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Simple binary Lie and non-Lie superalgebra has solvable even part

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ABSTRACT

We prove that every simple finite dimensional binary Lie superalgebra over the complex numbers field **C** with non-zero odd part is either a Lie superalgebra or has a solvable even part.

1. Introduction

Binary Lie algebras (*BL*-algebras) were introduced by A.I. Malcev [8] as anticommutative algebras in which any two elements generate a Lie subalgebra. This property is fulfilled in *Malcev algebras*, defined in the same paper (under the name of *Moufang-Lie algebras*) as the algebras satisfying the identities

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$$xy = -yx,$$
$$J(xy, z, x) = J(y, z, x)x,$$

where $J(x, y, z) = (xy)z + (yz)x + (zx)y$ denotes the *jacobian* of the elements x, y, z.

Laterly, A.T. Gainov [3] characterized binary Lie algebras by identities: an anticommutative algebra is a binary Lie algebra if and only if it satisfies the identity

$$J(xy, x, y) = 0. \tag{1}$$

It is clear that every Lie algebra is a Malcev algebra and every Malcev algebra is a BL-algebra. The most important example of a non-Lie Malcev algebra is the 7-dimensional algebra $sl(\mathbf{O})$ of octonions with zero trace under the product defined by the *commutator* $[x, y] = xy - yx$. The algebra $sl(\mathbf{O})$ is simple, and V.T. Filippov proved [2] that every simple non-Lie Malcev algebra (of any dimension and of characteristic $\neq 2, 3$) is isomorphic to $sl(\mathbf{O})$. Moreover, it was proved by the first author in [4] that every simple finite dimensional BL-algebra over a field of characteristic 0 is a Malcev algebra, that is, is a Lie algebra or is isomorphic to $sl(\mathbf{O})$.

The last author in [9] investigated prime Malcev superalgebras and proved that every non-trivial (that is, with nonzero odd part) prime Malcev superalgebra is a Lie one.

In this paper we continue the study of simple binary Lie superalgebras started in [6].

A $\mathbf{Z}_2$-graded algebra $B = B_0 \oplus B_1$ is called a *binary-Lie superalgebra (SBL-algebra)* if it satisfies the following super-identities:

$$\begin{aligned} xy &= -(-1)^{\bar{x}\bar{y}} yx, \\ SBL(x, y, z, t) &:= (xy.z)t - x(y.zt) \\ &+ (-1)^{\bar{x}\bar{y}}\{y(xz.t) + y(x.zt) - (y.xz)t)\} \\ &+ (-1)^{\bar{z}\bar{t}}\{x(yt.z) - (xy.t)z - (x.yt)z\} = 0, \end{aligned}$$

where $\bar{z} \in \{0, 1\}$ stands for the parity of a homogeneous element z: $\bar{z} = i$ iff $z \in B_i$.

The problem of classification of finite dimensional simple SBL-algebras over the field $\mathbf{C}$ is open. We know a unique example of simple non-Malcev SBL-algebra $B = B_0 \oplus B_1$. It has dimension two with $\dim_{\mathbf{C}} B_0 = \dim_{\mathbf{C}} B_1 = 1$ (see [1]).

Conjecture 1.1. *Let $B = B_0 \oplus B_1$ be a finite dimensional simple SBL-algebra over the field $\mathbf{C}$ and $\dim B_1 \neq 0$. Then B is a simple Lie superalgebra or $\dim B = 2$.*

We propose the following strategy for proving Conjecture 1.1 in four steps:

1. Reduction to the case when B_0 is solvable.
2. Reduction to the case when B_0 is nilpotent.
3. Reduction to the case when B_0 is abelian.

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4. To prove Conjecture 1.1 for abelian even part.

In this paper we prove that if $B_1 \neq 0$ and B is not a Lie superalgebra then the even part B_0 of B is solvable. Hence we realize the first step of the above strategy.

Note that in Conjecture 1.1 all the conditions: a) basic field is of characteristic 0; b) it is algebraically closed; c) the superalgebra has finite dimension, are important even in the case of abelian even part, as we showed in the paper [6].

2. Structure of B_0

Recall the results on the structure of finite dimensional BL-algebras from [4], [5].

