{qc}{=} b_k$, if the set of trajectories $T = \cup_{k\geq 1} T_k$ in which them a_k and b_k coalescing in some "generation k_n", T_{k_n}, has measure one.*

Lemma 1. $a_k \overset{\mathbf{P}}{=} b_k := \lim_{k\to\infty} \mathbf{P}(a_k = b_k) = 1$, *if and only if,* $a_k \overset{qc}{=} b_k$.

Proof. ($\Rightarrow$) Without loss of generality, let a_1 and b_1 chosen arbitrarily in X_1. From the condition that a_k coalesce in probability with b_k we have that:

$$\lim_{k\to\infty} \mathbf{P}_{a_1,b_1}(a_k = b_k) = 1 \Leftrightarrow \forall \epsilon = \epsilon(k_n) > 0 \; \exists k_n;\; \forall k \geq k_n \; \mathbf{P}(a_k = b_k) > 1 - \epsilon(k_n)$$

Let

- $\Omega = \{$ The set of all trajectories of the ancestors of a_1 and b_1 of the infinite realizations of each one $X_k\}$.
- $\epsilon(n) = \frac{1}{n}$ and $T_{k_n} := \{\omega \in \Omega; a_k(\omega) = b_k(\omega) \; \forall k \geq k_n\}$

It follows that $T = \bigcup_{n=1}^{\infty} T_{k_n}$ is such that $\mathbb{P}(T) = 1$, because

$$\forall n;\ \mathbb{P}(T) \geq \mathbb{P}(T_{k_n}) = 1 - \frac{1}{n} \Rightarrow \mathbb{P}(T) = 1$$

($\Leftarrow$) Immediate.

□

If the Poisson Point Processes are defined in $\mathbb{R}$ and $\liminf \lambda_k = 0$, the determination of a One-Ended Poissonian Tree is done with relative easiness, therefore in this in case, it is possible to determine the probability of coalescing in "generation k+1" given the position of k–th ancestors of a_1 and b_1, that is,

$$\mathbb{P}(D_{k+1} = 0|a_k, b_k) = e^{-2\lambda_{k+1}D_k}(1 + \lambda_{k+1}D_k)$$

On the other hand, if the processes X_k are defined in $\mathbb{R}^d$ $(d \geq 2)$, we will have only a lower bound for $\mathbb{P}(D_{k+1} = 0|a_k, b_k)$. In this case, the One-Ended Poissonian Tree will be obtained through the condition given in lema 6, which establishes that the coalescing conditional probability limit is uniformly bounded by a constant $\epsilon(d, \alpha, \beta)$ that depends on the dimension d, the ratio decay among the rates of the processes α and the value β associated to the "drift" of the rescaled process d_k.

2. One-Ended Poissonian Tree Constructed from Independent Point Processes in R

We saw in the section 1 that the sufficient condition for the determination of the One-Ended Poissonian Tree constructed from a sequence of Independent Poisson Point Processes is that given any two points in X_m, a_m and b_m, they coalesce in some point $\xi = \xi(a_m, b_m) \in X_k$ where $k = k(a_m, b_m) > m$. Obviously, the construction of the Poissonian Tree (One-Ended or not) will depend on the dimension where the processes are defined and the sequence of rates $(\lambda_k)_{k\geq 1}$. In theorem 1 we establish the construction of the One-Ended Poissonian Tree , when the processes X_k are defined in $\mathbb{R}$ and $\liminf \lambda_k = 0$.

Proposition 1. *Let $X_{a_{k+1}}$ the position of the ancestor of X_{a_k} in X_{k+1}. The distribution of $X_{a_{k+1}}|X_{a_k}$ is given by*

$$\begin{aligned} f_{X_{a_{k+1}}|X_{a_k}}(x) = {} & \lambda_{k+1}\exp(-2\lambda_{k+1}a_k)\exp(2\lambda_{k+1}x)\mathbb{1}_{(x<a_k)} + \\ & \lambda_{k+1}\exp(2\lambda_{k+1}a_k)\exp(-2\lambda_{k+1}x)\mathbb{1}_{(x>a_k)} \end{aligned} \tag{2.1}$$

Proof. Let X_{a_k} be the position of the k-th ancestor of a_1 in X_k, it follows by independence of $(X_k)_{k\geq 1}$ that $\forall t > 0$.

$$\mathbb{P}(|X_{a_{k+1}}|X_{a_k} - a_k| > t) = \mathbb{P}(X_{a_{k+1}}|X_{a_k} > a_k + t) + \mathbb{P}(X_{a_{k+1}}|X_{a_k} < a_k - t) =$$

$$\mathbb{P}(X_{a_{k+1}} \notin [a_k - t, a_k + t]) = \exp(-\lambda_{k+1}2t)$$

Since the processes are homogeneous

$$\mathbf{P}(X_{a_{k+1}}|X_{a_k} > a_k + t) = \mathbf{P}(X_{a_{k+1}}|X_{a_k} < a_k - t) = \frac{1}{2}\exp(-\lambda_{k+1}2t)$$

If $x = a_k + t$, then

$$\mathbf{P}(X_{a_{k+1}}|X_{a_k} > x) = \frac{1}{2}\exp(-\lambda_{k+1}2(x - a_k))\mathbb{1}_{(x>a_k)}, \Rightarrow$$

$$f_{X_{a_{k+1}}|X_{a_k}}(x) = \lambda_{k+1}\exp(2\lambda_{k+1}a_k)\exp(-2\lambda_{k+1}x)\mathbb{1}_{(x>a_k)}$$

Analogously, we have that

$$f_{X_{a_{k+1}}|X_{a_k}}(x) = \lambda_{k+1}\exp(-2\lambda_{k+1}a_k)\exp(2\lambda_{k+1}x)\mathbb{1}_{(x<a_k)}$$

