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Homogeneous Random Processes on Locally Compact Abelian Groups*

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1. INTRODUCTION

The purpose of this paper is to use the methodology of Brillinger and Rosenblatt [2, 3] to investigate the properties of the finite Fourier transform for the case of homogeneous random processes defined on locally compact Abelian groups. In this methodology the key role is played by the finite Fourier transform. If $\{X(t),\ t = 0, \pm 1, \pm 2, \ldots\}$ is a (weakly) stationary random process, then for the values $X(0), \ldots, X(T-1)$ of the process, the finite Fourier transform is defined by

$$d^{(T)}(\lambda) = (2\pi T)^{-1/2} \sum_{t=0}^{T-1} X(t) e^{-i\lambda t}, \quad -\infty < \lambda < +\infty. \tag{1.1}$$

In particular, the asymptotic properties of $d^{(T)}(\lambda)$ are fundamental to derive estimation procedures for the spectrum of $X(t)$, defined by

$$f(\lambda) = (2\pi)^{-1} \sum_{\tau=-\infty}^{\infty} R(\tau) e^{-i\lambda\tau}, \tag{1.2}$$

$-\infty < \lambda < \infty$, where $R(\tau)$ is the covariance function of the series supposed to "wear off" as $|\tau| \to \infty$.

In what follows we consider a homogeneous random process $\{X(t), t \in G\}$ on a locally compact Abelian group G which is second countable. We define the finite Fourier transform and prove a central limit theorem for it. Some remarks concerning the estimation of the spectrum are also made.

2. THE FOUNDATIONS

Let G be a locally compact Abelian group (LCAG) with addition as operation, and let m be the Haar measure of G. If $f \in L_1(G)$, the following invariance relation is valid:

$$\int_G f(x+a)\, dm(x) = \int_G f(x)\, dm(x), \quad a \in G. \tag{2.1}$$

Let us denote by $\hat{G}$ the *dual* group of G, that is, the set of all continuous characters of G, with composition defined by $(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x)$, $x \in G$, $\gamma_1, \gamma_2 \in \hat{G}$. $\hat{G}$ becomes a LCAG if it is endowed with a proper topology. See Rudin (1963) for details.

If G is a LCAG, for any $f \in L_1(G)$, the Fourier transform of f is defined by

$$\hat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}\, dm(x), \quad \gamma \in \hat{G}. \tag{2.2}$$

A function ϕ defined on G is called *positive definite* (p.d.) if and only if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \bar{c}_j \phi(x_i - x_j) \geq 0, \tag{2.3}$$

for any $x_1, \ldots, x_n \in G$ and any complex numbers $c_1, \ldots, c_n$. Bochner's theorem states that ϕ is p.d. if and only if there exists a non-

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negative measure $\mu \in M(\hat{G})$ such that

$$\phi(x) = \int_{\hat{G}} \gamma(x) d\mu(\gamma), \; x \in G. \qquad (2.4)$$

Here, $M(\hat{G})$ is the set of all complex measures on $\hat{G}$ which are regular and such that $|\mu|(\hat{G}) < \infty$, where $|\mu|$ denotes the total variation of μ.

We denote by $B(G)$ the set of all linear, finite combinations of continuous functions, p.d., on G; the inversion theorem establishes that, if $f \in L_1(G) \cap B(G)$ then $\hat{f} \in L_1(\hat{G})$ and

$$f(x) = \int_{\hat{G}} \hat{f}(\gamma)\gamma(x) d\gamma, \quad x \in G, \qquad (2.5)$$

where the Haar measure $d\gamma$ on G is appropriately normalized.

We shall denote by cum $(Y_1, \ldots, Y_p)$ the *cumulant of order* p of the p-dimensional random variable $(Y_1, \ldots, Y_p)$, with Y_j real or complex and $E\{|Y_j|^p\} < \infty$, $j = 1, \ldots, p$. In particular, $\mathrm{cum}\,(Y_i, \bar{Y}_j) = \mathrm{Cov}\,\{Y_i, Y_j\} = E\{[Y_i - E(Y_i)]\overline{[Y_j - E(Y_j)]}\}$. See Brillinger [2] for details on relevant properties of the cumulants.