Theorem 2.1. *Let P be a finite dimensional BL-algebra over the field $\mathbf{C}$ with the solvable radical $G = G(P)$. Then P contains a central ideal $R(P)$ such that*

(i) there exists a subalgebra S of P containing $R(P)$ such that

$$P/R(P) = S/R(P) \oplus G/R(P), \text{ a vector space direct sum,}$$

where $S/R(P)$ is a semisimple Malcev algebra and $G/R(P)$ is a completely reducible Malcev $S/R(P)$-module;

(ii) $R(P)$ annihilates every finite dimensional binary-Lie P-module.

Corollary 2.1. *Let $B = B_0 \oplus B_1$ be a finite dimensional simple SBL-algebra over the field $\mathbf{C}$ and $B_1 \neq 0$. Then*

(i) $R(B_0) = 0$,
(ii) $B_0 = P \oplus G(B_0)$, where P is a semisimple Malcev algebra and $G(B_0)$ is a completely reducible Malcev P-module.

Proof. Assume that $R(B_0) \neq 0$. Since B_1 is a finite dimensional binary-Lie B_0-module, by item (ii) of Theorem 2.1 we get $B_1 R(B_0) = 0$. Hence $R(B_0)$ is an abelian ideal of B and $R(B_0) = 0$. Now item (ii) of the Corollary follows from item (i) of Theorem 2.1. □

3. Supermodules over BL-algebra and its products

Recall the notion of a *tensor algebra of a bimodule* (see, for instance, [7]). Let A be a (super)algebra in a variety $\mathcal{M}$ and V be a (super)bimodule over A in the variety $\mathcal{M}$. Then the tensor algebra $A[V]$ of the bimodule V is defined as the quotient algebra $F_{\mathcal{M}}[A \oplus V]/I$, where $F_{\mathcal{M}}[A \oplus V]$ is the free algebra in $\mathcal{M}$ over the vector space $A \oplus V$ and I is its ideal generated by the set $\{a * b - ab,\ a * v - a \cdot v, v * a - v \cdot a \,|\, a, b \in A,\ v \in V\}$. Here $*$ and $\cdot$ stand for multiplication in the free algebra and action of A on V respectively.

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Observe that the generators of the ideal I are homogeneous with respect to V, hence we have

$$A[V] = \oplus_{i=0}^{\infty} V^{(i)},$$

where $V^{(0)} = A$, $V^{(1)} = V$, and $V^{(i)}$ is the A-submodule of $A[V]$ generated by all monomials that contain i elements from V.

Let now $V = U \oplus W$ be a direct sum of A-bimodules U and W. Then we have

$$V^{(2)} = U^{(2)} \oplus W^{(2)} \oplus (UW)_A \oplus (WU)_A,$$

where $(UW)_A$ denotes the A-subbimodule generated by the set UW. We will denote this subbimodule as $U \hat{\otimes} W$ and will call it *the tensor product* of the A-bimodules U and W.

Let A be an $\mathcal{M}$-algebra and V be an $\mathcal{M}$-bimodule over A. We can associate with V two $\mathcal{M}$-superbimodules over A: V_{even} and V_{odd}, where

$$(V_{even})_0 = V,\ (V_{even})_1 = 0;\ \ (V_{odd})_0 = 0,\ (V_{odd})_1 = V.$$

Clearly, $V_{even} \cong V_{odd} \cong V$ as A-bimodules.

Proposition 3.1. *Let S be a BL-algebra and V, W be BL-modules over S. Then we have the isomorphism of S-modules*

$$V_{even} \hat{\otimes} W_{odd} \cong V_{odd} \hat{\otimes} W_{even} \cong (V \hat{\otimes} W)_{odd},$$
$$V_{even} \hat{\otimes} W_{even} \cong V_{odd} \hat{\otimes} W_{odd} \cong (V \hat{\otimes} W)_{even},$$

where the first two tensor products in both lines are considered as products of supermodules.

Proof. Observe that in construction of the (super)product $V \hat{\otimes} W$ only the identities $SBL(x, y, z, t) = 0$ are used where at least two arguments are taken from S and at most one element from each of V and W is taken. Moreover, when we have $v \in V$ and $w \in W$ among the arguments x, y, z, t then the remaining elements, say, a, b belong to S, and due to super-anticommutativity our identity may be rewritten in such a way that in all the monomials v precede w. For example,

$$\begin{aligned} SBL(a, b, v, w) &= (ab.v)w - a(b.vw) \\ &\quad + b(av.w) + b(a.vw) - (b.av)w) \\ &\quad + a(bv.w) - (ab.v)w - (a.bv)w = 0, \\ SBL(w, a, v, b) &= (-1)^{\bar{v}\bar{w}}((va.w)b - v(a.wb) \\ &\quad + a(vw.b) + a(v.wb) - (a.vw)b) \\ &\quad + v(aw.b) - (va.w)b - (v.aw)b) = 0, \end{aligned}$$

