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Algebraic curves over $\bar{\mathbb{Q}}$ and deformations of complex structures

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1 Introduction

Belyi's theorem, [1], says that a complete non singular algebraic curve X defined over a field of zero characteristic can be defined over the algebraic closure $\bar{\mathbb{Q}}$ of the rationals if, and only if, there exists a morphism $\phi: X \longrightarrow \mathbb{P}^1$ ramified over three points.

Belyi's original contribution was to show, when X is defined over $\bar{\mathbb{Q}}$, how to build a covering of $\mathbb{P}^1$ ramified over three points. For the converse he essentially invokes Weil's criterion, [2], for descending the ground field.

It is not easy to find a good reference for Weil's criterion; and in fact Weil's paper originated a field of research. The interested reader can read Giraud, [3], or the series of Grothendieck's papers in the Bourbaki seminar, [4].

The purpose of this paper is to give an elementary proof of the converse quoted above, not using Weil's criterion nor Grothendieck's theorems. Our point of view is that of Shimura's note, [5], where he proves that if a projective nonsingular complex variety **V** is rigid, then **V** is biregularly equivalent to a complex variety defined over $\bar{\mathbb{Q}}$. We recall that **V** is rigid if $H^1(\mathbf{V}, \Theta) = 0$, where Θ is the sheaf of holomorphic sections of the tangent bundle of **V**.

We can now state the main result of this paper:

Theorem 1 *Let* **V** *be a projective nonsingular complex algebraic curve and* $\phi: \mathbf{V} \longrightarrow \mathbb{P}^1$ *a covering ramified only over points of* $\bar{\mathbb{Q}} \cup \infty$. *Then there is a nonsingular projective curve* $\mathbf{V}^0$ *defined over* $\bar{\mathbb{Q}}$ *and a biholomorphism* $\phi^0 : \mathbf{V}^0 \otimes \mathbb{C} \xrightarrow{\simeq} \mathbf{V}$, *where* $\mathbf{V}^0 \otimes \mathbb{C}$ *denotes the extension of scalars to* $\mathbb{C}$.

Although this result seems to belong to folklore, we believe that the elementary character of our proof justifies publication.

Belyi's original argument and the above theorem can be put together to give a complete proof of

Theorem 2 (Belyi) *A complete nonsingular algebraic curve* X *defined over* $\mathbb{C}$ *has a model over* $\bar{\mathbb{Q}}$ *if and only if there exists a morphism* $\phi: X \longrightarrow \mathbb{P}^1$ *ramified over three points.*

Since the fundamental group of the sphere minus three points is isomorphic to $SL_2(\mathbb{Z})$, theorem 2 has the important corollary:

Theorem 3 *Every compact Riemann surface* X *defined over* $\bar{\mathbb{Q}}$ *admits a hyperbolic uniformization, i.e.,there is a subgroup* Γ *of finite* *index in $SL_2(\mathbb{Z})$ such that X is biholomorphically equivalent to the compactification of $\Gamma \setminus \mathbb{H}$, where $\mathbb{H}$ is the complex upper half plane.*

2 Proof of theorem 1

Without loss of generality we shall suppose that the given covering $\phi: \mathbf{V} \longrightarrow \mathbb{P}^1$ is the projection on the x-coordinate and that the function field of $\mathbf{V}$ is $\mathbf{K} = k_0(x, y)$, where

$$y^n = \sum_{j=0}^{n-1} a_j(x) y^j \quad (a_j(x) \in k_0(x))$$

and k_0 is a finitelly generated extension of $\mathbb{Q}$.

Let L be the algebraic closure of $\mathbb{Q}$ in $\mathbf{K}$. So $L/\mathbb{Q}$ is a finite extension. We will construct a fibered variety Σ defined over L whose function field is isomorphic to $\mathbf{K}$ and whose generic fiber is $\mathbf{V}$. For this we shall transform the transcendental coefficients of the defining equation for $\mathbf{V}$ into variables: let $x_1, x_2, ..., x_{k+1}$ be a transcendence basis of $\mathbf{K}$ over L, where we consider $x_{k+1} = x$. If θ is a primitive element of $\mathbf{K}$ over $L(x_1, ..., x_{k+1})$ then we can define Σ as the projective closure of

$$\Sigma^a = Spec(L[X_1, ..., X_{k+1}, Y]/(f))$$

where $f = f(X_1, ..., X_{k+1}, Y)$ is the irreducible polynomial of θ.

As it is clear that to prove theorem 1 it is sufficient to consider the affine case, we make the extension of scalars to $\mathbb{C}$ and define

$$\phi: \Sigma^a \otimes \mathbb{C} \longrightarrow \mathbb{C}^k \times \mathbb{C}^1$$

by simply taking the projection in $((x_1, ..., x_k), x_{k+1})$.

