Hydrodynamic attractor and the fate of perturbative expansions in Gubser flow

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Perturbative expansions, such as the well-known gradient series and the recently proposed slow-roll expansion, have been recently used to investigate the emergence of hydrodynamic behavior in systems undergoing Bjorken flow. In this paper we determine for the first time the large order behavior of these perturbative expansions in relativistic hydrodynamics in the case of Gubser flow. While both series diverge, the slow-roll series can provide a much better overall description of the system's dynamics than the gradient expansion when both series are truncated at low orders. The truncated slow-roll series can also describe the attractor solution of Gubser flow as long as the system is sufficiently close to equilibrium near the origin (i.e., $\rho = 0$) in $dS_3 \otimes \mathbb{R}$. Differently than the case of Bjorken flow, here we show that the Gubser flow attractor solution is not solely a function of the effective Knudsen number $\tau_R \sqrt{\sigma_{\mu\nu}\sigma^{\mu\nu}} \sim \tau_R \tanh \rho$. Our results give further support to the idea that new *resummed* constitutive relations between dissipative currents and the gradients of conserved quantities can emerge in systems far from equilibrium that are beyond the regime of validity of the usual gradient expansion.

I. INTRODUCTION

Hydrodynamic behavior in a many-body system is usually characterized by the existence of constitutive relations between dissipative currents and gradients of conserved quantities [1]. When the gradients are small compared to the system's corresponding microscopic scales, i.e., when the Knudsen number is small, the system is close to local equilibrium and, in principle, dissipative corrections can be taken into account via a systematic expansion in powers of the Knudsen number [1]. Such an expansion, when truncated to first order, leads to the famous nonrelativistic Navier-Stokes (NS) equations [2,3].

The relativistic generalization of these equations was worked out by Eckart [4] and Landau [5] nearly a century ago and, with the advent of the quark-gluon plasma formed in ultrarelativistic heavy ion collisions, viscous relativistic hydrodynamics has become the main tool to describe the spacetime evolution of the hot and dense matter created in these collisions [6,7]. This led to a number of new theoretical approaches, e.g., [8–20] as well as computational/phenomenological developments [21–35] in viscous relativistic hydrodynamics, which have built upon Israel and Stewart's seminal work [36–38] and extended it in a number of ways (for a recent review, see [39]).

However, the extreme energy density and large spatial gradients expected to occur in the initial stages of the quark-gluon plasma formed in the collisions of large nuclei [26], together with the later measurement of large collective behavior also in small collision systems [40], have contributed in part to the question of whether hydrodynamic behavior can also appear when gradients are not small and more terms in the gradient series have to be taken into account. Reference [41] initiated the study of the large order behavior of the gradient series in the field, which was shown to have zero radius of convergence in the case of a strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills plasma undergoing Bjorken expansion [42]. Other examples later followed involving strongly coupled systems with different symmetries in the context of cosmology [43] and also in kinetic theory [44,45] (for a review, see [46]). To extract the properties and meaningfully handle these divergent gradient series, powerful mathematical techniques from resurgence theory (see, e.g., [47]) have been used to resum the large order behavior of the series in systems with a large degree of symmetry [18,43,45,48–51].

Now that it is known that the gradient series diverges and that it is still meaningful to look for hydrodynamic behavior even in situations where gradients are not necessarily small, one may ask if a different mathematical representation, which does not rely on the assumption of small gradients, can be formulated to describe the hydrodynamic regime. The first step in this direction was made in [18] with the proposal that hydrodynamic behavior may be meaningfully defined even far from equilibrium as long as a late time "attractor" structure is present. In this context, NS already

played the role of an attractor since the system, regardless of its initial conditions, approaches this limit when sufficiently close to equilibrium—the novelty of Ref. [18]'s proposal is that this may occur in the far-from-equilibrium regime as well. Several works have since then investigated this in Bjorken flow [52–59] but the question of what happens in systems with less symmetries still remains [55].

In Bjorken flow, an approximation to the attractor solution was obtained [18] using a method analogous to the slow-roll expansion developed in cosmology [60]. In the context of hydrodynamics, a given order in the slow-roll expansion contains derivative terms of all orders, which suggests that such an expansion can be useful in the formulation of hydrodynamics of far-from-equilibrium systems. The large order behavior of the slow-roll expansion was computed for the first time in [57] where it was shown that this series also diverges in Bjorken flow. Since this type of perturbative expansion is fairly recent in hydrodynamic applications, it is useful to check different types of systems and flow profiles where the slow-roll expansion can be systematically implemented to obtain a better understanding of its properties.

Divergent series are known to provide excellent approximations to the solution of several mathematical problems [61] and, thus, the fact that both series diverge is not an issue per se. The relevant question is which one of these divergent series provides the best approximation to the out-of-equilibrium dynamics of the system under consideration after truncation. The detailed analysis performed in [57] for Bjorken flow suggested that the slow-roll expansion may lead to a better overall description of the system's dynamics for a wider range of values of Knudsen number in comparison to the gradient series.

In this work we investigate the fate of both series in a simple system undergoing Gubser flow [62]. This type of flow describes a relativistic fluid that is not only longitudinally boost invariant but also undergoes a radially symmetric expansion in the transverse plane. For simplicity, we follow [18] and consider as our "microscopic" theory the Israel-Stewart formulation of transient fluid dynamics [36–38], whose properties in Gubser flow were first studied in [63]. We explain how to systematically construct the gradient and slow-roll perturbative series in this type of flow and we compute the large order behavior of both series for the first time in Gubser flow. Our results strongly indicate that both series are divergent, as happened for systems expanding following Bjorken flow. However, when comparing low order truncations of both series, we observe that the slow-roll expansion can provide a better overall description of exact solutions in a wider range of parameters in comparison to the gradient expansion. The truncated slow-roll series can also describe the attractor solution of Gubser flow as long as the shear stress tensor approximately vanishes near the origin of the timelike

coordinate (i.e., $\rho = 0$) in $dS_3 \otimes \mathbb{R}$. Differently than the case of Bjorken flow, we find that the Gubser flow attractor solution is not solely a function of the effective Knudsen number $\tau_R \sqrt{\sigma_{\mu\nu}\sigma^{\mu\nu}} \sim \tau_R \tanh\rho$. Our results support the idea that a new type of constitutive relations between dissipative currents (e.g., the shear stress tensor) and the gradients of conserved quantities can emerge in far-from-equilibrium systems which are, thus, beyond the regime of applicability of the gradient expansion.

This paper is organized as follows. In the next section we define the equations of motion of the Israel-Stewart-like theory considered in this paper while in Sec. III we define what Gubser flow is. Sections IV and V are devoted to explaining how to perform the same gradient series in two different ways while in Sec. VI we discuss a complementary perturbative series in powers of the inverse relaxation time. We develop the slow-roll expansion in Sec. VII and investigate the hydrodynamic attractor of Gubser flow in Sec. VIII. Our final remarks can be found in Sec. IX. The Appendix gives yet another way to develop the gradient series in Gubser flow.

Definitions: We use a mostly minus metric signature and natural units, $\hbar = c = k_B = 1$.

II. CONFORMAL ISRAEL-STEWART THEORY

Excluding the contribution from conserved charges, the main fluid-dynamical equations are the continuity equations related to the conservation of energy and momentum,

$$\nabla_\mu T^{\mu\nu} = 0. \tag{1}$$

The field $T^{\mu\nu}$ introduced above is the energy-momentum tensor, which is commonly decomposed in terms of the local fluid velocity u^μ as

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu} P + \pi^{\mu\nu}, \tag{2}$$

where ε is the energy density, P is the thermodynamic pressure and $\pi^{\mu\nu}$ is the shear stress tensor. The fluid velocity is constructed to be a normalized 4-vector, $u_\mu u^\mu = 1$, defined according to Landau's picture [5], $T^{\mu\nu}u_\nu \equiv \varepsilon u^\mu$, as an eigenvector of the energy-momentum tensor. Note that we also introduced the projection operator transverse to u^μ, $\Delta_{\mu\nu} \equiv g_{\mu\nu} - u_\mu u_\nu$, with $g_{\mu\nu}$ being the spacetime metric. In this paper, we only consider the dynamics of conformal fluids [9] and, consequently, there is no bulk viscous pressure and the equation of state of the fluid is given by $\varepsilon = 3P$ (i.e., the trace of $T^{\mu\nu}$ vanishes).

The conservation laws alone do not describe all the degrees of freedom of $T^{\mu\nu}$. They must be complemented by additional dynamical equations (or constitutive relations) that describe the time evolution of the shear stress tensor. For this purpose, we employ the transient hydrodynamic equations derived by Israel and Stewart [36–38] from kinetic theory (and later complemented by several

authors [64–73]). In this framework, the shear stress tensor satisfies the following relaxation-type equation,

$$\tau_R \Delta^{\mu\nu}_{\alpha\beta} D\pi^{\alpha\beta} + \delta_{\pi\pi}\Theta\pi^{\mu\nu} + \tau_{\pi\pi}\Delta^{\mu\nu}_{\alpha\beta}\pi^{\alpha\lambda}\sigma^\beta_\lambda - 2\tau_R\Delta^{\mu\nu}_{\alpha\beta}\pi^\alpha_\lambda\omega^{\beta\lambda}$$
$$+ \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}, \tag{3}$$

where $D = u^\mu\nabla_\mu$ is the comoving covariant derivative, $\Theta = \nabla_\mu u^\mu$ is the local expansion rate, $\sigma_{\mu\nu} = \Delta^{\alpha\beta}_{\mu\nu}\nabla_\alpha u_\beta$ is the shear tensor with $\Delta^{\alpha\beta}_{\mu\nu} = \frac{1}{2}(\Delta^\alpha_\mu\Delta^\beta_\nu + \Delta^\alpha_\nu\Delta^\beta_\mu - \frac{2}{3}\Delta^{\alpha\beta}\Delta_{\mu\nu})$, $\omega_{\mu\nu} = (\Delta^\lambda_\mu\nabla_\lambda u_\nu - \Delta^\lambda_\nu\nabla_\lambda u_\mu)/2$ is the vorticity tensor, η is the shear viscosity, and τ_R is the shear relaxation time.