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where the sign $(-1)^{\bar{v}\bar{w}}$ can be eliminated. All these identities are just versions of the full linearization of the identity

$$(xy \cdot x)y + (yx \cdot y)x = 0,$$

which is equivalent to (1). Therefore, the parity of the elements v, w do not matter, and all the considered tensor products are isomorphic, as S-modules, to $V\hat{\otimes}W$. This proves the proposition. □

From this proposition and [4, Lemmas 5,6], we get the following useful corollary.

Corollary 3.1. *Let B be a finite dimensional SBL-algebra over $\mathbf{C}$ and $S \cong sl_2(\mathbf{C}) \subset B_0$ be a subalgebra. Then for every homogeneous Lie S-submodules V and W of B we get $(vw)a = (va)w + v(wa)$ for any $a \in S$, $v \in V$, $w \in W$.*

Recall the structure of irreducible binary Lie modules over the Lie algebra $S = sl(2, \mathbf{C})$ with the basis $\{A, H, X \mid AX = H, AH = 2A, XH = -2X\}$ (see [4]).

Every finite dimensional irreducible S-module is either a Lie module L_n with a basis $\{v_{-n}, v_{2-n}, \dots, v_{n-2}, v_n\}$ and the following S-action for $i \geq -n, j > -n, k \geq -n$:

$$v_i \cdot H = iv_i,\ v_j \cdot X = v_{j-2},\ v_k \cdot A = \frac{(n+k+2)(k-n)}{4} v_{k+2},\ v_{-n} \cdot X = 0,$$

or is isomorphic to the 2-dimensional non-Lie Malcev module $M_2 = \mathbf{C} \cdot m_{-2} + \mathbf{C} \cdot m_2$ with the following action of S:

$$m_{-2} \cdot A = m_2 \cdot X = 0,\ m_{-2} \cdot X = 2m_2,\ m_2 \cdot A = -2m_{-2},\ m_i \cdot H = im_i.$$

We will also need the following binary Lie module over S from [4]. Let V, U be vector spaces, $\bar{V}$ be an isomorphic copy of V with the isomorphism $v \mapsto \bar{v}$, and let the following linear mappings be defined:

$$\alpha : V \oplus \bar{V} \to U, \quad \beta : V \oplus \bar{V} \to U.$$

Then the direct vector space sum $V \oplus \bar{V} \oplus U$ with the following action of S

$$vH = 2v,\ \bar{v}H = -2\bar{v},$$
$$vA = -2\bar{v} + \alpha(v),$$
$$\bar{v}X = 2v + \beta(\bar{v}),$$
$$\bar{v}A = \alpha(\bar{v}),\ vX = \beta(v),$$
$$U \cdot S = 0,$$

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for $v \in V$, forms a binary Lie module which is called *a module of type* (n, m, α, β), where $m = \dim U,\ n = \dim V$.

It is easy to see that if W is an S-module of type (m, n, α, β) with $\alpha = \beta = 0$ then W is a direct sum of n Malcev modules M_2 and of m one-dimensional modules L_0.

The following proposition follows from the results of [4] and Proposition 3.1.

Proposition 3.2. *Let $S = sl(2, \mathbf{C})$ and let $\alpha, \beta \in \{even,\ odd\}$. Then*

(i) $(L_n)_\alpha \hat{\otimes} (M_2)_\beta = 0$ *if* $n \neq 2$;
(ii) $(L_n)_\alpha \hat{\otimes} (L_m)_\beta$ *is a Lie* S*-module;*
(iii) $(L_2)_\alpha \hat{\otimes} (M_2)_\beta \cong (M_2)_\gamma$, *where* $\gamma \in \{even,\ odd\}$ *is uniquely defined by* α, β. *Moreover, if* v_{-2}, v_0, v_2 *and* m_{-2}, m_2 *are canonical bases of* L_2 *and* M_2*, respectively, then the elements* $t_{-2} = v_2 \hat{\otimes} m_2, t_2 = -v_{-2} \hat{\otimes} m_{-2}$ *form a canonical base of* $(L_2)_\alpha \hat{\otimes} (M_2)_\beta$ *as a module of type* M_2.