□

Lemma 2. *Let X_{k+1} be a Poisson Point Process of rate λ_{k+1} independent of X_k. If a_k and b_k are two points of X_k, then for all $k \geq 1$*

$$\mathbf{P}(D_{k+1} = 0|a_k, b_k) = e^{-2\lambda_{k+1}D_k}(1 + \lambda_{k+1}D_k) \tag{2.2}$$

Proof. According to the definitions explained in the first paragraph of the section 1, we have that for all $k \geq 1$, each particle determines its position. That is,

$$\mathbf{P}(D_{k+1} = 0|X_{a_k}, X_{b_k}) = \mathbf{P}(D_{k+1} = 0|X_{a_k} = a_k, X_{b_k} = b_k) = \mathbf{P}(D_{k+1} = 0|a_k, b_k)$$

Conditioning $\mathbf{P}(D_{k+1} = 0|a_k, b_k)$ in the position of the ancestor of a_k, $X_{a_{k+1}}$, and taking its expected value, it follows that

$$\mathbf{P}(D_{k+1} = 0|a_k, b_k) = \mathbf{E}(\mathbf{P}(D_{k+1} = 0|a_k, b_k, X_{a_{k+1}})) \tag{2.3}$$

For all $x < a_k$, $a_{k-1} < x < b_{k-1}$ and $x > b_{k-1}$ it holds respectively that

$$\mathbf{P}(D_{k+1} = 0|a_k, b_k, X_{a_{k+1}} = x) = \exp(-\lambda_{k+1}(2b_k - 2a_k)) = \exp(-2\lambda_{k+1}D_k) =$$

$$\mathbf{P}(D_{k+1} = 0|a_k, b_k, X_{a_{k+1}} = x) = \exp(-\lambda_{k+1}(2b_k - 2x))$$

$$\mathbf{P}(D_{k+1} = 0|a_k, b_k, X_{a_{k+1}} = x) = 1$$

From (2.1), with $X_{a_k} = a_k$, and (2.3) it follows that

$$\mathbf{P}(D_{k+1} = 0|a_k, b_k) = \mathbf{E}(\mathbf{P}(D_{k+1} = 0|a_k, b_k, X_{a_{k+1}})) =$$

$$\int_{-\infty}^{+\infty} f_{X_{a_{k+1}}|X_{a_k},X_{b_k}}(x)\mathbf{P}(D_{k+1} = 0|X_{a_k}, X_{b_k}, x)dx = \int_{-\infty}^{+\infty} f_{X_{a_{k+1}}|X_{a_k}}(x)\mathbf{P}(D_{k+1} = 0|X_{a_k}, X_{b_k}, x)dx =$$

$$\int_{-\infty}^{a_k} f_{X_{a_{k+1}}|X_{a_k}}(x)\exp(-\lambda_{k+1}(2b_k - 2a_k))\,dx + \int_{a_k}^{b_k} f_{X_{a_{k+1}}|X_{a_k}}(x)\exp(-\lambda_{k+1}(2b_k - 2x))\,dx +$$

$$\int_{b_k}^{+\infty} f_{X_{a_{k+1}}|X_{a_k}}(x)dx = \exp\left(-\lambda_{k+1}(2D_k)\right)\int_{-\infty}^{a_k} f_{X_{a_{k+1}}|X_{a_k}}(x)dx +$$

$$\int_{a_k}^{b_k} f_{X_{a_{k+1}}|X_{a_k}}(x)\exp\left(-\lambda_{k+1}(2b_k-2x)\right)dx + \int_{b_k}^{+\infty} f_{X_{a_{k+1}}|X_{a_k}}(x)dx \Rightarrow$$

$$\mathbf{P}(D_{k+1}=0|a_k,b_k) = e^{-2\lambda_{k+1}D_k}(1+\lambda_{k+1}D_k)$$

□

Proposition 2. *For all $k \geq 1$, $\mathbf{P}(D_k = 0) \leq \mathbf{P}(D_{k+1} = 0)$.*

Proof. By the calculation of $\mathbf{P}(D_{k+1} = 0)$ from the conditioning in D_k, we have that

$$\mathbf{P}(D_{k+1}=0) = \mathbf{E}(\mathbf{P}(D_{k+1}=0|D_k)) =$$

$$\mathbf{P}(D_k=0)\mathbf{P}(D_{k+1}=0|D_k=0) + \mathbf{P}(D_k\neq 0)\mathbf{P}(D_{k+1}=0|D_k\neq 0) =$$

$$\mathbf{P}(D_k=0) + \mathbf{P}(D_k\neq 0)\mathbf{P}(D_{k+1}=0|D_k\neq 0) \;\geq\; \mathbf{P}(D_k=0)$$

□

Proposition 3. *Let ξ_{k+1} be the ancestor of ξ_k. Then $\mathbf{E}(X_{\xi_{k+1}}|X_{\xi_k}) = X_{\xi_k}$.*

Proof. The result follows by proposition 1. From the conditional law given in (2.1), we have that

$$\mathbf{E}(X_{\xi_{k+1}}|X_{\xi_k}=\xi_k) = \int_{-\infty}^{\xi_k} x\lambda_{k+1}\exp(-2\lambda_{k+1}\xi_k)\exp(2\lambda_{k+1}x)dx +$$

$$\int_{\xi_k}^{\infty} x\lambda_{k+1}\exp(2\lambda_{k+1}\xi_k)\exp(-2\lambda_{k+1}x)dx = \xi_k$$

□

Proposition 4. *Let $(X_k)_{k\geq 1}$ be a sequence of Independent Poisson Point Processes defined in $\mathbf{R}$ with a_1 and b_1 arbitrarily chosen in X_1. For all $k \geq 1$, $\mathbf{E}(D_{k+1}) = \mathbf{E}(D_k) = D_1$.*

Proof. By the fact of a_1 and b_1 are chosen in a deterministic way, we have that $\mathbf{E}(D_1) = \mathbf{E}(X_{b_1} - X_{a_1}) = b_1 - a_1 = D_1$.

