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ON THE LOOP OF UNITS OF AN ALTERNATIVE LOOP RING

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ABSTRACT. An RA loop is a Moufang loop whose loop rings, in characteristic different from 2, are alternative but not associative. In this paper, we determine the class of RA loops in whose integral loop rings the torsion units form a subloop and show how this property relates to many other properties of the unit loop. For example, we determine the conditions under which the unit loop of the integral loop ring of an RA loop is also an RA loop.

1. Introduction. Scattered throughout the fascinating theory of integral group rings are theorems giving conditions under which the only units, or the only torsion units, are trivial. (A unit of $\mathbf{Z}G$ is *trivial* if it is of the form $\pm g$ for some $g \in G$.) The earliest of these is a theorem of Graham Higman which says that $\mathbf{Z}G$ has only trivial units if and only if G is an abelian group of exponent 2, 3, 4 or 6, or a Hamiltonian 2-group [12]. S. D. Berman proved a similar theorem giving conditions under which just the *torsion* units (those of finite order) are trivial [1]. Later, M. M. Parmenter and C. Polcino Milies showed that for finite G, the condition that $\mathbf{Z}G$ have only trivial torsion units is equivalent to several others, the most fundamental being that the torsion units of $\mathbf{Z}G$ should form a subgroup of the full unit group [14]. This theorem was later extended by the second author to arbitrary groups [13].

In the last ten years, there has developed a theory of alternative loop rings and it has become clear that many of the theorems of group rings hold also in the nonassociative setting. The theorems of Higman and Berman, for example, extend to alternative loop rings [9]. In a similar vein, it is the purpose of this paper to determine conditions under which the torsion units of a loop ring form a subloop and to explore the consequences of this property.

An alternative ring is a ring in which the alternative laws

$$x(xy) = x^2y \qquad \text{and} \qquad (yx)x = yx^2$$

are valid. For us, the most important property of an alternative ring is a theorem of E. Artin which says that the subring generated by any two elements of an alternative ring is associative [18]. If L is a loop whose loop ring RL over a ring of characteristic different from 2 is an

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alternative, but not associative, ring, we call L an *RA* (ring alternative) loop. The following theorem, which is implicit in [4] and clarified in [10], is fundamental.

Theorem 1.1. *A loop L is RA if and only if*

(i) *$L = G \cup Gu$ is the disjoint union of a nonabelian group G and a single coset Gu;*
(ii) *G has a unique nonidentity commutator, s, which is necessarily central and of order 2;*
(iii) *the map*

$$g \mapsto g^* = \begin{cases} g & \text{if } g \text{ is central} \\ sg & \text{otherwise} \end{cases} \tag{1.1}$$

is an involution of G (i.e., an antiautomorphism of order 2);
(iv) *multiplication in L is defined by*

$$\begin{aligned} g(hu) &= (hg)u \\ (gu)h &= (gh^*)u \\ (gu)(hu) &= g_0 h^* g \end{aligned}$$

where $g_0 = u^2$ is a central element of G. Moreover, the group G can be taken to be any group generated by the centre of L and two elements of L which do not commute.

The loop described in this theorem is denoted $M(G, *, u)$. It is a Moufang loop and therefore has the property that any two elements generate a group. (We refer the reader to the recent text by Pflugfelder [15] for an introduction to the theory of loops and, in particular, of Moufang loops.) In loop theory, a subloop N of a loop L is *normal* provided

$$N\ell = \ell N, \quad (N\ell_1)\ell_2 = N(\ell_1\ell_2), \quad (\ell_1 N)\ell_2 = \ell_1(N\ell_2) \text{ and } \ell_1(\ell_2 N) = (\ell_1\ell_2)N$$

for any $\ell, \ell_1, \ell_2 \in L$. We will have cause to use the fact that, if L is RA, these conditions are equivalent to the single condition $N\ell = \ell N$ for all ℓ. (Actually, this property holds more generally for *RA2 loops*; that is, for loops whose loop rings in characteristic 2 are alternative [5, Corollaries 2.4 and 2.11].)