4. Structure of B as an S-module

In this section we prove the following

Proposition 4.1. *Let $B = B_0 \oplus B_1$ be a simple finite dimensional SBL-algebra over the field $\mathbf{C}$ such that $B_1 \neq 0$, B is not a Lie superalgebra, and B_0 is not solvable. Then B_0 contains a simple Lie subalgebra $S \cong sl(2, \mathbf{C})$, and $B = (\sum_i \oplus V_i) \oplus (\sum_j \oplus W_j)$, where all $V_i \cong L_2$ and all $W_j \cong M_2$.*

We will need the following lemmas.

Lemma 4.1. *Let B be a finite dimensional SBL-algebra over $\mathbf{C}$ and $S \cong sl_2(\mathbf{C}) \subset B_0$ be a subalgebra. Then B is a completely reducible S-module.*

Proof. By [4, Theorem 3], every finite dimensional BL-module V over S has the form

$$V = V_l \oplus M,$$

where V_l is a Lie S-module and M is a module of type (n, m, α, β). The module V_l is completely reducible, and if $\alpha = \beta = 0$ then M is completely reducible as well. Assume that B is not completely reducible S-module, then by the above B contains an S-submodule I of type (n, m, α, β) with $\alpha \neq 0$ or $\beta \neq 0$. Denote $Z = \alpha(I) + \beta(I)$, then $Z \cdot S = 0$ by definition of α and β. It is also clear that $Z = Z_0 \oplus Z_1$, where $Z_i = Z \cap B_i,\ i = 0, 1$. By [4, lemmas 7, 8, 10] and Proposition 3.1 we have $Z_i \hat{\otimes} B_j = 0$ for all $i, j = 0, 1$. (Though lemmas 7, 8, 10 in [4] were proved for some particular values of n, m of modules of type (n, m, α, β), the proofs in fact are valid for arbitrary n, m.) As a corollary, we have $Z_i B_j = 0$ and eventually $ZB = 0$. Since B is simple, this implies $Z = 0$. Therefore, we have $\alpha = \beta = 0$ and B is completely reducible. □

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Lemma 4.2. *Let $B = B_0 \oplus B_1$ be a finite dimensional SBL-algebra over* $\mathbf{C}$*,* $S \simeq sl_2(\mathbf{C})$ *be a subalgebra of B_0 and B a Lie S-module. Then the ideal of B generated by S is a Lie superalgebra.*

Proof. Let $B = \sum_i \oplus B^{(i)}$ be a decomposition of B into a sum of eigen-superspaces with respect to H, that is, $B^{(i)} = \{v \in B \,|\, v \cdot H = iv\}$. Choose homogeneous $v \in B^{(i)}, w \in B^{(j)}$, $u \in B^{(k)}$, then for any $a \in S$ by the super-linearization of (1) we have

$$J_s(ua, v, w) + (-1)^{\bar{u}\bar{v}} J_s(va, u, w) + (-1)^{\bar{v}\bar{w}} J_s(uw, v, a) + (-1)^{\bar{u}(\bar{v}+\bar{w})} J_s(vw, u, a) = 0,$$

where $J_s(v, w, u) = vw \cdot u - (-1)^{\bar{w}\bar{u}} vu \cdot w - v \cdot wu$ is the *super-jacobian* of the elements v, w, u. By Corollary 3.1 we have

$$\begin{aligned} J_s(uw, v, a) &= (uw \cdot v)a + (a \cdot uw)v + (-1)^{\bar{v}(\bar{u}+\bar{w})}(va)(uw) \\ &= (ua \cdot w)v + (u \cdot wa)v + (uw)(va) \\ &+ (au \cdot w + u \cdot aw)v - (uw)(va) = 0. \end{aligned}$$

Therefore, we have

$$J_s(ua, v, w) + (-1)^{\bar{u}\bar{v}} J_s(va, u, w) = 0. \tag{2}$$

Assume first that among the numbers i, j, k there are at least two different, say, $i \neq k$. Substituting $a = H$ in (2), we get

$$0 = J_s(u \cdot H, v, w) - J_s(u, v \cdot H, w) = (k - i) J_s(v, w, u),$$

which implies $J_s(u, v, w) = 0$.

Let now $i = j = k > 0$. Then there exists $t \in B^{(i-2)}$ such that $u = t \cdot A$, and substituting $u = t, a = A$ in (2), we get

$$J_s(u, v, w) = J_s(t \cdot A, v, w) = J_s(t, v \cdot A, w).$$

Since $t \in B^{(i-2)}$, $v \cdot A \in B^{(i+2)}$, by the previous case $J_s(u, v, w) = 0$.

