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A nonlinear theory of generalized functions

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Abstract
Generalized Functions are crucial in the development of theories modeling physical reality. Starting with the linear theory, we give a partial account of what is known, focussing on the non-linear theory and some of its more recent achievements.

Keywords Generalized function · Equation · Distribution · Solution · Ultrametric

Mathematics Subject Classification Primary 46F30 · Secondary 46F20

1 Introduction

Let Ω be an open subset of $\mathbb{K}^n$, with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Schwartz was among the first to create a theory of generalized functions thus boosting research in the Theory of PDEs (see [1, 2]). Schwartz's Theory of Generalized Functions

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In honor of Alfredo Jorge Aragona Vallejo (In memoriam).

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is a linear theory and simple examples show that it does not permit consistent multiplication of Schwartz distributions. In fact, if multiplication existed and were consistent, then, for the Heaviside Function $H \in D'(\Omega)$, we would have that $H^2 = H$, $H' = \delta$ and thus $\delta = H' = 2HH' = 2H\delta$. Since also $H^3 = H$, it would follow that $\delta = 3H^2\delta = 3H\delta = \frac{3}{2}\delta$, a contradiction. The fact that $0 = (x \cdot \delta) \cdot vp[\frac{1}{x}] = \delta \cdot (x \cdot vp[\frac{1}{x}]) = \delta$, shows that there also does not exist an associative differential algebra, containing $C(\Omega)$ as a subalgebra, in which $D'(\Omega)$ embeds. These are very simple examples of nonlinearities which cannot be handled in $D'(\Omega)$. On the other hand, a simple example of Lewy, which was generalized by Hörmander (see [3–5]), shows that Schwartz's Theory can also not handle all linear PDEs. Grushin even showed that very simple linear equations can not have local solvability. In the case of first order linear PDEs, Nirenberg and Treves gave necessary and sufficient conditions for existence of solutions in $D'(\Omega)$ [6]. Not even generalizations such as the theory of hyperfunctions were able to handle all linear PDEs.

The search for a non-linear theory of generalized functions continued with work of Rosinger, Silva, Nachbin and many others (see [7–12]). It was J.F. Colombeau who proposed a theory, initially based on Silva Spaces (see [13, Page 46] and [14, 15]), which he later simplified to algebras of functions defined on Ω. These algebras are denoted by $\mathcal{G}(\Omega)$ and called Colombeau algebras of generalized functions. Colombeau's theory showed to be very consistent with physical reality and proved the existence of local generalized solutions for practically arbitrary linear PDE with C^∞ coefficients (see [10, Chapter 3], in particular page 170, and also [15–19]). Jorge Aragona, a student of L. Nachbin, inherited from the latter the passion for infinite dimensional holomorphy and locally convex spaces (see [7, 8, 11, 12]). Jorge Aragona and Hebe Biagioni, a student of Carvalho de Matos, M., improved substantially the theoretical basis of Colombeau's Theory, thus giving important contributions in the development of this new theory of generalized functions (see [20]). There are several Colombeau algebras constructed by Colombeau: the simplified and the full algebra. The full algebra contains a canonical copy of $D'(\Omega)$ and the the simplified algebra is very suitable for applications.

Jorge Aragona and his collaborators set the basis for the topological and algebraic theory of Colombeau algebras thus permitting further developments of the theory (see [21–30]). At the same time, progress was made applying the theory to nonlinear PDEs and Physics. A big achievement was the construction of a diffeomorphism invariant algebra, not constructed by Colombeau, thus permitting to extend Colombeau's Theory to manifolds [31, 32]. A more recent contribution by Alvarez et. al, shows existence of generalized step soliton solutions for the Shallow Water Equations (see [33]).

Since this survey is dedicated to the memory of Jorge Aragona, we shall focus on more recent developments of the theory having his contributions as a basis. With this we aim to remember the work, friendship and dedication of Jorge Aragona to the development of the Institute of Mathematics and Statistics of the University of São Paulo, IME-USP, Brazil.

2 Colombeau algebras

In this section we give a modern view of Colombeau algebras developed by Aragona-Fernandez-Juriaans. R. Fernandez had Jorge Aragona as her Ph.D. supervisor and the three were colleagues at IME-USP. The collaboration started around 1998 and was focused on developing topological, algebraic and analytical machinery for the theory with the purpose to make it as intrinsic as possible.

