Integration: Mathematical Theory and Applications ISSN xxxx-xxxx
Volume 1, Number 2, pp. 25–57

A FREDHOLM-TYPE THEOREM FOR LINEAR INTEGRAL EQUATIONS OF STIELTJES TYPE

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Abstract

We consider the linear integral equations of Fredholm and Volterra

$$x(t) - \int_a^b \alpha(t,s)\, x(s)\, dg(s) = f(t), \quad t \in [a,b],$$

and

$$x(t) - \int_a^t \alpha(t,s)\, x(s)\, dg(s) = f(t), \quad t \in [a,b],$$

in the frame of the Henstock-Kurzweil integral and we prove results on the existence and uniqueness of solutions. More precisely, we consider the above equations in the sense of Henstock-Kurzweil and we state a Fredholm Alternative theorem for the first equation and an existence and uniqueness result for the second equation for which the solution is given explicitly.

Keywords and phrases: Linear integral equations, Fredholm-Stieltjes integral equations, Volterra-Stieltjes integral equations, Henstock-Kurzweil integral, Riemann-Stieltjes integral, Fredholm Alternative, existence, uniqueness.

2000 Mathematical Subject Classification: 45A05, 45B05, 45C05, 26A39.

1. Introduction

The aim of this paper is mainly to disseminate C. S. Hönig's results and ideas on the theory of linear integral equations and the theory of Henstock-Kurzweil non-absolute integration. Because many of Hönig's works are not easily available for the public, we collect facts and proofs of the theory developed by him so that the presentation of them in the present paper is as self-contained as possible. In addition, we add some new results which generalize Hönig's ideas.

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We consider the following Fredholm and Volterra linear integral equations

$$x(t) - \int_a^b \alpha(t,s)\, x(s)\, dg(s) = f(t), \quad t \in [a,b], \tag{1}$$

and

$$x(t) - \int_a^t \alpha(t,s)\, x(s)\, dg(s) = f(t), \quad t \in [a,b], \tag{2}$$

in the frame of the Henstock-Kurzweil integral and prove a Fredholm Alternative theorem for (1) (see Theorem 5.2) and an existence and uniqueness theorem for (2) for which the solution is given explicitly (see Theorem 6.1).

Let $g : [a,b] \to \mathbb{R}$ be an element of a certain subspace of the space of continuous functions from $[a,b]$ to $\mathbb{R}$. Let $K_g([a,b],\mathbb{R})$ denote the space of all functions $f : [a,b] \to \mathbb{R}$ such that the integral $\int_a^b f(s)dg(s)$ exists in the sense of Henstock-Kurzweil (see [9], [10] and [15]). It is known that even when $g(s) = s$, an element of $K_g([a,b],\mathbb{R})$ can not only have many points of discontinuities, but it can also be of unbounded variation. Indeed, the space of Henstock-Kurzweil integrals encompasses Riemann, Lebesgue and Newton's integrals.

In the present paper, we prove a Fredholm Alternative theorem for equation (1), that is, we prove that

- either equation (1) has a unique solution $x_f \in K_g([a,b],\mathbb{R})$ and (1) has a resolvent with similar integral representation,
- or the corresponding homogeneous equation admits non-trivial solutions in $K_g([a,b],\mathbb{R})$ (see Theorem 5.2 in the sequell).

For equation (2) we prove that

- for every $f \in K_g([a,b],\mathbb{R})$, equation (2) admits a unique solution $x_f \in K_g([a,b],\mathbb{R})$ and the resolvent of (2) is given by Neumann series (Theorem 6.1).

Although the above results are proved in the case where α, x and f are real-valued, the auxiliary theory developed throughout this paper is presented in a general abstract space context.

The main obstacle encountered in obtaining the above results is the fact that the normed space of Henstock-Kurzweil integrable functions is not complete (it is ultrabornological however - see [8]). Therefore one can not apply usual fixed point theorems in order to obtain existence results. Such difficulty was faced by the authors in [3] and by Hönig in [11] when existence and uniqueness results were proved for the non-Stieltjes-type integral equations

$$x(t) - \int_a^b \alpha(t,s)\, x(s)\, ds = f(t), \quad t \in [a,b],$$

and

$$x(t) - \int_a^t \alpha(t,s)\, x(s)\, ds = f(t), \quad t \in [a,b],$$

in a general Henstock-Kurzweil integral setting.