The p-variate complex random variable $\underset{\sim}{X}$ is said to have a *multivariate complex normal distribution* $N_p^c(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ if $[\mathrm{Re}\,\underset{\sim}{X}, \mathrm{Im}\,\underset{\sim}{X}]'$ has a (2p-variate) multivariate normal distribution with mean $[\mathrm{Re}\,\underset{\sim}{\mu}, \mathrm{Im}\,\underset{\sim}{\mu}]'$ and covariance matrix

$$\frac{1}{2}\begin{bmatrix} \mathrm{Re}\,\underset{\sim}{\Sigma} & -\mathrm{Im}\,\underset{\sim}{\Sigma} \\ \mathrm{Im}\,\underset{\sim}{\Sigma} & \mathrm{Re}\,\underset{\sim}{\Sigma} \end{bmatrix},$$

for some p-vector $\underset{\sim}{\mu}$ and some $p \times p$ matrix $\underset{\sim}{\Sigma}$, Hermitian and non-negative definite. In the case $p=1$, if X is $N_1^c(\mu, \sigma^2)$, then ReX and ImX are independent random variables $N_1(\mathrm{Re}\mu, \sigma^2/2)$ and $N_1(\mathrm{Im}\mu, \sigma^2/2)$, respectively. For further properties of the complex normal distribution see Goodman [6].

Now let $(\Omega, \mathscr{A}, P)$ be a probability space and consider the random function $X = \{X(t,\omega) : t \in G, \omega \in \Omega\}$, with covariance function $R(t,s) = \mathrm{Cov}\,\{X(t,\omega), X(s,\omega)\}$ assumed to be continuous on $G \times G$. It follows that X is continuous in quadratic mean (q.m.).

DEFINITION: The random process X is *weakly homogeneous* if and only if

a) $E[X(t)] = c = \text{constant}$, for all $t \in G$;
b) $X(t) \in L_2(\Omega, \mathscr{A}, P)$, for all $t \in G$;
c) $E[X(t)\overline{X(s)}] = B(t-s)$. (2.6)

Here, $L_2(\Omega, \mathscr{A}, P)$ is the Hilbert space of all quadratic integrable random variables on $(\Omega, \mathscr{A}, P)$, with inner product $(U,V) = E(U\bar{V})$. Without loss of generality, we shall assume that $c=0$, hence $B(t-s)$ is the covariance function $R(t-s)$, and we have

$$R(\tau) = E\{X(t+\tau)\overline{X(t)}\}, \quad t, \tau \in G. \qquad (2.7)$$

DEFINITION: The random process X is *strictly homogeneous* iff for any $t, t_1, \ldots, t_k$ of G, the random variables $(X(t_1), \ldots, X(t_k))$ and $(X(t_1+t), \ldots, X(t_k+t))$ have the same distribution.

If X is strictly homogeneous, with $E|X(t)|^k < \infty$, we define the *k-th order cumulant function* of X as

$$c_k(t_1, \ldots, t_k) = \mathrm{cum}\,\{X(t_1), \ldots, X(t_k)\}, \qquad (2.8)$$

which is equal to $c_k(t_1+t, \ldots, t_k+t)$, for any $t_1, \ldots, t_k, t$ in G. We hall use frequently the asymmetric notation

$$c_k(t_1, \ldots, t_{k-1}) = c_k(t_1, \ldots, t_{k-1}, 0). \qquad (2.9)$$

The function $R(\tau)$ is easily seen to be positive definite and by Bochner's theorem (see 2.4), there exists a measure $F \in M(\hat{G})$ such that

$$R(\tau) = \int_{\hat{G}} \gamma(\tau) F(d\gamma). \qquad (2.10)$$

The measure F is uniquely determined by R and it is called the *spectral measure* of X.

A spectral representation of X is given by

$$X(t) = \int_{\hat{G}} \gamma(t)\, Z(d\gamma), \quad t \in G, \qquad (2.11)$$

where $Z(\Delta)$ is a *random measure* such that

$$E\{|Z(\Delta)|^2\} = F(\Delta), \qquad (2.12)$$

for every $\Delta \in \hat{G}$. For details, see Jajte [7].