Furthermore, let $i = j = k < 0$, then there exists $t \in B^{(i+2)}$ such that $u = t \cdot X$. Substituting $a = X, u = t$ in (2), we get

$$J_s(u, v, w) = J_s(t \cdot X, v, w) = J_s(t, v \cdot X, w).$$

Since $t \in B^{(i+2)}$, $v \cdot X \in B^{(i-2)}$, we again obtain $J_s(u, v, w) = 0$.

Finally, consider the case $i = j = k = 0$. We may write $u = u_1 + u_2$, where for u_1 there exists $t \in B^{(-2)}$ such that $u_1 = t \cdot A$, and $u_2 \cdot S = 0$. For u_1, as before, we

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have $J_s(u_1, v, w) = J_s(t \cdot A, v, w) = 0$. Therefore, it remains to consider the case when $u, v, w \in B^{(0)}$ and $u \cdot S = v \cdot S = w \cdot S = 0$.

Note that if I is the ideal, generated by S, then $I = \sum_i SB^i$. Here $SB^0 = S$, $SB^{i+1} = (SB^i)B$. Let $u \in SB^s \setminus SB^{s-1} \subseteq I$, we prove that $u = \sum_{j\neq 0} a_j b_{-j}$, where $a_j \in B^{(j)}$, $b_{-j} \in B^{(-j)}$.

We will use induction on s. It is clear that $s > 0$ since $uS = 0$. If $s = 1$ then $u = Ab_{-2} + Xb_2$, which gives the base of the induction. Hence $u = \sum_j a_j b_{-j}$, where $a_j \in SB^{s-1}$ and $a_j \in B^{(j)}$, $b_{-j} \in B^{(-j)}$. Assume first that $a_0 S = 0$, then by induction $a_0 = \sum_{j\neq 0} c_j d_{-j}$, where $c_j \in B^{(j)}$, $d_{-j} \in B^{(-j)}$. By the previous cases, $J_s(c_j, d_{-j}, B) = 0$, hence

$$a_0 b_0 = (\sum_{j\neq 0} c_j d_{-j}) b_0 = \sum_{j\neq 0} (-1)^{\bar{b}_j \bar{d}_{-j}} (c_j b_0) d_{-j} + \sum_{j\neq 0} c_j (d_{-j} b_0) \in \sum_{j\neq 0} B^{(j)} B^{(-j)}.$$

If $a_0 S \neq 0$, then without loss of generality we may assume that there exists $t \in B^{(-2)}$ such that $a_0 = t \cdot A$, and we have by (2)

$$a_0 b_0 = (tA) b_0 = -t(b_0 A) + (t b_0) A \in B^{(-2)} B^{(2)}.$$

Now we have by the super-linearization of (1)

$$\begin{aligned} J_s(u, v, w) &= \sum_{j\neq 0} J_s(a_j b_{-j}, v, w) \\ &= \sum_{j\neq 0} (\pm J_s(v b_{-j}, a_j, w) \pm J_s(a_j w, v, b_{-j}) \pm J_s(vw, a_j, b_{-j})) \\ &\in \sum_{j\neq 0} J_s(B^{(j)}, B^{(-j)}, B^{(0)}) = 0. \end{aligned}$$

Therefore, $J_s(I \cap B^{(0)}, B, B) = 0$. Since $I = I \cap B^{(0)} + \sum_{i\neq 0} B^{(i)}$, this finishes the proof of the lemma. □

Lemma 4.3. *In the notations of Lemma 4.2 (without assumption that B is a Lie S-module) let $V \subset B$ be an irreducible Lie S-module of type L_n, $n \neq 2$, and I be the ideal generated by V. Then I is a Lie S-module.*

Proof. First we prove that $I \cdot M_2 = 0$, where $M_2 \subset B$ is a non-Lie S-module of type M_2 with a canonical basis $\{m_2, m_{-2}\}$. By Proposition 3.2, $V \cdot M_2 = 0$. Hence, if $V \cdot W \neq 0$ for some irreducible S-module W, then W is a Lie S-module.