By the independence of $(X_k)_{k\geq 1}$ and the proposition 1, we obtain the following identities

$$\mathbf{E}(X_{b_{k+1}}|X_{a_k}, X_{b_k}) = \mathbf{E}(X_{b_{k+1}}|X_{b_k}) = X_{b_k}$$

$$\mathbf{E}(X_{a_{k+1}}|X_{a_k}, X_{b_k}) = \mathbf{E}(X_{a_{k+1}}|X_{a_k}) = X_{a_k}$$

Thus, for all $k \geq 1$

$$\mathbf{E}(D_{k+1}) = \mathbf{E}(\mathbf{E}(D_{k+1}|X_{a_k}, X_{b_k})) = \mathbf{E}(\mathbf{E}(X_{b_{k+1}} - X_{a_{k+1}}|X_{a_k}, X_{b_k})) =$$

$$\mathbf{E}(\mathbf{E}(X_{b_{k+1}}|X_{a_k}, X_{b_k})) - \mathbf{E}(\mathbf{E}(X_{a_{k+1}}|X_{a_k}, X_{b_k})) = \mathbf{E}(X_{b_k} - X_{a_k}) = \mathbf{E}(D_k) \Rightarrow$$

$$\mathbf{E}(D_k) = \mathbf{E}(D_1) = D_1, \ \forall k \geq 1$$

□

Theorem 1. *Let* $(X_k)_{k\geq 1}$ *be a sequence of Independent Poisson Point Processes defined in* $\mathbf{R}$ *with respective rates* λ_k *such that* $\liminf \lambda_k = 0$. *Then there is a One-Ended Poissonian Tree consisting of the points of all processes.*

Proof. Taking the expected value in (2.2) it follows that

$$\mathbf{P}(D_{k+1} = 0) = \mathbf{E}(\mathbf{P}(D_{k+1} = 0|a_k, b_k)) = \mathbf{E}\left(e^{-2\lambda_{k+1}D_k}\left[1 + \lambda_{k+1}D_k\right]\right) \geq$$

$$\mathbf{E}\left(e^{-2\lambda_{k+1}D_k}\right) \geq e^{-2\lambda_{k+1}\mathbf{E}(D_k)} = e^{-2\lambda_{k+1}D_1} \Rightarrow \liminf_{k\to\infty} \mathbf{P}(D_{k+1} = 0) = 1 \Rightarrow$$

$$\lim_{k\to\infty} \mathbf{P}(D_{k+1} = 0) = 1$$

□

Remark 2. *The second inequality follows by Jensen inequality applied to the convex function* $e^{-2\lambda_{k+1}D_k}$. *The third equality and the implication follow by the propositions 4 and 2.*

Remark 3. *The proof of the existence of a unique end for the graph with the algorithm described in section 1, is also sufficient to guaranty that the infinite points of each one of the infinite Independent Poisson Point Process build a connected graph.*

2.1. Determination Criteria for a One-Ended Poissonian Tree as a Function of the Rates Sequence.

From the result established in the equation (2.2), it is easy to see that for all $k \geq 1$, the coalescing conditional probability in the k-th "iteration" decays with the increase of the rate λ_{k+1} for any positive fixed value u of D_k. The theorem 1 gives a sufficient condition on the sequence of rates $(\lambda_k)_{k\geq 1}$ for the existence of a connected graph, without cycles and one-ended. Is it possible to obtain another condition on the rates and get the same result of theorem 1? Is there a sequence of rates for which the graph is not connected? Is it possible to obtain lower bounds for the probability that the graph is not connected?

The answers for these three questions will be given in theorem 2, theorem 3 and corollary 1. For any $k \geq 1$, let $G(\lambda_{k+1}) : \mathbf{R}^+ \to [0, 1]$ defined in the following manner.

$$G(\lambda_{k+1}) := \mathbf{P}(A_{k+1} = B_{k+1}|D_k \neq 0) = \int_{0^+}^{\infty} e^{-2\lambda_{k+1}u}\left(1 + \lambda_{k+1}u\right) f_{D_k}(u)du$$

$$\Leftrightarrow$$

$$\mathbf{P}(A_{k+1} \neq B_{k+1}|D_k \neq 0) = 1 - \int_{0^+}^{\infty} e^{-2\lambda_{k+1}u}\left(1 + \lambda_{k+1}u\right) f_{D_k}(u)du$$

The events related to the non coalescing among the ancestors of a_1 and b_1 in each one of the stages, weighed for the distribution of the distance in the past generation, are independent. Hence, the probability of not having coalescing in all stages between all the ancestors of a_1 and b_1(and therefore that the graph is not connected) is given by:

$$\left[1 - e^{-2\lambda_2 D_1}(1 + \lambda_2 D_1)\right] \prod_{k=2}^{\infty} \left(1 - \int_{0^+}^{\infty} e^{-2\lambda_{k+1} u} (1 + \lambda_{k+1} u) f_{D_k}(u) du\right)$$

But for all $k \geq 1$, $G(\lambda_{k+1})$ is a continuous function for all $\lambda_{k+1} \in \mathbf{R}^+$. Moreover, $G(\lambda_{k+1} = 0) = 1$ and $G(\lambda_{k+1} = \infty) = 0$, therefore there exists $\lambda_{k+1} \in \mathbf{R}^+$ such that $G(\lambda_{k+1}) \in (0, 1)$.

