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A RESULT ON HAMILTON-CONNECTED GRAPHS

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ABSTRACT

A graph is hamilton-connected if every pair of distinct vertices are extremes of a hamilton path. We prove that if G is a non-bipartite, n-regular graph with 2n vertices then G is hamilton-connected.

All graphs considered here are simple. The vertex set of a graph G is denoted by *VG* and the edge set by *EG*. An edge $e \in EG$ with ends u and v is denoted by *uv*. If $v \in VG$, $N(v)$ denotes the set of vertices in $VG \setminus \{v\}$ which are adjacent with v.

A *path* in G is a finite non-null sequence $P = v_0 e_1 v_1 e_2 \ldots e_k v_k$, whose terms are alternately vertices (all distinct) and edges, such that, for $1 \le i \le k$, the ends of e_i are v_{i-1} and v_i. The set of vertices of P is denoted by *VP*. The vertices v_0 and v_k are called *origin* and *terminus* of P, respectively. If w is a vertex in P, w_s denotes the successor of w in P, w_p denotes the predecessor of w in P and w_{pp} denotes the predecessor of w_p. If $X \subseteq VP$ then $X_p = \{v \in VP: v_s \in X\}$ and $X_s = \{v \in VP: v_p \in X\}$. A *section* of P is a path that is a subsequence $v_i e_{i+1} v_{i+1} \ldots e_j v_j$ of consecutive terms of P and is denoted by $P(v_i, v_j)$. The path $v_j e_j \ldots v_{i+1} e_{i+1} v_i$ obtained by reversing $P(v_i, v_j)$ is denoted by $P^{-1}(v_j, v_i)$. The concatenation of two paths P and Q is denoted by $P \circ Q$. If $Q = u\ e\ v$ then we may refer to $P \circ Q$ as $P \circ uv$.

<u>THEOREM</u>. If G is a non-bipartite, n-regular graph with 2n vertices, then G is hamilton-connected.

<u>Proof</u>. Suppose G is not hamilton-connected. Then, there exist vertices u,v and there is no hamilton path with ends u and v.

Let $\mathcal{P}$ be the set of hamilton paths with origin u or v and $\mathcal{P}^*$ be the set of paths in $\mathcal{P}$ whose section with ends u and v has maximum length.

Without loss of generality, let P be a path with origin v and terminus x, $P \in \mathcal{P}$.

The results R1, R2,..., R7 listed below refer to the path P and will be used to show that G is bipartite.

R1. There is no vertex in $P(v,u_p)$ adjacent with x whose predecessor is adjacent with u_p.

R2. There is no vertex in $P(u,x)$ adjacent with x whose sucessor is adjacent with u_p.

R3. A vertex in $P(v,u_p)$ is adjacent with x if and only if its predecessor is not adjacent with u_p.

R4. A vertex is adjacent with v if and only if its predecessor is adjacent with x.

R5. If $P \in \mathcal{P}^*$ then x is not adjacent with two consecutive vertices of $P(v,u)$.

R6. If $P \in \mathcal{P}^*$ then v is adjacent with u_p.

R7. If $P \in \mathcal{P}^*$ then u is the predecessor of x.

The diagram shown in figure 1 relates the results listed, according to the proof which will be presented subsequently. An arc from a node i to a node j indicates that the result in i will be used in the proof of the result in j.