Since, for $L = M(G, *, u)$, the elements of L are those of $G \cup Gu$, it is easy to see that any element $\alpha = \sum \alpha_\ell \ell$ in the loop ring RL can be expressed in the form $\alpha = x + yu$, where x and y are in the group ring RG. Also, the involution on G extends to L by setting $(gu)^* = s(gu)$, and then to an involution on RL which can be expressed in equivalent forms:

$$(\sum \alpha_\ell \ell)^* = \sum \alpha_\ell \ell^* \quad \text{or} \quad (x + yu)^* = x^* + syu. \tag{1.2}$$

In this paper, $\mathcal{Z}(X)$ denotes the centre of a group or ring X. It is important to remember that if X is not associative, the centre of X is the set of elements which commute with all other elements *and associate with all other pairs of elements.* The centre of an alternative loop ring RL is

$$\mathcal{Z}(RL) = \{\alpha \in RL \mid \alpha^* = \alpha\} \tag{1.3}$$

[10]. The group or loop of units (invertible elements) in a ring R is denoted $\mathcal{U}(R)$ while $T\mathcal{U}(R)$ is the group or loop of *torsion units* in R (i.e., those units which have finite order). The commutator subloop of a loop L is denoted L'.

If $L = M(G,*,u)$ is an RA loop, then $\mathcal{Z}(L) = \mathcal{Z}(G)$ and $L' = G' = \{1, s\}$. Consequently, for any $x, \ell \in L$,

$$x^{-1}\ell x \in \{\ell, \ell^*\} \subseteq \{\ell, s\ell\}. \tag{1.4}$$

2. Some Basic Lemmas. The following lemma is not generally true for groups, but it holds for RA loops, as a consequence of their rather special nature.

Lemma 2.1. *Let T denote the set of torsion elements of an RA loop L. Then T is a normal subloop of L. If L is finitely generated, then T is finite.*

Proof. It is sufficient to prove that T is closed under inverses and products. Clearly it is closed under inverses. Now if a and b are elements of T which commute, then certainly $ab \in T$. On the other hand, if a and b do not commute, then ab is again of finite order since, for $n \geq 0$,

$$(ab)^n = \begin{cases} a^n b^n & \text{if } n \equiv 0 \text{ or } 1 \pmod 4 \\ s a^n b^n & \text{if } n \equiv 2 \text{ or } 3 \pmod 4 \end{cases}$$

and s is central of order 2. So T is a subloop. Next, note that for $t \in T$ and $\ell \in L$, $\ell^{-1}t\ell$ has the same order as t, so $\ell^{-1}T\ell \subseteq T$ and $T\ell = \ell T$ which, as we have mentioned, implies that T is normal. Now suppose that L is finitely generated. Since $L' \subseteq T$ (because $s^2 = 1$) and since s is both a unique nonidentity commutator and associator in L [8, Theorem 3], T/L' is a subgroup of the finitely generated abelian group L/L'. So T/L' is a finitely generated torsion abelian group and hence finite. □

It is an open question whether or not an integral group ring has idempotent elements other than 0 and 1, but the situation is clear with RA loops.

Lemma 2.2. *Suppose $L = M(G,*,u)$ is an RA loop. Then the idempotents of $\mathbf{Z}L$ are trivial; i.e., equal to 0 or 1.*

Proof. We first claim that the idempotents of $\mathbf{Z}G$ are trivial. For this, let $e \in \mathbf{Z}G$ be idempotent. Replacing G by the subgroup generated by the support of e, we may assume that G is finitely generated. Let $T(G)$ and $T(L)$ denote the torsion units of G and L respectively. Since $T(G) \subseteq T(L) \cap G$, the previous lemma shows that $T(G)$ is a finite subgroup of G. Since $G' \subseteq T(G)$, $G/T(G)$ is a finitely generated abelian group, hence polycyclic. By [17, Theorem I.2.20], e is trivial. Now suppose e is an idempotent in $\mathbf{Z}L$. Write $e = x + yu$ with x and y in the group ring $\mathbf{Z}G$ and observe that

$$e^2 = (x^2 + g_0 yy^*) + (yx + yx^*)u = e \implies y(x + x^*) = y \implies y(x + x^*)^n = y$$

for all $n \geq 1$. Let $\mathcal{Z} = \mathcal{Z}(L)$ and express x in the form $x = \sum_{g \in \mathcal{Z}} \alpha_g g + \sum_{g \notin \mathcal{Z}} \alpha_g g$. Remembering that $g^* = sg$ for $g \notin \mathcal{Z}$, it is easy to see that $x + x^* = 2x_1 + (1+s)x_2$ for some central element $x_1 \in \mathbf{Z}G$. It follows that the coefficients of $(x + x^*)^2$, when expressed as a linear combination of group elements, are even and hence, for any $k > 0$, $(x + x^*)^{2^k}$ has coefficients divisible by 2^k. Thus $y = y(x + x^*)^{2^k}$ has coefficients divisible by 2^k for any $k > 0$. This cannot happen unless $y = 0$ in which case $e = x \in \mathbf{Z}G$ where we have already shown that idempotents are trivial. □

In what follows, we need to be sure that a lemma of Sehgal remains true in the nonassociative setting [17, Lemma VI.3.22].