Now, assume that $R(\tau) \in L_1(G)$, that is,

$$\int_G |R(\tau)| dm(\tau) < \infty. \qquad (2.13)$$

Then, we define the *spectrum* of X by

$$f(\gamma) = \int_G R(t)\overline{\gamma(t)}dm(t), \quad \gamma \in \hat{G}. \qquad (2.14)$$

By (2.13), $f(\gamma)$ is bounded in G and continuous as a map from a part of $L_1(G)$ (corresponding to the class of homogeneous random functions) in the set of all complex-valued, bounded and continuous functions on $\hat{G}$, which vanish at infinity. Since $R(\tau) \in L_1(G) \cap B(G)$, the inversion theorem (2.5) is applicable and we have

$$R(\tau) = \int_{\hat{G}} f(\gamma)\gamma(\tau)d\gamma, \quad \tau \in G, \qquad (2.15)$$

where $d\gamma$ is the Haar measure of $\hat{G}$, appropriately normalized.

If the condition,

$$\int_G \cdots \int_G |c_k(t_1,\ldots,t_{k-1})| dm(t_1)\ldots dm(t_{k-1}) < \infty \qquad (2.16)$$

is satisfied we define the *k-th order cumulant spectrum* of X by

$$f_k(\gamma_1,\ldots,\gamma_{k-1}) = $$
$$= \int_G \cdots \int_G c_k(t_1,\ldots,t_{k-1})\gamma_1(t_1)\ldots$$
$$\ldots\gamma_{k-1}(t_{k-1})dm(t_1)\ldots dm(t_{k-1}),$$
$$\gamma_1,\ldots,\gamma_{k-1} \in \hat{G}. \qquad (2.17)$$

3. THE FINITE FOURIER TRANSFORM

Suppose that G is a non compact LCAG which is second countable. Let $X = \{X(t), t \in G\}$ be a homogeneous, continuous in quadratic mean, real-valued random process on G, with mean zero and covariance function $R(\tau)$. Consider a sequence of subsets E_n of G, such that:

(i) $E_n \subset E_{n+1}$, $n = 1,2,\ldots$, $\bigcup_{n=1}^{\infty} E_n = G$; (3.1)

(ii) E_n is compact and $m(E_n) \to +\infty$, $n = 1,2,\ldots$, where $m(E_n) = \nu_n$ is the Haar measure of E_n.

DEFINITION: The *finite Fourier transform* of $X(t)$, $t \in E_n$ is defined to be

$$d^{(n)}(\gamma) = \nu_n^{-1/2} \int_{E_n} X(t)\overline{\gamma(t)}dm(t), \quad \gamma \in \hat{G} \quad (3.2)$$

It follows that $d^{(n)}(\gamma)$ is a random variable with mean zero. Our purpose is to derive the asymptotic distribution of $d^{(n)}(\gamma)$. Let us set

$$\Delta^{(n)}(\gamma) = \int_{E_n} \gamma(t)dm(t). \qquad (3.3)$$

It is easy to see that there exists a subsequence $\{E_{n_k}\}$ such that

$$\frac{\Delta^{(n_k)}(\gamma)}{\nu_{n_k}} \to 0, \qquad (3.4)$$

for almost all $\gamma \in \hat{G}$.

See [1] for details. We shall write $\int_G f(t)dt$ to denote $\int_G f(t)dm(t)$. We assume that X satisfies

ASSUMPTION A: The process $X = \{X(t), t \in G\}$ is real-valued, strictly homogeneous, with finite moments of all orders and satisfies (2.17) for $k = 2,3,\ldots$.

We see that the cumulant spectra of all orders exist for X satisfying Assumption A. If X is Gaussian, (2.17) reduces to (2.13).

Denote by F_n the complement of E_n relative to G.

THEOREM 3.1. Let $d^{(n)}(\gamma)$ be given by (3.2) and assume that for all $s \in G$ and $n \to \infty$,

$$\int_{F_n} |R(t-s)|dt \le k\,\nu_n^{-1}, \qquad (3.5)$$

where k is a positive constant. Then,

$$\mathrm{Cov}\{d^{(n)}(\gamma_1), d^{(n)}(\gamma_2)\} = $$
$$= \nu_n^{-1}\Delta^{(n)}(\gamma_2 - \gamma_1)f(\gamma_1) + 0(\nu_n^{-1}), \qquad (3.6)$$

the error being uniform in γ_1, γ_2.