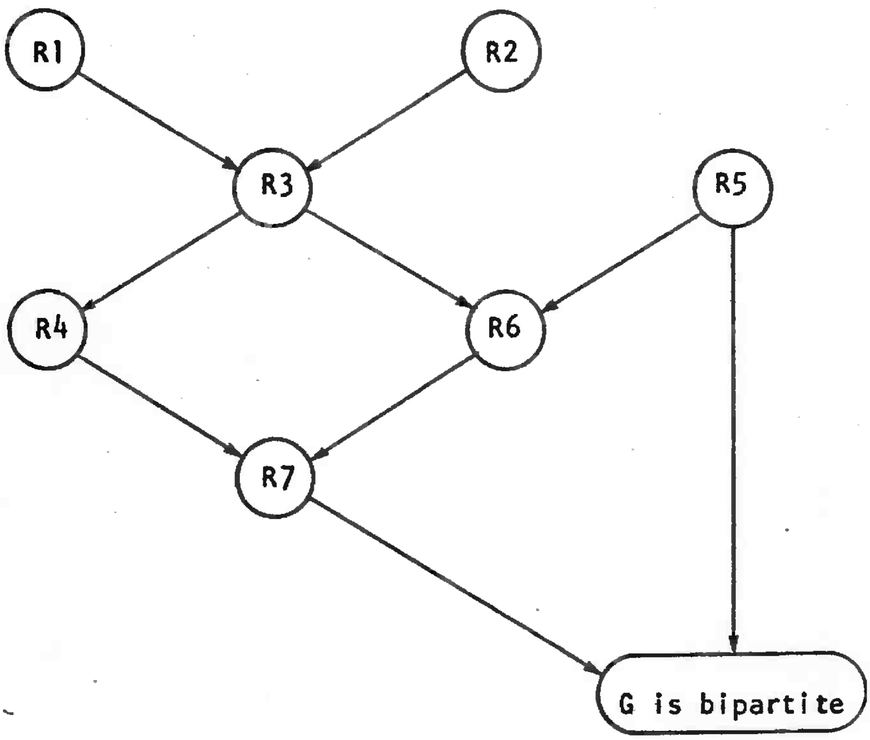

Figure 1

R1. $VP(v,u_p) \cap N(x) \cap (N(u_p))_s = \phi$

Suppose R1 is false. Thus, there is a vertex $z \in VP(v,u_p)$ such that $zx \in EG$ and $z_p u_p \in EG$. In this case,

$$P(v,z_p) \bullet z_p u_p \bullet P^{-1}(u_p,z) \bullet zx \bullet P^{-1}(x,u)$$

is a hamilton path with ends u and v, a contradiction.

R2. $VP(u,x) \cap (N(u_p))_p \cap N(x)=\phi$

Suppose R2 does not hold. Thus, there is a vertex $z \in VP(u,x)$ such that $zx \in EG$ and $u_p z_s \in EG$. But then,

$$P(v,u_p) \bullet u_p z_s \bullet P(z_s,x) \bullet xz \bullet P^{-1}(z,u)$$

is a hamilton path with ends u and v, a contradiction.

R3. $VP(v,u_p) \cap N(x) = VP(v,u_p)\backslash(N(u_p))_s$

From R1,

$$VP(v,u_p) \cap N(x) \subseteq VP(v,u_p)\backslash(N(u_p))_s. \qquad (1)$$

From R2,

$$VP(u,x) \cap N(x) \subseteq VP(u,x_p)\backslash(N(u_p))_p. \qquad (2)$$

Thus,

$$n = |N(x)| \le |VP(v,u_p)\backslash(N(u_p))_s| + |VP(u,x_p)\backslash(N(u_p))_p|. \qquad (3)$$

But,

$$|VP(v,u_p)\backslash(N(u_p))_s| = |VP(v,u_p)| - |VP(v,u_p) \cap (N(u_p))_s| =$$

$$= |VP(v,u_p)| - |VP(v,u_{pp}) \cap N(u_p)|. \qquad (4)$$

Analogously,

$$|VP(u,x_p)\backslash(N(u_p))_p| = |VP(u,x_p)| - |VP(u_s,x) \cap N(u_p)|. \qquad (5)$$

From (3), (4) and (5),

$$n = |N(x)| \le 2n-1 - (n-1) = n.$$

This implies that equality holds in (3) and therefore, equality holds in (1) and (2).

R4. $N(v) = (N(x))_s$.