Lemma 2.3. *Let L be a finitely generated RA loop and T its normal torsion subloop. Suppose that $\mathbf{Q}T \cong D_1 \oplus \cdots \oplus D_n$ is the direct sum of division rings and that every idempotent of $\mathbf{Q}T$ is central in $\mathbf{Q}L$. Then*

(i) *Every unit $\mu \in \mathbf{Z}L$ can be written in the form $\mu = \sum d_i \ell_i$, with $d_i \in D_i$, $\ell_i \in L$, and*
(ii) *$\mathcal{U}(\mathbf{Z}L) = [\mathcal{U}(\mathbf{Z}T)]L$.*

Proof. Write $1 = \sum e_i$, for idempotents $e_1, e_2, \ldots, e_n$ of $\mathbf{Q}T$. Since $D_i = (\mathbf{Q}T)e_i$, since e_i is central in $\mathbf{Q}L$ (by hypothesis) and since T is normal in L, the division ring D_i is also *normal* in $\mathbf{Q}L$ in the sense that

$$D_i\alpha = \alpha D_i, \quad (D_i\alpha)\beta = D_i(\alpha\beta), \quad (\alpha D_i)\beta = \alpha(D_i\beta), \text{ and } \alpha(\beta D_i) = (\alpha\beta)D_i$$

for any $\alpha, \beta \in \mathbf{Q}L$. These facts are not hard to see. By linearity, it is sufficient to establish each in the case that α and β are elements of L. So, to obtain $(D_i\ell_1)\ell_2 = D_i(\ell_1\ell_2)$ for $\ell_1, \ell_2 \in L$, for example, let $d \in D_i$, write $d = (\sum d_t t)e_i = \sum d_t(te_i)$, $d_t \in \mathbf{Q}, t \in T$ and use

$$\begin{aligned} [(te_i)\ell_1]\ell_2 &= [t(e_i\ell_1)]\ell_2 = [t(\ell_1 e_i)]\ell_2 = [(t\ell_1)e_i]\ell_2 = (t\ell_1)(e_i\ell_2) = (t\ell_1)(\ell_2 e_i) = [(t\ell_1)\ell_2]e_i \\ &= [t'(\ell_1\ell_2)]e_i, \qquad \text{for some } t' \in T, \\ &= t'[(\ell_1\ell_2)e_i] = t'[e_i(\ell_1\ell_2)] = (t'e_i)(\ell_1\ell_2) \end{aligned}$$

to conclude that $(d\ell_1)\ell_2 = d'(\ell_1\ell_2)$ for some $d' \in D_i$. Since T is normal in L, L has a decomposition as the disjoint union of cosets of T [3, §V.1]; i.e.,

$$L = \bigcup_{q \in \mathcal{Q}} Tq$$

for some transversal $\mathcal{Q}$. Note that if $q \in L \setminus T$, then $Tq \cap Tq^{-1} = \emptyset$ since q^2 has infinite order. Thus we may assume that $q \in \mathcal{Q} \implies q^{-1} \in \mathcal{Q}$ as well.

Now let μ be a unit in $\mathbf{Z}L$. Then μ can be written $\mu = \sum \mu_j q_j$ with $\mu_j \in \mathbf{Z}T$ and the q_j distinct elements of $\mathcal{Q}$ and, for any i,

$$e_i\mu = \sum e_i(\mu_j q_j) = \sum_{j=1}^{u} d_j q_j \tag{2.1}$$

where $d_j = e_i\mu_j \in D_i$, since e_i is central. Since L/T is a finitely generated torsion free abelian group, it can be ordered. Thus, in (2.1), we may assume that the q_j are such that $Tq_1 < Tq_2 < \cdots < Tq_u$. Similarly, we can write

$$e_i\mu^{-1} = \sum_{j=1}^{v} d'_j q'_j \tag{2.2}$$

with $d'_j \in D_i$ and $Tq'_1 < Tq'_2 < \cdots < Tq'_v$.