Proof: Since $E[X(t)] = 0$, we have

$$\mathrm{Cov}\{d^{(n)}(\gamma_1), d^{(n)}(\gamma_2)\} = $$
$$= \nu_n^{-1} \cdot \int_{E_n}\int_{E_n} \overline{\gamma_1(t)}\gamma_2(s)E[X(t)X(s)]dtds =$$
$$= \nu_n^{-1} \cdot \int_{E_n}\int_{E_n} \overline{\gamma_1(t)}\gamma_2(s)R(t-s)dt\,ds,$$

since the modulus of this last integral is

$$\leq \nu_n^{-1}\int_{E_n}\int_{E_n}|R(t-s)|dt\,ds \leq$$

$$\leq \nu_n^{-1}\int_{E_n}\int_{G}|R(t)|dt\,ds < \infty.$$

Taking in account that $\gamma_1[(t-s)+s] = \gamma_1(t-s)\cdot\gamma_1(s)$ and $\gamma_1(u) = -\gamma_1(u)$, the covariance in question becomes

$$\nu_n^{-1}\cdot\int_{E_n}\left[\int_{E_n}\overline{\gamma_1(t-s)}R(t-s)dt\right](\gamma_2-\gamma_1)(s)ds =$$

$$=\nu_n^{-1}\int_{E_n}\left[\int_{G}\overline{\gamma^1(t)}R(t)dt\right](\gamma_2-\gamma_1)(s)ds + \varepsilon_n$$

using the invariance relation (2.1), and where

$$|\varepsilon_n| \leq \nu_n^{-1}\int_{E_n}\int_{F_n}|\overline{\gamma_1(t-s)}\,R(t-s)(\gamma_2-\gamma_1)(s)|ds =$$

$$= \int_{E_n}\int_{F_n}|R(t-s)|dtds \leq k.\nu_n^{-1}$$

by (4.5). The theorem follows noting that $\int_G R(t)\overline{\gamma_1(t)}dt = f(\gamma_1)$ and $\int_{E_n}(\gamma_2-\gamma_1)(s)ds = \Delta^{(n)}(\gamma_2-\gamma_1)$.

REMARKS: (a) Condition (3.5), which is a "mixing" condition implies that $\int_{F_n} R(t-s)dt = 0(\nu_n^{-1})$, $s \in G$;

(b) For $\gamma_1 = \gamma_2$, the theorem gives the variance of $d^{(n)}(\gamma)$,

$$\text{Var}\{d^{(n)}(\gamma)\} = E\{|d^{(n)}(\gamma)|^2\} = \\ = f(\gamma)+0(\nu_n^{-1}), \tag{3.7}$$

since $\Delta^{(n)}(0) = \nu_n$. This relation suggests that an estimate for the spectrum $f(\gamma)$ is given by

$$|d^{(n)}(\gamma)|^2 = \nu_n^{-1}\left|\int_{E_n}X(t)\overline{\gamma(t)}dt\right|^2. \tag{3.8}$$

The theorem 3.1 can be generalized for cumulants of order k. Let A_n denote $E_n x \ldots x E_n = E_n^{k-1}$ and B_n denote the complement of A_n in G^{k-1} (which is a topological group when endowed with the product topology).

THEOREM 3.2. Let X satisfying Assumption A and $f_k(\gamma_1,\ldots,\gamma_{k-1})$ be defined by (2.17). Then, for $k > 2$,

$$\text{cum}\{d^{(n)}(\gamma_1),\ldots,d^{(n)}(\gamma_k)\} = \\ = \nu_n^{-k/2}.\,\Delta^{(n)}(\gamma_1+\ldots+\gamma_k)f_k(\gamma_1,\ldots,\gamma_{k-1}) + \\ + 0(\nu_n^{-k/2+1}). \tag{3.9}$$

Proof. The proof follows along the same lines as Theorem 3.1. and it will be sketched.