The equations above may be derived from the Boltzmann equation using the 14-moment approximation or the relaxation time approximation, as shown in Refs. [14,67,69]. For a massless gas in the 14-moment approximation, it was demonstrated that $\delta_{\pi\pi} = 4/3\tau_R$, $\tau_{\pi\pi} = 10/21\tau_R$ and $\eta = (\varepsilon + P)\tau_R/5$. Since we are dealing with a conformal fluid, the shear relaxation time must be inversely proportional to the temperature

$$\tau_R = \frac{c}{T}, \tag{4}$$

where c is a constant that determines the magnitude of the shear viscosity to entropy density ratio, $\eta/s = c/5$. For the sake of simplicity, we neglect the transport coefficient $\tau_{\pi\pi}$ in this work. Furthermore, the nonlinear term that contains the vorticity vanishes in the flow profile investigated in this paper (see next section) and does not have to be considered as well. This leads to the so-called (simplified) conformal Israel-Stewart equations [63]

$$\Delta^{\mu\nu}_{\alpha\beta} D\pi^{\alpha\beta} + \frac{4}{3}\pi^{\mu\nu}\Theta + \frac{\pi^{\mu\nu}}{\tau_R} = 2\frac{\eta}{\tau_R}\sigma^{\mu\nu}. \tag{5}$$

III. GUBSER FLOW

Following [62], we look for solutions of the hydrodynamic equations with $SO(3) \otimes SU(1,1) \otimes Z_2$ symmetry in flat spacetime. This is more naturally implemented by performing a Weyl transformation to $dS_3 \otimes \mathbb{R}$ spacetime (where dS_3 stands for the three-dimensional de Sitter spacetime [74]) and assuming that the fluid is homogeneous in this curved geometry. This spacetime is described by the line element [62]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = d\rho^2$$
$$- (\cosh^2\rho\, d\theta^2 + \cosh^2\rho\sin^2\theta d\phi^2 + d\eta^2). \tag{6}$$

The nonzero Christoffel symbols of this metric are $\Gamma^\rho_{\theta\theta} = \cosh\rho\sinh\rho$, $\Gamma^\rho_{\phi\phi} = (\sin\theta)^2\cosh\rho\sinh\rho$, $\Gamma^\theta_{\rho\theta} = \Gamma^\phi_{\rho\phi} = \tanh\rho$, $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$, $\Gamma^\phi_{\theta\phi} = (\tan\theta)^{-1}$ and its determinant is $\sqrt{-g} = \sin\theta(\cosh\rho)^2$. Since the system is homogeneous,

all fields depend only on the timelike variable ρ, without displaying any dependence on θ, ϕ, and η. If we transform back to Minkowski spacetime in hyperbolic coordinates (τ, r, ϕ, η), where the line element is $ds^2 = d\tau^2 - dr^2 - r^2 d\phi^2 - \tau^2 d\eta$ [62], this homogeneous system is mapped into a longitudinally boost invariant fluid whose expansion in the transverse plane is radially symmetric. In flat spacetime, this type of fluid displays a more complex pattern of expansion in comparison to the Bjorken solutions [42] considered in previous works, e.g., [18,41,44,50,57] where no radial expansion is allowed.

The assumption that the system is homogeneous in $dS_3 \otimes \mathbb{R}$ leads to a trivial velocity field, $u^\mu = (1,0,0,0)$, which automatically satisfies the momentum conservation continuity equations. Nevertheless, in this curved spacetime, the fluid still has a nonzero expansion rate and shear tensor, which are given by

$$\Theta \equiv \nabla_\mu u^\mu = 2\tanh\rho, \tag{7}$$

$$\sigma_{\mu\nu} \equiv \frac{1}{2}\Delta^\alpha_\mu\Delta^\beta_\nu(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - \frac{1}{3}\Delta_{\mu\nu}\Theta \tag{8}$$

$$= \mathrm{diag}(0, g_{\theta\theta}, g_{\phi\phi}, -2g_{\eta\eta}) \times \frac{1}{3}\tanh\rho, \tag{9}$$

where $\nabla_\alpha u_\beta = \partial_\alpha u_\beta - \Gamma^\lambda_{\alpha\beta}u_\lambda$ is the covariant derivative of the flow velocity. As already stated, the vorticity tensor vanishes for this flow profile, $\omega^{\mu\nu} = 0$, and plays no role in the results obtained in this paper. Since the shear tensor is diagonal, the shear stress tensor will also be diagonal, $\pi^{\mu\nu} = \mathrm{diag}(0, \pi^{\theta\theta}, \pi^{\phi\phi}, \pi^{\eta\eta})$, as long as it is initially diagonal. The equation of motion for the energy density, ε, then becomes

$$u_\nu\nabla_\mu T^{\mu\nu} = \frac{d\varepsilon}{d\rho} + \frac{8}{3}\varepsilon\tanh\rho - \frac{1}{3}\pi^{\eta\eta}\tanh\rho = 0,$$

where we used that $\pi^{\mu\nu}\sigma_{\mu\nu} = \pi^{\eta\eta}\tanh\rho$. The equations of motion for $\pi^{\mu\nu}$ are

$$\frac{d\pi^\theta_\theta}{d\rho} + \frac{8}{3}\pi^\theta_\theta\tanh\rho + \frac{\pi^\theta_\theta}{\tau_R} = \frac{2\eta}{3\tau_R}\tanh\rho, \tag{10}$$

$$\frac{d\pi^\phi_\phi}{d\rho} + \frac{8}{3}\pi^\phi_\phi\tanh\rho + \frac{\pi^\phi_\phi}{\tau_R} = \frac{2\eta}{3\tau_R}\tanh\rho, \tag{11}$$

$$\frac{d\pi^\eta_\eta}{d\rho} + \frac{8}{3}\pi^\eta_\eta\tanh\rho + \frac{\pi^\eta_\eta}{\tau_R} = -\frac{4\eta}{3\tau_R}\tanh\rho. \tag{12}$$

For the purposes of this paper, it is convenient to reexpress these equations in terms of the temperature, T (the system is conformal, $\varepsilon \sim T^4$), and a variable π defined as

$$\pi \equiv -\frac{\pi_\eta^\eta}{\varepsilon + P}. \tag{13}$$

With this choice of variables, the simplified conformal hydrodynamical equations of Israel-Stewart theory become [63]

$$\frac{1}{T}\frac{dT}{d\rho} + \frac{2}{3}\tanh\rho - \frac{1}{3}\pi\tanh\rho = 0, \tag{14}$$

$$\frac{d\pi}{d\rho} + \frac{\pi}{\tau_R} + \frac{4}{3}\pi^2\tanh\rho = \frac{4}{15}\tanh\rho. \tag{15}$$

These are the equations that we investigate throughout this paper.

IV. GRADIENT EXPANSION

We develop our calculations following the procedure outlined in Ref. [57]. We convert the original problem into a perturbation theory problem by introducing a dimensionless parameter ϵ into the differential equation satisfied by π as follows,

$$\epsilon\frac{d\pi}{d\rho} + \frac{\pi T}{c} + \frac{4}{3}\pi^2\epsilon\tanh\rho = \frac{4}{15}\epsilon\tanh\rho, \tag{16}$$

where we already used that $\tau_R = c/T$. Now, in this problem the shear stress tensor and the temperature are functions of ρ and the new variable ϵ, i.e., $T = T(\rho, \epsilon)$ and $\pi = \pi(\rho, \epsilon)$. The parameter ϵ was introduced in every term which contains a derivative or $\tanh\rho$ and, hence, it becomes a book-keeping parameter to count orders or powers of gradients. Thus, an expansion in powers of ϵ naturally leads to an expansion in powers of gradients. Note that while in Bjorken scaling it was straightforward to deduce that powers of gradients correspond to inverse powers of the time coordinate τ [57], here the situation is not that simple and the corresponding terms are obtained from the perturbative procedure itself.

Next, we look for a solution for π that can be represented as a series in powers of ϵ,

$$\pi \sim \sum_{n=0}^{\infty} \pi_n(\rho)\epsilon^n. \tag{17}$$

This reduces the problem to solving an infinite number of simpler equations (in this case, algebraic equations), which are given by recurrence relations. At the end of the calculation, one sets $\epsilon = 1$ to recover the parameters of the original problem. We note that such perturbative procedure is only useful when the first few terms of the series contain at least some basic properties of the exact solution, i.e., if a low order truncation of the series can capture basic trends of the solution. This does not necessarily mean that the series must converge. As a matter of fact, in practice convergent series quite often do not offer useful representations of functions since they may be slowly convergent

and require a great number of terms to provide a good approximation in a given domain [61]. On the other hand, truncations of divergent series are known to provide very good approximations of certain functions (as in the case of the error function).

Naturally, this procedure does not lead to a general solution of Israel-Stewart theory since the equations obtained in this type of perturbative approach do not contain any free parameter associated with the initial condition for π (the initial condition for the temperature remains a free parameter, even in the perturbative problem). In this sense, what is obtained with this approach is just one solution for π that cannot be adjusted to an arbitrary boundary condition. However, this solution is expected to have physical meaning, reflecting the long-time, slow evolution of the system when all transient, initial-state dynamics is lost and the system enters a universal, hydrodynamical regime (assumed in this section to be described by the gradient expansion).

We now continue the perturbative calculation, substituting the proposed series solutions in powers of ϵ into the equations of motion for π. One then obtains the following result:

$$c\sum_{n=0}^{\infty}\frac{d\pi_n}{d\rho}\epsilon^{n+1} + T\sum_{n=0}^{\infty}\pi_n\epsilon^n$$
$$+ \frac{4}{3}c\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\pi_n\pi_m\epsilon^{n+m+1}\tanh\rho = \frac{4}{15}c\epsilon\tanh\rho. \tag{18}$$

The problem in solving this equation, i.e., in collecting all terms that are of the same power in ϵ, is that the term $d\pi_n/d\rho$ also has an ϵ-dependence that must be considered when grouping the terms. This can be taken into account by noticing that the coefficient π_n depends on ρ through two different variables, $\pi_n = \pi_n(\tanh\rho, T(\rho))$ and, thus, its derivative can be mathematically reexpressed in the following way:

$$\frac{d\pi_n}{d\rho} = \left.\frac{\partial\pi_n}{\partial\rho}\right|_T + \left.\frac{\partial\pi_n}{\partial T}\right|_{\tanh\rho}\frac{dT}{d\rho}. \tag{19}$$

Above, the derivatives are taken as if $\tanh\rho$ and T were two independent variables. It is the temperature derivative that carries the ϵ-dependence and, thus, using Eq. (14) we can rewrite Eq. (18) as

$$c\sum_{n=0}^{\infty}\left(\left.\frac{\partial\pi_n}{\partial\rho}\right|_T - \frac{2}{3}T\tanh\rho\left.\frac{\partial\pi_n}{\partial T}\right|_{\tanh\rho}\right)\epsilon^{n+1}$$
$$+ \frac{c}{3}\tanh\rho\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\pi_m T\left.\frac{\partial\pi_n}{\partial T}\right|_{\tanh\rho}\epsilon^{n+m+1} + T\sum_{n=0}^{\infty}\pi_n\epsilon^n$$
$$+ \frac{4}{3}c\sum_{n=0}^{\infty}\pi_n\pi_m\epsilon^{n+m+1}\tanh\rho - \frac{4}{15}c\epsilon\tanh\rho = 0. \tag{20}$$

Now that the terms are properly organized in powers of ϵ we can group together the terms that are of the same order and obtain the set of recurrence relations that must be solved to obtain π_n. The zeroth order term must satisfy

$$T\pi_0 = 0, \tag{21}$$

which describes a system that is in local equilibrium, as expected. Collecting the terms that are of first order in ϵ one obtains

$$\pi_1(\rho) = \frac{4}{15}\tau_R \tanh\rho, \tag{22}$$

which corresponds to relativistic NS theory. This expression also reflects the fact that the shear stress tensor in NS theory is linear in the Knudsen number $K_N \sim \tau_R \sqrt{\sigma_{\mu\nu}\sigma^{\mu\nu}}$ for Gubser flow. Finally, the terms that are of second order or higher in ϵ, ($n \geq 2$), satisfy the equations

$$\frac{T}{c}\pi_{n+1} = -\frac{4}{3}\sum_{m=0}^{n}\pi_{n-m}\pi_m \tanh\rho$$
$$-\frac{1}{3}\tanh\rho \sum_{m=0}^{n}\pi_m T\frac{\partial\pi_{n-m}}{\partial T}\bigg|_{\tanh\rho}$$
$$+\left(\frac{2}{3}\tanh\rho\right)T\frac{\partial\pi_n}{\partial T}\bigg|_{\tanh\rho} - \frac{\partial\pi_n}{\partial\rho}\bigg|_T. \tag{23}$$

Since we already know π_0 and π_1, we can calculate all the π_n's that follow. For the sake of completeness, we write down below the answer up to third order,

$$\pi = \frac{4}{15}\epsilon\tau_R \tanh\rho - \frac{4}{15}(\epsilon\tau_R)^2\left(1 - \frac{1}{3}\tanh^2\rho\right)$$
$$+\frac{8}{45}(\epsilon\tau_R)^3\left(\tanh\rho - \frac{1}{15}\tanh^3\rho\right) + \mathcal{O}(\epsilon^4). \tag{24}$$

Therefore, we can write the solution as a series in powers of $\epsilon\tau_R$, with all of the temperature contribution to the shear stress tensor contained in the relaxation time. Also, we note that while the NS result depends solely on the combination $\tau_R \tanh\rho$, the same is not true for the higher order terms in the gradient expansion, which depend separately on τ_R and $\tanh\rho$.