Now, for $a, b \in D_i$ and $q, q' \in L$,

$$aq_j \cdot bq'_k = a_1(q_j \cdot bq'_k) = a_1(q_j b_1 \cdot q'_k) = a_1(b_2 q_j \cdot q'_k) = a_1(b_3 \cdot q_j q'_k) = a_2 b_3 \cdot q_j q'_k$$

where, by the normality of D_i, the elements a_1, a_2, b_1, b_2, b_3 all belong to D_i. From (2.1) and (2.2), it follows that $e_i\mu \cdot e_i\mu^{-1}$ can be written

$$e_i\mu \cdot e_i\mu^{-1} = \sum_{j=1}^{u}\sum_{k=1}^{v} d_{jk} \cdot q_j q'_k, \quad \text{where } d_{jk} \in D_i.$$

On the other hand, Artin's Theorem gives $e_i\mu \cdot e_i\mu^{-1} = e_i$; so, in fact,

$$e_i = \sum_{j=1}^{u}\sum_{k=1}^{v} d_{jk} \cdot q_j q'_k.$$

Now the left hand side of this expression has support in T, while the terms of the right hand side have supports in

$$Tq_1q'_1,\ Tq_1q'_2, \ldots, Tq_uq'_v.$$

The first coset here is the unique smallest and the last the unique largest, and, if either $u > 1$ or $v > 1$, these are different. Thus $u = v = 1$ so that $e_i\mu = d_iq_i$, $d_i \in D_i$, and

$$\mu = (\sum_1^n e_i)\mu = \sum_1^n d_iq_i, \qquad d_i \in D_i,\ q_i \in L.$$

This gives part (i) of the lemma.

Continuing, we can also write

$$\mu^{-1} = \sum_1^n q'_i d'_i, \tag{2.3}$$

$d'_i \in D_i$. It is important to realize that there is no reason here to expect the q_i or q'_i to be distinct. We can and do assume, however, that

$$Tq_1 \leq Tq_2 \leq \cdots \leq Tq_n \qquad \text{and} \qquad Tq'_1 \leq Tq'_2 \leq \cdots \leq Tq'_n$$

Since for $q, q' \in L$, $a \in D_i$, $b \in D_j$ and $i \neq j$,

$$q'a \cdot bq = q'(a_1 \cdot bq) = q'(a_2b \cdot q),$$

$a_1, a_2 \in D_i$, this product is 0, so

$$1 = \mu^{-1}\mu = \sum_1^n q'_i d'_i \cdot d_i q_i.$$

The left hand side has support in T while the terms of the right hand side have supports in $Tq'_1q_1, \ldots, Tq'_nq_n$ which are ordered $Tq_1q'_1 \leq \cdots \leq Tq_nq'_n$. If the first and the last of these cosets are different, we have a contradiction. Thus each $Tq'_iq_i = T$, so each $q'_iq_i \in T$ and, by choice of the transversal $\mathcal{Q}$, each $q'_i = q_i^{-1}$. In the expression $\mu = \sum d_iq_i$, collect equal q_i and write

$$\mu = \sum_{q \in \mathcal{Q}} \mu_q q$$

where each $\mu_q \in \mathbf{Z}T$ is a sum of certain d_i. Note that $\mu_q\mu_{q'} = 0$ if $q \neq q'$ since the division rings represented by the d_i in μ_q are all different from those represented in μ'_q. Since $q'_i = q_i^{-1}$, the

q''s in the expression (2.3) for μ^{-1} will collect exactly as the q's in the expression for μ, so that we have

$$\mu^{-1} = \sum_{q \in \mathcal{Q}} q^{-1}\nu_q$$

where both $\mu_q \nu_{q'} = 0$ and $\nu_q \nu_{q'} = 0$ if $q \neq q'$. Also,

$$1 = \mu\mu^{-1} = \sum_{q,r \in \mathcal{Q}} \mu_q q \cdot r^{-1}\nu_r.$$

The left side has support in T while $\mu_q q \cdot r^{-1}\nu_r$ does not, unless $r = q$. So

$$1 = \sum_{q \in \mathcal{Q}} \mu_q q \cdot q^{-1}\nu_q. \tag{2.4}$$

Choose some $\mu_{q_0} \neq 0$. Then we claim that $(\mu_q q \cdot q^{-1}\nu_q)\mu_{q_0} q_0 = 0$ if $q \neq q_0$. To see why, observe first that the element in question can be expressed as a linear combination of terms of the form $(aq \cdot q^{-1}b)cq_0$ where b and c come from different D_i, that

$$aq \cdot q^{-1}b = (aq \cdot q^{-1})b_1 = ab_1 \tag{2.5}$$

where b_1 and b come from the same D_i, and hence $(aq \cdot q^{-1}b)cq_0 = ab_1 \cdot cq_0 = (ab_1 \cdot c_1)q_0 = (a_1 \cdot b_1c_1)q_0$, with a and a_1 in the same division algebra, and c and c_1 in the same division algebra. Thus $(\mu_q q \cdot q^{-1}\nu_q)\mu_{q_0} q_0 = 0$ since $b_1c_1 = 0$.