So $\Sigma^a \otimes \mathbf{C}$ is a fibered variety and to each $\mathbf{x} = (x_1, ..., x_k) \in \mathbf{C}^k$ we can consider the restriction $\phi_{\mathbf{x}}$ of ϕ to the fiber $\Sigma_{\mathbf{x}}$:

$$\phi_{\mathbf{x}} : \Sigma_{\mathbf{x}} \longrightarrow \mathbf{C}^1 \quad .$$

It is also clear that if $\mathbf{x}_0 \in \mathbf{C}^k$ is the point corresponding to the coefficients of the original curve $\mathbf{V}$, then there is an isomorphism $\varrho : \Sigma_{\mathbf{x}_0} \longrightarrow \mathbf{V}$ such that

$$\begin{array}{ccc} \Sigma_{\mathbf{x}_0} & \xrightarrow{\cong} & \mathbf{V} \\ \phi_{\mathbf{x}_0} \searrow & & \swarrow \phi \\ & \mathbf{C}^1 & \end{array}$$

is commutative. The function field of $\Sigma_{\mathbf{x}_0}$ and V are the same , $\phi_{\mathbf{x}_0} : \Sigma_{\mathbf{x}_0} \longrightarrow \mathbf{C}$ has the same ramification points as those of $\mathbf{V}$, and there is an open set U containing $\mathbf{x}_0$ in $\mathbf{C}^k$ such that the fiber $\Sigma_{\mathbf{y}}$, for $\mathbf{y}$ in U is nonsingular. [1] We shall prove that the family $\Sigma^a \otimes \mathbf{C}$ is rigid near the fiber $\Sigma_{\mathbf{x}_0}$. To see this, consider the polynomial

$$\sum_{j=0}^{n} a_j(x) y^j$$

in two variables that defines $\mathbf{V}$ and it's partial derivative with respect to y . Considering both polynomials as polynomials in the variable y and forming the discriminant of the first one, we see that this discriminant is a polynomial in x which vanishes if, and only if, x is a ramification point (recall that we are considering the x-projection!)

[1] In fact we don't need to find a nonsingular equation for V in order to make things work. In a forthcoming paper we shall prove an analogous criterion for varieties of arbitrary dimension, only in terms of function fields.

We can write this discriminant as

$$\Delta(x) = c_0 \prod (x - \zeta_i)^{m_i}$$

where ζ_i are algebraic numbers and c_o is a complex nonzero constant. As this happens in the generic fiber, we have $c_0 = c_0(x_1, ..., x_k)$ and so there is an open set $U' \subset U$ with $\mathbf{x}_0 \in U'$ such that $c_0(\mathbf{y}) \neq 0$ for $\mathbf{y} \in U'$. Then all the fibers corresponding to points in U' have the same ramification points as those of $\mathbf{V}$ and by Riemann's existence theorem all the fibers are biholomorphic. We can finish the proof of theorem 1 by taking a point $\mathbf{y}^0$ in U' all of whose coordinates are algebraic numbers and define $\mathbf{V}^0 = \Sigma_{\mathbf{y}^0}$. It is clear that $\mathbf{V}^0$ is defined over $L(\mathbf{y}^0)$ and is biholomorphic to $\mathbf{V}$.

3 Proof of Belyi's theorem

We now give Belyi's original argument for reducing the number of ramification points. Let X be the complex projective nonsingular curve defined over $\bar{\mathbf{Q}}$ and t an arbitrary non constant rational function $t: X \longrightarrow \mathbf{P}^1$, ramified over $r_1, ..., r_n, \infty$, with $r_i \in \bar{\mathbf{Q}}$. Let $h_1 \in \mathbf{Q}[X]$ be the anulator of the r_i's. So $h_1(t)$ ramifies over $\infty, 0, h_1(\vartheta_i)$, where ϑ_i are the roots of h_1'. Let $h_2 \in \mathbf{Q}[X]$ be the anulator of the values $h_1(\vartheta_i)$. Since $Gal(\bar{\mathbf{Q}})$ leaves the set $h_1(\vartheta_i)$ invariant, we have $degree(h_2) \leq degree(h_1) - 1$ and if we continue this way, in at most $degree(h_1)$ steps we reach a linear polynomial $h_l \in \mathbf{Q}[X]$. The composition $\phi' = h_l(h_{l-1}(...(h_1(t))...)$ is a covering ramified only over points of $\mathbf{Q} \cup \infty$. By conveniently changing ϕ' by $c_1\phi' + c_2$ we can suppose that $0, 1$, and ∞ are among the ramification points. Let $r \in \mathbf{Q}$

another ramification point. We can suppose that $0 < r < 1$, that is, $r = m/m+n$ for positive integers m, n. We want a polynomial $g(X)$ such that

$$g(\infty) = \infty, \quad g(0) = 0, \quad g(1) = 0, \quad g(r) = 1$$

and

$$g'(w) = 0 \Rightarrow w \in \{0, 1, \infty\}$$

If we try $g(w) = c_0 w^{n_1}(1-w)^{n_2}$ and impose the above conditions, we obtain

$$g(w) = \frac{(m+n)^{m+n}}{m^m n^n} w^m (1-w)^n$$

Then, changing ϕ' by $g(\phi')$ and applying induction we reduce the ramification points to $0, 1, \infty$. We can now prove Belyi's theorem:

Theorem 4 *A complete nonsingular algebraic curve X defined over $\mathbb{C}$ can be defined over $\bar{\mathbb{Q}}$ if and only if there exists a morphism $\phi: X \longrightarrow \mathbb{P}^1$ ramified over three points.*

Proof: If X is defined over $\bar{\mathbb{Q}}$ then the above argument applies. If ϕ ramifies over three points, by composing with a linear fracional transformation we can suppose that the points are $0, 1, \infty$ and then apply theorem 1.