Theorem 2. *If $G(\lambda_{k+1}) > \frac{1}{k}$, then with probability one, there is a One-Ended Poissonian Tree consisting of all the points of the processes $(X_k)_{k\geq 1}$.*

Proof.

$$\mathbf{P}(a_k \neq b_k \; \forall k | D_1 \neq 0) =$$

$$\left[1 - e^{-2\lambda_2 D_1}(1 + \lambda_2 D_1)\right] \prod_{k=2}^{\infty} \left(1 - \int_{0^+}^{\infty} e^{-2\lambda_{k+1} u} (1 + \lambda_{k+1} u) f_{D_k}(u) du\right) \leq$$

$$\prod_{k=2}^{\infty} \left(1 - \frac{1}{k}\right) = 0 \Rightarrow \exists \, n_o; \; \forall n > n_0 \; \mathbf{P}(a_n = b_n | D_1 \neq 0) = 1.$$

□

Theorem 3. *If $G(\lambda_{k+1}) < \frac{1}{e^k}$, then there is a positive probability larger than γ of having a graph which is not connected and has no cycles.*

Proof.

$$\mathbf{P}(a_k \neq b_k \; \forall k | D_1 \neq 0) =$$

$$\left[1 - e^{-2\lambda_2 D_1}(1 + \lambda_2 D_1)\right] \prod_{k=2}^{\infty} \left(1 - \int_{0^+}^{\infty} e^{-2\lambda_{k+1} u} (1 + \lambda_{k+1} u) f_{D_k}(u) du\right) >$$

$$\left[1 - e^{-2\lambda_2 D_1}(1 + \lambda_2 D_1)\right] \prod_{k=2}^{\infty} \left(1 - \frac{1}{e^k}\right) = \gamma$$

□

Corollary 1. *For any $p \in (0, 1)$ it is always possible to obtain a sequence of rates $(\lambda_k)_{k\geq 1}$ such that the probability of not having a One-Ended Poissonian Tree is larger than p.*

Proof. For any $p \in (0, 1)$, there exists a sequence $\{a_k \in (0, 1), k \geq 1\}$ that satisfies:

$$\prod_{k=1}^{\infty} (1 - a_k) = p$$

But if $G(\lambda_{k+1}) < a_k$ and $\left[1 - e^{-2\lambda_2 D_1}(1 + \lambda_2 D_1)\right] < a_1$, then:

$$\mathbf{P}(a_k \neq b_k\ \forall k | D_1 \neq 0) = \left[1 - e^{-2\lambda_2 D_1}(1 + \lambda_2 D_1)\right] \prod_{k=2}^{\infty} (1 - G(\lambda_{k+1})) >$$

$$\prod_{k=1}^{\infty} (1 - a_k) = p$$

□

3. The case of R^d, $d \geq 2$

We present in this section the construction of a One-Ended Poissonian Tree when the processes $(X_k)_{k \geq 1}$ are defined in $\mathbf{R}^d$, for $d \geq 2$ and $\lambda_k = (\alpha)^k$. In comparison to the problem of determining the construction of the One-Ended Poissonian Tree , described in section 1, the main difficulties that appear now are:

(1) There is not a "closed expression" for $\mathbf{P}(D_{k+1} = 0 | a_k, b_k)$, differently of what happens in (2.2).
(2) The distribution of D_k does not have the property described in proposition 4.

The attainment of a lower bound for the coalescing conditional probability, $\mathbf{P}(D_{k+1} = 0 | a_k, b_k)$, will be the alternative to the result presented in the equation (2.2). With respect to the second item, we will consider a deterministic rescale of the process D_k, such that this new process, whose distances will be denoted by d_k, will have the following characteristic: whenever a_k and b_k are "sufficiently distant", $d_k > L_d = f(\alpha, \beta, d)$, then $\mathbf{E}(d_{k+1} | d_k) < \beta d_k$. From this, theorem 4 will ensure that $d_k \leq L_d$ for infinite values of k. This fact and the condition given in lemma 6, which establishes a positive lower bound for the limit of coalescing conditional probability , will be enough to prove the existence of the One-Ended Poissonian Tree.

Remark 4. *As will be shown in (3.3), L_d is a constant value that depends on the ratio decay of the rates of the processes α, the dimension d and the value β which is related to the "mean drift" of the rescaled process d_k. As α and β can be chosen "a priori" satisfying (3.3) for any sequence $(X_k)_{k \geq 1}$, we will indicate only the index d in L_d.*

We will determine a lower bound for the coalescing conditional probability, lemma 4, from the conditioning given in (3.1), in which $R_{a_k}^{a_{k+1}}$ is the distribution of the distance from a point a_k of X_k to its ancestor a_{k+1} of X_{k+1} and whose density is given in corollary 2 of lemma 2.

$$\mathbf{P}(D_{k+1} = 0 | a_k, b_k) = \mathbf{E}(\mathbf{P}(D_{k+1} = 0 | a_k, b_k, R_{a_k}^{a_{k+1}})) \tag{3.1}$$

Lemma 3. *For all $k \geq 1$, $(R_{a_k}^{a_{k+1}})^d \sim Exp(\lambda_{k+1} \nu_d(1))$.*

Proof. Let $r > 0$, then

$$\left\{R_{a_k}^{a_{k+1}} > r\right\} \Leftrightarrow \left\{(R_{a_k}^{a_{k+1}})^d > r^d\right\} \Rightarrow$$

$$\mathbf{P}(R_{a_k}^{a_{k+1}} > r) = \mathbf{P}((R_{a_k}^{a_{k+1}})^d > r^d) = \exp(-\lambda_{k+1}\nu_d(1)r^d)$$

□

Corollary 2. *$R_{a_k}^{a_{k+1}}$ has a probability density function given by:*

$$f_{R_{a_k}^{a_{k+1}}}(r) = d\lambda_{k+1}\nu_d(1)r^{d-1}\exp(-\lambda_{k+1}\nu_d(1)r^d)\mathbb{1}_{(r\geq 0)}$$

Proof.

$$F_{R_{a_k}^{a_{k+1}}}(r) = 1 - \mathbf{P}(R_{a_k}^{a_{k+1}} > r) = 1 - \exp(-\lambda_{k+1}\nu_d(1)r^d) \Rightarrow$$

$$f_{R_{a_k}^{a_{k+1}}}(r) = d\lambda_{k+1}\nu_d(1)r^{d-1}\exp(-\lambda_{k+1}\nu_d(1)r^d)\mathbb{1}_{(r\geq 0)}$$

□

Remark 5. *For any $R_{a_k}^{a_{k+1}} = r > 0$, if there is no point of X_{k+1} in the shaded region in figure 2, then necessarily a_{k+1} is the nearest point of a_k and b_k, determining the coalescing of those points in X_{k+1}.*

FIGURE 2. Region which determines the coalescing between two points for a given distance from one of them to its ancestor.