The obstacle mentioned above can be overcome if some ideas due to Hönig are applied ([11]). See [3] and [5], for instance. In the present paper, we adapt such ideas to the Stieltjes case. The crucial point is

- to obtain representation theorems which, together with integration by parts and substitution formulas, will enable us to transform integral equations in the Henstock-Kurzweil sense into integral equations with respect to the usual Riemann-Stieltjes integral.

Then we apply the Fredholm Alternative for the Riemann-Stietjes integral (proved in [3], Theorems 2.4 and 2.5) and a result on the existence and uniqueness of a solution of a Volterra-Stieltjes integral equation (proved in [12], Theorems 3.8 and 3.4) in order to obtain results on the existence and uniqueness of solutions of these Stieltjes-type integral equations. As a consequence, we obtain a Fredholm Alternative for equation (1), which is presented in Section 5. We also get an existence and uniqueness result for equation (2), which is presented in Section 6. The other sections are organized as follows. Section 2 is devoted to the fundamental theory of the Riemann-Stieltjes integral in Banach spaces, where we present basic results, representation theorems and the Fredholm Alternative. In Section 3, we give some basic definitions of the Henstock-Kurzweil integration theory. In Section 4, we present auxiliary results for the Henstock-Kurzweil integral such as the fundamental theorem of calculus and a substitution formula.

2. The Riemann-Stieltjes integral in Banach spaces

2.1. Functions of Bounded $\mathcal{B}$-variation, of Bounded Semi-variation and of Bounded Variation

A *bilinear triple* (we write BT) is a set of three vector spaces E, F and G, where F and G are normed spaces with a bilinear mapping $\mathcal{B}: E \times F \to G$. For $x \in E$ and $y \in F$, we write $xy = \mathcal{B}(x,y)$ and we denote the BT by $(E,F,G)_{\mathcal{B}}$ or simply by (E,F,G). A *topological BT* is a BT (E,F,G) where E is also a normed space and $\mathcal{B}$ is continuous. We suppose that $\|\mathcal{B}\| \leq 1$.

If E and F are normed spaces, then we denote by $L(E,F)$ the space of all linear continuous functions from E to F. We write $E' = L(E,\mathbb{R})$ and $L(E) = L(E,E)$, where $\mathbb{R}$ denotes the real line.

Throughout this paper, X, Y and Z will always denote Banach spaces.

Example 2.1. *As an example of a BT we can consider $E = L(X,Y)$, $F = L(Z,X)$, $G = L(Z,Y)$ and $\mathcal{B}(v,u) = v \circ u$. In particular, when $Z = \mathbb{R}$, we have $E = L(X,Y)$, $F = X$, $G = Y$ and $\mathcal{B}(u,x) = u(x)$; when $X = \mathbb{R}$, we have $E = Y$, $F = Y'$, $G = \mathbb{R}$ and $\mathcal{B}(y,y') = \langle y,y' \rangle$; when $X = Z = \mathbb{R}$, we have $E = G = Y$, $F = \mathbb{R}$ and $\mathcal{B}(y,\lambda) = \lambda y$.*

Given a BT $(E,F,G)_{\mathcal{B}}$, for every $x \in E$, we define

$$\|x\|_{\mathcal{B}} = \sup\{\|\mathcal{B}(x,y)\|; \|y\| \leq 1\}$$

and

$$E_{\mathcal{B}} = \{x \in E; \|x\| < \infty\}.$$

When we endow $E_{\mathcal{B}}$ with the norm $\|\cdot\|_{\mathcal{B}}$, we say that the topological BT $(E_{\mathcal{B}},F,G)$ is associated to the BT (E,F,G).

Let E be a vector space and Γ_E be a set of seminorms defined on E such that $p_1,\ldots,p_m \in \Gamma_E$ implies

$$\sup[p_1,\ldots,p_m] \in \Gamma_E.$$

Then Γ_E defines a topology on E and the sets

$$V_{p,\varepsilon} = \{x \in E;\, p(x) < \varepsilon\}, \quad p \in \Gamma_E,\ \ \varepsilon > 0,$$

form a basis of neighborhoods of 0. The sets $x_0 + V_{p,\varepsilon}$ form a basis of the neighborhood of $x_0 \in E$. Moreover, when endowed with this topology, E is called a locally convex space (see [13], p. 3, 4).

