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RESEARCH

On the Braid Group Action on Exceptional Sequences for Weighted Projective Lines

Edson Ribeiro Alvares[1] · Eduardo Nascimento Marcos[2] · Hagen Meltzer[3]

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Abstract
We give a new and intrinsic proof of the transitivity of the braid group action on the set of full exceptional sequences of coherent sheaves on a weighted projective line. We do not use the corresponding result of Crawley-Boevey for modules over hereditary algebras. As an application we prove that the strongest global dimension of the category of coherent sheaves on a weighted projective line $\mathbb{X}$ does not depend on the parameters of $\mathbb{X}$. Finally we prove that the determinant of the matrix obtained by taking the values of n $\mathbb{Z}$-linear functions defined on the Grothendieck group $\mathrm{K}_0(\mathbb{X}) \simeq \mathbb{Z}^n$ of the elements of a full exceptional sequence is an invariant, up to sign.

Keywords Braid group · Exceptional sheaf · Exceptional sequence · Weighted projective line · Tilting sheaf · Tilting complex · Strong global dimension · Grothendieck group · Diophantine equation

Mathematics Subject Classification (2010) Primary 14H05 · Secondary 16G20 · 16G99

1 Introduction

Let $\mathbb{X}$ be a weighted projective line in the sense of Geigle and Lenzing [6]. The braid group B_n on n strings acts on the set of full exceptional sequences in the category $\mathrm{coh}(\mathbb{X})$ of coherent

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We dedicate this work to the memory of Andrzej Skowroński

Presented by: Christof Geiß

✉ Edson Ribeiro Alvares
rolo1rolo@gmail.com

Eduardo Nascimento Marcos
enmarcos@gmail.com

Hagen Meltzer
hagen.meltzer@usz.edu.pl

1 Departamento de Matemática Universidade Federal do Parana, Curitiba, Brazil

2 Departamento de Matemática , IME, Universidade de Sao Paulo Brazil, Sao Paulo, Brazil

3 Instytut Matematyki, Uniwersytet Szczeciński, 70451 Szczecin, Poland

sheaves on $\mathbb{X}$, where n denotes the rank of the Grothendieck group $K_0(\mathbb{X})$ of $\mathrm{coh}(\mathbb{X})$. This action is given by mutations in the sense of Gorodentsev and Rudakov [8]. The following result was proved in [15].

Theorem 1.1 *The action of the braid group on the set of full exceptional sequences in the category of coherent sheaves on a weighted projective line* $\mathbb{X}$ *is transitive.*

The proof was based on induction on the rank of the Grothendieck group of $\mathrm{coh}(\mathbb{X})$ and on the rather strong result of Crawley-Boevey [5] which states that the braid group acts transitively on the set of full exceptional sequences in the category of finitely generated modules over a hereditary algebra over an algebraically closed field.

It is desirable to have in the geometric situation a purely sheaf-theoretical proof for the transitivity of the braid group operation. In this short note we show that this in fact can be done using perpendicular calculus of exceptional pairs. For this we calculate the left perpendicular category of the sum of two line bundles $L \oplus L(\vec{c})$ formed in the sheaf category, where L is a line bundle and $\vec{c}$ the canonical element of the grading group of $\mathbb{X}$. For the convenience of the reader we also state the unchanged parts of the original proof.

Furthermore, we give two applications of the transitivity of the braid group action. First we show that the strong global dimension of a weighted projective line $\mathbb{X}$ is independent of the parameters of $\mathbb{X}$. This means that if $\mathbb{X} = \mathbb{X}(\mathbf{p}, \lambda)$ and $\mathbb{X}' = \mathbb{X}(\mathbf{p}, \lambda')$, are weighted projective lines with the same weight sequence $\mathbf{p}$ and different parameter sequences λ and λ' then the strong global dimensions for $\mathbb{X}$ and $\mathbb{X}'$ are the same.

Second we prove that the determinant of the matrix obtained by applying n additive functions defined on the Grothendieck group of $\mathrm{coh}(\mathbb{X})$ to the sheaves of a full exceptional sequence on $\mathbb{X}$ is independent of the exceptional sequence, up to sign. Finally, we calculate this invariant for taking the rank function, the degree function and $n-2$ Euler forms with respect to simple exceptional sheaves.

2 Preliminaries

2.1 Weighted projective lines were introduced by Geigle and Lenzing in 1987 in order to give a geometric approach to Ringel's canonical algebras [17]. We recall some of the basic facts and refer for details to [6].

Let k be an algebraically closed field. A weight sequence $\mathbf{p} = (p_1, \dots, p_t)$ is a sequence of natural numbers p_i with $p_i \geq 2$. For a weight sequence $\mathbf{p}$ denote by $\mathbb{L}(\mathbf{p})$ the abelian group with generators $\vec{x}_1, \dots, \vec{x}_t$ and relations $p_1\vec{x}_1 = \cdots = p_t\vec{x}_t := \vec{c}$. The element $\vec{c}$ is called the canonical element. $\mathbb{L}(\mathbf{p})$ is an ordered group with $\sum_{i=1}^{t} \mathbb{N}\vec{x}_i$ as cone of non-negative elements. Furthermore, each element $\vec{x}$ can be written in normal form $\vec{x} = l\vec{c} + \sum_{i=1}^{t} l_i\vec{x}_i$ with $l \in \mathbb{Z}$ and $0 \leq l_i < p_i$. Consider further a sequence of parameters $\lambda = (\lambda_3, \dots, \lambda_t)$, that is the λ_i are non-zero and pairwise distinct elements from k. We denote $\mathrm{S} = \mathrm{S}(\mathbf{p}, \lambda) = \mathrm{K}[\mathrm{X}_1, \dots, \mathrm{X}_t]/(\mathrm{X}_i^{p_i} + \mathrm{X}_1^{p_1} + \lambda_i \mathrm{X}_2^{p_2},\ i = 3, \dots, t)$. The algebra $\mathrm{S}(\mathbf{p}, \lambda)$ is $\mathbb{L}(\mathbf{p})$-graded by defining $\deg(X_i) = \vec{x}_i$. Then the weighted projective line $\mathbb{X} = \mathbb{X}(\mathbf{p}, \lambda)$ is defined to be the $\mathbb{L}(\mathbf{p})$-graded projective scheme $\mathrm{Proj}^{\mathbb{L}(\mathbf{p})}(\mathrm{S}(\mathbf{p}, \lambda))$ and the category $\mathrm{coh}(\mathbb{X})$ of coherent sheaves on $\mathbb{X}$ is the quotient of the category of finitely generated $\mathbb{L}(\mathbf{p})$-graded S modules modulo the $\mathbb{L}(\mathbf{p})$-graded S modules of finite length. The category $\mathrm{coh}(\mathbb{X})$ is abelian, hereditary, that is $\mathrm{Ext}^i(A, B) = 0$ for all A and B in $\mathrm{coh}(\mathbb{X})$ and $i \geq 2$, and it has finite dimensional Hom and Ext^1 spaces. Moreover, $\mathrm{coh}(\mathbb{X})$ admits Serre duality in the form $\mathrm{Ext}^1(A, B) \simeq \mathrm{D}\,\mathrm{Hom}(B, A(\vec{\omega}))$, where $\vec{\omega}$ denotes the dualizing element $(t-2)\vec{c} - \sum_{i=1}^{t} \vec{x}_i$, and consequently $\mathrm{coh}(\mathbb{X})$ has Auslander-Reiten sequences.