Let $z \in N(x)$. Then,

$$P' = P(v,z) \bullet zx \bullet P^{-1}(x,z_s)$$

is a hamilton path with origin v and terminus z_s.

From R3,

$$VP'(v,u_p) \cap N(z_s) = VP'(v,u_p) \setminus (N(u_p))_s,$$

and thus, z_s is adjacent with v.

Therefore,

$$(N(x))_s \subseteq N(v).$$

As $|(N(x))_s| = n = |N(v)|$, the result follows.

R5. If $P \in \mathcal{P}^*$ then $VP(v,u) \cap N(x) \cap (N(x))_s = \phi$

Suppose there is a vertex $z \in VP(v,u)$ such that $zx \in EG$ and $z_p x \in EG$. Thus,

$$P(v,z_p) \bullet z_p x \bullet xz \bullet P(z,x_p)$$

is a hamilton path with origin v whose section with ends v and u has length greater than the length of $P(v,u)$, a contradiction.

R6. If $P \in \mathcal{P}^*$ then $v \in N(u_p)$.

From R3, $v \in N(x)$. Thus, from R5, $v_s \notin N(x)$.

From R3, $v_s \notin VP(v,u_p) \setminus (N(u_p))_s$.

As $v \neq u_p$, it follows that $v_s \in VP(v,u_p)$. Thus, $v_s \in (N(u_p))_s$ and therefore, $v \in N(u_p)$.

R7. If $P \in \mathcal{P}^*$ then $x = u_s$.

From R6, $u_p \in N(v)$. Thus, from R4, $u_{pp} \in N(x)$. Then,

$Q = P(v,u_{pp}) \bullet u_{pp}x \bullet P^{-1}(x,u_p)$

is a hamilton path in $\mathcal{P}^*$ such that $|VQ(v,u)| = 2n-1$. Since $P \in \mathcal{P}^*$ it follows that u is the predecessor of x in P.

Finally, let us show that G is bipartite.

Let $P = x_1e_1x_2 \ldots e_{2n}x_{2n}$ be a path in $\mathcal{P}^*$ where $x_1 = v$.

Let $S = \{x_1, x_3, \ldots, x_{2n-1}\}$ and

$T = \{x_2, x_4, \ldots, x_{2n}\}$.

From R7, $x_{2n-1} = u$ and therefore, $|VP(x_1, x_{2n-1})| = 2n-1$. Since $|N(x_{2n})| = n$, from R5 it follows that $N(x_{2n}) = S$.

Suppose there exist vertices x_i and x_k in T, $2 \le i < k \le 2n$, such that $x_ix_k \in EG$. Thus, $x_k \neq x_{2n}$ and

$P(x_1,x_i) \bullet x_ix_k \bullet P^{-1}(x_k,x_{i+1}) \bullet x_{i+1}x_{2n} \bullet x_{2n}x_{k+1} \bullet P(x_{k+1},x_{2n-1})$

is a hamilton path with ends v and u, a contradiction. Therefore, $N(t) \subseteq S$ for any t in T.

Since $n = |N(t)| \le |S| = n$, it follows that $N(t) = S$ for any t in T. As for any s in S, $|N(s)| = n$, it follows that $N(s) = T$, and therefore G is bipartite, a contradiction.

∎

REMARK. The topic of hamilton-connected graphs was introduced in 1963 by Ore [1]. He proved the following two results:

Theorem 1. If G is a graph such that $|N(u)|+|N(v)| \geq |VG|+1$ for every pair of non-adjacent vertices u and v, then G is hamilton-connected.

Theorem 2. If G is a graph with $|VG| \geq 3$ and $|EG| \geq (|VG|-1)(|VG|-2)/2 + 3$ then G is hamilton-connected.

Ore also showed that the bounds presented in both theorems are best possible.

The theorem we proved shows that with the regularity hypothesis, Ore's result can be improved for a graph with an even number of vertices.

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