Example 2.2. *Every normed or seminormed space E is a locally convex space.*

For other examples, the reader may want to consult [5].

Let $[a,b]$ be a compact interval of $\mathbb{R}$. Any finite set of closed non-overlapping subintervals $[t_{i-1},t_i]$ of $[a,b]$ such that the union of all intervals $[t_{i-1},t_i]$ equals $[a,b]$ is called a *division* of $[a,b]$. In this case we write $d = (t_i) \in D_{[a,b]}$, where $D_{[a,b]}$ denotes the set of all divisions of $[a,b]$. By $|d|$ we mean the number of subintervals in which $[a,b]$ is divided through a given $d \in D_{[a,b]}$.

Given a BT $(E,F,G)_{\mathcal{B}}$ and a function $\alpha : [a,b] \to E$, for every division $d = (t_i) \in D_{[a,b]}$ we define

$$SB_d(\alpha) = SB_{[a,b],d}(\alpha) = \sup\left\{ \left\| \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})]\, y_i \right\| ;\, y_i \in F,\, \|y_i\| \leq 1 \right\}$$

and

$$SB(\alpha) = SB_{[a,b]}(\alpha) = \sup\left\{ SB_d(\alpha);\, d \in D_{[a,b]} \right\}.$$

Then $SB(\alpha)$ is the *$\mathcal{B}$-variation* of α on $[a,b]$. We say that α is a function of *bounded $\mathcal{B}$-variation* whenever $SB(\alpha) < \infty$. When this is the case, we write $\alpha \in SB([a,b],E)$.

The following properties are not difficult to prove:

(SB1) $SB([a,b],E)$ is a vector space and the mapping $\alpha \in SB([a,b],E) \mapsto SB(\alpha) \in \mathbb{R}_+$ is a seminorm;

(SB2) Given $\alpha \in SB([a,b],E)$, the function $t \in [a,b] \mapsto SB_{[a,t]}(\alpha) \in \mathbb{R}_+$ is monotonically increasing;

(SB3) Given $\alpha \in SB([a,b],E)$ and $c \in\,]a,b[$, $SB_{[a,b]}(\alpha) \leq SB_{[a,c]}(\alpha) + SB_{[c,b]}(\alpha)$

Consider the BT $(L(X,Y),X,Y)$. In this case we replace $SB([a,b],L(X,Y))$ and $SB(\alpha)$ by $SV([a,b],L(X,Y))$ and $SV(\alpha)$ respectively. Any element of $SV([a,b],L(X,Y))$ is called a function of *bounded semi-variation*.

Given a function $\alpha : [a,b] \to E$, E a normed space, and $d = (t_i) \in D_{[a,b]}$, we define

$$V_d(\alpha) = V_{d,[a,b]}(\alpha) = \sum_{i=1}^{|d|} \|\alpha(t_i) - \alpha(t_{i-1})\|$$

and the *variation* of α is given by

$$V(\alpha) = V_{[a,b]}(\alpha) = \sup\left\{V_d(\alpha); d \in D_{[a,b]}\right\}.$$

If $V(f) < \infty$, then α is called a function of *bounded variation*. In this case, we write $\alpha \in BV([a,b],E)$. We also have

$$BV([a,b],L(E,F)) \subset SV([a,b],L(E,F))$$

and

$$SV([a,b],L(E,\mathbb{R})) = BV\left([a,b],E'\right).$$

Remark 2.1. *Consider a BT (E,F,G). The definition of variation of a function $\alpha : [a,b] \to E$, where E is a normed space, can also be considered as a particular case of the B-variarion of α in two different ways.*

- *Let $E = F'$, $G = \mathbb{R}$ or $G = \mathbb{C}$ and $B(x',x) = \langle x, x'\rangle$. By the definition of the norm in $E = F'$, we have*

$$\begin{aligned} V_d(\alpha) &= \sum_{i=1}^{|d|} \|\alpha(t_i) - \alpha(t_{i-1})\| \\ &= \sup\left\{\left|\sum_{i=1}^{|d|}\langle x_i, \alpha(t_i) - \alpha(t_{i-1})\rangle\right|; x_i \in F, \|x_i\| \leq 1\right\} = SB_d(\alpha). \end{aligned}$$

  Thus when we consider the BT $(Y',Y,\mathbb{R})$, we write $BV(\alpha)$ and $BV([a,b],Y')$ instead of $SB(\alpha)$ and $SB([a,b],Y')$ respectively.