We denote the structure sheaf on $\mathbb{X}$ by $\mathcal{O}$. It is well known that the group of line bundles on $\mathbb{X}$ is isomorphic to the group $\mathbb{L}(\mathbf{p})$ via the map $\vec{x} \mapsto \mathcal{O}(\vec{x})$ where $\mathcal{O}(\vec{x})$ is the twisted by $\vec{x}$ structure sheaf. Moreover, the homomorphism space between two line bundles can be calculated by the formula $\mathrm{Hom}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) \simeq \mathrm{S}_{\vec{y}-\vec{x}}$ and if $\vec{z} = l\vec{c} + \sum_{j=1}^{t} l_i \vec{x}_i$ is in normal form, then $\dim \mathrm{S}_{\vec{z}} = l + 1$ provided $l \geq -1$. For coherent sheaves on $\mathbb{X}$ we have the rank and the degree function. The sheaves of rank 0 are those of finite length. One of the key results in [6] is that the sheaf $\bigoplus_{0 \leq \vec{x} \leq \vec{c}} \mathcal{O}(\vec{x})$ is a tilting sheaf such that its endomorphism algebra is a canonical algebra.

2.2 Recall that an object in a hereditary k-category $\mathcal{H}$ is called exceptional if $\mathrm{End}(E) = k$ and $\mathrm{Ext}^1(E, E) = 0$. Moreover, a sequence of exceptional objects $\epsilon = (E_1, \ldots, E_r)$ is called an exceptional sequence if $\mathrm{Hom}(E_j, E_i) = 0 = \mathrm{Ext}^1(E_j, E_i)$ for all $j > i$. If $r = 2$ then ϵ is called an exceptional pair and if r equals the rank of the Grothendieck group $\mathrm{K}_0(\mathcal{H})$ then ϵ is called a full exceptional sequence.

Gorodentsev and Rudakov defined mutations of exceptional sequences on $\mathbb{P}^n$ which give rise to an operation of the braid group $\mathrm{B_r} = \langle \sigma_1, \ldots \sigma_{\mathrm{r}-1} | \sigma_{\mathrm{i}}\sigma_{\mathrm{j}} = \sigma_{\mathrm{j}}\sigma_{\mathrm{i}} \text{ for } \mathrm{i} - \mathrm{j} \geq 2 \text{ and } \sigma_{\mathrm{i}}\sigma_{\mathrm{i}+1}\sigma_{\mathrm{i}} = \sigma_{\mathrm{i}+1}\sigma_{\mathrm{i}}\sigma_{\mathrm{i}+1} \rangle$ on the set of (isomorphism classes) of exceptional sequences of length r [8]. For a categorical treatment we refer to [4].

We will study the action of the braid group $\mathrm{B_n}$ on the set of full exceptional sequences in $\mathrm{coh}(\mathbb{X})$. In this case each line bundle is exceptional. Moreover, the simple exceptional sheaves of rank 0 fit in exact sequences

$$0 \longrightarrow \mathcal{O}(j\vec{x}_i) \longrightarrow \mathcal{O}((j+1)\vec{x}_i) \longrightarrow S_{i,j} \longrightarrow 0.$$

For an exceptional pair (A, B) in $\mathrm{coh}(\mathbb{X})$ we have $\mathrm{Hom}(A, B) = 0$ or $\mathrm{Ext}^1(A, B) = 0$ ([16, Lemma 3.2.4]). Furthermore, if the space $\mathrm{Hom}(A, B)$ is non-zero then the canonical map $\mathrm{can} : \mathrm{Hom}(A, B) \otimes_k A \longrightarrow B$ is surjective or injective but not bijective, the proof for this fact is similar as the proof of [10, Lemma 4.1].

Then the left mutation of (A, B) is the exceptional pair $(\mathrm{L}_A B, A)$, where $\mathrm{L}_A B$ is given by one of the following three exact sequences: if $\mathrm{Hom}(A, B) \neq 0$ then

$$0 \longrightarrow \mathrm{L}_A B \longrightarrow \mathrm{Hom}(A, B) \otimes_k A \overset{\mathrm{can}}{\longrightarrow} B \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Hom}(A, B) \otimes_k A \overset{\mathrm{can}}{\longrightarrow} B \longrightarrow \mathrm{L}_A B \longrightarrow 0,$$

and if $\mathrm{Ext}^1(A, B) \neq 0$ then

$$0 \longrightarrow B \longrightarrow \mathrm{L}_A B \longrightarrow \mathrm{Ext}^1(A, B) \otimes_k A \longrightarrow 0,$$

where the third sequence is a universal extension. If $\mathrm{Hom}(A, B) = 0 = \mathrm{Ext}^1(A, B)$ then $\mathrm{L}_A B = B$ and the left mutation of the pair (A, B) is called a transposition. Now, the generators of $\mathrm{B_r}$ act on the set of full exceptional sequences in $\mathrm{coh}(\mathbb{X})$ as follows

$$\sigma_i \cdot (E_1, \ldots E_{i-1}, E_i, E_{i+1}, E_{i+2}, \ldots, E_r) = (E_1, \ldots E_{i-1}, L_{E_i} E_{i+1}, E_i, E_{i+2}, \ldots, E_r).$$

Further the right mutation of an exceptional pair (A, B) is the exceptional pair $(B, \mathrm{R}_B A)$, where $\mathrm{R}_B A$ is given by one of the following three exact sequences

$$0 \longrightarrow A \overset{\mathrm{cocan}}{\longrightarrow} \mathrm{DHom}(A, B) \otimes_k B \longrightarrow \mathrm{R}_B A \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{R}_B A \longrightarrow A \overset{\mathrm{cocan}}{\longrightarrow} \mathrm{DHom}(A, B) \otimes_k B \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{DExt}^1(A, B) \otimes_k B \longrightarrow \mathrm{R}_B A \longrightarrow A \longrightarrow 0,$$

where $\mathrm{D} = \mathrm{Hom}_k(-, k)$, cocan denotes the co-canonical map and the third sequence is a universal extension. Then σ_i^{-1} acts in the following way:

$$\sigma_i^{-1}\cdot(E_1,\ldots E_{i-1}, E_i, E_{i+1}, E_{i+2},\ldots,E_r)=(E_1,\ldots E_{i-1}, E_{i+1},\mathrm{R}_{E_i+1}E_i,E_{i+2},\ldots,E_r).$$

Lemma 2.1 *We have*

(i) $\sigma_1\ldots\sigma_{n-1}(E_1, E_2,\ldots,E_n) = (E_n(\vec{\omega}), E_1, E_2,\ldots,E_{n-1})$.
(ii) $\sigma_{n-1}\ldots\sigma_1(E_1, E_2,\ldots,E_n) = (E_2,\ldots,E_{n-1}, E_1(-\vec{\omega}))$.
(iii) In the orbit of an exceptional sequence $(E_1,\ldots E_a, E_{a+1},\ldots)$ *there is an exceptional sequence of the form* $(E_a, E_{a+1},\ldots)$.

The proof for (i) and (ii) is given in [15, Proposition 2.4] and (iii) is a consequence of (i) and (ii).

2.3 Recall that for an object X in a hereditary category $\mathcal{H}$ the left perpendicular category with respect to X is defined as the full subcategory of all objects Y satisfying $\mathrm{Hom}(Y, X) = 0$ and $\mathrm{Ext}^1(Y, X) = 0$ [7]. The right perpendicular category is defined dually.

3 Proof of Theorem 1.1

In this section we will prove Theorem 1.1. Let $\mathbb{X}$ be a weighted projective line of weight type $\mathbf{p} = (p_1,\ldots,p_t)$ and rank of $\mathrm{K}_0(\mathbb{X})$ equals n. The item (a) of the next Proposition can be seen as a consequence of [1]. We give a proof, for the sake of completeness.