From (2.4), we now obtain $\mu_{q_0} q_0 = (\mu_{q_0} q_0 \cdot q_0^{-1}\nu_{q_0})\mu_{q_0} q_0$ and, since $x = xyx$ implies that xy is an idempotent, we see that $\mu_{q_0} q_0 \cdot q_0^{-1}\nu_{q_0}$ is a (nonzero) idempotent, and as a calculation similar to (2.5) (using normality of T) shows, it's in $\mathbf{Z}T$. Since idempotents in $\mathbf{Z}T$ are trivial (Lemma 2.2), $\mu_{q_0} q_0 \cdot q_0^{-1}\nu_{q_0} = 1$. So $q_0^{-1}\nu_{q_0}$, and therefore ν_{q_0}, are invertible. Similarly, μ_{q_0} is invertible. Since $\mu_q \nu_{q_0} = 0$ for $q \neq q_0$, it follows that $\mu_q = 0$ for $q \neq q_0$ and $\mu = \mu_{q_0} q_0 \in [\mathcal{U}(\mathbf{Z}T)]L$. This completes the lemma. □

3. Main Results. A *Hamiltonian 2-loop* is a nonabelian loop all of whose elements have order a power of 2 and in which all subloops are normal. It is well-known that the only Hamiltonian 2-loops which are Moufang are Q (the group of quaternions), C (the Cayley loop) and direct products of Q and C with elementary abelian 2-groups. The following theorem shows that a result of Polcino Milies for group rings [13] holds also for alternative loop rings.

Theorem 3.1. *Let L be an RA loop and T its torsion subloop. Then the torsion units in the integral loop ring of L form a subloop of $\mathcal{U}(\mathbf{Z}L)$ if and only if T is an abelian group or a Moufang Hamiltonian 2-loop and, for every $\ell \in L$ and $t \in T$, $\ell^{-1}t\ell \in \langle t \rangle$.*

Proof. Suppose the torsion units of $\mathbf{Z}L$ form a subloop. If T is associative, it is a group and the torsion units of the group ring $\mathbf{Z}T$ form a subgroup of the unit group of $\mathbf{Z}T$ since they are contained in the torsion units of $\mathbf{Z}L$. By the known result for groups, the set of torsion units of T (that is to say, T itself) is an abelian or a Hamiltonian 2-group. Suppose that T is not associative and hence an RA loop. Let a, b and u be three elements of T which do not associate and let $G = \langle a, b, \mathcal{Z} \rangle$, where $\mathcal{Z}$ is the centre of L. Note that a and b do not commute, since commuting elements in an RA loop associate with every third element [8, Theorem 3]. Then $L = M(G, *, u)$ and we claim that $T = M(T(G), *, u)$, where $T(G)$ is the torsion subgroup of G. The only point at issue here is the fact that T is the set union $T(G) \cup T(G)u$. For this, we first note that $T(G) \subseteq T$ and $T(G)u \subseteq T$ because the elements of $T(G)u$ are the products of torsion units (which form a loop). On the other hand, if $x \in T$ and also $x \in G$, then $x \in T(G)$, while if $x \in T$ but $x \notin G$, then $x = gu$ for some $g \in G$, in which case $g = xu^{-1}$ is the product of torsion units, hence in $T(G)$, giving $x \in T(G)u$. So $T = M(T(G), *, u)$ as asserted.

Now the torsion units of $\mathbf{Z}T(G)$ form a subgroup of $\mathcal{U}(\mathbf{Z}T(G))$, so the set of torsion units of $T(G)$ (i.e., $T(G)$ itself) is an abelian group or a Hamiltonian 2-group (by the known result for groups), but not abelian because it contains a and b. Thus $T(G) = Q$ or $Q \times E$, for some elementary abelian 2-group E. In the case $T(G) = Q$, we claim that $T = M(Q, *, u) = C$, the Cayley loop. For this first note that, in the case of Q, the involution $*$ defined by (1.1), is just the map $g \mapsto g^{-1}$. Secondly, since u and b are part of a triple of elements which do not associate, u and b do not commute, so $\langle b, u\rangle$ is a nonabelian group in an integral group ring where the torsion units form a subgroup. Thus $\langle b, u\rangle$ is Q or $Q \times E$, E some elementary abelian 2-group, and, in either case, u^2 is the generator of the centre of $\langle b, u\rangle$. In particular, $u^2 \neq 1$. Thus $u^2 = s$ is the unique nonidentity commutator of Q, so $T = M(Q, *, u) = C$ [8, preamble to Theorem 1]. It remains to consider the possibility that $T(G) = Q \times E$, where E is an elementary abelian 2-group. As before, we note that $*$ is necessarily the inverse map on $Q \times E$ and that $u^2 = (s, 1)$, s the generator of the centre of Q. Thus $T(G) = (Q \times E, -1, u) = M(Q, -1, u_1) \times E = C \times E$, where $u = (u_1, u_2) \in Q \times E$, [5, Lemma 5.2]. This completes the proof in one direction.