These equations have the form that is traditionally associated with a gradient expansion: the zeroth and first order truncations obtained in this section correspond to well-known results, ideal hydrodynamics and Navier-Stokes theory, respectively. Both of these examples were already studied extensively in the literature [62]. In the Appendix we derive the gradient expansion using another equivalent method.

We end this section with a comment about the meaning of the gradient expansion used in this work. Hydrodynamics is usually understood as an effective theory constructed via the gradient expansion [1]. In the relativistic regime, the energy-momentum tensor is expanded in a series of gradients of the temperature, flow, and other macroscopic quantities, and this series is characterized by transport coefficients that can be determined by matching this long wavelength description to the underlying microscopic theory. In this language, Israel-Stewart theory can be viewed as a particular example of second order hydrodynamics. Alternatively, the Israel-Stewart equations written above (where the shear stress tensor is a dynamical variable that obeys a differential equation) may be understood as a particular resummation of the gradient expansion that is correct to second order in gradients; see Ref. [9]. Following Refs. [18,52], in this work the same equations are used to define a simple "toy model" microscopic theory for which coarse-grained descriptions can be systematically derived using different approaches such as, for instance, the gradient expansion and also the slow-roll series discussed in Sec. VII.

V. GRADIENT EXPANSION—REVISITED

In the previous section we constructed the gradient expansion solution of Israel-Stewart theory following a perturbative scheme. We demonstrated that the result can be expressed as an expansion in powers of $\epsilon\tau_R$. In this section we construct once again the gradient expansion solution of Israel-Stewart theory, but now using the knowledge that the temperature appears in the series only through powers of the relaxation time, $\tau_R \sim 1/T$. Therefore, it is more straightforward to simply assume an expansion of π just in powers of τ_R from the very beginning (the parameter ϵ can also be included, as before, but it will not make any difference),

$$\pi = \sum_{n=0}^{\infty}\hat{\pi}_n(\rho)\tau_R^n. \tag{25}$$

The calculations become simpler this way (even though less general) as they allow us to determine the large order behavior of the gradient series. In order to avoid confusion with the variables from the previous section, here we change the notation of the expansion coefficients by adding a hat, i.e., $\hat{\pi}_n$. We note that the coordinates in the line element in (6) are dimensionless[1] and so are all the variables computed in $dS_3 \otimes \mathbb{R}$. Therefore, an expansion in powers of τ_R is again an expansion in terms of a dimensionless parameter.

Replacing this expansion in the equation of motion for π leads to

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[1]Naturally, an energy scale (called q in [62]) is introduced when going from $dS_3 \otimes \mathbb{R}$ back to Minkowski. In this paper we set this scale to unity.

$$\sum_{n=0}^{\infty} \tau_R^{n+1} \frac{d\hat{\pi}_n}{d\rho} + \frac{2}{3} \tanh\rho \sum_{n=0}^{\infty} n\tau_R^{n+1}\hat{\pi}_n + \sum_{n=0}^{\infty} \hat{\pi}_n \tau_R^n$$

$$+ \frac{1}{3}\tanh\rho \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (4-n)\hat{\pi}_n\hat{\pi}_m\tau_R^{n+m+1} = \frac{4}{15}\tau_R\tanh\rho. \tag{26}$$

We see that the terms can be grouped together according to their power of τ_R. If we collect all the terms of the same order, we obtain the equations satisfied by each expansion coefficient $\hat{\pi}_n$. We also note that this procedure of collecting powers of τ_R completely removes the temperature from the perturbation theory problem. As expected, the zeroth and first order coefficients satisfy

$$\hat{\pi}_0 = 0, \tag{27}$$

$$\hat{\pi}_1(\rho) = \frac{4}{15}\tanh\rho. \tag{28}$$

The higher order coefficients obey the equations

$$\hat{\pi}_{n+1} + \frac{d\hat{\pi}_n}{d\rho} + \frac{2n}{3}\tanh\rho\hat{\pi}_n$$

$$+ \frac{1}{3}\tanh\rho \sum_{m=0}^{n}(4-n+m)\hat{\pi}_{n-m}\hat{\pi}_m = 0. \tag{29}$$

More specifically, the second order coefficient is given by

$$\hat{\pi}_2(\rho) = -\frac{d\hat{\pi}_1}{d\rho} - \frac{2}{3}\hat{\pi}_1\tanh\rho = -\frac{4}{15} + \frac{4}{45}\tanh^2\rho. \tag{30}$$

We note that the solutions obtained for $\hat{\pi}_1$ and $\hat{\pi}_2$ are exactly the same as the ones obtained in the previous section, provided one multiplies each coefficient by the appropriate power of the relaxation time and sets $\epsilon = 1$ in the results obtained in the previous section. However, the procedure described in this section is much simpler to implement than the one constructed in the previous section. Nevertheless, it is important to remember that the formalism constructed before is more general since it is not always possible to rearrange the problem in terms of a simpler (though equivalent) perturbative expansion. When dealing with more general flow configurations or with more complicated perturbative series (see Sec. VII), one must follow the general procedure outlined in the previous section.

When $\rho \to \infty$ all derivatives of $\tanh\rho$ vanish and the recurrence relations satisfied by $\hat{\pi}_n$ considerably simplify

$$\hat{\pi}_{n+1} + \frac{2n}{3}\hat{\pi}_n + \frac{1}{3}\sum_{m=0}^{n}(4-n+m)\hat{\pi}_{n-m}\hat{\pi}_m = 0. \tag{31}$$

In this case, the coefficients $\hat{\pi}_n$ are just pure numbers that satisfy recurrence relations that are very similar to those
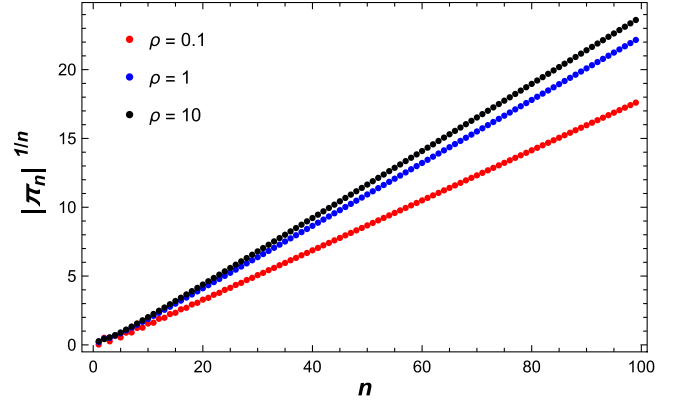

FIG. 1. $|\pi_n|^{1/n}$ as a function of n for $\rho = 0.1$ (red), $\rho = 1$ (bule), and $\rho = 10$ (black).

obtained in the Bjorken case (see, e.g., [57]). In this case, it is straightforward to see that when $n \gg 1$ this expression leads to $\hat{\pi}_n \sim n!$ and, consequently, to a divergent series.

A. Divergence of the gradient expansion in Gubser flow

With the expansion constructed above we are able to determine the large order behavior of the gradient expansion for arbitrary values of ρ. The recurrence relations derived above were solved with Wolfram's *Mathematica*, up to $n = 100$, and plotted in Fig. 1 for $\rho = 0.1$, $\rho = 1$, and $\rho = 10$. In Fig. 2, we also show the corresponding result when $\rho = 0$ and $\rho \to \infty$. We do not plot results for negative values of ρ since the modulus of each coefficient does not depend on the sign of ρ. Both figures clearly display the factorial growth of the coefficients of the series, indicating that the gradient expansion has zero radius of convergence. We also remark that the results vary very little when ρ is changed.

It is interesting to see that when $\rho = 0$ (i.e., when the shear tensor is 0) the gradient expansion still diverges. Moreover, since the first order term in the expansion is
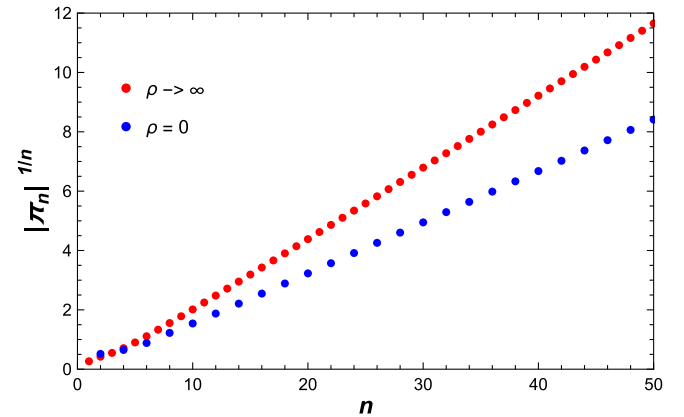

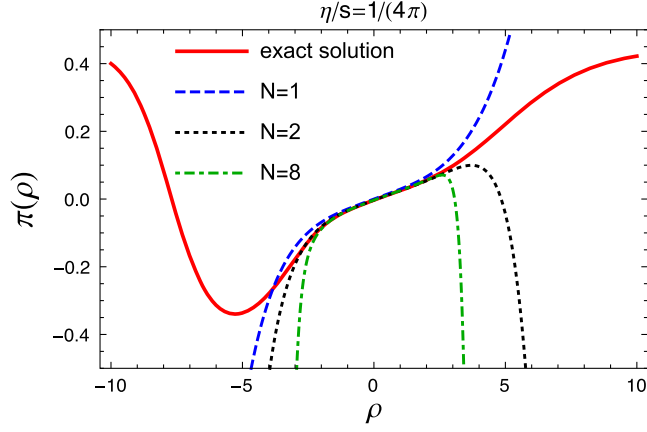FIG. 2. $|\pi_n|^{1/n}$ as a function of n for $\rho = 0$ and $\rho \to \infty$.

FIG. 3. Comparison between the exact result for π defined in (13) obtained by solving Eqs. (14) and (15) and the gradient expansion truncated at different orders. In this plot $\eta/s = 1/(4\pi)$.

FIG. 4. Comparison between the exact result for π defined in (13) obtained by solving Eqs. (14) and (15) and the gradient expansion truncated at different orders. In this plot $\eta/s = 1$.

proportional to $\tanh\rho$, this term vanishes when $\rho = 0$ and, as a matter of fact, one can show that all odd terms in the series vanish in this case. However, the coefficients $\hat{\pi}_{2n}(0)$ do not vanish and these terms alone display factorial growth.