Lemma 4 (Lower Bound for the Coalescing Conditional Probability).

$$\mathbf{P}(D_{k+1} = 0|D_k) = \mathbf{P}(a_{k+1} = b_{k+1}|D_k) \geq \exp(-\lambda_{k+1}{D_k}^d\nu_d(1)) +$$

$$\sum_{j=1}^{d-1}\binom{d-1}{j} d\left(-D_k\nu_d(1)^{\frac{1}{d}}\right)^j {\lambda_{k+1}}^{\frac{1}{d}}{\lambda_{k+1}}^{\frac{j}{d}} \int_{(\lambda_{k+1}{D_k}^d\nu_d(1))^{\frac{1}{d}}}^{\infty} \exp(-w^d)w^{d-1-j}dw$$

Proof. Appendix [A1]. □

Despite the difficulty cited in item 2, namely, that we do not have anymore that $\forall k \geq 1$ $\mathbf{E}(D_k) = D_1$, we can get an upper bound for $\mathbf{E}(D_{k+1})$ which depends on $\mathbf{E}(D_k)$ and λ_{k+1} from (3.2) and the following corollary.

$$D_{k+1}|D_k \leq D_k + R_{a_k}^{a_{k+1}} + R_{b_k}^{b_{k+1}} \tag{3.2}$$

Corollary 3. *For all $k \geq 1$,*

$$\mathbf{E}(R_{a_k}^{a_{k+1}}) = \frac{1}{(\lambda_{k+1}\nu_d(1))^{\frac{1}{d}}}\Gamma\left(1 + \frac{1}{d}\right)$$

Proof. From lemma 3, with $u = \lambda_{k+1}\nu_d(1)x^d$ it follows that

$$\mathbf{E}(R_{a_k}^{a_{k+1}}) = \int_0^\infty \mathbf{P}(R_{a_k}^{a_{k+1}} > x)dx = \int_0^\infty \exp(-\lambda_{k+1}\nu_d(1)x^d)dx =$$

$$\frac{1}{d\,(\lambda_{k+1}\nu_d(1))^{\frac{1}{d}}}\int_0^\infty e^{(-u)}u^{\left(\frac{1}{d}-1\right)}du = \frac{1}{d\,(\lambda_{k+1}\nu_d(1))^{\frac{1}{d}}}\Gamma\left(\frac{1}{d}\right) =$$

$$\frac{1}{(\lambda_{k+1}\nu_d(1))^{\frac{1}{d}}}\Gamma\left(1+\frac{1}{d}\right)$$

□

From this

$$\mathbf{E}(D_{k+1}|D_k) \leq D_k + \frac{1}{(\lambda_{k+1})^{\frac{1}{d}}}c(d) \text{ where } c(d) = 2\frac{\Gamma\left(1+\frac{1}{d}\right)}{(\nu_d(1))^{\frac{1}{d}}}$$

Consider the following rescaling for the distance between the ancestors of a_1 and b_1 in k-th generation. The reescalated distance $(d_k, k \geq 2)$ is given by $d_k = \left(\prod_{j=2}^{k} l_j\right) D_k = (\alpha)^{\frac{k-1}{d}} D_k$, where $l_j = (\alpha)^{\frac{1}{d}}\ \forall j \geq 2$.

Thus, this implies that

$$L_d = \frac{c(d)}{(\alpha)^{\frac{1}{d}}\left(\beta-(\alpha)^{\frac{1}{d}}\right)} = \frac{2\frac{\Gamma(1+\frac{1}{d})}{(\nu_d(1))^{\frac{1}{d}}}}{(\alpha)^{\frac{1}{d}}\left(\beta-(\alpha)^{\frac{1}{d}}\right)} \text{ where } \beta \in (\alpha^{\frac{1}{d}}, 1) \tag{3.3}$$

$$\mathbf{E}(d_{k+1}|d_k) = \begin{cases} <\beta d_k & \text{if } d_k \in (L_d, \infty) \\ \leq \beta L_d & \text{if } d_k \in [0, L_d] \end{cases} \tag{3.4}$$

(1) L_d is a positive constant that as said before, depends on: the dimension d where the processes are defined, the ratio α of decay rates of the processes, and the value β given in (3.4) associated to the "mean drift" of the rescaled process d_k.
(2) From (3.4), it follows that the "mean drift" of d_{k+1} with respect to d_k is at least $(1-\beta)$, if $d_k \in (L_d, \infty)$.
(3) For a given value of α, the bigger the dimension, the smaller the "mean drift" will be.
(4) For a given dimension d, in order to establish a "mean drift" of at least $1-\beta$, the bigger the rate α, the bigger the value of L_d must be.
(5) If $d_k \in [0, L_d]$, then $D_k \in \left[0, \frac{L_d}{\prod_{j=2}^{k} l_j}\right]$.
(6) $\{d_k = 0\} \Leftrightarrow \{D_k = 0\}\ \forall k \geq 2$.