In the other direction, we assume that the torsion subloop T of L is a loop which is either an abelian group or a Moufang Hamiltonian 2-loop and that

$$\ell^{-1} t \ell \in \langle t\rangle. \tag{3.1}$$

for every $t \in T$ and $\ell \in L$. We first argue that in each case here, idempotents of $\mathbf{Q}T$ are central in $\mathbf{Q}L$. If T is an abelian group, the idempotents of $\mathbf{Q}T$ are *group determined*, that is, $\mathbf{Q}$-linear combinations of idempotents of the form $\widehat{K} = \frac{1}{|K|}\sum_{k \in K} k$, K a subgroup of T [17, Proposition VI.1.16]. Since every subgroup of T is normal in L by (3.1), every $\widehat{K}$ is central and so every idempotent of $\mathbf{Q}T$ is central in $\mathbf{Q}L$ as claimed. If $T = Q$ or $Q \times E$, Coelho and Polcino Milies have explicitly determined the idempotents of T and noted that (3.1) implies that each is central in $\mathbf{Q}L$ [6]. If $T = C$, then $T = M(Q, -1, u)$ where $u^2 = s$ is the generator of the centre of Q. The group algebra $\mathbf{Q}Q \cong 4\mathbf{Q} \oplus H$, H the division algebra of quaternions over $\mathbf{Q}$, and the primitive idempotents corresponding to this decomposition are, respectively,

$$\begin{aligned}
e_1 &= \tfrac{1}{8}(1 + a + a^2 + a^3 + b + ab + a^2 + a^3 b) = \widehat{Q} \\
e_2 &= \tfrac{1}{8}(1 + a + a^2 + a^3 - b - ab - a^b - a^3 b) = \widehat{a} - \widehat{Q} \\
e_3 &= \tfrac{1}{8}(1 - a + a^2 - a^3 + b - ab + a^2 b - a^3 b) = \widehat{b} - \widehat{Q} \\
e_4 &= \tfrac{1}{8}(1 - a + a^2 - a^3 - b + ab - a^2 b + a^3 b) = \widehat{ab} - \widehat{Q} \\
e_5 &= \tfrac{1}{2}(1 - a^2).
\end{aligned}$$

Here Q is presented as $\langle a, b \mid a^4 = 1, b^2 = a^2, ba = a^3 b\rangle$ and we have written $\widehat{x}$ for $\widehat{\langle x\rangle}$. By expressing each idempotent as a linear combination of $\widehat{K}$'s for various subgroups K of Q it is very clear why each is central. Now, remembering that the elements of C are those of $Q \cup Qu$, it is easy to see that in $\mathbf{Q}C$, for $i = 1, 2, 3, 4$, e_i is the sum $e_{i1} + e_{i2}$ of orthogonal idempotents $e_{i1} = \frac{1}{2}(1 + u)e_i$ and $e_{i2} = \frac{1}{2}(1 - u)e_i$ and that $(\mathbf{Q}C)e_{ij} \cong \mathbf{Q}$. (The important thing to note is that $u^2 = a^2$ and $a^2 e_i = e_i$.) Thus $\mathbf{Q}C \cong 8\mathbf{Q} \oplus C$ where $C \cong (\mathbf{Q}C)e_5$ is the Cayley division algebra. (See also [7, p. 3983].) Since e_5 and each of the e_{ij} are of the form $x+yu$, with x central in $\mathbf{Q}Q$ and $a^2 y = y$, and since $a^2 = s$, the unique commutator in C, it follows that e_5 and all the e_{ij} are central in $\mathbf{Q}L$ (see (1.2) and (1.3)). Since $\mathbf{Q}C$ is the direct sum of division algebras with identities e_5 and the e_{ij}, the only idempotents in $\mathbf{Q}C$ are linear combinations of the primitive ones and hence central. The case $T = C \times E$ is similar. Here, the loop algebra $\mathbf{Q}T$ is the loop algebra of C with coefficients in $\mathbf{Q}E$ which is the direct sum of fields Q_i with identities ϵ_i. In this case, Coelho and Polcino Milies have shown that the primitive idempotents of $(\mathbf{Q}E)Q$ are $\epsilon_i e_j$, with the e_j, $j = 1, \ldots, 5$, as above. It follows that the primitive idempotents of $\mathbf{Q}(C \times E)$ are $\epsilon_i e_5$ and $\frac{1}{2}(1 \pm u)\epsilon_i e_j$, $j = 1, 2, 3, 4$. These are central in $\mathbf{Q}L$ and hence all idempotents are. Note that our arguments here have also shown that if T is a Moufang Hamiltonian 2-loop, the loop algebra $\mathbf{Q}T$ is the direct sum of division algebras, something which is also certainly true if T is abelian. Thus the hypotheses of Lemma 2.3 are satisfied.