B. Determining the domain of applicability of the gradient expansion

As already mentioned, low order truncations of a divergent expansion can still be used to provide reasonable approximations for solutions of the theory, at least in some domains. In Figs. 3 and 4 we compare several truncations of this divergent series with exact solutions obtained by numerically solving the Israel-Stewart equations (14) and (15), for two values of $\eta/s - \eta/s = 1/(4\pi)$ and 1. The initial conditions for the numerical problem where chosen to be $T(\rho_0) = 0.0057$ and $\pi(\rho_0) = 0.4$, with $\rho_0 = -30$. We remark that our results do not depend strongly on the value chosen for $\pi(\rho_0)$, but they do depend on the choice of $T(\rho_0)$.[2] In this comparison, we take the exact temperature profile $T(\rho)$ and insert it into the corresponding constitutive relations obtained for π using the gradient expansion.

We find that, when the viscosity is small ($\eta/s = 1/4\pi$), the first and second order truncations of the gradient expansion provide a good approximation to the exact solution in a wide region around $\rho = 0$. We remark that the best approximation to the exact solution, for this value of viscosity, is obtained by truncating the expansion at second order, as also happened when performing this analysis assuming Bjorken flow [57]. However, the divergent nature of the series is manifest by the fact that the

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[2]It is straightforward to see from the equations of motion for T and π that rescaling the initial value of the temperature is equivalent to changing the shear viscosity of the system. This happens in such a way that reducing the initial temperature of the system corresponds to effectively increasing the value of η/s.
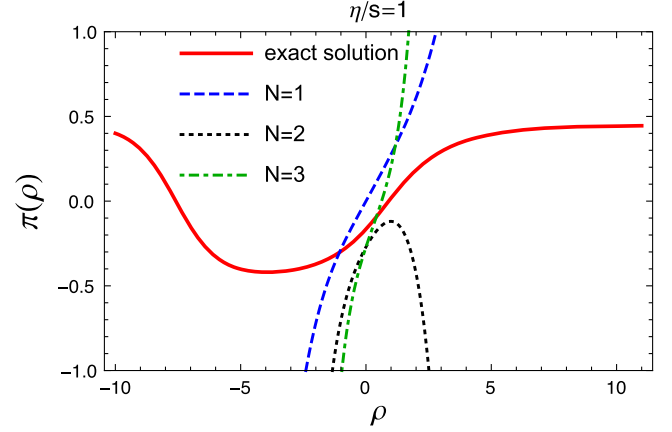
eighth order truncation is significantly worse than the lower orders, indicating that the optimal truncation of the series is indeed at a lower order.

Meanwhile, for a larger value of viscosity, $\eta/s = 1$, none of the different truncations of the gradient series is able to provide a reasonable description of the exact solution. In fact, in this case one even finds that $\pi(0)$ deviates significantly from 0 in the exact solution—a result that is very hard to describe using the gradient expansion, since it implies that the NS limit is not approached at all even when $\rho \approx 0$. In general, our results suggest that truncations of the gradient expansion can provide a good description of solutions of Israel-Stewart theory around $\rho = 0$, though how far in ρ this occurs (or if this occurs at all) clearly depends on the value of η/s.

Furthermore, we remark that the gradient expansion cannot describe the quantitative and qualitative behavior of the solution when $|\rho|$ is very large. In fact, it is known from [63] that the exact solution for π in Israel-Stewart theory asymptotes to a constant when $|\rho| \gg 1$—a result that can never be obtained within a gradient expansion. However, this is not the limit where one would expect the gradient expansion to be useful since in this case all powers of $\tanh\rho$ become of the same order. In fact, in this regime a complementary perturbative expansion must be developed, which is the subject of the next section.

VI. EXPANSION IN POWERS OF THE INVERSE RELAXATION TIME

Previously, we showed that the gradient expansion in Israel-Stewart theory undergoing Gubser flow corresponds to a series in powers of the relaxation time. The large order behavior of this expansion, investigated for the first time in the last section, suggests that it has a zero radius of convergence. From the nature of the equations, it is obvious that if a series in powers of τ_R is possible, then a series in

powers of $1/\tau_R$ should also be possible, though its regime of applicability should be complementary to the former. Such a series is interesting in its own way and here we study the properties of this other type of perturbative expansion. We note that in Sec. 4.2.1 of Ref. [75] a similar series to the one studied in this section was investigated.

Let us now consider again the Israel-Stewart equations and develop an expansion of π in powers of τ_R^{-1},

$$\pi = \sum_{n=0}^{\infty} \tilde{\pi}_n \tau_R^{-n}. \tag{32}$$

Again, we change our notation for the expansion coefficient to $\tilde{\pi}_n$ in order to avoid confusion with the variables employed in the previous sections. Substituting this ansatz into the simplified Israel-Stewart equation for π (15) leads to

$$\sum_{n=0}^{\infty} \tau_R^{-n} \frac{d\tilde{\pi}_n}{d\rho} + \sum_{n=0}^{\infty} \tau_R^{-(n+1)} \tilde{\pi}_n - \frac{2}{3} \tanh \rho \sum_{n=0}^{\infty} \tau_R^{-n} n \tilde{\pi}_n$$
$$+ \frac{1}{3} \tanh \rho \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \tau_R^{-(n+m)} (n+4) \tilde{\pi}_n \tilde{\pi}_m - \frac{4}{15} \tanh \rho = 0. \tag{33}$$

We now follow the same procedure as before and collect the terms that are of the same order in the inverse relaxation time. At zeroth order we find the following differential equation of motion for $\tilde{\pi}_0$:

$$\frac{d\tilde{\pi}_0}{d\rho} + \frac{4}{3} \tilde{\pi}_0^2 \tanh \rho = \frac{4}{15} \tanh \rho.$$

This equation corresponds to the cold plasma limit solution first found and studied in Ref. [63], whose analytical solution is

$$\tilde{\pi}_0(\rho) = \frac{\sqrt{5}}{5} \tanh \left[\frac{\sqrt{5}}{5} \left(\frac{4}{3} \ln \cosh \rho - 5b \right) \right],$$

where b is a free parameter that is fixed by the initial condition chosen for π. Approximating π by this zeroth order solution, one then obtains the following solution for the temperature,

$$T_0(\rho) = a \frac{\exp(5b/2)}{(\cosh \rho)^{2/3}} \cosh^{1/4} \left[\frac{\sqrt{5}}{5} \left(\frac{4}{3} \ln \cosh \rho - 5b \right) \right],$$

where a is a free parameter that fixes the initial temperature.

The equation satisfied by the first order coefficient is

$$\frac{d\tilde{\pi}_1}{d\rho} + \tilde{\pi}_0 + \left(3\tilde{\pi}_0 - \frac{2}{3} \right) \tilde{\pi}_1 \tanh \rho = 0. \tag{34}$$

This equation does not have a simple analytical solution but it can be easily solved numerically. Finally, the equation of motion for $\tilde{\pi}_n$, $n \geq 1$, is

$$\frac{d\tilde{\pi}_n}{d\rho} + \tilde{\pi}_{n-1} - \left(\frac{2n}{3} \tanh \rho \right) \tilde{\pi}_n$$
$$+ \tanh \rho \sum_{m=0}^{n} \frac{n - m + 4}{3} \tilde{\pi}_{n-m} \tilde{\pi}_m = 0.$$

One can see that this type of expansion is qualitatively different than the gradient expansion as it requires solving a differential equation at every order, which makes it harder to determine its large order behavior. In fact, this introduces back into the problem the initial condition for π, which now can be taken into account by the 0th order term of the expansion. Moreover, differently than the gradient expansion, we note that in this case there is no approximate constitutive relation between the shear stress tensor and the hydrodynamic variables.

As this series is defined by powers of the inverse relaxation time, if one keeps the initial condition for the temperature fixed, the regime of applicability of the series is controlled by how large η/s is. In Fig. 5 we set $\eta/s = 100$ and compare the exact solution of the Israel-Stewart equations with the same initial conditions as before to different truncations of this new series in powers of the inverse relaxation time. One can see that the 0th order term already correctly describes the asymptotic regime at very large $|\rho|$, though the agreement worsens for $0 < \rho < 10$. The inclusion of higher order terms improves the agreement near $\rho \sim 0$ though for $5 < \rho < 10$ not even the 4th order truncation can describe the numerical solution.

We finish this section with the remark that in flow situations with less symmetry, the type of series developed in this section basically requires solving a problem as hard
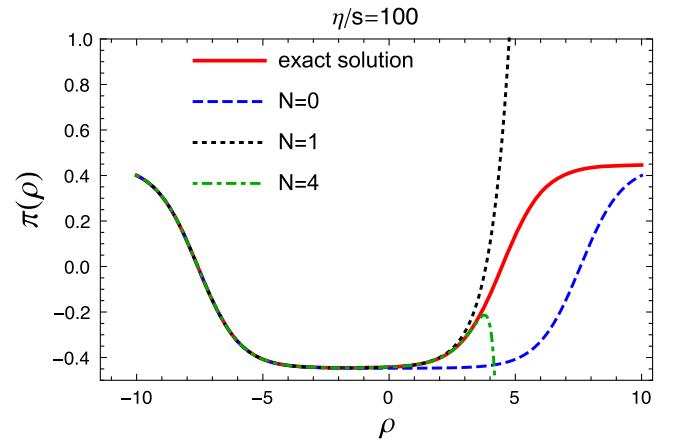
FIG. 5. Comparison between the exact result for π defined in (13) obtained by solving Eqs. (14) and (15) and the series in inverse powers of the relaxation truncated at different orders. In this plot $\eta/s = 100$.

as the original problem (coupled, nonlinear partial differential equations for the shear stress tensor) at each order. Therefore, we expect its use in practical applications to be more limited than the more easily implementable gradient expansion. In the next section we develop a perturbative expansion that can describe not only the $\rho \sim 0$ regime but also the asymptotic values of the solution when $\rho \to \infty$.