- *Let $F = E'$, $G = \mathbb{R}$ or $G = \mathbb{C}$ and $B(x,x') = \langle x, x'\rangle$. By the Hahn-Banach Theorem, we have*

$$\|\alpha(t_i) - \alpha(t_{i-1})\| = \sup\left\{\left|\langle\alpha(t_i) - \alpha(t_{i-1}), x_i'\rangle\right|; x_i' \in E', \|x_i'\| \leq 1\right\}$$

  and hence

$$\begin{aligned} V_d(\alpha) &= \sum_{i=1}^{|d|} \|\alpha(t_i) - \alpha(t_{i-1})\| \\ &= \sup\left\{\left|\sum_{i=1}^{|d|}\langle\alpha(t_i) - \alpha(t_{i-1}), x_i'\rangle\right|; x_i' \in E', \|x_i'\| \leq 1\right\} = SB_d(\alpha). \end{aligned}$$

Given $c \in [a,b]$, we define the spaces

$$BV_c([a,b],X) = \{f \in BV([a,b],X); f(c) = 0\}$$

and

$$SV_c([a,b],L(X,Y)) = \{\alpha \in SV([a,b],L(X,Y)); \alpha(c) = 0\}.$$

Such spaces are complete when endowed, respectively, with the norm given by the variation $V(f)$ and the norm given by the semi-variation $SV(\alpha)$. See [17], for instance.

The following properties are not difficult to prove:

(V1) Every function $\alpha \in BV([a,b],E)$ is bounded and $\|\alpha(t)\| \leq \|\alpha(a)\| + V_{[a,t]}(\alpha)$.

(V2) Given $\alpha \in BV([a,b],E)$ and $c \in]a,b[$, we have $V_{[a,b]}(\alpha) = V_{[a,c]}(\alpha) + V_{[c,b]}(\alpha)$.

The next results are borrowed from [14]. We include the proofs here, since this reference is not easily available. Lemmas 2.1 and 2.2 below are respectively Theorems I.2.7 and I.2.8 from [14].

Lemma 2.1. *Let* $\alpha \in BV([a,b],X)$. *Then*

(i) For all $t \in]a,b]$, *there exists* $\alpha(t-) = \lim_{\varepsilon \downarrow 0} \alpha(t-\varepsilon)$.

(ii) For all $t \in [a,b[$, *there exists* $\alpha(t+) = \lim_{\varepsilon \downarrow 0} \alpha(t+\varepsilon)$.

Proof. We will prove (i). The proof of (ii) follows analogously.

Consider a strictly increasing sequence $\{t_n\}$ in $[a,t[$ converging to t. Then

$$\sum_{i=1}^{n} \|\alpha(t_i) - \alpha(t_{i-1})\| \leq V_{[a,t]}(\alpha), \quad \text{for all } n.$$

Hence

$$\sum_{i=1}^{\infty} \|\alpha(t_i) - \alpha(t_{i-1})\| \leq V_{[a,t]}(\alpha).$$

Then $\{\alpha(t_n)\}$ is a Cauchy sequence, since

$$\|\alpha(t_m) - \alpha(t_n)\| \leq \sum_{i=n+1}^{m} \|\alpha(t_i) - \alpha(t_{i-1})\| \leq \varepsilon,$$

for sufficiently large m,n. The limit $\alpha(t-)$ of $\{\alpha(t_n)\}$ is independent of the choice of $\{t_n\}$ and we finish the proof. □

Lemma 2.2. *Let* $\alpha \in BV([a,b],X)$. *For every* $t \in [a,b]$, *let* $v(t) = V_{[a,t]}(\alpha)$. *Then*

(i) $v(t+) - v(t) = \|\alpha(t+) - \alpha(t)\|$, $t \in [a,b[$.

(ii) $v(t) - v(t-) = \|\alpha(t) - \alpha(t-)\|$, $t \in]a,b]$.