Theorem 4. *Let $S_0 > C$ and, for some $\epsilon > 0$ and for all $n \geq 0$,*

$$\mathbf{E}(\widetilde{S}_{n+1}|\mathbf{F}_n) \leq \widetilde{S}_n - \epsilon 1\{\tau > n\}\ a.s. \tag{3.5}$$

Then:

$$\mathbf{E}(\tau) < \frac{S_0}{\epsilon} < \infty$$

Proof. See (Fayolle et all, 1995). □

Remark 6. *From theorem 4, it can be established that for all $k \geq 1$, the variable d_k which measures the distance between the k-th ancestors of a_1 and b_1, will assume a value less or equal than L_d in a finite time. Since the processes $(X_k)_{k\geq 1}$ are independent, in particular are the processes in which $d_k \in [0, L_d]$.*

Remark 7. *The comments about the distribution of D_k extend to d_k, since d_k is a deterministic rescale of D_k.*

Lemma 5. *If $L \geq l$, then for all $k \geq 1$*

$$\mathbf{P}(D_{k+1} = 0 | D_k = L) \geq \mathbf{P}(D_{k+1} = 0 | D_k = l)$$

Proof. Let a_k, b_k, a'_k and b'_k points of X_k defined in $\mathbf{R}^d$ such that $L = |a_k - b_k|$ and $l = |a'_k - b'_k|$.

By the properties of invariance by rotation and translation of the Poisson Point Process, without loss of generality, we may consider the four points in the same line of a plane $\pi \subset \mathbf{R}^2$ with a_k and a'_k in the same position. Let $\xi_{k+1}^{a_k, b_k}$ be the common ancestor of a_k and b_k in X_{k+1}, that is, a_k and b_k belongs to the Voronoi Cell of $\xi_{k+1}^{a_k, b_k}$, implying the same for a'_k and b'_k, that is, the coalescing of a_k and b_k will imply the coalescing between a'_k and b'_k. □

The remark 6 tell us that the processes with the rescaled distances d_k always assume values less or equals a L_d in a finite time, "it visits a box of length L_d in a finite time". **From lemma 5, it follows that in each one of these "visits" we can replace the value assumed by $d_k \in (0, L_d)$ by $d_k = L_d$, that is, we change a random variable of unknown distribution by a constant that depends only the dimension, keeping the inequality.**

Lemma 6. *The coalescing conditional probability limit is larger than a positive constant ϵ that depends on d, α and β.*

$$\lim_{k \to +\infty} \mathbf{P}(d_{k+1} = 0 | d_k \in [0, L_d]) \geq \exp(-\alpha^2 (L_d)^d v_d(1)) = \epsilon(\alpha, \beta, d) > 0$$

Proof. Appendix A2. □

Theorem 5. *Let $(X_k)_{k\geq 1}$ be a sequence of independent Poisson Point Processes defined in $\mathbf{R}^d$ with respective rates $\lambda_k = \alpha^k$. Then there is a One-Ended Poissonian Tree whose vertices are all the points of the processes.*

Proof. From observation 6 and lemma 6 it follows that there are infinitely many times in which the event that represents the coalescing between the k-th ancestors of a_1 and b_1 has probability uniformly lower bounded and all these events are independent. This implies the result. □

APPENDIX

[A1]

$$\mathbb{P}(D_{k+1}=0|D_k)=\mathbb{P}(a_{k+1}=b_{k+1}|D_k)\geq \exp(-\lambda_{k+1}{D_k}^d\nu_d(1))+$$
$$\sum_{j=1}^{d-1}\binom{d-1}{j}d\left(-D_k\nu_d(1)^{\frac{1}{d}}\right)^j{\lambda_{k+1}}^{\frac{1}{d}}{\lambda_{k+1}}^{\frac{j}{d}}\int_{(\lambda_{k+1}D_k{}^d\nu_d(1))^{\frac{1}{d}}}^{\infty}\exp(-w^d)w^{d-1-j}dw$$

Proof. From observation 5 it follows that

$$\mathbb{P}(D_{k+1}=0|D_k,R_{a_k}^{a_{k+1}}=r)\geq \exp\left(-\lambda_{k+1}\nu_d(1)\left[(r+D_k)^d-r^d\right]\right) \tag{3.6}$$

From the conditioning given in (3.1), (3.6) and corollary 2, it follows that

$$\mathbb{P}(a_{k+1}=b_{k+1}|D_k)\geq\int_0^\infty \exp\left(-\lambda_{k+1}\nu_d(1)\left[(r+D_k)^d-r^d\right]\right)\lambda_{k+1}\nu_d(1)dr^{d-1}\exp\left(-\lambda_{k+1}\nu_d(1)r^d\right)dr$$

Let $\nu=\nu_d(1)r^d\Rightarrow d\nu=dr^{d-1}\nu_d(1)dr$, this implies that

$$\mathbb{P}(a_{k+1}=b_{k+1}|D_k)\geq\ \lambda_{k+1}\int_0^\infty \exp\left[-\lambda_{k+1}\left(\nu^{\frac{1}{d}}+\left((D_k)^d\nu_d(1)\right)^{\frac{1}{d}}\right)^d\right]d\nu$$

Consider the following change of variables:

$$u^{\frac{1}{d}}=\nu^{\frac{1}{d}}+\left((D_k)^d\nu_d(1)\right)^{\frac{1}{d}}\ \Rightarrow d\nu=\left(\frac{u}{\nu}\right)^{\left(\frac{1}{d}-1\right)}du$$

From this, we have that,

$$\mathbb{P}(a_{k+1}=b_{k+1}|D_k)\geq \exp(-\lambda_{k+1}(D_k)^d\nu_d(1))+$$
$$\sum_{j=1}^{d-1}\binom{d-1}{j}\int_{(D_k)^d\nu_d(1)}^{\infty}\lambda_{k+1}\exp\left(-\lambda_{k+1}u\right)(-1)^j\left(\frac{(D_k)^d\nu_d(1)}{u}\right)^{\frac{j}{d}}du$$