VII. SLOW-ROLL EXPANSION

The slow-roll expansion was first used in the context of hydrodynamics in [18] in the case of Bjorken flow. The systematic implementation of this series in that problem was discussed in [54] and later in [57]. The slow-roll expansion in Gubser flow is defined by the following perturbative problem

$$\epsilon c \frac{d\pi}{d\rho} + T\pi + \frac{4}{3} c\pi^2 \tanh \rho = \frac{4}{15} c \tanh \rho, \qquad (35)$$

where now the book-keeping parameter ϵ multiplies only the derivative term of the equation. As before, we look for perturbative solutions of the form

$$\pi \sim \sum_{n=0}^{\infty} \bar{\pi}_n \epsilon^n, \qquad (36)$$

and, at the end of the calculation, set $\epsilon = 1$, recovering, in principle, a solution of the original equation of motion. Replacing this expansion into Eq. (35), we obtain the following set of relations:

$$c \sum_{n=0}^{\infty} \frac{d\bar{\pi}_n}{d\rho} \epsilon^{n+1} + T \sum_{n=0}^{\infty} \bar{\pi}_n \epsilon^n + \frac{4}{3} c \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \bar{\pi}_n \bar{\pi}_m \epsilon^{n+m} \tanh \rho$$
$$= \frac{4}{15} c \tanh \rho. \qquad (37)$$

As before, we write the derivative of $\bar{\pi}_n$ as

$$\frac{d\bar{\pi}_n}{d\rho} = \frac{\partial \bar{\pi}_n}{\partial \rho}\bigg|_T + \frac{\partial \bar{\pi}_n}{\partial T}\bigg|_{\tanh \rho} \frac{dT}{d\rho},$$

which allows us to properly collect all powers of ϵ and leads to the following equations,

$$c \sum_{n=0}^{\infty} \left(\frac{\partial \bar{\pi}_n}{\partial \rho}\bigg|_T - \frac{2}{3} \tanh \rho \, T \frac{\partial \bar{\pi}_n}{\partial T}\bigg|_{\tanh \rho} \right) \epsilon^{n+1} + \frac{c}{3} \tanh \rho \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \bar{\pi}_m T \frac{\partial \bar{\pi}_n}{\partial T}\bigg|_{\tanh \rho} \epsilon^{n+m+1}$$
$$+ T \sum_{n=0}^{\infty} \bar{\pi}_n \epsilon^n + \frac{4}{3} c \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \bar{\pi}_n \bar{\pi}_m \epsilon^{n+m} \tanh \rho = \frac{4}{15} c \tanh \rho. \qquad (38)$$

The zeroth order solution is the Gubser flow generalization of the well-known result first derived for Bjorken flow in [18],

$$\frac{4}{3} \bar{\pi}_0^2 \tau_R \tanh \rho + \bar{\pi}_0 = \frac{4}{15} \tau_R \tanh \rho,$$

$$\Rightarrow \bar{\pi}_0^{\pm}(\rho) = \frac{-3 \pm \sqrt{9 + \frac{4}{5}(4\tau_R \tanh \rho)^2}}{8\tau_R \tanh \rho}. \qquad (39)$$

We remark that the 0th order term of the slow-roll expansion for Israel-Stewart theory shown above was first computed in [75]. Out of the two possible solutions obtained above, we chose the one that asymptotes to the Navier-Stokes solution in the limit $\rho \to 0$; i.e., we consider only the solution $\bar{\pi}_0^+$. In fact, one may expand (39) in powers of τ_R to find

$$\bar{\pi}_0^+(\rho) = \frac{4}{15} \tau_R \tanh \rho - \frac{64}{675} (\tau_R \tanh \rho)^3 + \mathcal{O}(\tau_R^5). \qquad (40)$$

Thus, we see that the 0th order term in the slow-roll expansion recovers the NS result (i.e., it matches the

gradient expansion truncated at 1st order). However, it is important to notice that the higher order terms generated by Taylor expanding $\bar{\pi}_0^+$ differ from the higher order terms present in the gradient expansion.

Also, it is interesting to notice that the 0th order slow-roll term $\bar{\pi}_0^+$ is solely a function of the Knudsen number $\tau_R \tanh \rho \sim \tau_R \sqrt{\sigma_{\mu\nu} \sigma^{\mu\nu}}$ for Gubser flow, as also happened with the first order truncation of the gradient expansion (NS theory). A similar situation was found in Bjorken flow, where the zeroth-order truncation of the slow–roll expansion was shown to depend solely on the combination $T\tau$, the inverse Knudsen number for this flow configuration [18]. In Bjorken flow this feature persisted to all orders in the slow-roll expansion [57] though here we see that this does not hold for the slow-roll expansion in Gubser flow (the same was observed for the gradient expansion in Gubser flow in Sec. IV). As a matter of fact, we explicitly show in the following that the higher order terms of the slow-roll series in Gubser flow are functions of both τ_R and $\tanh \rho$, separately.

Using Eq. (39), higher order solutions are obtained by solving the recurrence relation

$$\left.\frac{\partial\bar{\pi}_n}{\partial\rho}\right|_T - \frac{2T}{3}\tanh\rho\left.\frac{\partial\bar{\pi}_n}{\partial T}\right|_{\tanh\rho} + \frac{T}{3}\tanh\rho\sum_{m=0}^{n}\bar{\pi}_m\left.\frac{\partial\bar{\pi}_{n-m}}{\partial T}\right|_{\tanh\rho} + \frac{T}{c}\bar{\pi}_{n+1} + \frac{4}{3}\sum_{m=0}^{n+1}\bar{\pi}_{n+1-m}\bar{\pi}_m\tanh\rho = 0, \tag{41}$$

which can be simplified to

$$\left(\frac{1}{\tau_R} + \frac{8}{3}\bar{\pi}_0^+\tanh\rho\right)\bar{\pi}_{n+1} = -\frac{2}{3}\tanh\rho\,\tau_R\left.\frac{\partial\bar{\pi}_n}{\partial\tau_R}\right|_{\tanh\rho} - \left.\frac{\partial\bar{\pi}_n}{\partial\rho}\right|_{\tau_R} + \frac{1}{3}\tanh\rho\sum_{m=0}^{n}\bar{\pi}_m\tau_R\left.\frac{\partial\bar{\pi}_{n-m}}{\partial\tau_R}\right|_{\tanh\rho}$$

$$- \frac{4}{3}\sum_{m=1}^{n}\bar{\pi}_{n+1-m}\bar{\pi}_m\tanh\rho. \tag{42}$$

Solving this recurrence relation, one can determine the first order solution to be

$$\bar{\pi}_1(\rho) = \frac{15\bar{\pi}_0^+}{\tau_R\tanh\rho}\left[\frac{(1+\bar{\pi}_0^+)(\tau_R\tanh\rho)^2 - 3\tau_R^2}{45 + 64(\tau_R\tanh\rho)^2}\right]. \tag{43}$$

This result clearly demonstrates that $\bar{\pi}_1$ depends on both τ_R and $\tanh\rho$, separately, a fact that remains true for all higher order coefficients of the slow-roll expansion.

Furthermore, expanding now the truncated series at 1st order (i.e., $\bar{\pi} \to \bar{\pi}_0^+ + \bar{\pi}_1$) in powers of τ_R leads to

$$\bar{\pi}(\rho) = \frac{4}{15}\tau_R\tanh\rho - \frac{4}{15}(\tau_R)^2\left(1 - \frac{1}{3}\tanh^2\rho\right)$$

$$- \frac{16}{225}(\tau_R\tanh\rho)^3 + \mathcal{O}(\tau_R^4), \tag{44}$$

which shows that the 1st order truncation of the slow-roll series is able to recover the result obtained from the gradient expansion truncated at 2nd order; see (24). In general, a Taylor expansion of the slow-roll expansion truncated at the Nth term reduces to the correct expression for the gradient expansion truncated at order $N + 1$. In this sense, the slow-roll expansion can be seen as a type of reorganization of the gradient series that contains an infinite resummation of gradients at a given order.

A. Divergence of the slow-roll series in Gubser flow

Now we solve Eq. (42) numerically to determine the behavior of the slow-roll series at higher orders. In Fig. 6 we show how the coefficients of the series change with n when $\rho \to 0$ and for the following values of relaxation time, $\tau_R = 0.1, 1$. This plot indicates that, in this limit, the magnitude of the terms grows larger than $n!$ when n is large.[3] To illustrate the fact that the large order behavior of the slow-roll series now depends on two parameters (i.e., ρ and τ_R), we show in Fig. 7 what happens when $\rho = 1$. Even

though the series still appears to diverge, in this case $|\bar{\pi}_n|^{1/(n+1)}$ only grows linearly with n (the same qualitative result appears for other values of τ_R and also when ρ is negative). Therefore, in Israel-Stewart theory both the gradient and the slow-roll expansions generally diverge in Gubser flow, just as it occurred in Bjorken flow [57].
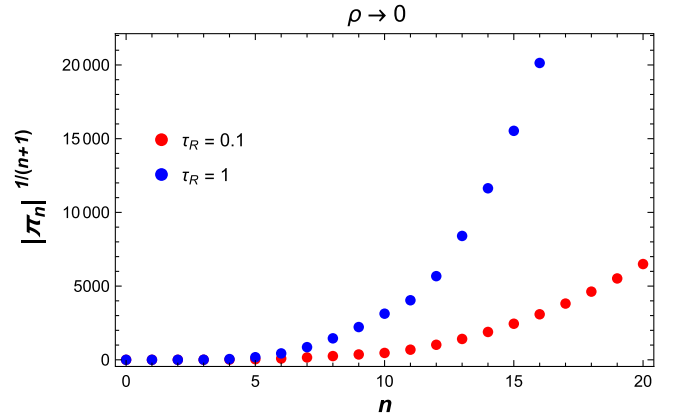
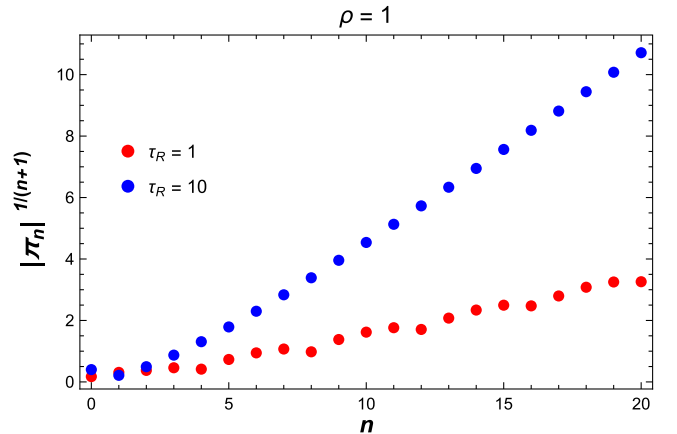FIG. 6. Large order behavior of the slow-roll series when $\rho \to 0$ and $\tau_R = 0.1, 1$.

FIG. 7. Large order behavior of the slow-roll series when $\rho = 1$ and $\tau_R = 1, 10$.

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[3]We checked that $|\pi_n|$ does not grow larger $(n!)^\alpha$ where $\alpha < 1.3$ in the range considered (this result is robust with respect to the choice of values for τ_R).
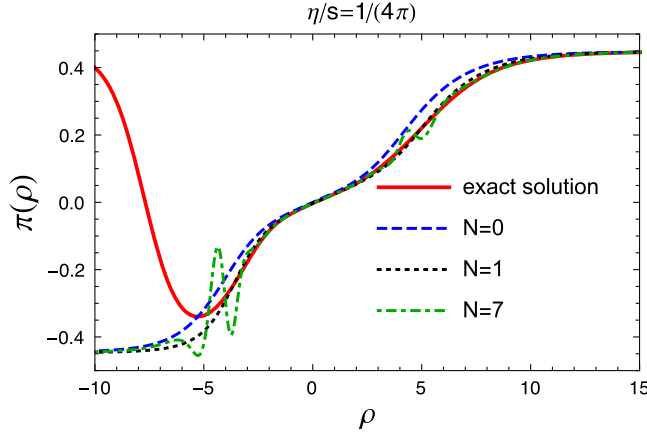
FIG. 8. Comparison between the exact solution of Israel-Stewart (IS) equations and different truncations of the slow-roll series in Gubser flow for $\eta/s = 1/(4\pi)$.

FIG. 10. Comparison between the exact solution of IS equations and different truncations of the slow-roll series in Gubser flow for $\eta/s = 1$.

The only exception occurs when $|\rho| \to \infty$. Since the temperature vanishes in this limit one must also take $\tau_R \to \infty$, which implies that $\bar{\pi}_0^{\pm} \to \pm \text{sign}\rho/\sqrt{5}$ and Eq. (42) gives $\bar{\pi}_{n>0} = 0$. This shows that the slow-roll expansion in fact converges in this limit to $\bar{\pi}_0^{\pm}$. As discussed in [63], solutions of the Israel-Stewart equation for the shear stress tensor (15) do display the same behavior and, thus, one can see that the slow-roll series necessarily converges to solutions of the Israel-Stewart equation when $|\rho| \to \infty$.