Now let μ be a torsion unit in $\mathbf{Z}L$. Replacing L by the subloop generated by the support of μ, we can assume that L is finitely generated. By Lemma 2.3, $\mu = \nu\ell$ where $\nu \in \mathbf{Z}T$ and $\ell \in L$. Since $\mu^n = 1$ for some n and since (3.1) implies that $(\nu\ell)^n$ can be written in the form $\gamma\ell^n$ where $\gamma \in \mathbf{Z}T$, we have $\ell^n = \gamma^{-1}$. So ℓ^n has support in T and, since it's an element of L, $\ell^n \in T$. Thus $\ell \in T$ and $\mu \in \mathcal{U}(\mathbf{Z}T)$. Since T is abelian or a Hamiltonian 2-loop, the torsion units of $\mathbf{Z}T$ are trivial [9] and hence form a subloop. This completes the proof. □

The following corollary is the nonassociative analogue of a lemma of Polcino Milies [13].

Corollary 3.2. *The torsion units of an alternative loop ring form a subloop if and only if they are trivial.*

Proof. One direction is obvious and the other became obvious in the last steps of our proof. □

We shall now show that several properties of the loop of units in $\mathbf{Z}L$, known to be equivalent for groups which are finite [14], are always equivalent when L is an RA loop (finite or infinite). In doing so, we shall use some ideas of Vikas Bist [2].

In general, in an arbitrary loop, it is not true that conjugacy is an equivalence relation because it need not be transitive. However, as in group theory, we shall say that a loop L is an *FC loop* if, for each $\ell \in L$, the set $\{x^{-1}\ell x \mid x \in L\}$ is finite. From (1.4) we see that RA loops are FC loops.

A loop L is *n-Engel* if for any $x, y \in L$, the extended commutator $(\ldots((x,y),y),\ldots,y)$ (with y repeated n times) is the identity. Since commutators of an RA loop are central, an RA loop is 2-Engel (and hence n-Engel for all $n \geq 2$).

Theorem 3.3. *Let L be an RA loop with torsion subloop T. Then the following are equivalent:*

(i) *$\mathcal{U}(\mathbf{Z}L)$ is an RA loop.*
(ii) *$\mathcal{U}(\mathbf{Z}L)$ is FC.*
(iii) *$\mathcal{U}(\mathbf{Z}L)$ is nilpotent.*
(iv) *$\mathcal{U}(\mathbf{Z}L)$ is nilpotent of class 2.*
(v) *$[\mathcal{U}(\mathbf{Z}L)]'$ is a torsion loop.*
(vi) *$[\mathcal{U}(\mathbf{Z}L)]'$ is a group of order 2.*
(vii) *$\mathcal{U}(\mathbf{Z}L)$ is n-Engel for some $n \geq 2$.*
(viii) *$\mathcal{U}(\mathbf{Z}L)$ is 2-Engel.*

(ix) *T is an abelian group or a Moufang Hamiltonian 2-loop and, for any $t \in T$ and any $x \in L$, we have $x^{-1}tx = t^{\pm 1}$. Moreover, if T is abelian and $x \in L$ is an element that does not centralize T, then $x^{-1}tx = t^{-1}$ for all $t \in T$.*