Let us prove that $(V \cdot W) M_2 = 0$. Recall that B is a completely reducible S-module (see Lemma 4.1). Hence by Proposition 3.2(ii), $V \cdot W = \sum_i \oplus V_i$, where V_i are irreducible

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Lie S-modules. If $V_i \ncong L_2$ then $V_i \cdot M_2 = 0$ by Proposition 3.2(i). Let $V_k \cong L_2$. Then for some $v \in V, u \in W$, such that $vH = iv, uH = ju, i + j = 2$, by Proposition 3.2(iii) we have $vu \cdot m_2 = t_{-2}$, where t_{-2}, $t_{-2}X$ form a basis of an irreducible non-Lie S-module of type M_2. Observe that by Proposition 3.2 we have $vw_2 = uw_2v = 0$. Now by applying the defining identity $SBL(x, y, z, t) = 0$ we have

$$\begin{aligned}0 &= vuw_2H \pm vw_2uH \pm Huw_2v \pm Hw_2uv \\ &\quad + (-1)^{\bar{v}\bar{u}}(uvHw_2 + uHvw_2) \pm w_2Hvu \pm w_2vHu \\ &\quad = -2vuw_2 + (-1)^{\bar{v}\bar{u}}(2uvw_2 + juvw_2) = -(j+4)t_{-2}; \\ 0 &= vuHw_2 + vHuw_2 \pm w_2uHv \pm w_2Huv \\ &\quad + (-1)^{\bar{v}\bar{u}}uvw_2H \pm uw_2vH \pm w_2vu \pm vw_2u \\ &\quad = 2vuw_2 + ivuw_2 - 2(-1)^{\bar{v}\bar{u}}uvw_2 = (i+4)t_{-2}.\end{aligned}$$

Hence $i = j = -4$, a contradiction.

We proved that $(VW)M_2 = 0$ for any irreducible S-submodule W of B. Assume that $(...(VW_1)...)W_n)M_2 = 0$ for any irreducible S-submodules $W_1, ..., W_n$ of B, $n > 0$. Let $(...(VW_1)...)W_n) = \sum_k \oplus U_k$, where U_k are irreducible Lie S-modules and $U_kM_2 = 0$. Hence every U_k has the same property as V, and we can prove as above that $(U_kW_{n+1})M_2 = 0$ for every irreducible S-submodule W_{n+1} of B. Then $((\sum_k \oplus U_k)W_{n+1})M_2 = (...(VW_1)...)W_n)W_{n+1}))M_2 = 0$.

Hence $IM_2 = 0$. Since $I = \sum_n (...(VW_1)...)W_n)$ where all W_i are Lie S-modules, then I is a Lie S-module. □

Proof of Proposition 4.1. Assume that B as S-module contains an irreducible Lie submodule V of type L_n, $n \neq 2$. Then the ideal I generated by V would be a non-zero ideal of B. Since B is simple then $I = B$, and by Lemma 4.3 B is a Lie S-module. Then by Lemma 4.2 B is a Lie superalgebra, a contradiction. □

5. The main theorem

Theorem 5.1. *Let $B = B_0 \oplus B_1$ be a finite dimensional simple SBL-algebra over the field* $\mathbf{C}$. *If $B_1 \neq 0$ and B is not a Lie superalgebra, then the even part B_0 of B is solvable.*

Proof. Assume that B_0 is not solvable and B is not a Lie superalgebra, then by Proposition 4.1 we get that B_0 contains a simple Lie subalgebra $S \cong sl(2, \mathbf{C})$ such that $B = (\sum_i \oplus V_i) \oplus (\sum_j \oplus W_j)$, where $V_i \cong L_2$ and $W_j \cong M_2$ for all i, j. Consider the Grassmann envelope $M = \Gamma(B) = B_0 \otimes \Gamma_0 \oplus B_1 \otimes \Gamma_1$, of the superalgebra B, where $\Gamma = \Gamma_0 \oplus \Gamma_1$ is the Grassmann algebra. Then $\Gamma(B)$ is a binary Lie algebra. By construction, the BL-algebra M has a subalgebra $S_0 = S \otimes \mathbf{C1} \simeq sl_2(\mathbf{C})$, where $\mathbf{1}$ is the unit element of the Grassmann algebra Γ. Moreover, the algebra M as an S_0-module has a decomposition into a direct sum of 3- and 2-dimensional S_0-modules. By Theorem 1 of

[5], M is a Malcev algebra. Hence B is a Malcev superalgebra, but by [9] any non-trivial simple Malcev superalgebra is a Lie superalgebra. □

Data availability

No data was used for the research described in the article.

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