Proposition 3.1 *(a) Let* (L, L') *be an exceptional pair of line bundles in* $\mathrm{coh}(\mathbb{X})$ *with* $\dim_k \mathrm{Hom}(L, L') \geq 2$. *Then* $L' \simeq L(\vec{c}))$ *and* $\dim \mathrm{Hom}(L, L') = 2$.
(b) The left perpendicular category with respect to $L \oplus L(\vec{c})$ *for a line bundle* L*, formed in* $\mathrm{coh}(\mathbb{X})$*, consists only of finite length sheaves. Moreover, this perpendicular category is equivalent to the category of finite dimensional modules over the path algebra of the disjoint union of linearly oriented quivers of type* A_{p_i-1}, $i = 1,\ldots,t$.

Proof (a) We have $L' = L(\vec{x})$ for some $\vec{x}$. We can assume that $\vec{x}$ is in normal form and, after renumbering if necessary the indices, $\vec{x} = l\vec{c} + \sum_{j=1}^{r} l_j\vec{x}_j$, where $l_1 \neq 0,\ldots,l_r \neq 0$ for some r. Since $\dim \mathrm{Hom}_k(L, L') \geq 2$ we have $l \geq 1$. Using Serre duality and the fact that $(L, L(\vec{x}))$ is an exceptional pair we have $0 = \mathrm{Ext}^1(L(\vec{x}), L) \simeq \mathrm{Hom}(L, L(\vec{x} + \vec{\omega})) \simeq \mathrm{Hom}(\mathcal{O}, \mathcal{O}(\vec{x} + \vec{\omega}))$. Now $\vec{x} + \vec{\omega} = l\vec{c} + \sum_{j=1}^{r} l_j\vec{x}_j + (t-2)\vec{c} - \sum_{i=1}^{t}\vec{x}_i = (l-2+r)\vec{c} + \sum_{j=1}^{r}(l_j-1)\vec{x}_j + \sum_{i=r+1}^{t}(p_i-1)\vec{x}_i$. This element is in normal form and it follows that $l-2+r < 0$, hence $l = 1$ and $r = 0$. Consequently $\vec{x} = \vec{c}$.

(b) After renumbering, if necessary, the indices for the simple exceptional sheaves in the tubes we can assume that $\mathrm{Ext}^1(S_{i,0}, L) \neq 0$ for $i = 1,\ldots,t$. Denote by ${}^{[j]}S_{i,1}$ the indecomposable sheaf in the tubes with socle $S_{i,1}$ and quasi-length j. Following a similar argument as in [3, 8.2], we conclude that $L \oplus L(\vec{c}) \bigoplus_{i=1}^{t}\bigoplus_{j=1}^{p_j-1} {}^{[j]}S_{i,1}$ is a tilting object in $\mathrm{coh}(\mathbb{X})$. Then $\bigoplus_{i=1}^{t}\bigoplus_{j=1}^{p_j-1} {}^{[j]}S_{i,1}$ is a tilting object in the left perpendicular category $\mathcal{H} = {}^{\perp}(L \oplus L(\vec{c}))$ and consists of $n-2$ indecomposable direct summands. It is well-known that $\mathcal{H}$ is an hereditary category [12]. Therefore T is a tilting sheaf in $\mathcal{H}$ and consequently $\mathcal{H}$ consists of the objects of the wings for ${}^{[j]}S_{i,p_i-1}$, $i = 1,\ldots t$. This also shows that the endomorphism algebra of T is the disjoint union of linear quivers of type A_{p_i-1}, $i = 1,\ldots,t$. □

3.2 We will use the following three results of [15]

Lemma 3.2 *[15, Lemma 2.7] If $(E_1, ...E_n)$ and $(F_1, ..., F_n)$ are complete exceptional sequences in* $\mathrm{coh}(\mathbb{X})$ *which differ in at most one place, say $E_j \cong F_j$ for $j \neq i$, then also $E_i \cong F_i$.*

3.3 An exceptional sequence $(E_1, ..., E_n)$ in $\mathrm{coh}(\mathbb{X})$ is called orthogonal if $\mathrm{Hom}_{\mathcal{A}}(E_i, E_j) = 0$ for all $i \neq j$.

Proposition 3.3 *[15, Proposition 2.8] There are no orthogonal complete exceptional sequences in* $\mathrm{coh}(\mathbb{X})$.

Lemma 3.4 *[15, Lemma 3.1] Let $E_1, \ldots, E_n$ be an exceptional sequence in* $\mathrm{coh}(\mathbb{X})$ *such that* $\dim_k \mathrm{Hom}_{\mathbb{X}}(E_1, E_2) \geq 2$.

(i) Suppose that $\mathrm{L}E_2 = \mathrm{L}_{E_1}E_2$ *is defined by an exact sequence*

$$0 \to \mathrm{L}E_2 \to \mathrm{Hom}_{\mathbb{X}}(E_1, E_2) \otimes E_1 \to E_2 \to 0.$$

Then morphisms $0 \neq h \in \mathrm{Hom}_{\mathbb{X}}(\mathrm{L}E_2, E_1)$ *and* $0 \neq f \in \mathrm{Hom}_{\mathbb{X}}(E_1, E_2)$ *are either both monomorphisms or both epimorphisms.*

(ii) Suppose that $\mathrm{R}E_1 = \mathrm{R}_{E_2}E_1$ *is defined by an exact sequence*

$$0 \to E_1 \to \mathrm{D}\,\mathrm{Hom}_{\mathbb{X}}(E_1, E_2) \otimes E_2 \to \mathrm{R}E_1 \to 0.$$

Then morphisms $0 \neq h \in \mathrm{Hom}_{\mathbb{X}}(E_2, \mathrm{R}E_1)$ *and* $0 \neq f \in \mathrm{Hom}_{\mathbb{X}}(E_1, E_2)$ *are either both monomorphisms or both epimorphisms.*

3.5 Remark: The Meltzer's proof of the Theorem 1.1 is contained in [15, Proposition 4.3.6, Lemma 4.3.7, Proposition 4.3.8]. We follow the philosophy of that proof and briefly blended them. Moreover we give the necessary arguments, without using the result of Crawley-Boevey [5].

For an exceptional sequence $\epsilon = E_1, \ldots, E_n$ we define

$$\|\epsilon\| = (\mathrm{rk}\,(E_{\pi(1)}), ..., \mathrm{rk}\,(E_{\pi(n)})),$$

where π is a permutation of $1, ..., n$ such that $\mathrm{rk}\,(E_{\pi(1)}) \geq ... \geq \mathrm{rk}\,(E_{\pi(n)})$.

Proposition 3.5 *Let $\mathbb{X}$ be a weighted projective line with at least one weight, i.e. $\mathbb{X} \neq \mathbb{P}^1$. Then in each orbit under the braid group action there is a complete exceptional sequence containing a simple sheaf of rank* 0.

Proof We show first that if ϵ is a complete exceptional sequence in $\mathrm{coh}(\mathbb{X})$ with $\mathrm{rk}\, E_i \geq 1$ for all i then there exists $\sigma \in \mathrm{B}_n$ such that $\|\sigma \cdot \epsilon\| < \|\epsilon\|$ or that $\sigma \cdot \epsilon$ contains a rank zero sheaf.

Let ϵ be a complete exceptional sequence in $\mathrm{coh}(\mathbb{X})$. We know from (3.3) that ϵ is not orthogonal. Choose $a < b$ such that $\mathrm{Hom}(E_a, E_b) \neq 0$, but $\mathrm{Hom}(E_i, E_j) = 0$ for the remaining $a \leq i < j \leq b$.