Make a final change of variables

$$w=(u)^{\frac{1}{d}}\ \Rightarrow dw=\frac{1}{d}(u)^{\left(\frac{1}{d}-1\right)}du$$

$$\mathbb{P}(a_{k+1}=b_{k+1}|D_k)\geq \exp(-\lambda_{k+1}(D_k)^d\nu_d(1))+$$
$$\sum_{j=1}^{d-1}\binom{d-1}{j}d\left(-D_k\nu_d(1)^{\frac{1}{d}}\right)^j\int_{((D_k)^d\nu_d(1))^{\frac{1}{d}}}^{\infty}\lambda_{k+1}\exp\left(-\lambda_{k+1}w^d\right)w^{d-1-j}dw \tag{3.7}$$

Let the following integral:

$$F(y)=\int_0^y \exp(-w^d)w^{d-1-j}dw \Rightarrow \frac{d}{dy}F(\lambda^{\frac{1}{d}}y)=\exp\left(-\lambda y^d\right)(\lambda)^{1-\frac{1}{d}-\frac{j}{d}}\, y^{d-1-j} \tag{3.8}$$

The integral in (3.7) with the notation given in (3.8), can be expressed as

$$\lambda_{k+1}\int_{((D_k)^d v_d(1))^{\frac{1}{d}}}^{\infty} \exp\left(-\lambda_{k+1}w^d\right)w^{d-1-j}dw =$$

$$(\lambda_{k+1})^{\frac{1}{d}}(\lambda_{k+1})^{\frac{j}{d}}\int_{((D_k)^d v_d(1))^{\frac{1}{d}}}^{\infty} \exp\left(-\lambda_{k+1}w^d\right)(\lambda_{k+1})^{1-\frac{1}{d}-\frac{j}{d}}w^{d-1-j}dw =$$

$$(\lambda_{k+1})^{\frac{1}{d}}(\lambda_{k+1})^{\frac{j}{d}}\int_{((D_k)^d v_d(1))^{\frac{1}{d}}}^{\infty} \frac{d}{dw}F((\lambda_{k+1})^{\frac{1}{d}}w) =$$

$$(\lambda_{k+1})^{\frac{1}{d}}(\lambda_{k+1})^{\frac{j}{d}}\left[F(\infty)-F((\lambda_{k+1})^{\frac{1}{d}}\left((D_k)^d v_d(1)\right)^{\frac{1}{d}})\right] =$$

$$(\lambda_{k+1})^{\frac{1}{d}}(\lambda_{k+1})^{\frac{j}{d}}\int_{(\lambda_{k+1})^{\frac{1}{d}}((D_k)^d v_d(1))^{\frac{1}{d}}}^{\infty} \exp(-w^d)w^{d-1-j}dw \Rightarrow$$

$$(\lambda_{k+1})\int_{((D_k)^d v_d(1))^{\frac{1}{d}}}^{\infty} \exp\left(-(\lambda_{k+1})w^d\right)w^{d-1-j}dw =$$

$$(\lambda_{k+1})^{\frac{1}{d}}(\lambda_{k+1})^{\frac{j}{d}}\int_{(\lambda_{k+1})^{\frac{1}{d}}((D_k)^d v_d(1))^{\frac{1}{d}}}^{\infty} \exp(-w^d)w^{d-1-j}dw \tag{3.9}$$

Substituting (3.9) in (3.7) the result follows.

□

[A2]

$$\lim_{k\to+\infty} \mathbf{P}(d_{k+1}=0|d_k\in[0,L_d]) \geq \exp(-\alpha^2(L_d)^d\nu_d(1)) = \epsilon(\alpha,\beta,d) > 0$$

Proof. Let $l^{\bullet}_{k,d}$ be the value assumed by D_k when $d_k = L_d$, that is,

$$l^{\bullet}_{k,d} = \frac{L_d}{\prod_{j=2}^{k} l_j} = \frac{L_d}{\alpha^{\frac{k-1}{d}}} \tag{3.10}$$

from lemma 5 and by the fact of $D_k = 0 \Leftrightarrow d_k = 0$, item (6), it follows that

$$\mathbf{P}(d_{k+1}=0|d_k\in[0,L_d]) \geq \mathbf{P}(d_{k+1}=0|d_k=L_d) = \mathbf{P}(D_{k+1}=0|D_k=l^{\bullet}_{k,d}) \tag{3.11}$$

The lower bound for the coalescing conditional probability of given in lemma 4 with $D_k = l^{\bullet}_{k,d}$ and equation (3.11), yield that

$$\lim_{k\to+\infty} \mathbf{P}(d_{k+1}=0|d_k\in[0,L_d]) \geq \lim_{k\to+\infty} \mathbf{P}(d_{k+1}=0|D_k=l^{\bullet}_{k,d}) \geq$$

$$\lim_{k\to+\infty}\sum_{j=1}^{d-1}\binom{d-1}{j} d\left(-l^{\bullet}_{k,d}\nu_d(1)^{\frac{1}{d}}\right)^j {\lambda_{k+1}}^{\frac{1}{d}} {\lambda_{k+1}}^{\frac{j}{d}} \int_{\left(\lambda_{k+1}(l^{\bullet}_{k,d})^d\nu_d(1)\right)^{\frac{1}{d}}}^{\infty} \exp(-w^d)w^{d-1-j}dw +$$

$$\lim_{k\to+\infty} \exp(-\lambda_{k+1}(l^{\bullet}_{k,d})^d\nu_d(1)) = \exp(-\alpha^2(L_d)^d\nu_d(1)) +$$

$$\lim_{k\to+\infty}\sum_{j=1}^{d-1}\binom{d-1}{j} d\left(-l^{\bullet}_{k,d}\nu_d(1)^{\frac{1}{d}}\right)^j {\lambda_{k+1}}^{\frac{1}{d}} {\lambda_{k+1}}^{\frac{j}{d}} \int_{\left(\lambda_{k+1}(l^{\bullet}_{k,d})^d\nu_d(1)\right)^{\frac{1}{d}}}^{\infty} \exp(-w^d)w^{d-1-j}dw \tag{3.12}$$