Proof. By property (SB2), v is monotonically increasing and hence $v(t+)$ and $v(t-)$ exist. By Lemma 2.1, $\alpha(t+)$ and $\alpha(t-)$ also exist. We will prove (i). The proof of (ii) follows analogously.

Suppose $s > t$. Then property (V2) implies $V_{[a,s]}(\alpha) = V_{[a,t]}(\alpha) + V_{[t,s]}(\alpha)$. Therefore $\|\alpha(s) - \alpha(t)\| \leq V_{[t,s]}(\alpha) = V_{[a,s]}(\alpha) - V_{[a,t]}(\alpha)$ and hence $\|\alpha(t+) - \alpha(t)\| \leq v(t+) - v(t)$.

Conversely, given $d \in D_{[a,t]}$, let $v_d(t) = V_d(\alpha)$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that $v(t+\sigma) - v(t+) \leq \varepsilon$ and $\|\alpha(t+\sigma) - \alpha(t+)\| \leq \varepsilon$ and there exists $d : a = t_0 < t_1 < \ldots < t_n = t < t_{n+1} = t+\sigma$ such that $v(t+\sigma) - v_d(t+\sigma) \leq \varepsilon$ whenever $0 < \sigma \leq \delta$. Then

$$\begin{aligned} v(t+\sigma) - v(t) &\leq v_d(t+\sigma) + \varepsilon - v_d(t) = \|\alpha(t+\sigma) - \alpha(t)\| + \varepsilon \\ &\leq \|\alpha(t+) - \alpha(t)\| + 2\varepsilon \end{aligned}$$

and hence $v(t+) - v(t) \leq \|\alpha(t+) - \alpha(t)\|$. This completes the proof. □

Let $\alpha \in BV([a,b],X)$. Since $\|\alpha(t)\| \leq \|\alpha(a)\| + V_{[a,t]}$, then Lemma 2.2 implies that the sets

$$\{t \in [a,b[; \|\alpha(t+) - \alpha(t)\| \geq \varepsilon\} \qquad \text{and} \qquad \{t \in]a,b]; \|\alpha(t) - \alpha(t-)\| \geq \varepsilon\}$$

are finite for every $\varepsilon > 0$. Thus we have the next result which can be found in [14] (Proposition I.2.10 there).

Proposition 2.1. *Let $\alpha \in BV([a,b],X)$. Then the set of points of discontinuity of α is countable (and all discontinuities are of the first kind).*

Let us define

$$BV_a^+([a,b],X) = \{\alpha \in BV_a([a,b],X); \alpha(t+) = \alpha(t), t \in]a,b[\}.$$

A proof that $BV_a^+([a,b],X)$ with the variation norm is complete can be found in [14], Theorem I.2.11. We reproduce it next.

Theorem 2.1. *$BV_a^+([a,b],X)$ is a Banach space when endowed with the variation norm.*

Proof. For every $t \in]a,b[$, we have $\|\alpha(t)\| \leq \|\alpha(a)\| + V(\alpha) = V(\alpha)$. Hence the mappings

$$T_t : \alpha \in BV_a([a,b],X) \mapsto \alpha(t) \in X$$

and

$$T_{t+} : \alpha \in BV_a([a,b],X) \mapsto \alpha(t+) \in X$$

are continuous. Therefore $BV_a^+([a,b],X)$ is a closed subspace of $BV_a([a,b],X)$, since it is given by the continuous mappings T_t and T_{t+}, $t \in]a,b[$, and the result follows from the fact that $BV_a([a,b],X)$ is a Banach space with the variation norm. □

Given $u \in L(X,Z')$ and $z \in Z$, we denote an element of X' by $z \circ u$ which is given by

$$\langle z \circ u, x\rangle = \langle z, u(x)\rangle, \qquad x \in X.$$

We have $|\langle z \circ u, x\rangle| = |\langle z, u(x)\rangle| \leq \|z\| \, \|u(x)\| \leq \|z\| \, \|u\| \, \|x\|$.

We denote by $u^* \in L(Y',X')$ the adjoint or transposed operator of $u \in L(X,Y)$ which is defined by

$$\left\langle x, u^*\left(y'\right)\right\rangle = \left\langle u(x), y'\right\rangle, \qquad x \in X, \; y' \in Y'.$$

Then

$$y' \circ u = u^*(y'), \qquad y' \in Y'$$

since $(y' \circ u)(x) = \langle y', u(x)\rangle = \langle u^*(y'), x\rangle$ for every $x \in X$.