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $I = [0, 1]$ and consider $\mathcal{E}_M(\mathbb{K})$ the ring of germs $\hat{x}(\epsilon)$ at $x_0 = 0$ for which there exists $a < 0$ such that $|\hat{x}(\epsilon)| = o(\epsilon^a)$. These are called moderate germs. The set $\mathcal{N}(I)$ of germs $\hat{x}$ for with $|\hat{x}(\epsilon)| = o(\epsilon^n)$, $\forall n \in \mathbb{N}$, called null germs, is an ideal in $\mathcal{E}_M(\mathbb{K})$. The Colombeau Ring of Generalized Numbers is the quotient algebra $\overline{\mathbb{K}} = \mathcal{E}_M(\mathbb{K})/\mathcal{N}(\mathbb{K})$. The ultra metric $\|\_\|$ of [26] turns $(\overline{\mathbb{K}}, \|\ \ \|)$ into a topological Hausdorff algebra. The prime ideals of $\overline{\mathbb{K}}$ are uniquely linked to ultrafilters in the parameter space I and elements of $\overline{\mathbb{K}}$ are either units or zero divisors [26]. Denoting by $\alpha = [\epsilon \to \epsilon]$ and $\alpha_r := \alpha^r = [\epsilon \to \epsilon^r]$, these elements are units and $\|\alpha^r\| = e^{-r}, r \in \mathbb{R}$. Given $\Omega \subset \mathbb{K}^n$, let $\widetilde{\Omega}_c = \{x = (x_1, \cdots, x_n) \in \overline{\mathbb{K}}^n \ : \ \exists\, \eta > 0, \{\hat{x}(\epsilon) \ : \ \epsilon < \eta\} \subset\subset \Omega\}$. A function $f \in \mathcal{F}(\widetilde{\Omega}_c, \overline{\mathbb{K}})$ is said to be differentiable if there exists $z_0 \in \overline{\mathbb{K}}$ such that $\lim_{x \to x_0} \frac{f(x) - f(x_0) - z_0(x - x_0)}{\alpha_{-\log(\|x - x_0\|)}} = 0$. Denoting this limit by $f'(x_0)$, one shows that this is a derivation on $\mathcal{F}(\widetilde{\Omega}_c, \overline{\mathbb{K}})$ which satisfies almost all properties of classical differential calculus and we have the following embedding theorem.

Theorem 2.1 (Aragona-Fernandez-Juriaans) *There exists a continuous $\mathbb{K}-$ linear injection $\kappa : D' \to C^\infty(\widetilde{\Omega}_c, \overline{\mathbb{K}})$ such that $\kappa\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial(\kappa(f))}{\partial x_i}$, $\forall f \in D', \forall i$.*

This proves the existence of an environment, with an underlying algebraic structure, containing $D'(\Omega)$ while maintaining all of its differential properties. Multiplication of distributions reduces to classical multiplication but with a gain: in this environment we have a strong and weak equality, the former being the classical one and the latter, called association, is denoted by $\approx$. As shown before, the strong equality does not hold between H and H^2 but they are weakly equal. Also, in this new environment $x \cdot \delta \neq 0$. This has a fundamental impact on how one sees differential equations in this environment. It is this distinction that permits solutions for the shallow water equation, equations of Brezis-Friedman, solutions in the form of shock waves, the response of nonlinear systems to step inputs and Physics and many other nonlinear PDEs (see [13, 31–44]). Infinitesimals not necessarily cancel out in this new environment thus making a link with modern non-Archimedean function theory started by J. Tate (see [45]). In this direction, a more recent result is that $D'(\Omega)$ can be embedded in a differential algebra $C^\infty(X, \mathbb{L})$, where $X \subset \mathbb{L}^n$ and $\mathbb{L}$ is a non-Archimedean field whose topology is defined by a valuation (see [46, 47]). More precisely, the image of $D'(\Omega)$ is contained in a quotient of $\mathcal{G}(\Omega)$, called *Aragona Algebra*. Using Aragona algebras, the maximal ideals of $\mathcal{G}(\Omega)$ were characterized in a similar way as done in the Gelfand-Kolmogoroff Theorem on maximal ideals of $C(X, \mathbb{R})$ (see [46, 48, 49]). The minimal prime and maximal ideals of $\overline{\mathbb{K}}$ were classified by Aragona-Juriaans in [26]. So we have a very good understanding of the algebra $\mathcal{G}(\Omega)$ and its ring of constant functions $\overline{\mathbb{K}}$ together with a differential calculus which extends, but is similar to, classical calculus. This provides algebraic, analytical and topological tools plus an environment to deal with nonlinearities in PDEs.

3 Holomorphic generalized functions

This section is based on four, more recent, papers of Jorge Aragona. These papers are [21, 22, 34, 50] where the interested reader can find precise definitions and more details. It is important to mention [51] which is the first paper of Aragona dealing with generalized holomorphic functions. Since then, he and his collaborators have obtained significant results in this direction. It is important to notice that holomorphic functions were among the favorite subjects Jorge Aragona liked to study.