Proof. We show that each of conditions (i) through (viii) is equivalent to (ix). First, notice that each of conditions (i) to (vi) guarantees that $T\mathcal{U}(\mathbf{Z}L)$ is a subloop, hence that T is an abelian group or a Moufang Hamiltonian 2-loop, by Theorem 3.1. In the case that $\mathcal{U}(\mathbf{Z}L)$ is RA, Lemma 2.1 gives this to us. If $\mathcal{U}(\mathbf{Z}L)$ is FC, and if α and β are torsion units of $\mathbf{Z}L$, then the subgroup which they generate is FC (a subloop of an FC loop is clearly FC) and it contains the torsion units α and β. Since the torsion units of an FC group form a subgroup [16, Chapter 15], $\alpha\beta$ is a torsion unit. The argument in the case $\mathcal{U}(\mathbf{Z}L)$ is nilpotent is essentially the same. If $\mathcal{U}(\mathbf{Z}L)$ is nilpotent and α and β are torsion units of $\mathbf{Z}L$, then the group which they generate is nilpotent. Since the torsion elements of a nilpotent group form a subgroup, it follows again that $\alpha\beta$ is a torsion unit. If $[\mathcal{U}(\mathbf{Z}L)]'$ is a torsion subloop, and if α and β are torsion units, then for any n, $(\alpha\beta)^n = \alpha^n\beta^n c$, for some c in the commutator subloop of $\mathcal{U}(\mathbf{Z}L)$. Thus, for some n, $(\alpha\beta)^n$ is in $[\mathcal{U}(\mathbf{Z}L)]'$ and hence of finite order. So again we see that $\alpha\beta$ has finite order. In the case that $\mathcal{U}(\mathbf{Z}L)$ is n-Engel, we argue differently. In this case, for each element $x \in L$ and each $t \in T$, we know that $\langle x, t\rangle$ is a group contained in L whose group ring is therefore n-Engel. So we can apply the argument in [2, §2] to $\mathbf{Z}\langle x, t\rangle$ and obtain directly that every subloop of T is normal in L; so T is an abelian group or a Moufang Hamiltonian loop which is necessarily a 2-loop (otherwise, its loop ring would contain a free noncyclic group [11]). Hence, in all cases, T is an abelian group or a Moufang Hamiltonian 2-loop. Also, if $x \in L$ and $t \in T$, by applying another argument from §2 of [2] to the group $\langle x, t\rangle$, we see, in addition, that, $x^{-1}tx = t^{\pm 1}$. If T is abelian and $x \in L$ does not centralize T, there exists at least one element $t_1 \in T$ such that $x^{-1}t_1x = t_1^{-1}$. Then, given any other $t \in T$, since t_1 and t commute, they associate with every other element $x \in L$, so $\langle x, t_1, t\rangle$ is a group. Further arguments in [2] now give $x^{-1}tx = t^{-1}$. Thus, each of conditions (i) through (viii) implies (ix). Now we show that condition (ix) implies all the other conditions of the theorem.

Assuming (ix), we first notice that if T is a Moufang Hamiltonian 2-loop, then $\mathcal{U}(\mathbf{Z}T) = \pm T$ [9] so $\mathcal{U}(\mathbf{Z}L) = \pm L$ by Lemma 2.3. Suppose T is commutative. Here the arguments in [2, §3] show that $\mathcal{U}(\mathbf{Z}L) = \Gamma L$, $\Gamma = \{\gamma \in \mathcal{U}(\mathbf{Z}T) \mid \gamma = \gamma'\}$ where, for $\gamma = \sum \gamma_t t$, γ' is defined to be $\sum \gamma_t t^{-1}$. We claim that every $\gamma \in \Gamma$ is central. For this, let $\gamma \in \Gamma$ and note that γ is certainly central if every t in the support of γ is central. On the other hand, if some t in the support of γ is not central, then for some $x \in L$, $x^{-1}tx = t^{-1}$ for every $t \in T$. Since $x^{-1}tx \in \{t, t^*\}$ (see (1.4)), we have $t^{-1} \in \{t, t^*\}$ for all t (since T is commutative). If $t^{-1} = t$, then t is central (because $x^{-1}tx = t^{\pm 1}$ for all x), so $t^* = t$ and $t^{-1} = t = t^*$. Thus, for any t, we have $t^{-1} = t^*$, so $\gamma' = \gamma^*$ (see (1.2)) and again we have that $\gamma \in \Gamma$ is central, by (1.3).

Summarizing, we have shown that whether T is an abelian group or a Moufang Hamiltonian 2-loop, we have

$$\mathcal{U}(\mathbf{Z}L) = \Gamma L$$

where Γ is a group contained in $\mathcal{Z}(\mathcal{U}(\mathbf{Z}L))$. From this fact, it is clear that $\mathcal{U}(\mathbf{Z}L)$ is nilpotent and 2-Engel. It is also clear that this loop has a unique nonidentity commutator and the *limited commutativity* property (see [4]) and hence is RA, thus FC (as remarked earlier) and with a commutator subloop of order 2. Thus condition (ix) implies all of conditions (i) through (viii) and the theorem is complete. □

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