Let $f : E_a \to E_b$ a nonzero morphism. We know that f is a monomorphism or an epimorphism, thus we distinguish two cases.

Case 1: f is a monomorphism.
Then f induces epimorphisms $\mathrm{Ext}^1(E_b, E_i) \twoheadrightarrow \mathrm{Ext}^1(E_a, E_i)$ for all i. Since the first Ext-group is zero for $i \leq b$ the second Ext-group also vanishes for these i. We see that both

$\mathrm{Hom}_{\mathbb{X}}(E_a, E_i) = 0$, and $\mathrm{Ext}^1(E_a, E_i) = 0$ for all $a < i < b$, therefore applying transpositions we obtain that

$$\sigma_{b-2}^{-1}...\sigma_{a+1}^{-1}\sigma_a^{-1}\epsilon = (E_1, ..., E_{a-1}, E_{a+1}, ..., E_{b-1}, E_a, E_b, ..., E_n).$$

Moreover, using Lemma 2.1, we can assume that $a = 1$ and $b = 2$.

Now, the left mutation $\mathrm{L}E_2 = \mathrm{L}_{E_1} E_2$ is defined by an exact sequence being of the form

$$\text{(i)} \qquad 0 \to \mathrm{Hom}_{\mathbb{X}}(E_1, E_2) \otimes E_1 \to E_2 \to \mathrm{L}E_2 \to 0$$

or

$$\text{(ii)} \qquad 0 \to \mathrm{L}E_2 \to \mathrm{Hom}_{\mathbb{X}}(E_1, E_2) \otimes E_1 \to E_2 \to 0.$$

In the case (i) we have $\mathrm{rk}\,(\mathrm{L}E_2) < \mathrm{rk}\,(E_2)$, hence $\|\sigma_1\epsilon\| < \|\epsilon\|$ and we are done.

In the case (ii) there exists a nonzero morphism $h : \mathrm{L}E_2 \to E_1$. Again, h is a monomorphism or an epimorphism. Because f is a monomorphism we infer that $\dim_k \mathrm{Hom}_{\mathbb{X}}(E_1, E_2) \geq 2$. But then, in view of Lemma 3.4, h is a monomorphism. Thus

$$\mathrm{rk}\,(\mathrm{L}E_2) \leq \mathrm{rk}\,(E_1) \leq \mathrm{rk}\,(E_2).$$

If $\mathrm{rk}\,(\mathrm{L}E_2) < \mathrm{rk}\,(E_2)$ we apply σ_1^{-1} as before and obtain $\|\sigma_1\epsilon\| < \|\epsilon\|$.

Assume otherwise that $\mathrm{rk}\,(\mathrm{L}E_2) = \mathrm{rk}\,(E_2)$. Then also $\mathrm{rk}\,(E_1) = \mathrm{rk}\,(E_2)$ and therefore $\dim_k \mathrm{Hom}_{\mathbb{X}}(E_1, E_2) = 2$.

Consider an exact sequence

$$0 \to E_1 \xrightarrow{f} E_2 \to C \to 0$$

where $C = \mathrm{coker(f)}$. Clearly $\mathrm{rk}\,(C) = 0$. Furthermore, applying the functor $\mathrm{Hom}_{\mathbb{X}}(E_i, -)$ we conclude that $\dim_k \mathrm{Hom}_{\mathbb{X}}(E_i, C) = 1$, for $i = 1, 2$. Finally, applying the functor $\mathrm{Hom}_{\mathbb{X}}(-, C)$ we obtain $\mathrm{Hom}(C, C) = k$ and $\mathrm{Ext}^1(C, C) = k$, in particular C is indecomposable.

We have to consider two cases. First assume that C is a finite length sheaf concentrated at an ordinary point. Now $\mathrm{End}(C)) = k$ which implies that C is a simple sheaf. The Riemann Roch formula yields $\mathrm{Hom}_{\mathbb{X}}(L, C) = k$ for each line bundle L. Thus using a line bundle filtration for E_1 we obtain $\dim_k \mathrm{Hom}_{\mathbb{X}}(E_1, C) = \mathrm{rk}\,(E_1)$. We have shown before that $\dim_k \mathrm{Hom}_{\mathbb{X}}(E_1, C) = 1$. Thus we obtain $\mathrm{rk}\,(E_1) = \mathrm{rk}\,(E_2) = 1$ and we have also $\dim \mathrm{Hom}(E_1, E_2) = 2$. But then we have $\mathrm{rk}\,(E_i) = 0$ for $i > 2$ by Lemma 3.1.

Now, assume that C is a sheaf of finite length concentrated at an exceptional point, say λ_i of weight p_i. It follows from $\mathrm{Hom}(C, C) = k$ and $\mathrm{Ext}^1(C, C) = k$ that the length of C is p_i, and therefore for the classes in the Grothendieck group $K_0(\mathbb{X})$ we have $[C] = \sum_{j=0}^{p_i-1}[S_{i,j}]$ where $S_{i,j}$ are the objects on the mouth of the tube. From the exact sequences stated in 2.1. we infer that $[S_{i,j}] = [\mathcal{O}(j+1)\vec{x}_i)] - [\mathcal{O}(j)\vec{x}_i]$ for $i = 1, \cdots, t$, $j = 1, \cdots, p_i$. Hence $\sum_{j=0}^{p_i-1}[S_{i,j}] = [\mathcal{O}(\vec{c})] - [\mathcal{O}]$. On the other hand there is an exact sequence $0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(\vec{c}) \longrightarrow S \longrightarrow 0$ where S is a simple finite length sheaf concentrated in an ordinary point and consequently $[C] = [S]$. We conclude that $1 = \dim_k \mathrm{Hom}_{\mathbb{X}}(E_1, C) = \chi([E_1], [C]) = \chi([E_1], [S]) = \dim_k \mathrm{Hom}_{\mathbb{X}}(E_1, S) = \mathrm{rk}\,(E_1)$, where χ is the Euler form. Then we have $\mathrm{rk}\,(E_1) = \mathrm{rk}\,(E_2) = 1$ and again $\dim \mathrm{Hom}(E_1, E_2) = 2$, and consequently $\mathrm{rk}\,(E_i) = 0$ for $i > 2$ by Lemma 3.1.

Case 2 f is an epimorphism.

Then f induces epimorphisms $\mathrm{Ext}^1(E_i, E_a) \twoheadrightarrow \mathrm{Ext}^1(E_i, E_b)$. The first Ext-group is zero for $i \geq a$, thus also the second Ext-group vanishes for these i. We see that both

$\mathrm{Hom}_{\mathbb{X}}(E_i, E_b) = 0$ and $\mathrm{Ext}^1(E_i, E_b) = 0$ for all $a < i < b$, and again applying transpositions we have

$$\sigma_{a+1}^{-1} ... \sigma_{b-1}^{-1} \epsilon = (E_1, ..., E_{a-1}, E_a, E_b, E_{a+1}, ..., E_n).$$

As before we can assume $a = 1$ and $b = 2$. Then $\mathrm{R}E_1 = \mathrm{R}_{E_2} E_1$ is defined by an exact sequence

$$\text{(i)} \qquad 0 \to \mathrm{R}E_1 \to E_1 \to \mathrm{D}\,\mathrm{Hom}_{\mathbb{X}}(E_1, E_2) \otimes E_2 \to 0$$

or

$$\text{(ii)} \qquad 0 \to E_1 \to \mathrm{D}\,\mathrm{Hom}_{\mathbb{X}}(E_1, E_2) \otimes E_2 \to \mathrm{R}E_1 \to 0$$

In the first case we have $\mathrm{rk}\,(\mathrm{R}E_1) < \mathrm{rk}\,(E_1)$, and consequently $\|\sigma_1 \epsilon\| < \|\epsilon\|$. In the second case there is a nonzero map $h : E_2 \to \mathrm{R}E_1$, which again is a monomorphism or an epimorphism. Since f is an epimorphism we conclude that $\mathrm{Hom}(E_1, E_2) \geq 2$ and therefore h is an epimorphism by Lemma 3.4.