The cumulant in question is equal to

$$\nu_n^{-k/2}.\int_{E_n}\cdots\int_{E_n}\overline{\gamma_1(t_1)}\ldots\overline{\gamma_k(t_k)}c_k(t_1,\ldots,t_k)dt_1\ldots dt_k) =$$

$$= \nu_n^{-k/2}.\int_{E_n}(\gamma_1+\ldots+\gamma_k)(s)\cdot$$

$$\cdot\left[\int_{E_n}\cdots\int_{E_n}\gamma_1(t-s)\ldots\gamma_{k-1}(t-s)\cdot\right.$$

$$\left.\cdot c_k(t_1-s,\ldots,t_{k-1}-s)dt_1\ldots dt_{k-1}\right]ds =$$

$$= \nu_n^{-k/2}.\,\Delta^{(n)}(\gamma_1+\ldots+\gamma_k)f_k(\gamma_1,\ldots,\gamma_{k-1}) + R_n,$$

where

$$|R_n| \leq \nu_n^{-k/2}.\int_{E_n}\left[\int_{B_n}|c_k(t-s,\ldots,t_{k-1}-s)|\cdot\right.$$

$$\left.\cdot\, dt_1\ldots dt_{k-1}\right]ds \leq C.\nu_n^{-k/2+1},$$

a finite constant, by (3.13).

REMARK: If $\gamma_1+\ldots+\gamma_k = 0$, then the cumulant of the left handside of (3.9) is approximately equal to $\nu_n^{-k/2+1}f_k(\gamma_1,\ldots,\gamma_{k-1})$, which suggests that we can estimate $f_k(\gamma_1,\ldots,\gamma_{k-1})$ using $d^{(n)}(\gamma_1),\ldots,d^{(n)}(\gamma_k)$, with $\sum_{i=1}^{k}\gamma_i = 0$. The following lemma is well-known (see [2], Chapter 4).

LEMMA 3.1. Let $\{\underset{\sim}{Z}^{(n)}, n \geq 1\}$ be a sequence of $p \times 1$ random vectors with complex components. Assume that the cumulants of the r.v. $(Z_1^{(n)},\overline{Z}_1^{(n)},\ldots,Z_p^{(n)},\overline{Z}_p^{(n)})$ exists and converge to the corresponding cumulants of a r.v. $(Z_1,\overline{Z}_1,\ldots,Z_p,\overline{Z}_p)$, which is determined by its moments. Then $\{\underset{\sim}{Z}^{(n)}\}$ converges in distribution to $\underset{\sim}{Z} = (Z_1,\ldots,Z_p)$.

Then, we have the following

THEOREM 3.3. Under the conditions of Theorem 3.2., if $\gamma \neq 0$ $d^{(n)}(\gamma)$ is asymptotically $N_1^c(0,f(\gamma))$ and if $\gamma=0$, $d^{(n)}(\gamma)$ is asymptotically $N_1(0,f(0))$. Moreover, there exists a subsequence $\{E_{n_k}\}$ such that $\{d^{(n_k)}(\gamma_j)\}$ are asymptotically independent.

REMARKS: (a) For $\gamma=0$ we obtain a central limit theorem for $v_n^{-1/2}\int_{E_n} X(t)dt$, that is

$$v_n^{-1/2}\cdot \int_{E_n} X(t)dt \xrightarrow{\mathscr{D}} N_1(0,f(0)).$$

(b) If $X(t)$ is Gaussian, then $d^{(n)}(\gamma)$ is a complex normal random variable, hence it is enough to prove that the asymptotic variance if $f(\gamma)$ or $f(0)$, for $\gamma\neq 0$ or $\gamma=0$, respectively.

(c) It would be desirable to prove the asymptotic independence of the $d^{(n)}(\gamma_j)$ for the entire sequence $\{E_n\}$. This is possible for particular groups. It is easy to see that

$$E\{d^{(n)}(\alpha)\overline{d^{(n)}(\beta)}\} = \int_{\hat{G}} \Delta^{(n)}(\alpha-\alpha')\overline{\Delta^{(n)}(\beta-\alpha')}dF(\alpha'),$$

and the behaviour of the integral depends on the particular form of the kernel $\Delta^{(n)}(\gamma)$.