B. Determining the domain of applicability of the slow-roll expansion

In Figs. 8–11 we investigate how different truncations of the slow-roll series for π fare in comparison to the same exact solution of the Israel-Stewart equations used in previous sections. These exact solutions were constructed using the same initial condition for T and π employed in the previous sections. We can see in Fig. 8 that, for

$\eta/s = 1/(4\pi)$, the 0th order truncation deviates only slightly from the exact solution though the 1st order truncation of the slow-roll expansion gives a reasonably accurate description of the solution for $\rho > -5$. We also show the results for the 7th order truncation, which are found to display oscillatory behavior compatible with the divergent character of the series when ρ is finite (when $|\rho| \to \infty$, however, the series converges and this is why the oscillations do not appear in that regime). Figure 9 shows that the agreement with the exact solution improves at 2nd order for $\rho > 0$.

Figures 10 and 11 show that the agreement between the truncated slow-roll series and the exact solution considerably worsens when $\eta/s = 1$ (the same occurred in the gradient expansion investigated in Sec. V). As already mentioned, in this case $\pi(0)$ deviates considerably from 0 in the exact solution and the 0th order approximation of the slow-roll series is not accurate even at $\rho = 0$. This affects the overall ability of the truncated series to describe the
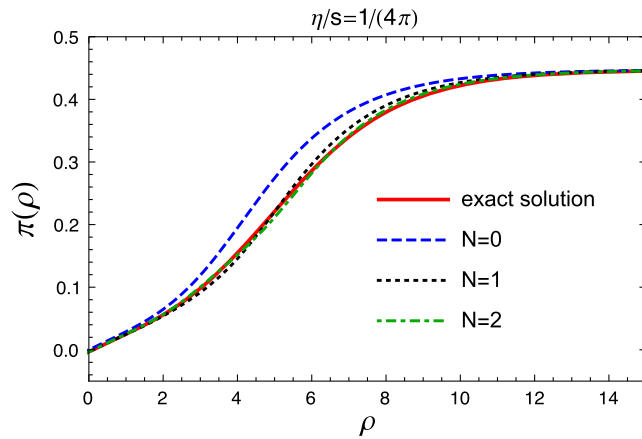

FIG. 9. Detailed comparison when $\rho > 0$ between the exact solution of the IS equations and the low order truncations of the slow-roll series in Gubser flow for $\eta/s = 1/(4\pi)$.
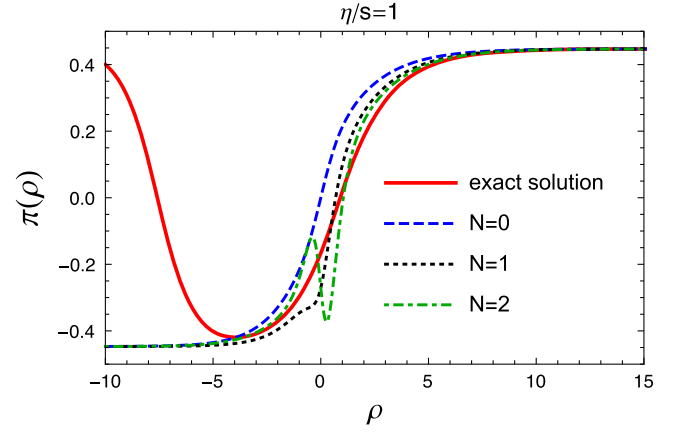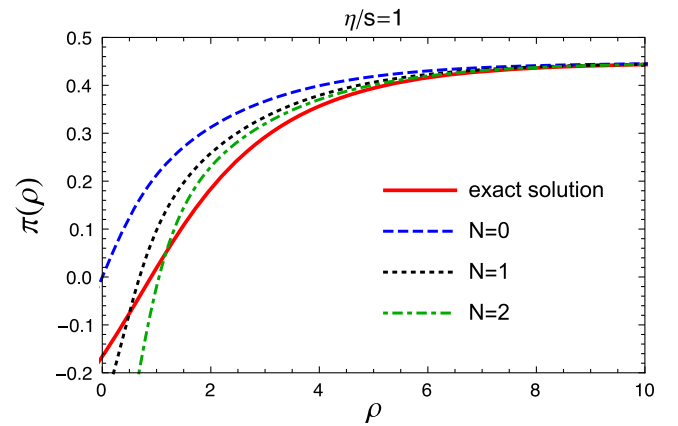

FIG. 11. Detailed comparison when $\rho > 0$ between the exact solution of the IS equations and the low order truncations of the slow-roll series in Gubser flow for $\eta/s = 1$.
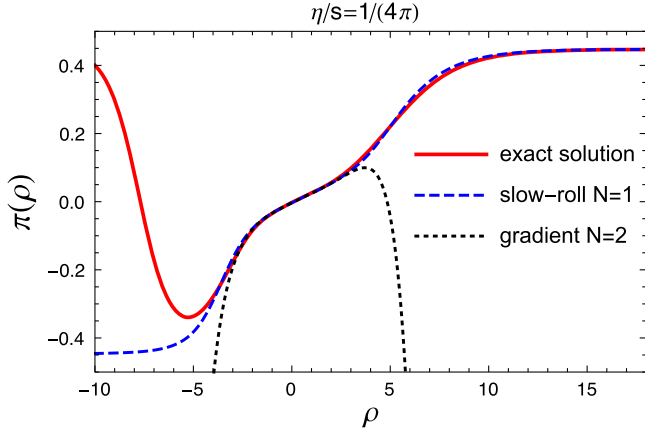
FIG. 12. Comparison between the exact solution of IS equations undergoing Gubser flow for $\eta/s = 1/(4\pi)$ and the $N = 1$ and $N = 2$ truncations of the slow-roll and gradient series, respectively.

FIG. 14. Comparison between the exact solution of IS equations undergoing Gubser flow for $\eta/s = 1$ and the $N = 1$ and $N = 2$ truncations of the slow-roll and gradient series, respectively.

solution and one can see in Fig. 10 that oscillations now appear already at 2nd order. Figure 11 shows the results for the $\rho > 0$ region in detail. Even though the higher order truncations become closer to the exact solution, the agreement is generally poor in comparison to what was found in Fig. 9, where $\eta/s = 1/(4\pi)$. Overall, since T has a maximum at $\rho = 0$ in Israel-Stewart theory [63], the relaxation time has a minimum at the same location and we find that the slow-roll series is not a good proxy for the exact solutions when $\tau_R(0) \gtrsim 1$, which occurs in this example when $\eta/s = 1$.

C. Comparison between the perturbative expansions

We mentioned before that a Taylor series in powers of τ_R of the Nth order truncation of the slow-roll expansion reduces to the result obtained from the $(N + 1)$th order

truncation of the gradient series. In order to assess how these two perturbative series fare in comparison to the exact solution of Israel-Stewart equations, we plot in Fig. 12 the 2nd order gradient expansion result and the $N = 1$ truncation of the slow-roll series for $\eta/s = 1/(4\pi)$. One can see that the $N = 1$ slow-roll expansion indeed matches the $N = 2$ gradient series result as long as the latter still provides a good approximation to the exact solution (approximately when $-3 < \rho < 3$). However, outside this regime the gradient series fails to describe the solution while the slow-roll result nicely continues to provide a very good description of the system's dynamics towards larger values of ρ. A detailed comparison between the truncated series and the numerical result when $\rho > 0$ can be found in Fig. 13. The 1st order slow-roll series is much more accurate in this regime than the gradient expansion, which fails around $\rho \sim 3$.

Figure 14 shows that, when $\eta/s = 1$, both expansions have difficulties in describing the behavior of the exact solution, as expected from the results of the previous sections. However, we remark that the 1st order truncation of the slow-roll series still behaves much better than the 2nd order gradient expansion. In general, perturbative series such as the gradient or the slow-roll expansions can only be accurate if their lowest order terms are not too far from the exact solution. This is the case when $\eta/s = 1/(4\pi)$, since $\pi(0)$ is very small and can be well approximated by the Navier-Stokes solution. The same does not happen when $\eta/s = 1$, in which case the solution for $\pi(0)$ considerably deviates from 0.
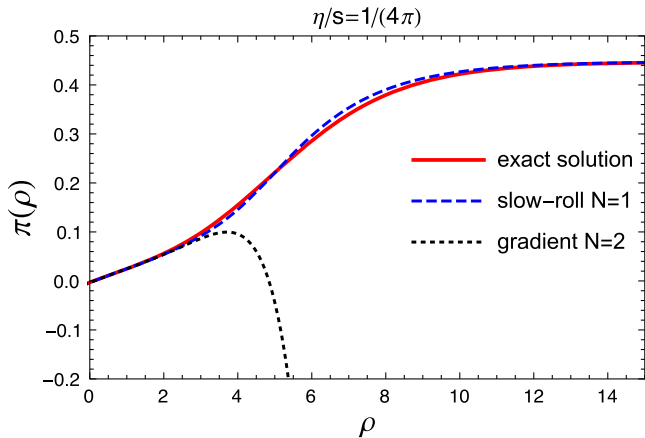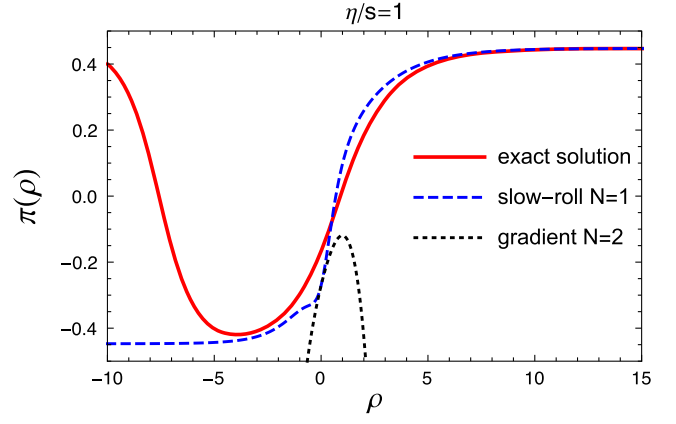

FIG. 13. Detailed comparison when $\rho > 0$ between the exact solution of IS equations undergoing Gubser flow for $\eta/s = 1/(4\pi)$ and the $N = 1$ and $N = 2$ truncations of the slow-roll and gradient series, respectively.