Now, in this case,

$$\mathrm{rk}\,(E_1) > \mathrm{rk}\,(E_2) > \mathrm{rk}\,(\mathrm{R}E_1)$$

and therefore again $\|\sigma^{-1}\epsilon\| < \|\epsilon\|$.

After successive applying the norm reduction above, if necessary, we can shift by a braid group element any full exceptional sequence to a sequence containing an exceptional sheaf of rank 0.

Now by the same arguments as in [15] it follows that the orbit contains an exceptional sequence with a simple sheaf and we refer to that paper.

Now let s be the minimal number with the property that the orbit of ϵ contains an exceptional sequence with a rank 0 sheaf F of length s. By Lemma 2.1 we can assume that this exceptional sequence is of the form $(E_1, ..., E_{n-1}, F)$.

We have to show that $s = 1$. Assume contrary that F is not simple and denote by S the socle of F. We claim that $(E_1, ..., E_{n-1}, S)$ is an exceptional sequence, too. Indeed, we have $\mathrm{Ext}(S, E_i) = 0$ for $1 \leq i \leq n-1$, because the embedding $S \hookrightarrow F$ induces epimorphisms $\mathrm{Ext}(F, E_i) \twoheadrightarrow \mathrm{Ext}(S, E_i)$ and the first Ext-group vanishes by assumption. On the other hand, $\mathrm{Hom}(S, E_i) = 0$ for $1 \leq i \leq n-1$, because the existence of a nonzero morphism from S to some E_i implies that E_i also has finite length, and equals therefore ${}^{[r]}S$, for some r, the unique indecomposable finite length sheaf with socle S and length r. Then $r \geq s$ by minimality of s. But this implies $\mathrm{Hom}(F, E_i) \neq 0$, contrary to the fact that $(E_1, ..., E_{n-1}, F)$ is an exceptional sequence. Thus we have two exceptional sequences which coincide in the first $n-1$ terms but are different in the last one. By Lemma 3.2 this is impossible. □

3.6 Proof of Theorem 1.1 The proof is by induction on the rank n of $\mathrm{K}_0(\mathbb{X})$ and it is similar to the arguments in [15, 4.3.9]. □

4 The strong global dimension of coh($\mathbb{X}$)

The strong global dimension of a finite dimensional algebra A was introduced by Skowroński and is by definition the maximum of the width of indecomposable bounded differential complexes of finite dimensional projective A-modules (see [20]). The strong global dimension of A will be denoted by s.gl.dim. A. Happel-Zacharia have shown that s.gl.dim. $A < \infty$ if and only if A is piecewise hereditary. An algebra A is said to be piecewise hereditary, if there exists a hereditary, abelian category $\mathcal{H}$, such that the bounded derived categories $\mathcal{D}^b(A)$ and

$\mathcal{D}^b(\mathcal{H})$ are equivalent, as triangulated categories. The categories $\mathcal{H}$ occurring in this situation have been described by Happel and Happel-Reiten in [11, 12]. They proved the following: $\mathcal{H}$ is derived equivalent to mod H for some finite dimensional hereditary k-algebra H or derived equivalent to coh$(\mathbb{X})$ for some weighted projective line $\mathbb{X}$.

Hence, when $\mathcal{D}^b(A) \simeq \mathcal{D}^b(\mathcal{H})$, then there exists a tilting object $T \in \mathcal{D}^b(\mathcal{H})$ such that $A \simeq \mathrm{End(T)}^{\mathrm{op}}$ as k-algebras. In particular, there exists a $\ell \in \mathbb{N}$ such that T lies in the additive closure $\bigvee_{i=0}^{\ell} \mathcal{H}[i]$ of the union $\bigcup_{i=0}^{\ell} \mathcal{H}[i]$. If A is a piecewise hereditary algebra which is not an hereditary algebra, then there exist such a pair $(\mathcal{H}, \ell)$ such that s.gl.dim. $A = \ell + 2$ (see [2]). An upper bound of the strong global dimension was found by Happel-Zacharia. It is rk $\mathrm{K}_0(A) + 1$ where rk $\mathrm{K}_0(A)$ is the rank of the Grothendieck group of the algebra (see [13]). Therefore, it is natural define the maximum of the strong global dimension.

We point out that the bound given in [13] is not, in general optimal. For example, if $\mathcal{D}^b(A) \simeq \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ and $\mathbb{X}$ has weight type $(2, 2, \dots 2)$, (t entries) then s.gl.dim. $A \leq 3$ (see [16]). For tubular weighted projective lines $\mathbb{X}$, a more detailed analysis provides better bounds of the strong global dimension. For example, if $\mathbb{X}$ has weight type $(3, 3, 3)$, $(2, 4, 4)$ or $(2, 3, 6)$ then, s.gl.dim. $A \leq 6$, 8 and 9 respectively (see [19]).

Definition 4.1 Let $\mathbb{X}(\mathbf{p}, \lambda)$ be a weighted projective line. The strongest global dimension is the maximum of the strong global dimension of all algebras which are derived equivalent to coh$(\mathbb{X})$. The strongest global dimension of $\mathbb{X}$ will be denoted by st.gl.dim. $\mathbb{X}$.

As an application of the transitivity of the braid group action in [16] it was shown that several data are independent of the parameters of the weighted projective line. Here we are going to study the strongest global dimension of weighted projective lines.

We have the following characterization of the strongest global dimension.

Proposition 4.2 *The strongest global dimension of a weighted projective line $\mathbb{X}$ is one if $\mathbb{X} = \mathbb{P}^1$ or is the maximal number $m+2$ such that there exists a tilting complex T in the derived category of* coh$(\mathbb{X})$ *of the form $\bigoplus_{i=0}^{m} T_i[i]$ with $T_i \in \mathrm{coh}(\mathbb{X})$, $i \in \mathbb{Z}$ and $T_0 \neq 0 \neq T_m$.*

Proof. See Theorem 1 in [2]. □

It follows from the definition that if the bounded derived category of an algebra A is triangular equivalent to the bounded derived category of coh$(\mathbb{X})$, then the s.gl.dim. $A \leq$ st.gl.dim. $\mathbb{X}$.

Before the main theorem in this section we have the following remarks and facts. Recall that, if (A, B) is an exceptional pair in coh$(\mathbb{X})$, then $\mathrm{Hom}_{\mathbb{X}}(A, B) = 0$ or $\mathrm{Ext}^1(A, B) = 0$. First we assume that $\mathrm{Hom}(A, B) \neq 0$. We have then two cases for the left mutation to consider:

$$(\alpha) : 0 \longrightarrow \mathrm{L}_A B \longrightarrow \mathrm{Hom}(A, B) \otimes_k A \xrightarrow{\text{can}} B \longrightarrow 0,$$

$$(\beta) : 0 \longrightarrow \mathrm{Hom}(A, B) \otimes_k A \xrightarrow{\text{can}} B \longrightarrow \mathrm{L}_A B \longrightarrow 0.$$

It is important to note that the surjectivity of the canonical map depends only on rk (A), rk (B) and on the dimension of the spaces $\mathrm{Hom}_{\mathbb{X}}(A, B)$. We have that

- if rk $(A) \neq 0$ then
  can is surjective $\Longleftrightarrow \dim_k \mathrm{Hom}(A, B) \cdot \mathrm{rk}\,(A) > \mathrm{rk}\,(B)$.
- if rk $(A) = 0$
  can is surjective $\Longleftrightarrow \dim_k \mathrm{Hom}(A, B) \cdot \dim_k A > \dim_k B$.