Next we present [14], Proposition I.3.5 and the corollary that follows it.

Proposition 2.2. *Given a function $\alpha : [a,b] \to L(X,Z')$. Then*

(i) $SV(\alpha) = \sup\{V(z \circ \alpha); z \in Z, \|z\| \leq 1\}$*;*

(ii) $\alpha \in SV([a,b],L(X,Z'))$ *if and only if* $z \circ \alpha \in BV([a,b],X')$*, for every* $z \in Z$*.*

Proof. In order to prove (i), it is enough to observe that, if $d = (t_i) \in D_{[a,b]}$, then

$$\begin{aligned} SV_d(\alpha) &= \sup\left\{ \left\| \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})]x_i \right\| ; x_i \in X, \|x_i\| \leq 1 \right\} \\ &= \sup\left\{ \sup_{\|z\|\leq 1} \left| \sum_{i=1}^{|d|} \langle z, [\alpha(t_i) - \alpha(t_{i-1})]x_i \rangle \right| ; x_i \in X, \|x_i\| \leq 1 \right\} \\ &= \sup_{\|z\|\leq 1} \sup\left\{ \sum_{i=1}^{|d|} \langle x_i, z\circ\alpha(t_i) - z\circ\alpha(t_{i-1}) \rangle ; x_i \in X, \|x_i\| \leq 1 \right\} \\ &= \sup\{V_d(z\circ\alpha); z \in Z, \|z\| \leq 1\}. \end{aligned}$$

Now we will prove (ii). By (i), if $\alpha \in SV([a,b], L(X,Z'))$, then $z\circ\alpha \in BV([a,b],X')$, $z \in Z$. The converse follows by the uniform boundedness principle. Indeed. Let us define $\mathcal{D} = \{(d,x); d \in D_{[a,b]}, x = (x_1,\ldots,x_{|d|}), x_i \in X, \|x_i\| \leq 1\}$. Then for each $(d,x) \in \mathcal{D}$, we define $F_{d,z} \in Z'$ by

$$F_{d,z}(z) = \sum_{i=1}^{|d|} \langle z, [\alpha(t_i) - \alpha(t_{i-1})]x_i \rangle, \qquad z \in Z.$$

Then the set $\{F_{d,z}; (d,z) \in \mathcal{D}\} \subset Z'$ is simply bounded on Z, since $|F_{d,z}(z)| \leq V_d(z\circ\alpha) \leq V(z\circ\alpha)$, for all $(d,z) \in \mathcal{D}$, and all $z \in Z$. Therefore by uniform boundedness principle, there exists $M \geq 0$ such that $\|F_{d,x}\| \leq M$, $(d,z) \in \mathcal{D}$, that is for all $(d,z) \in \mathcal{D}$, we have $\sup\{|F_{d,z}(z)|; z \in Z, \|z\| \leq 1\} \leq M$. Hence $SV(\alpha) \leq M$. □

Corollary 2.1. *Suppose* $\alpha \in SV([a,b], L(X,Z'))$. *Then*

(i) For every $t \in]a,b]$, *there exists* $\alpha(t-) \in L(X,Z')$ *in the sense that*

$$\lim_{\varepsilon\to 0_+} (z\circ\alpha)(t-\varepsilon) = z\circ\alpha(t-), \qquad z \in Z;$$

(ii) For every $t \in [a,b[$, *there exists* $\alpha(t+) \in L(X,Z')$ *in the sense that*

$$\lim_{\varepsilon\to 0_+} (z\circ\alpha)(t+\varepsilon) = z\circ\alpha(t+), \qquad z \in Z.$$

Proof. We will prove (i). The proof of (ii) follows analogously.

We may suppose, without loss of generality, that $\alpha(a) = 0$. By Lemma 2.1 (i) and Proposition 2.2 (ii), given $z \in Z$, there exists

$$T_z = \lim_{\varepsilon\to 0_+} (z\circ\alpha)(t-\varepsilon) \in X'.$$

Then the mapping $T : z \in Z \mapsto T_z \in X'$ is linear. It is also continuous, since $\|z\circ\alpha(t)\| \leq \|z\circ\alpha(a)\| + V_{[a,t]}(z\circ\alpha)$. Besides, Proposition 2.2 (i) implies $\|(z\circ\alpha)(t-\varepsilon)\| \leq V(z\circ\alpha) \leq \|z\| SV(\alpha)$. Hence $\|T\| \leq SV(\alpha)$.

