Model Theory of Sheaves of Metric Spaces of Analytic Functions

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We have started to study from a logical perspective a heuristic principle in complex analysis, the so called *Bloch's Principle*, that is, *there is nothing in the infinite that was not previously in the finite; every proposition in whose statement the actual infinity occurs can be always viewed as a corollary, almost immediate, of a proposition where it does not occur, a proposition in finite terms*, proposed by André Bloch in his article [3, p. 84] in 1926, and with a first (non logical) approach obtained by Lawrence Zalcman in [11], following a suggestion of Abraham Robinson, [10, §8, pp. 508–510]. The logical framework we intend to apply is of sheaves of metric structures as described below.

Model Theory of Metric Structures is a recent version of Continuous Logic where the structures are complete metric spaces with bounded diameter, that is, they are endowed with a metric with values in the real interval $[0,1]$, also considered a metric spaces with the usual metric $|y-x|$ (see [1, 2].) This logic generalizes classical logic in the sense that we view classical structures as discrete metric spaces, whose metrics assume values in the set $\{0,1\}$, but with 0 as "truth" and 1 as "false". A metric structure is a complete metric space (M,d), whose metric is the function $d: M\times M\to[0,1]$, n-ary uniformly continuous functions $f: M^n\to M$, and n-ary predicates as uniformly continuous functions $P: M^n\to[0,1]$, together with their "moduli of uniform continuity" $\Delta_f,\Delta_P:[0,1]\to[0,1]$, considered as continuous crescent functions (here we diverge a little from the current literature). The logical part comprises continuous functions $f:[0,1]^n\to[0,1]$ as n-place propositional connectives, and the operators $\sup_x$ and $\inf_x$ as quantifiers.

Sheaves of metric structures were studied in [8, 9]. We propose to study the (pre-)sheaf of analytic functions over $\mathbb{C}$. Choose an enumeration ρ_n for all positive rational numbers and an enumeration ξ_n of $\mathbb{Q}+\boldsymbol{i}\mathbb{Q}\subset\mathbb{C}$. The set of closed disks $B(\xi_m,r_n)=\{z\in\mathbb{C}:|z-\xi_m|\le r_n\}$ generates by unions the topology of $\mathbb{C}$, and we fix an enumeration B_k of such disks. With this we define the metric on the set $\Gamma(U)$ of sections os analytic functions on the open set $U\subseteq\mathbb{C}$ as follows. The chordal distance in $\mathbb{C}$ comes from the stereographical projection of $\mathbb{C}$ into $\mathbb{R}^3$ and reduces to $\chi(z_0,z_1)=\dfrac{|z_0-z_1|}{\sqrt{|z_0|^2+1}\sqrt{|z_1|^2+1}}\in[0,1]$. For each open set $U\subseteq\mathbb{C}$, let $\mathcal{B}_U=\{B_{k_m}:m\in\omega\}$ be the set of all closed balls from $\mathcal{B}$ contained in U, with the enumeration derived from that of $\mathcal{B}$, and let $K_{U,n}=\bigcup_{j=1}^n B_{k_j}$, an exhausting sequence of compact subsets of U. The $[0,1]$-valued metric in $\Gamma(U)$ is $d_U(f,g)=\sum_{j\in\omega}2^{-j-1}\sup\{\chi(f(z),g(z)):z\in K_j\}$. These metrics satisfy $U\subseteq V$ and $f,g\in\Gamma(V)$ imply $d_U(f\restriction_U,g\restriction_U)\le d_V(f,g)$.

We recall that a normal family of analytic functions on an open set $U\subseteq\mathbb{C}$ is a relatively compact subset of $\Gamma(U)$. We intend to formalize this notion in the appropriate language and try to find a common feature of the questions treated in [12], and particularly in [11] [*each property of analytic functions which implies that entyre functions to be constant locally implies that the family with such property is normal*], with the techniques from [4–9].

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