Applying $\mathrm{Hom}_{\mathbb{X}}(\ ,A)$ in (α) we have $\mathrm{Hom}_{\mathbb{X}}(\alpha, A)$:

$$\begin{aligned}0\longrightarrow &\mathrm{Hom}(B,A) \longrightarrow \mathrm{Hom}(\mathrm{Hom}(A,B)\otimes_k A, A) \longrightarrow \mathrm{Hom}(\mathrm{L}_A B, A)\longrightarrow\\ \longrightarrow &\mathrm{Ext}^1(B,A) \longrightarrow \mathrm{Ext}^1(\mathrm{Hom}(A,B)\otimes_k A, A) \longrightarrow \mathrm{Ext}^1(\mathrm{L}_A B, A)\longrightarrow 0\end{aligned}$$

Applying $\mathrm{Hom}_{\mathbb{X}}(\ ,A)$ in (β) we have $\mathrm{Hom}_{\mathbb{X}}(\beta, A)$:

$$\begin{aligned}0\longrightarrow &\mathrm{Hom}(\mathrm{L}_A B,A) \longrightarrow \mathrm{Hom}(B, A) \longrightarrow \mathrm{Hom}(\mathrm{Hom}(A,B)\otimes_k A, A)\longrightarrow\\ \longrightarrow &\mathrm{Ext}^1(\mathrm{L}_A B,A) \longrightarrow \mathrm{Ext}^1(B, A) \longrightarrow \mathrm{Ext}^1(\mathrm{Hom}(A,B)\otimes_k A, A)\longrightarrow\end{aligned}$$

Now, the following remarks follows from both long exact sequences:

Remark 4.3 The mutation of (A, B) is the exceptional pair $(\mathrm{L}_A B, A)$ and:

- In the case (α), the conditions $\mathrm{Hom}_{\mathbb{X}}(B, A) = 0 = \mathrm{Ext}^1(B, A)$ imply that $\dim_k\mathrm{Hom}(\mathrm{L}_A B, A) = \dim_k\mathrm{Hom}(A, B)$ and $\dim_k\mathrm{Ext}^1(\mathrm{L}_A B, A) = 0$.
- In the case (β), the conditions $\mathrm{Hom}_{\mathbb{X}}(B, A) = 0 = \mathrm{Ext}^1(B, A)$ imply that $\dim_k\mathrm{Hom}(\mathrm{L}_A B, A) = 0$ and $\dim_k\mathrm{Ext}^1(\mathrm{L}_A B, A) = \dim_k\mathrm{Hom}(A, B)$.

Now assume that $\mathrm{Ext}(A, B) \neq 0$. Then we have the universal extension

$$(\gamma) : 0\longrightarrow B\longrightarrow \mathrm{L}_A B\longrightarrow \mathrm{Ext}(A, B)\otimes_k A\longrightarrow 0.$$

Applying $\mathrm{Hom}_{\mathbb{X}}(\ ,A)$ in (γ) we have $\mathrm{Hom}_{\mathbb{X}}(\gamma, A)$:

$$\begin{aligned}0\longrightarrow &\mathrm{Hom}(\mathrm{Ext}(A,B)\otimes_k A, A) \longrightarrow \mathrm{Hom}(\mathrm{L}_A B, A) \longrightarrow \mathrm{Hom}(B, A)\longrightarrow\\ \longrightarrow &\mathrm{Ext}^1(\mathrm{Ext}(A,B)\otimes_k A, A) \longrightarrow \mathrm{Ext}^1(\mathrm{L}_A B, A) \longrightarrow \mathrm{Ext}^1(B, A)\longrightarrow\ 0.\end{aligned}$$

The mutation of (A, B) is the exceptional pair $(\mathrm{L}_A B, A)$ and:

- In the case (γ), the conditions $\mathrm{Hom}_{\mathbb{X}}(B, A) = 0 = \mathrm{Ext}^1(B, A)$ imply that $\dim_k\mathrm{Hom}(\mathrm{L}_A B, A) = \dim_k\mathrm{Ext}^1_{\mathbb{X}}(A, B)$ and $\dim_k\mathrm{Ext}^1(\mathrm{L}_A B, A) = 0$.

Summarizing, from (4.3) if (A, B) is an exceptional pair, then on the mutation pair $(\mathrm{L}_A B, A)$ we can compute the dimensions $\dim_k\mathrm{Hom}((\mathrm{L}_A B, A)$, $\dim_k\mathrm{Ext}^1(\mathrm{L}_A B, A)$, and $\mathrm{rk}\,(\mathrm{L}_A B)$ without using the parameters λ.

Lemma 4.4 *Let* $\epsilon = (E_1, \ldots, E_n)$ *be a complete exceptional sequences in* $\mathrm{coh}(\mathbb{X})$ *and* σ *be the generator of the braid group* B_n *such that* $\sigma\cdot\epsilon = (E_1, \ldots, E_{l-1}, \mathrm{L}E_{l+1}, E_l, E_{l+2}, \ldots, E_n)$, *where we write shortly* $\mathrm{L}E_{l+1}$ *instead of* $\mathrm{L}_{E_l}E_{l+1}$. *Then the respective dimensions* $\dim_k\mathrm{Hom}(E_i, \mathrm{L}E_{l+1})$, $\dim_k\mathrm{Ext}^1(E_i, \mathrm{L}E_{l+1})$ *for* $1 \le i \le l-1$, $\dim_k\mathrm{Hom}_{\mathbb{X}}(\mathrm{L}E_{l+1}, E_i)$, $\dim_k\mathrm{Ext}^1(\mathrm{L}E_{l+1}, E_i)$ *for* $i \in \{l, l+2, \ldots, n\}$ *and* $\mathrm{rk}\,(\mathrm{L}E_{l+1})$ *depend only on the dimensions of the* Hom, Ext^1 *and the ranks of the elements in* ϵ.

Proof In Remark 4.3 we have seen that the dimensions $\dim_k\mathrm{Hom}(\mathrm{L}E_{l+1}, E_l)$, $\dim_k\mathrm{Ext}^1(\mathrm{L}E_{l+1}, E_l)$ depend only of the dimension of $\mathrm{Hom}_{\mathbb{X}}(E_l, E_{l+1})$ or $\mathrm{Ext}^1(E_l, E_{l+1})$. Now we will prove the claim for $\dim_k\mathrm{Hom}(E_j, \mathrm{L}E_{l+1})$, $\dim_k\mathrm{Ext}^1(E_j, \mathrm{L}E_{l+1})$ for $1 \le j \le l-1$.