4 Belyi's theorem and Galois representations

To conclude, we give an interpretation of Belyi's theorem in terms of a certain representation of the absolute Galois group of $\mathbb{Q}$. This representation is not well understood but the reader can see Grothendieck, [6], Deligne, [7], and Ihara, [8], [9].

Let M be the maximal extension of $\bar{\mathbf{Q}}(t)$ unramified outside $0, 1, \infty$. So $M/\mathbf{Q}(t)$ is also Galois and, identifying naturally $G_{\mathbf{Q}} = Gal(\bar{\mathbf{Q}}/\mathbf{Q})$ with $Gal(\bar{\mathbf{Q}}(t)/\mathbf{Q}(t))$, then the following sequence is exact:

$$1 \longrightarrow Gal(M/\bar{\mathbf{Q}}(t)) \longrightarrow Gal(M/\mathbf{Q}(t)) \longrightarrow G_{\mathbf{Q}} \longrightarrow 1$$

If we put $H = Gal(M/\bar{\mathbf{Q}}(t))$, $G = Gal(M/\mathbf{Q}(t))$ then we have the sequence of natural maps:

$$G \longrightarrow Int(G) \xrightarrow{Res} Aut(H) \longrightarrow Out(H) = Aut(H)/Int(H)$$

where $Int(G)$ is the group of inner authomorphisms of G, Res is the restriction to H. Since the composition Φ of the above maps is trivial in H, we have a homomorphism

$$\phi_{\mathbf{Q}}: Gal(\bar{\mathbf{Q}}/\mathbf{Q}) \longrightarrow Out(Gal(M/\bar{\mathbf{Q}}(t)))$$

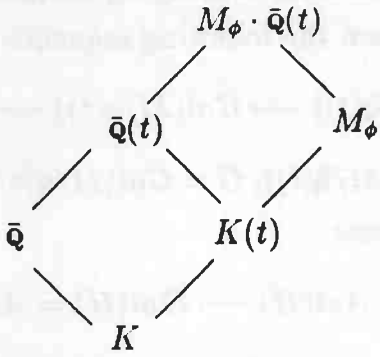
Belyi's theorem is equivalent to the injectivity of $\phi_{\mathbf{Q}}$. To see this we consider the fixed field K corresponding to the kernel of $\phi_{\mathbf{Q}}$. By our earlier identification we can write $\mathrm{Ker}\phi_{\mathbf{Q}} = G(\phi_{\mathbf{Q}})/H$ for a certain subgroup $G(\phi_{\mathbf{Q}})$ of G. It is then clear that $\phi_{\mathbf{Q}}$ is injective if and only if $G(\phi_{\mathbf{Q}}) = H$, (or equivalently, $K = \bar{\mathbf{Q}}$). Since $G(\phi_{\mathbf{Q}}) = Gal(M/K(t))$, the natural homomorphism

$$G(\phi_{\mathbf{Q}}) \longrightarrow Aut(H)$$

has its image in $Int(H)$. The kernel C of this map is the centralizer of H in $G(\phi_{\mathbf{Q}})$. Since H is a profinite proup free in two generators, it is well known that the center of H is trivial and so $C \cap H = \{1\}$ and $H \cong Int(H)$. consequently

$$G(\phi_{\mathbf{Q}}) \cong H \times C$$

If M_ϕ is the fixed field of C we have



and so $M_\phi \cap \bar{\mathbf{Q}} = K$ and $M = M_\phi \cdot \bar{\mathbf{Q}}$. So $K/\mathbf{Q}$ is an algebraic Galois extension such that M has a model over K, that is, the map $S \to S \cdot \bar{\mathbf{Q}}$ between subfields of M_ϕ/K and subfields of $M/\bar{\mathbf{Q}}(t)$ is an isomorphism of the lattices of the corresponding fields. Then every subfield of $M/\bar{\mathbf{Q}}(t)$ of finite degree over $\bar{\mathbf{Q}}(t)$ can be defined by an equation with coefficients in K. But for every $j \in \bar{\mathbf{Q}}$ we consider an elliptic curve E with invariant j given by a minimal Weierstrass equation defined over $\mathbf{Q}(j)$ and apply Belyi's theorem to see the function field of E as a subfield of $M/\bar{\mathbf{Q}}(t)$ of finite degree. So $j \in K$ and consequently $K = \bar{\mathbf{Q}}$.

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