When studying differential equations it is important to consider boundary conditions. This was fully addressed by Aragona in [50] where the notion of quasi-regular sets is introduced. A subset $X \subset \mathbb{R}^n$ is a quasi-regular set if it is non-void and contained in the closure of its interior points. It basically is the union of the interior of a subset $\Omega \subset \mathbb{R}^n$ and a subset of $\partial\Omega$, the boundary of Ω. This shows the importance of considering generalized functions defined on $\partial\Omega$. This is done in [50] Sects.1, 2, 3. For what follows, we refer the reader to [20] for the definition of the full algebra of Colombeau generalized functions.

One of the problems addressed in [50] is to give a condition for the boundary value problem $\Delta u = 0$, $u_{|\partial\Omega} = f$ to have a solution. The condition given is that f must be regular meaning that, using the Poison kernel, we can extend it to $\overline{\Omega}$ and this extension defines a Colombeau generalized function. Let $\Omega_{n0}(\xi, z)$ denote the Ramirez-Henkin differential form (see [52]), then we have:

Theorem 3.1 *Let $\Omega \subset \mathbb{C}^n$ be a strictly pseudo convex domain and f a generalized holomorphic function of $\overline{\Omega}$. If g is a representative of f then the function $f_* : A_0(2n) \times \Omega \to \mathbb{C}$ defined by $f_*(\phi, z) = a \int_{\partial\Omega} g(\phi, \xi)\Omega_{n0}(\xi, z)$, with $a = (2\pi i)^{-n}(-1)^{\frac{1}{2}n(n-1)}$, is a representative of the restriction $f_{|\Omega}$.*

Theorem 3.2 *Let B denote the open unit euclidean ball in $\mathbb{C}^n$ and let $V \subset \overline{V} \subset B$ be such that $B/\partial V$ is connected and ∂V is not a real analytic manifold. Then, for each generalized function f defined on $\overline{B}$, the following are equivalent: (i) f is generalized holomorphic; (ii) $\int_{\partial V} f = 0$ and $\frac{\partial f}{\partial \bar{z}_j}(z) = 0$, $\forall\, j$ and $z \in K \subset\subset \overline{B}$.*

A question which is not obvious is the following: given a generalized holomorphic function f does it has a representative given by holomorphic functions? This question was answered in [53] in case f is defined on a subset of $\mathbb{C}$. This was used in [22] to prove that such a function also has a power series. This was the stepping stone for a theory of generalized functions over membranes developed in [21, 22]. This proved that this new theory behaved very similar to the classical theory and gave connection with non-standard analysis which was carried out in [54]. This last development set the stage for other applications. Among these are the notion of internal sets (see [54]) which generalizes the notion of membranes defined in [21]. A saturation principle and also an overspill and underspill principles are proved and interesting applications are given.

In search for an Identity Theorem for generalized holomorphic functions, Colombeau and Galé proved that if such an f, defined on a domain Ω of $\mathbb{C}$, is zero on an open subset $W \subset \Omega$, then f is identically zero (see [55, 56]). Note that such functions can have accumulation points of zeros as the example $f(z) = \chi_A z$, with A belonging to an ultrafilter of [0, 1] and $\chi_A \notin \{0, 1\}$, shows. Since we are dealing with nets, it must be checked if there exists flexibility in the topological property that the set of zeros of a generalized holomorphic function can have. A more general criterion than the one of Colombeau-Galé was given in [57] where it is shown that W can be substituted by an arc of a differentiable curve contained in Ω. These results were recently extended in [58] where it is shown that W can be replaced by a Carleson set of uniqueness. Carleson sets of uniqueness play an important role in Hardy spaces and it turns out that they are minimal sets on which a generalized holomorphic function can be zero in the sense that if f is a generalized holomorphic function such that its zero set contains a Carleson set of uniqueness, then f must be identically zero. So this is the Identity Theorem for these functions.

The spaces associated to the Theory of Colombeau Generalized Functions are ultrametric spaces and hence we have lack of compactness. This can be an obstacle in applications. To address the need, in [34], new algebras were constructed using holomorphic functions. These algebras are smaller than the original Colombeau algebras but have several compactness properties. Important classes of distributions can be embedded in these algebras and hence they are suitable for studying problems coming from reality. As mentioned in [34], in contrast with the original introduction of nonlinear generalized functions by Colombeau and the exposition of his theory in various expository texts, in [34] is put in evidence a very rich hidden locally convex topological structure in subalgebras of Colombeau algebras of generalized functions that permits the use of the classical deep tools of topology and functional analysis. Many variants of the original construction introduced in this paper are possible.