Suppose that the mutation is given by type (α), then we have the exact sequence

$$0\longrightarrow \mathrm{L}E_{l+1}\longrightarrow \mathrm{Hom}(E_l, E_{l+1})\otimes_k E_l \xrightarrow{\text{can}} E_{l+1}\longrightarrow 0,$$

which induces a long exact sequence

$$\begin{aligned}0\longrightarrow &\mathrm{Hom}(E_j, \mathrm{L}E_{l+1}) \longrightarrow \mathrm{Hom}(E_j, \mathrm{Hom}(E_l, E_{l+1})\otimes_k E_l) \longrightarrow \mathrm{Hom}(E_j, E_{l+1})\longrightarrow\\ \longrightarrow &\mathrm{Ext}^1(E_j, \mathrm{L}E_{l+1}) \longrightarrow \mathrm{Ext}^1(E_j, \mathrm{Hom}(E_l, E_{l+1})\otimes_k E_l) \longrightarrow \mathrm{Ext}^1(E_j, E_{l+1})\longrightarrow 0\end{aligned}$$

for $1 \le j \le l-1$.

Since by [15, Lemma 3.2.4] $\mathrm{Hom}_{\mathbb{X}}(E_j, E_{l+1}) = 0$ or $\mathrm{Ext}^1(E_j, E_{l+1}) = 0$ we have either

$$\dim_k \mathrm{Hom}(E_j, \mathrm{L}E_{l+1}) = \dim_k \mathrm{Hom}(E_j, E_l) \cdot \dim_k \mathrm{Hom}(E_l, E_{l+1})$$

and

$$\dim_k \mathrm{Ext}^1(E_j, \mathrm{L}E_{l+1}) = \dim_k \mathrm{Ext}^1(E_j, E_l) \cdot \dim_k \mathrm{Hom}(E_l, E_{l+1}) - \dim_k \mathrm{Ext}^1(E_j, E_{l+1})$$

or

$$\dim_k \mathrm{Hom}(E_j, E_l) \cdot \dim_k \mathrm{Hom}(E_l, E_{l+1}) + \dim_k \mathrm{Ext}^1(E_j, \mathrm{L}E_{k+1}) =$$

$$\dim_k \mathrm{Hom}(E_j, \mathrm{L}E_{l+1}) + \dim_k \mathrm{Hom}(E_j, E_{l+1}).$$

Since $(E_j, \mathrm{L}E_{l+1})$ is an exceptional pair, we have $\mathrm{Hom}_{\mathbb{X}}(E_j, \mathrm{L}E_{l+1}) = 0$ or $\mathrm{Ext}^1(E_j, \mathrm{L}E_{l+1}) = 0$.

Each one gives us that $\dim_k \mathrm{Hom}(E_j, \mathrm{L}E_{l+1})$ and $\mathrm{Ext}^1(E_j, \mathrm{L}E_{l+1})$ depend only of the dimensions of the Hom, Ext spaces of ϵ.

In the cases that the left mutation is given by type (β) or type (γ) the proof is similar. □

We have as a consequence of the previous discussion the following:

Corollary 4.5 *Suppose that* $\epsilon = (E_1, \ldots, E_n)$ *and* $\epsilon' = (E_1^{'}, \ldots, E_n^{'})$ *are complete exceptional sequences in* $\mathrm{coh}(\mathbb{X})$ *such that the following formulas are valid* $\dim_k \mathrm{Hom}_{\mathbb{X}}(E_j, E_l) = \dim_k \mathrm{Hom}_{\mathbb{X}}(E_j^{'}, E_l^{'})$, $\dim_k \mathrm{Ext}^1_{\mathbb{X}}(E_j, E_l) = \dim_k \mathrm{Ext}^1_{\mathbb{X}}(E_j^{'}, E_l^{'})$ *and* $\mathrm{rk}\,(E_j) = \mathrm{rk}\,(E_j^{'})$ *for all* $1 \le j, l \le n$. *Let* $\sigma \in \mathrm{B}_n$ *and* $\sigma \cdot \epsilon = (F_1, \cdots, F_n)$, $\sigma \cdot \epsilon' = (F_1^{'}, \cdots, F_n^{'})$. *Then* $\dim_k \mathrm{Hom}_{\mathbb{X}}(F_j, F_l) = \dim_k \mathrm{Hom}_{\mathbb{X}}(F_j^{'}, F_l^{'})$, $\dim_k \mathrm{Ext}^1_{\mathbb{X}}(F_j, F_l) = \dim_k \mathrm{Ext}^1_{\mathbb{X}}(F_j^{'}, F_l^{'})$ *and* $\mathrm{rk}\,(F_j) = \mathrm{rk}\,(F_j^{'})$ *for all* $1 \le j, l \le n$. □

Theorem 4.6 *Let* $\mathbb{X} = (\mathbf{p}, \lambda)$ *and* $\mathbb{X}' = (\mathbf{p}, \lambda')$ *be weighted projective lines with the same weight type. Then* $\mathrm{st.gl.dim.}\,\mathbb{X} = \mathrm{st.gl.dim.}\,\mathbb{X}'$.

Proof Let m be maximal such that there exists a tilting complex T of the form $\bigoplus_{i=0}^m T_i[i]$ with $T_i \in \mathrm{coh}(\mathbb{X})$. and $T_0 \neq 0 \neq T_m$. Write $T = \bigoplus E_j[n_j]$ with indecomposable sheaves E_j and $n_j \in \mathbb{Z}$. The E_j can be ordered in such a way that they form a full exceptional sequence ϵ in $\mathrm{coh}(\mathbb{X})$. By Theorem 1.1. there exists a braid group element $\sigma \in \mathrm{B}_n$ such that $\epsilon = \sigma \cdot \kappa$ where $\kappa = (\mathcal{O}_{\mathbb{X}}, \mathcal{O}_{\mathbb{X}}(\vec{x}_1), \ldots, \mathcal{O}_{\mathbb{X}}((p_1-1)\vec{x}_1), \ldots, \mathcal{O}_{\mathbb{X}}(\vec{x}_t), \ldots, \mathcal{O}_{\mathbb{X}}((p_t-1)\vec{x}_t), \mathcal{O}_{\mathbb{X}}(\vec{c}))$ is the exceptional sequence obtained from the canonical tilting sheaf $\bigoplus_{0 \le \vec{x} \le \vec{c}} \mathcal{O}_{\mathbb{X}}(\vec{x})$ on $\mathbb{X}$.

Now the application of the same braid group element σ to the exceptional sequence $\kappa' = (\mathcal{O}_{\mathbb{X}'}, \mathcal{O}_{\mathbb{X}'}(\vec{x}_1), \ldots, \mathcal{O}_{\mathbb{X}'}((p_1-1)\vec{x}_1), \ldots, \mathcal{O}_{\mathbb{X}'}(\vec{x}_t), \ldots, \mathcal{O}_{\mathbb{X}'}((p_t-1)\vec{x}_t), \mathcal{O}_{\mathbb{X}'}(\vec{c}))$ obtained from the canonical tilting sheaf $\bigoplus_{0 \le \vec{x} \le \vec{c}} \mathcal{O}_{\mathbb{X}'}(\vec{x})$ on $\mathbb{X}'$ yields a full exceptional sequence ϵ' for the weighted projective line $\mathbb{X}'$. The sequence ϵ' is constructed from κ' using successively the same kind of mutations as in the construction of ϵ from κ. Therefore the exceptional sheaves E_j' of ϵ' satisfy the same dimension formulas for the Hom and Ext^1 spaces as the exceptional sheaves E_j of ϵ. Hence the E_j' can be shifted in the derived category of $\mathrm{coh}(\mathbb{X}')$ as the E_j which yields a tilting complex $\bigoplus_{i=0}^m T_i'[i]$ with $T_i' \in \mathrm{coh}(\mathbb{X}')$ for $\mathbb{X}'$ and with $T_0' \neq 0 \neq T_m'$. Consequently $\mathrm{st.gl.dim.}\,\mathbb{X} = m \le \mathrm{st.gl.dim.}\,\mathbb{X}'$. By symmetry, $\mathrm{st.gl.dim.}\,\mathbb{X}' \le \mathrm{st.gl.dim.}\,\mathbb{X}$ and consequently $\mathrm{st.gl.dim.}\,\mathbb{X} = \mathrm{st.gl.dim.}\,\mathbb{X}'$. □

Note that from Corollary 4.5 we obtain that the ordinary quivers of the algebras End T and End T' in the former theorem are the same which was already stated in [16].