VIII. ATTRACTOR SOLUTION IN GUBSER FLOW

In this section we investigate the attractor solution of Israel-Stewart equations for a system expanding according to Gubser flow, which was first studied in Ref. [75]. In this work, we interpret the attractor solution as a resummed slow-roll expansion. The analysis presented in this section
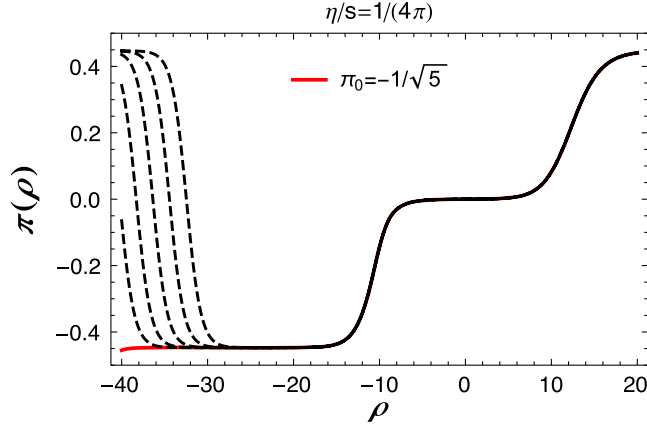
FIG. 15.   Attractor solution of IS equations undergoing Gubser flow (solid red) compared to other solutions of the equations (dashed black curves) for $\eta/s = 1/(4\pi)$.
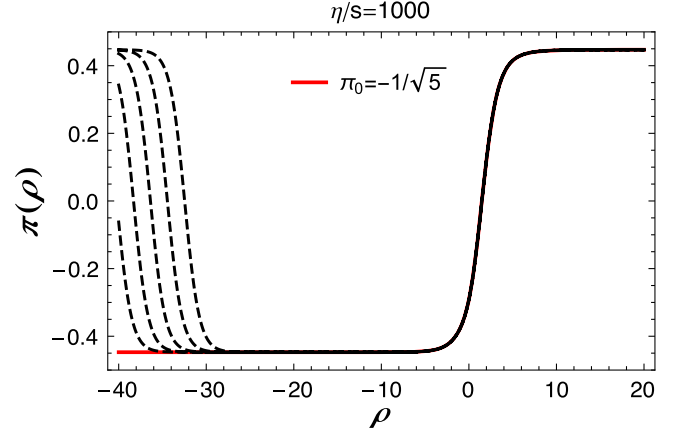

FIG. 16.   Attractor solution of IS equations undergoing Gubser flow (solid red) compared to other solutions of the equations (dashed black curves) for the very large value of $\eta/s = 1000$.

aims to explore two fundamental aspects of the attractor: its possible functional dependence on the Knudsen number and its approximate description using the slow-roll series truncated at higher orders.

In the previous section, we found that the slow-roll series converges when $|\rho| \to \infty$ and we can use this fact to numerically construct a resummed version of the slow-roll solution. We found that there are two possible convergent solutions for π when $|\rho| \to \infty$: $1/\sqrt{5}$ and $-1/\sqrt{5}$. It is natural to expect that one of these boundary conditions leads to the exact solution related to the slow-roll expansion. In practice, we observe that the vast majority of solutions (if not all of them) of Israel-Stewart theory actually converge to $1/\sqrt{5}$ when $\rho \to -\infty$ (for $\rho \to \infty$, all known numerical solutions of Israel-Stewart theory also converge to $1/\sqrt{5}$). So far, the only case in which we are able to obtain a solution that is equal to $-1/\sqrt{5}$, when $\rho \to -\infty$, is when we give exactly this boundary condition at a very small value of ρ. Any small deviation from $-1/\sqrt{5}$ makes the solution tend to $1/\sqrt{5}$ when we decrease the value of ρ. This behavior suggests that the boundary condition $\pi(-\infty) = -1/\sqrt{5}$ defines a unique solution of the equation and that such a unique solution can be identified as the resummed result for the slow-roll series. On the other hand, the other boundary condition at $\rho = -\infty$, $\pi(-\infty) = 1/\sqrt{5}$, is satisfied by an infinite number of solutions and cannot be used to define any specific solutions of the equations.

This is illustrated in Figs. 15 and 16 for two vastly different values of η/s. In these plots, the solid red curve depicts the solution of Israel-Stewart equations assuming $T(-30) = 9.222 \times 10^{-8}$ and $\pi(-30) = -1/\sqrt{5}$. The dashed curves are computed keeping the initial condition for the temperature fixed while considering very small variations of $\pi(-30)$, of at most 1% around $-1/\sqrt{5}$. We see that any small variation of the value of π at $\rho = -30$ makes the

solution converge to $1/\sqrt{5}$ when ρ is decreased. This only does not happen when we fix π to be exactly $-1/\sqrt{5}$ (red curve). Furthermore, one can see that all the solutions converge extremely rapidly to the solution where $\lim_{\rho \to -\infty} \pi(\rho) = -1/\sqrt{5}$. This solution, represented here by the solid red curve in these plots, corresponds to the hydrodynamic attractor solution first discussed in [18] in Bjorken flow, which was later investigated in more detail in [54].

Comparing Figs. 15 and 16 we see that the profile of the attractor depends on the value of η/s, becoming closer to a step function as η/s is increased even further. Such a behavior is very hard to describe using the slow-roll series, as one can see in Fig. 17. In this figure we compare the attractor solutions (solid red) with truncations of the slow-roll series. For $\eta/s = 1/(4\pi)$ we used the 2nd order truncation while for the extremely large value of $\eta/s = 1000$ we only took the 0th order term in the series. When η/s is small, the truncated slow-roll series provides
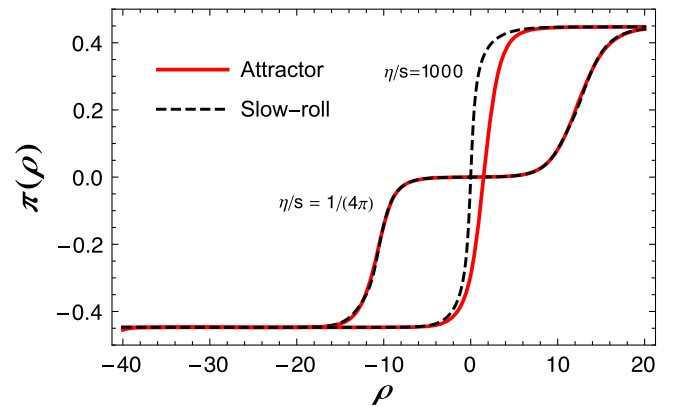

FIG. 17.   Comparison between the attractor solutions of IS equations undergoing Gubser flow, for $\eta/s = 1/(4\pi)$ and $\eta/s = 1000$, and the corresponding approximations using the slow-roll series (dashed).
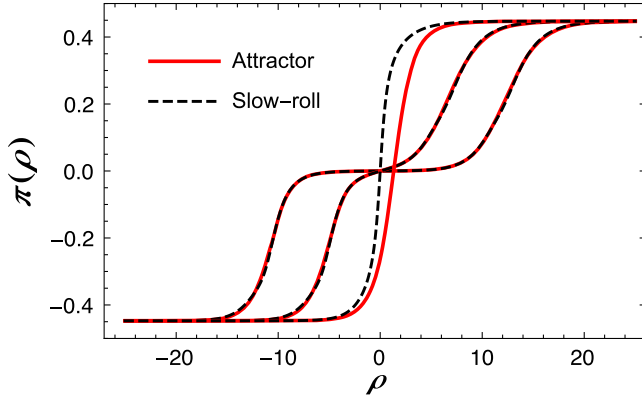
FIG. 18. Comparison between the attractor solutions of IS equations undergoing Gubser flow, computed using different initial conditions for the temperature, and the corresponding approximations using the slow-roll series (dashed). In this plot $\eta/s = 1/(4\pi)$.
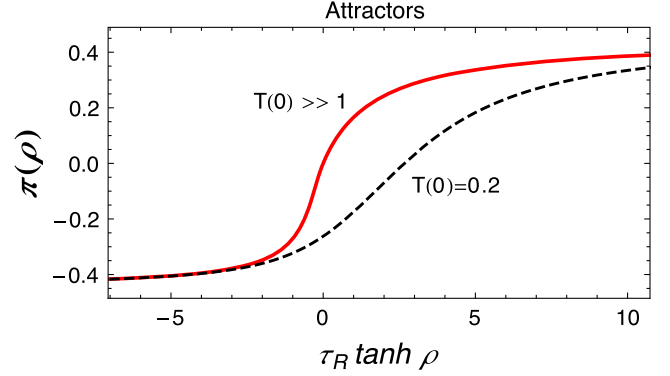

FIG. 19. Attractor solutions of IS equations undergoing Gubser flow, computed using different initial conditions for the temperature, vs the Knudsen numberlike quantity $\tau_R \tanh \rho$. The fact that these curves are distinct imply that in Gubser flow the attractor solution of IS theory is not just a simple function of $\tau_R \tanh \rho$. Rather, the attractor depends on both τ_R and $\tanh \rho$, separately. In this plot $\eta/s = 1/(4\pi)$.

an excellent description of the attractor while for very large values of η/s this perturbative approach provides a poor description, even though still qualitatively accurate, of the attractor away from the asymptotic regime, as expected.

Now we explore a different feature, so far exclusive to Gubser flow, which is the fact that the attractor solution *per se* depends on the temperature. This can be seen in Fig. 18 where we now fix $\pi(-30) = -1/\sqrt{5}$ (attractor solutions) and consider three very different values for the initial temperature $T(-30) = 9.222 \times 10^{-8}$, $T(-30) = 9.222 \times 10^{-10}$, and $T(-30) = 9.222 \times 10^{-12}$ and $\eta/s = 1/(4\pi)$. These variations in initial temperature lead to very different values for the temperature at $\rho = 0$ in the attractor solutions. One obtains three different profiles for the attractor, which are also compared to their corresponding slow-roll series truncated at 2nd order for the first two solutions and at 0th order for the last one. Even though η/s is small, by significantly decreasing the initial value of the temperature one can again recover the large $\tau_R(0)$ regime where the slow-roll series does not work well.

Finally, in Fig. 19 we explore another key difference between the Gubser flow attractor and the Bjorken flow attractor. In this figure we picked two of the attractor solutions discussed above where $T(0) = 0.2$ and $T(0) \gg 1$, with $\eta/s = 1/(4\pi)$, and plotted them against the Knudsen number combination $\tau_R \tanh \rho$. The fact that these curves are different show that, in contrast to the Bjorken flow case, the attractor solution of Israel-Stewart equations undergoing Gubser flow is not solely a function of $\tau_R \tanh \rho$, even though the 0th order truncation of the slow-roll series is. This illustrates that characterizing attractor solutions by the 0th order slow-roll series can be misleading as this special solution of Israel-Stewart equations (the attractor) displays a more complex dependence on ρ than the simplest truncated series. Since already the 1st order truncation of the slow-roll series depends on both τ_R and $\tanh \rho$, we see

that higher order truncations of the slow-roll series are better suited to properly characterize the attractor.

IX. CONCLUSIONS

In this paper we developed a perturbative scheme to construct asymptotic solutions, such as the gradient and slow-roll expansions, of Israel-Stewart theory undergoing Gubser flow. We then determine for the first time the large order behavior of these perturbative expansions in the case of Gubser flow. We demonstrated numerically that the expansion coefficients in both cases grow factorially, indicating that these series have a zero radius of convergence. Even though these series appear to diverge, their low order truncations can still offer a reasonable description of exact solutions of Israel-Stewart theory near the origin of the Gubser coordinate system. However, we emphasize that such agreement is only possible when the relaxation time is sufficiently small at $\rho = 0$, i.e., $\tau_R(\rho = 0) \ll 1$.

When comparing both asymptotic solutions, we found that the slow-roll expansion provides a much better overall description of exact solutions of Israel-Stewart, when truncated at low orders. In particular, a truncated slow-roll expansion can even describe qualitative and, often, quantitative, aspects of exact solutions when $\rho \to \infty$—a region in which gradients cannot be considered small. This suggests the existence of nontrivial constitutive relations satisfied by the shear stress tensor that are valid even when gradients are large (i.e., the far-from-equilibrium regime).