Proof of Theorem 3.3. By Lemma 3.1., it is sufficient to prove that the cumulants, appropriately normalized, tend to the cumulants of the asserted normal distribution. We have that $E\{d^{(n)}(\gamma)\} = 0$ and by Theorem 3.1.,

$$\begin{aligned}&\mathrm{Cov}\,\{d^{(n)}(\gamma_1),d^{(n)}(\gamma_2)\} = \\ &= v_n^{-1}\cdot\Delta^{(n)}(\gamma_2-\gamma_1)f(\gamma_1)+0(v_n^{-1});\end{aligned}$$

this tends fo $f(\gamma)$ for $\gamma_1=\gamma_2=\gamma\neq 0$; the asymptotic independence follows from (3.4). By Theorem 3.2.,

$$\begin{aligned}&\mathrm{cum}\,\{d^{(n)}(\gamma_1),\ldots,d^{(n)}(\gamma_k)\} = \\ &= v_n^{-k/2}\Delta^{(n)}(\gamma_1+\ldots+\gamma_k)f_k(\gamma_1,\ldots,\gamma_{k-1})+ \\ &\quad + 0(v_n^{-k/2+1}),\end{aligned}$$

which tends to zero, if $k>2$ since $\Delta^{(n)}(\cdot)|v_n$ is bounded with respect to n. If $\gamma=0$, $\overline{d^{(n)}(0)}=d^{(n)}(0)$, hence $d^{(n)}(0)$ is real and the stated limit distribution follows. The theorem is proved, since it is easily seen that the above argument holds for the conjugates of the involved variables.

4. FURTHER COMMENTS

We conclude this work with some final remarks.

a) Relation (3.7.) suggests a way to estimate the spectrum $f(\gamma)$.

Let us define

$$I^{(n)}(\gamma) = v_n^{-1}|d^{(n)}(\gamma)|^2,\quad \gamma\in\hat{G}. \tag{4.1}$$

$I^{(n)}(\gamma)$ is called the *periodogram* of $X(t)$, $t\in E_n$. It is easily seen from (3.7.) that

$$E\{I^{(n)}(\gamma)\} = f(\gamma)+0(v_n^{-1}), \tag{4.2}$$

that is, $I^{(n)}(\gamma)$ is an asymptotically unbiased estimate for $f(\gamma)$. From Theorem 3.3. we see immediately that $I^{(n)}(\gamma)$ is a random variable with an asymptotic distribution $f(\gamma)\chi^2(2)/2$, if $\gamma\neq 0$, where $\chi^2(2)$ is a random variable with a chi-square distribution with two degrees of freedom. Moreover, the asymptotic variance of $I^{(n)}(\gamma)$ is $f^2(\gamma)$, which shows that the periodogram is not consistent. For $\gamma=0$, the asymptotic distribution is $f(0)\chi^2(1)$ and the asymptotic variance is $2f^2(0)$.

b) The case of the group $G=\mathbb{R}$ of reals is considered by Brillinger [2] and Brillinger and Rosenblatt [3]. In this case condition (3.5.) is satisfied if we assume

$$\int_{-\infty}^{\infty} |\tau|\,|R(\tau)|d\tau < \infty.$$

The case of the group $G=D$, the dyadic group with operation addition modulo 2 componentwise is considered by Morettin [8].

c) The problem of computing the fast Fourier transform on finite Abelian groups is considered by Cairns [4]. It is shown that the generalized Fourier coefficients given by

$$Z(j) = \sum_{k=0}^{N-1} X(t_k)\overline{\gamma_j(t_k)} \tag{4.3}$$

can be computed with $N(r+s)$ complex multiplications. Here, N is the order (number of elements) of G and γ_j are the characters of G. For an arbitrary subgroup H of G, of order s, let $A(\hat{G},H) = \{\gamma\in\hat{G}:\gamma(t)=1, \forall t\in H\}$. The conclusion stated depends on this annihilator of H in $\hat{G}$, which is a subgroup of $\hat{G}$; r is the order of $G|H$.

d) The development of section 3 may be extended for multivariate processes $\underset{\sim}{X}(t) = [X_1(t),\ldots,X_p(t)]'$, $t \in G$. In particular, the asymptotic distribution for $\underset{\sim}{d}^{(n)}(\gamma)$ will be a multivariate complex normal with dimension p.

e) Problems related to ergodicity are considered by Blum and Eisenberg [1] and Jajte [7]. Generalized homogeneous random processes are discussed by Ponomarenko [9] and for the questions of prediction and interpolation of homogeneous processes on groups see Weron [12].

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6. SUMMARY

The purpose of this paper is to investigate the properties of the finite Fourier transform for the case of homogeneous processes defined on locally compact Abelian groups. A central limit theorem for this transform is proved and some remarks concerning the estimation of the spectrum are also made.

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