To conclude the proof of lemma 6, we will show that the limit given in (3.12) is zero. Observe that for $u = w^d$, it holds that

$$\sum_{j=1}^{d-1}\binom{d-1}{j} d\left(-l^{\bullet}_{k,d}\nu_d(1)^{\frac{1}{d}}\right)^j {\lambda_{k+1}}^{\frac{1}{d}} {\lambda_{k+1}}^{\frac{j}{d}} \int_{\left(\lambda_{k+1}(l^{\bullet}_{k,d})^d\nu_d(1)\right)^{\frac{1}{d}}}^{\infty} \exp(-w^d)w^{d-1-j}dw =$$

$$(\lambda_{k+1})^{\frac{1}{d}}\sum_{j=1}^{d-1}\binom{d-1}{j}(-1)^j\left(\lambda_{k+1}\left(l^{\bullet}_{k,d}\right)^d\nu_d(1)\right)^{\frac{j}{d}} \int_{\lambda_{k+1}(l^{\bullet}_{k,d})^d\nu_d(1)}^{\infty} e^{-u}u^{-\frac{j}{d}}du \Rightarrow$$

$$|\sum_{j=1}^{d-1}\binom{d-1}{j} d\left(-l^{\bullet}_{k,d}\nu_d(1)^{\frac{1}{d}}\right)^j {\lambda_{k+1}}^{\frac{1}{d}} {\lambda_{k+1}}^{\frac{j}{d}} \int_{\left(\lambda_{k+1}(l^{\bullet}_{a,d})^d\nu_d(1)\right)^{\frac{1}{d}}}^{\infty} \exp(-w^d)w^{d-1-j}dw| \leq$$

$$(\lambda_{k+1})^{\frac{1}{d}}|\sum_{j=1}^{d-1}\binom{d-1}{j}(-1)^j\left(\lambda_{k+1}\left(l^{\bullet}_{k,d}\right)^d\nu_d(1)\right)^{\frac{j}{d}} \int_{\lambda_{k+1}(l^{\bullet}_{k,d})^d\nu_d(1)}^{\infty} e^{-u}u^{-\frac{j}{d}}du| <$$

$$(\lambda_{k+1})^{\frac{1}{d}} \left| \sum_{j=1}^{d-1} \binom{d-1}{j} \left(\left(\lambda_{k+1} \left(l^{*}_{k,d} \right)^{d} v_d(1) \right)^{\frac{1}{d}} \right)^{j} \int_0^{\infty} e^{-u} u^{-\frac{j}{d}} du \right| =$$

$$(\lambda_{k+1})^{\frac{1}{d}} \sum_{j=1}^{d-1} \binom{d-1}{j} \left(\left(\lambda_{k+1} \left(l^{*}_{k,d} \right)^{d} v_d(1) \right)^{\frac{1}{d}} \right)^{j} \Gamma\left(1 - \frac{j}{d}\right) <$$

$$(\lambda_{k+1})^{\frac{1}{d}} \sum_{j=1}^{d-1} \binom{d-1}{j} \left(\left(\lambda_{k+1} \left(l^{*}_{k,d} \right)^{d} v_d(1) \right)^{\frac{1}{d}} \right)^{j} <$$

$$(\lambda_{k+1})^{\frac{1}{d}} \sum_{j=0}^{d-1} \binom{d-1}{j} \left(\left(\lambda_{k+1} \left(l^{*}_{k,d} \right)^{d} v_d(1) \right)^{\frac{1}{d}} \right)^{j} =$$

$$(\lambda_{k+1})^{\frac{1}{d}} \left(1 + (\lambda_{k+1})^{\frac{1}{d}} (l^{*}_{k,d}) (v_d(1))^{\frac{1}{d}} \right)^{d-1} \Rightarrow$$

$$-(\lambda_{k+1})^{\frac{1}{d}} \left(1 + (\lambda_{k+1})^{\frac{1}{d}} (l^{*}_{k,d}) (v_d(1))^{\frac{1}{d}} \right)^{d-1} \leq$$

$$\sum_{j=1}^{d-1} \binom{d-1}{j} d \left(-l^{*}_{k,d} v_d(1)^{\frac{1}{d}} \right)^{j} \lambda_{k+1}{}^{\frac{j}{d}} \lambda_{k+1}{}^{\frac{1}{d}} \int_{(\lambda_{k+1} (l^{*}_{k,d})^{d} v_d(1))^{\frac{1}{d}}}^{\infty} \exp(-w^{d}) w^{d-1-j} dw \leq$$

$$(\lambda_{k+1})^{\frac{1}{d}} \left(1 + (\lambda_{k+1})^{\frac{1}{d}} (l^{*}_{k,d}) (v_d(1))^{\frac{1}{d}} \right)^{d-1}$$

It happens that

$$\lim_{k \to \infty} \left[(\lambda_{k+1})^{\frac{1}{d}} \left(1 + (\lambda_{k+1})^{\frac{1}{d}} (l^{*}_{k,d}) (v_d(1))^{\frac{1}{d}} \right)^{d-1} \right] =$$

$$\lim_{k \to \infty} (\lambda_{k+1})^{\frac{1}{d}} \lim_{k \to \infty} \left(1 + (\lambda_{k+1})^{\frac{1}{d}} (l^{*}_{k,d}) (v_d(1))^{\frac{1}{d}} \right)^{d-1} =$$

$$\lim_{k \to \infty} (\lambda_{k+1})^{\frac{1}{d}} \left(1 + \alpha^{\frac{3}{d}} L_d (v_d(1))^{\frac{1}{d}} \right)^{d-1} = 0$$

Using the sandwich theorem we finish the proof that the limit in (3.12) equals zero and thus lemma 6.

□

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