The former proof also suggests the following:
Conjecture: Let T be a tilting complex of the form $\bigoplus_{i=0}^{m} T_i[i]$ with $T_i \in \operatorname{coh}(\mathbb{X})$ and $T_0 \neq 0 \neq T_m$ and $A = \operatorname{End} T$. The strong global dimension of A does not depend on the parameters.

The validity of this conjecture implies the statement of the Theorem 4.6.

5 Determinants for exceptional sequences

Let $f_1, \dots f_n$ be group homomorphisms defined on the Grothendieck group $\mathrm{K}_0(\mathbb{X})$ of a weighted projective line with values in $\mathbb{Z}$. For a full exceptional sequence $\epsilon = (E_1, \dots, E_n)$ on $\mathbb{X}$ we form the $n \times n$ matrix $M(\epsilon)$ whose coefficient at the place (i, j) equals $f_i(E_j)$ and we consider the determinant of that matrix $\det(M(\epsilon))$. We show next that the determinant of $\det(M(\epsilon))$, is invariant up to sign.

Theorem 5.1 *There exists a constant $c \in k$ such that $\det(M(\epsilon)) = c$ or $-c$ for all full exceptional sequences ϵ in $\operatorname{coh}(\mathbb{X})$.*

Proof We are going to show that the determinant of the matrix does not change if we apply to the exceptional sequence in $\operatorname{coh}(\mathbb{X})$ the left mutation σ_i. For right mutations the proof is analogous.

For a full exceptional sequence $\epsilon = (E_1, E_2, \dots E_n)$ we denote $\dim_k \operatorname{Hom}(E_i, E_{i+1}) = h$ and $\dim_k \operatorname{Ext}^1(E_i, E_{i+1}) = e$. Now, $\sigma_i \cdot \epsilon$ equals $(E_1, \dots, E_{i-1}, \mathrm{L}E_{i+1}, E_i, E_{i+2}, \dots E_n)$ and we have $[\mathrm{L}_{E_i} E_{i+1}] = h[E_i] - [E_{i+1}]$, $[\mathrm{L}_{E_i} E_{i+1}] = [E_{i+1}] - h[E_i]$ or $[\mathrm{L}_{E_i} E_{i+1}] = e[E_i] + [E_{i+1}]$ depending on the type of the left mutation of the pair (E_i, E_{i+1}) (see Section 2). The matrix for the exceptional sequence $\sigma_i \cdot \epsilon$ is obtained from that of ϵ by replacing the values in the i-th column by $f_j(E_{i+1}) - hf_j(E_i)$, $-f_j(E_{i+1}) + hf_j(E_i)$ or $f_j(E_{i+1}) + ef_j(E_i)$, $j = 1, \dots, n$ and by replacing the values in the $i+1$-th column by $f_j(E_i)$, $j = 1, \dots n$. Then the statement follows from the known rules for determinants. □

As an example we can apply the method above to the rank function, the degree function and the $n-2$ Euler forms $\langle -, S_{i,j} \rangle$, $j = 1, \dots p_i - 1$, $i = 1, \dots t$.

Corollary 5.2 *For each full exceptional sequence $\epsilon = (E_1, E_2, \dots, E_n)$ in $\operatorname{coh}(\mathbb{X})$ the determinant of the matrix*

$$M(\epsilon) = \begin{pmatrix} \operatorname{rk} E_1 & \operatorname{rk} E_2 & \operatorname{rk} E_3 & \dots & \operatorname{rk} E_{p_1+1} & \dots & \operatorname{rk} E_n \\ \deg E_1 & \deg E_2 & \deg E_3 & \dots & \deg E_{p_1+1} & \dots & \deg E_n \\ \langle E_1, S_{1,1} \rangle & \langle E_2, S_{1,1} \rangle & \langle E_3, S_{1,1} \rangle & \dots & \langle E_{p_1+1}, S_{1,1} \rangle & \dots & \langle E_n, S_{p_1,1} \rangle \\ \vdots & & & & & & \\ \langle E_1, S_{1,p_1-1} \rangle & \langle E_2, S_{1,p_1-1} \rangle & \langle E_3, S_{1,p_1-1} \rangle & \dots & \langle E_{p_1+1}, S_{1,p_1-1} \rangle & \dots & \langle E_n, S_{1,p_1-1} \rangle \\ \vdots & & & & & & \\ \langle E_1, S_{t,1} \rangle & \langle E_2, S_{t,1} \rangle & \langle E_3, S_{t,1} \rangle & \dots & \langle E_{p_1+1}, S_{t,1} \rangle & \dots & \langle E_n, S_{t,1} \rangle \\ \vdots & & & & & & \\ \langle E_1, S_{t,p_t-1} \rangle & \langle E_2, S_{t,p_t-1} \rangle & \langle E_3, S_{t,p_t-1} \rangle & \dots & \langle E_{p_1+1}, S_{t,p_t-1} \rangle & \dots & \langle E_n, S_{t,p_t-1} \rangle \end{pmatrix}$$

equals p or $-p$. Recall that p denotes the least common multiple of the weights $p_1, \dots, p_t$.

Proof The determinant is easily calculated to be p or $-p$ for the exceptional sequence $(\mathcal{O}, \mathcal{O}(\vec{c}), S_{1,1}, \ldots, S_{1,p_1-1}, \ldots S_{t,1}, \ldots, S_{t,p_1-1})$ using the block structure of the matrix and the fact that $\text{rk}\,\mathcal{O} = \text{rk}\,\mathcal{O}(\vec{c}) = 1$, $\deg \mathcal{O} = 0$ and $\deg \mathcal{O}(\vec{c}) = p$. Then the statement follows from Theorem 5.1. □

Remark. The determinantal equation obtained in the way above can be interpreted as a diophantine equation for the weighted projective line $\mathbb{X}$. Diophantine equations expressed for data in terms of exceptional sequences seems to be typical. So Rudakov showed that the ranks of the vector bundles of an exceptional triple on the projective plane satisfy the Markov equation $X^2 + Y^2 + Z^2 = 3XYZ$ [18]. Diophantine equations for partial tilting sequences on weighted projective lines were given in [16, Chapter 10.2].

Example. For an weighted projective line and an exceptional pair (E_1, E_2) the equation is

$$\text{rk}\,(E_1)\deg\,(E_2) - \text{rk}\,(E_2)\deg\,(E_1) = 1.$$

For an exeptional triple (E_1, E_2, E_3) on a weighted projective line of type (2) the equation is

$$\text{rk}\,(E_1)\deg\,(E_2)\langle E_3, S\rangle + \text{rk}\,(E_2)\deg\,(E_3)\langle E_1, S\rangle + \text{rk}\,(E_3)\deg\,(E_1)\langle E_2, S\rangle$$

$$-\text{rk}\,(E_3)\deg\,(E_2)\langle E_1, S\rangle - \text{rk}\,(E_1)\deg\,(E_3)\langle E_2, S\rangle - \text{rk}\,(E_2)\deg\,(E_1)\langle E_3, S\rangle = 2.$$

Here S denotes simple exceptional sheaf of finite length.

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