We also demonstrated that the slow-roll series converges to $\pm 1/\sqrt{5}$ when $|\rho| \to \infty$ and used this fact to numerically construct a resummed version of the slow-roll expansion. We showed numerically that such a solution displays the basic properties expected of an attractor, with solutions obtained using different initial conditions always converging to such a resummed solution. Differently than the case

of Bjorken flow, we found that the Gubser flow attractor solution cannot be expressed just as a function of the effective Knudsen number $\sim \tau_R \tanh \rho$. Instead, it depends on the relaxation time and $\tanh \rho$ separately, suggesting the existence of a class of attractor solutions and not just a single universal function. Nevertheless, we emphasize that when the relaxation time is sufficiently small at $\rho = 0$, $\tau_R(0) \ll 1$, the attractor can be approximately expressed as a function of $\tau_R \tanh \rho$. Furthermore, the truncated slow-roll series can also describe the attractor solution of Gubser flow as long as the system is sufficiently close to equilibrium near the origin (i.e., $\rho = 0$).

Finally, our results give further support to the idea that new *resummed* constitutive relations between dissipative currents and the gradients of conserved quantities can emerge in systems far from equilibrium that are beyond the regime of validity of the usual gradient expansion.

APPENDIX: YET ANOTHER IMPLEMENTATION OF THE GRADIENT EXPANSION

In this Appendix we consider a possible solution for T and π that is represented as a series in powers of ϵ,

$$T \sim \sum_{n=0}^{\infty} T_n(\rho) \epsilon^n, \qquad (A1)$$

$$\pi \sim \sum_{n=0}^{\infty} \pi_n(\rho) \epsilon^n. \qquad (A2)$$

Substituting the proposed series solutions in powers of ϵ into the equations of motion for π and T one obtains the following set of *coupled* equations:

$$\sum_{n=0}^{\infty} \partial_\rho T_n \epsilon^n + \frac{2}{3} \sum_{n=0}^{\infty} T_n \epsilon^n \tanh \rho$$
$$- \frac{1}{3} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \pi_n T_m \epsilon^{n+m} \tanh \rho = 0, \qquad (A3)$$

$$c \sum_{n=0}^{\infty} \partial_\rho \pi_n \epsilon^{n+1} + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \pi_n T_m \epsilon^{n+m}$$
$$+ \frac{4}{3} c \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \pi_n \pi_m \epsilon^{n+m+1} \tanh \rho = \frac{4}{15} c \epsilon \tanh \rho. \quad (A4)$$

We now group together the terms that are of the same power in ϵ, obtaining the set of recurrence relations that must be solved to obtain T_n and π_n. The terms that are of zeroth order in ϵ satisfy

$$\partial_\rho T_0 + \frac{2}{3} T_0 \tanh \rho - \frac{1}{3} \pi_0 T_0 \tanh \rho = 0, \qquad (A5)$$

$$\pi_0 T_0 = 0, \qquad (A6)$$

leading to the solution

$$\partial_\rho T_0 + \frac{2}{3} T_0 \tanh \rho = 0, \qquad (A7)$$

$$\pi_0 = 0. \qquad (A8)$$

Note that the equations above are exactly the same as those of an ideal fluid and, consequently, the lowest order truncation of the series leads to ideal hydrodynamics, as expected.

Collecting the terms that are of first order in ϵ, one obtains

$$\partial_\rho T_1 + \frac{2}{3} T_1 \tanh \rho - \frac{1}{3} \pi_1 T_0 \tanh \rho = 0, \qquad (A9)$$

$$\pi_1 = \frac{4c}{15 T_0} \tanh \rho, \qquad (A10)$$

leading to the following equation for T_1:

$$\partial_\rho T_1 + \frac{2}{3} T_1 \tanh \rho = \frac{4c}{45} \tanh^2 \rho. \qquad (A11)$$

Furthermore, the terms that are of second order or higher in ϵ, $(n \geq 2)$, satisfy the equations

$$\partial_\rho T_n + \frac{2}{3} T_n \tanh \rho - \frac{1}{3} \sum_{m=0}^{n} T_{n-m} \pi_m \tanh \rho = 0, \quad (A12)$$

$$c \partial_\rho \pi_n + T_0 \pi_{n+1} + \sum_{m=1}^{n} T_m \pi_{n-m+1}$$
$$+ \frac{4c}{3} \sum_{m=1}^{n} \pi_{n-m} \pi_m \tanh \rho = 0. \qquad (A13)$$

The equations/solutions obtained above have a rather disturbing feature. So far, it was not possible to derive a constitutive equation for π that is expressed solely in terms of T and $\tanh \rho$ (gradients of velocity), as would be expected in a gradient expansion. Instead, the solution appears in terms of the temperature expansion coefficients, T_n. Nevertheless, we later demonstrate that this solution, truncated up to a given order in ϵ, can be resummed and reexpressed solely in terms of the full temperature T.

For the first order truncation, this procedure is rather obvious: one simply takes the $1/T_0$ dependence, that appears in the solution for π_1, and reexpresses it as $1/T_0 \sim 1/T + \mathcal{O}(\epsilon)$. Therefore, neglecting terms of second order or higher in ϵ, we have that

$$\pi = \pi_0 + \epsilon \pi_1 + \mathcal{O}(\epsilon^2) = \frac{4c\epsilon}{15T} \tanh \rho + \mathcal{O}(\epsilon^2). \quad \text{(A14)}$$

Similarly, the equation of motion for the temperature up to second order, $T_0 + \epsilon T_1$, can be written as

$$\partial_\rho (T_0 + \epsilon T_1) + \frac{2}{3}(T_0 + \epsilon T_1) \tanh \rho = \frac{4c}{45} \epsilon \tanh^2 \rho, \quad \text{(A15)}$$

and, consequently, we have that

$$\partial_\rho T + \frac{2}{3} T \tanh \rho = \frac{4c}{45} \epsilon \tanh^2 \rho + \mathcal{O}(\epsilon^2). \quad \text{(A16)}$$

If we set $\epsilon = 1$, the equation above becomes the Navier-Stokes equation under Gubser flow. This equation can be solved analytically, as was shown in Ref. [62], even though this solution displays unphysical features such as negative temperatures when $\rho \to -\infty$.

Next, we obtain the second and third order solutions and show that these can also be expressed in terms of the full temperature (up to the corresponding order). The second order equations (for T_2 and π_2) are

$$\partial_\rho T_2 + \frac{2}{3} T_2 \tanh \rho - \frac{1}{3} \pi_2 T_0 \tanh \rho - \frac{1}{3} \pi_1 T_1 \tanh \rho = 0, \quad \text{(A17)}$$

$$c \partial_\rho \pi_1 + \pi_2 T_0 + \pi_1 T_1 = 0. \quad \text{(A18)}$$

Using the solutions/equations already obtained for π_1, T_1, and T_0, we find the constitutive equation satisfied by π_2,

$$\pi_2 = \frac{4c^2}{15T_0^2} \left(-1 - \frac{T_1}{c} \tanh \rho + \frac{1}{3} \tanh^2 \rho \right). \quad \text{(A19)}$$

One can see that π_2 depends separately on T_0 and T_1, while the equation of motion for T_2 is coupled to T_0, T_1, π_1, and π_2.

The third order equations are

$$\partial_\rho T_3 + \frac{2}{3} T_3 \tanh \rho - \frac{1}{3} T_2 \pi_1 \tanh \rho - \frac{1}{3} T_1 \pi_2 \tanh \rho$$
$$- \frac{1}{3} T_0 \pi_3 \tanh \rho = 0, \quad \text{(A20)}$$

$$c \partial_\rho \pi_2 + \pi_3 T_0 + \pi_2 T_1 + \pi_1 T_2 + \frac{4}{3} c \pi_1^2 \tanh \rho = 0. \quad \text{(A21)}$$

Using all the equations obtained for the lower order coefficients, the constitutive equation satisfied by π_3 can be simplified to

$$\pi_3 = \frac{4c^2}{15T_0^2} \frac{T_1}{T_0} \left(1 + \frac{T_1}{c} \tanh \rho - \frac{1}{3} \tanh^2 \rho \right) - \frac{4c}{15T_0} \frac{T_2}{T_0} \tanh \rho \quad \text{(A22)}$$

$$+ \frac{4c^3}{15T_0^3} \left(\frac{T_1}{c} + \frac{2}{3} \tanh \rho - \frac{1}{3} \frac{T_1}{c} \tanh^2 \rho - \frac{2}{45} \tanh^3 \rho \right). \quad \text{(A23)}$$

The second and third order coefficients, π_2 and π_3, are complicated and contain terms that have mixed contributions from T_0, T_1, and T_2.

We now reexpress all these contributions solely in terms of T. In order to perform this task, one should first note that

$$\pi_1 = \frac{4\tau_R}{15} \left(1 + \epsilon \frac{T_1}{T} + \epsilon^2 \frac{T_2}{T} + \epsilon^2 \frac{T_1^2}{T^2} \right) \tanh \rho + \mathcal{O}(\epsilon^3), \quad \text{(A24)}$$

$$\pi_2 = -\frac{4}{15} \tau_R^2 \left(1 - \frac{1}{3} \tanh^2 \rho + \frac{T_1}{c} \tanh \rho \right)$$
$$- \frac{8 T_1}{15 T} \epsilon \tau_R^2 \left(1 - \frac{1}{3} \tanh^2 \rho + \frac{T_1}{c} \tanh \rho \right) + \mathcal{O}(\epsilon^2), \quad \text{(A25)}$$

$$\pi_3 = \frac{4\tau_R^3}{15} \left[\frac{2}{3} \tanh \rho - \frac{2}{45} \tanh^3 \rho + \frac{T_1}{c} \left(1 - \frac{1}{3} \tanh^2 \rho \right) \right]$$
$$- \frac{4\tau_R}{15} \frac{T_2}{T} \tanh \rho \quad \text{(A26)}$$

$$+ \frac{4\tau_R^2}{15} \left(1 - \frac{1}{3} \tanh^2 \rho + \frac{T_1}{c} \tanh \rho \right) \frac{T_1}{T} + \mathcal{O}(\epsilon). \quad \text{(A27)}$$

All that was done above was to rewrite T_0 in terms of T, up to a given order in ϵ. Combining all these expressions, $\pi = \epsilon \pi_1 + \epsilon^2 \pi_2 + \epsilon^3 \pi_3 + \mathcal{O}(\epsilon^4)$, one can verify that all contributions including T_1 and T_2 cancel each other and only the dependence on the full temperature is left. The resumed answer is

$$\pi = \frac{4}{15} \epsilon \tau_R \tanh \rho - \frac{4}{15} (\epsilon \tau_R)^2 \left(1 - \frac{1}{3} \tanh^2 \rho \right) \quad \text{(A28)}$$

$$+ \frac{8}{45} (\epsilon \tau_R)^3 \left(\tanh \rho - \frac{1}{15} \tanh^3 \rho \right) + \mathcal{O}(\epsilon^4). \quad \text{(A29)}$$

Therefore, we are able to rewrite the previous series as a power series in $\epsilon \tau_R$, with all the temperature contributions

contained in the powers of the relaxation time. Note that the equation of motion for the (full) temperature can also be written in the form

$$\partial_\rho T + \frac{2}{3} T \tanh \rho = \frac{1}{3} \pi T \tanh \rho + \mathcal{O}(\epsilon^4). \qquad \text{(A30)}$$

Finally, following this scheme we obtain equations of motion for the temperature that are complemented by constitutive equations for the shear stress tensor. These equations have the form that is traditionally associated with a gradient expansion, as discussed in the main text.

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