4 Nonlinear equations

As mentioned before, very simple equations cannot be handled by Schwartz distribution theory. On the other hand, there are important nonlinear problems which one knows must have a solution since they model real world phenomena. One of the most important features of Colombeau's theory is that it permits to deal with nonlinearities. A good example of how to deal with nonlinear equations which have solutions in the form of shockwaves can be found in recent papers coauthored by Aragona (see [35, 37]). In the isothermal case, and taking into account some other simplifications (no gravitational effects and no transfer of momentum between the two fluids), the standard one pressure model used to describe mathematically a mixture of two immiscible fluids is given by the system

$$(\rho_l\alpha_l)_t + (\rho_l\alpha_l u_l)_x = 0, \tag{1}$$

$$(\rho_g\alpha_g))_t + (\rho_g\alpha_g u_g))_x = 0, \tag{2}$$

$$(\rho_l\alpha_l u_l)_t + (\rho_l\alpha_l (u_l)^2)_x + \alpha_l p_x = 0, \tag{3}$$

$$(\rho_g\alpha_g u_g)_t + (\rho_g\alpha_g (u_g)^2)_x + \alpha_g p_x = 0, \tag{4}$$

$$\alpha_l + \alpha_g = 1, \tag{5}$$

$$p = \mathcal{P}_l(\rho_l), p = \mathcal{P}_g(\rho_g), \tag{6}$$

where the two fluids are denoted by the indices l and g respectively for "liquid" and "gas". In the case of variable temperature, the state laws involve the energy and velocity so that the two energy equations cannot be dissociated from the other Eqs. 1, 2, 3, 4. The system is then nonconservative, i.e. the derivatives cannot be transferred to test functions. In the case of shockwaves the variables p and α_i are discontinuous and these terms appear as undefined products of the form of the Heaviside function multiplied by the Dirac delta distribution. Not only these products do not make sense mathematically but also one cannot obtain formulas for the jump conditions since formal calculations give contradictory results depending on the way they are done. Correspondingly, numerical methods become ambiguous in that it has been observed that slight modifications in the formulas of the scheme can give very different results, which can be explained in the nonlinear theory of generalized functions by the existence of different Heaviside functions. This same observation, led to step soliton generalized solutions of the Shallow Water Equations in [33]. In the flat bottom case these equations are

$$\begin{aligned} h_t + (hu)_x &\approx 0 \\ u_t + \frac{1}{2}(u^2)_x + gh_x &\approx 0 \end{aligned}$$

where $\approx$ is the dynamical equality. The solutions obtained match with those obtained by simulations and correspond to physical reality.

The last equation we consider is one studied by Brézis-Friedman (see [38]). They proved that if certain initial conditions are considered, then these equations do not have a solutions. To be more specific, let $\Omega \subset \mathbb{R}^n$ be a bounded domain , $T > 0$ and let $Q = \Omega \times [0, T]$. In [38] they proved that the following equation has no weak solution in Q.

$$\begin{aligned} u_t - \Delta u + u^3 &= 0 \text{ in } Q \\ u(x,t) &= 0 \text{ in } \partial\Omega\times]0,T[\\ u(x,0) &= \delta(x) \text{ in } \Omega \end{aligned}$$

These non-existence results were considered rather surprising as carefully explained by them. An explanation for these non-existence results was given in [19] by Colombeau and Langlais. Even more, they proved that the Brézis-Friedman equations do have a unique solution in the Colombeau algebra, as long as the initial data, which could be a distribution or Colombeau generalized function, had compact support. Natural questions are: Is there still a solution if the initial data has non-compact support? Is this solution unique? Both questions were answered in the affirmative by Aragona-Garcia-Juriaans (see [23, 36]).

Theorem 4.1 *Let $u_0 \in \mathcal{G}_f(\Omega)$ be an element of the full algebra of Colombeau generalized functions. Then*

$$\begin{aligned} u_t - \Delta u + u^3 &= 0 \text{ in } \mathcal{G}_f(Q) \\ u(x,t) &= 0 \text{ in } \mathcal{G}_f(\partial\Omega \times [0,T]) \\ u(x,0) &= u_0 \text{ in } \widetilde{\Omega}_c \end{aligned}$$

has a unique solution in $\mathcal{G}_f(\overline{Q})$.

The interesting thing about this last result is its proof. All the algebraic, analytical and topological machinery developed by Aragona and his collaborators, together with ideas resulting in the notion of quasi regular sets, piece together to make the proof of the theorem intrinsic, i.e., all calculations are done within the Colombeau algebras involved without appealing to representatives. The interested reader can also find more about the theory in [1, 2, 24–27, 29, 30, 37, 44, 59–67]. We apologize for references that are important but have been left out.

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