Let ${}^tT$ be the transposed mapping of T, that is, ${}^tT : x \in X \mapsto {}^tTx \in Z'$ is defined by $\langle z, {}^tTx\rangle = \langle T_z, x\rangle$, where $z \in Z$. Then ${}^tT \in L(X, Z')$ and $\|{}^tT\| = \|T\| \leq SV(\alpha)$. Also ${}^tT = \alpha(t-)$ in the sense of (i), since for every $x \in X$, we have

$$\langle z \circ {}^tT, x\rangle = \langle z, {}^tTx\rangle = \langle T_z, x\rangle = \langle \lim_{\varepsilon \to 0_+} (z \circ \alpha)(t - \varepsilon), x\rangle = \lim_{\varepsilon \to 0_+} \langle (z \circ \alpha)(t - \varepsilon), x\rangle$$

and we finished the proof. □

In view of Corollary 2.1, we define the space

$$SV_a^+([a,b], L(X,Z')) = \left\{\alpha \in SV_a([a,b], L(X,Z'));\, z \circ \alpha \in BV_a^+([a,b], X'),\, z \in Z\right\}$$

which is complete when equipped with the semi-variation norm. This result can be found in [14], Theorem I.3.7. We include it here.

Theorem 2.2. *$SV_a^+([a,b], L(X,Z'))$ is a Banach space with the semi-variation norm.*

Proof. By Theorem 2.2 (i), for every $z \in Z$, the mapping

$$F_z : \alpha \in SV_a([a,b], L(X,Z')) \mapsto z \circ \alpha \in BV_a([a,b], X')$$

is continuous. By Theorem 2.1, $BV_a^+([a,b], X')$ is a closed subspace of $BV_a([a,b], X')$ and therefore $SV_a^+([a,b], L(X,Z')) = \cap\left\{(F_z)^{-1}(BV_a([a,b], X'));\, z \in Z\right\}$ is a closed subspace of the Banach space $SV_a([a,b], L(X,Z'))$ which implies the result. □

2.2. Riemann-Stieltjes Integration

For the next results we need the concept of the Riemann-Stieltjes integral which we define by means of tagged divisions.

A *tagged division* of $[a,b]$ is any set of pairs (ξ_i, t_i) such that $(t_i) \in D_{[a,b]}$ and $\xi_i \in [t_{i-1}, t_i]$ for every i. In this case we write $d = (\xi_i, t_i) \in TD_{[a,b]}$, where $TD_{[a,b]}$ denotes the set of all tagged divisions of $[a,b]$. Any subset of a tagged division of $[a,b]$ is a *tagged partial division* of $[a,b]$ and, in this case, we write $d \in TPD_{[a,b]}$.

A *gauge* of a set $E \subset [a,b]$ is any function $\delta : E \to]0, \infty[$. Given a gauge δ of $[a,b]$, we say that $d = (\xi_i, t_i) \in TPD_{[a,b]}$ is δ-*fine*, if $[t_{i-1}, t_i] \subset \{t \in [a,b]\,; |t - \xi_i| < \delta(\xi_i)\}$ for every i, that is,

$$\xi_i \in [t_{i-1}, t_i] \subset]\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)[, \qquad i = 1, 2, \ldots |d|.$$

Now we will define the Riemann-Stieltjes integrals by means of tagged divisions $d = (\xi_i, t_i)$ of $[a,b]$ and constant gauges δ (i.e., there is a $\delta_0 > 0$ such that $\delta(\xi) = \delta_0$ for every $\xi \in [a,b]$).

Let (E, F, G) be a BT. Any function $\alpha : [a,b] \to E$ is said to be Riemann integrable with respect to a function $f : [a,b] \to F$ if there exists an $I \in G$ such that for every $\varepsilon > 0$, there is a constant gauge δ of $[a,b]$ such that for every δ-fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$\left\| \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})]\, f(\xi_i) - I \right\| < \varepsilon.$$

In this case, we write $I=\int_a^b d\alpha(t)\,f(t)$.

By $R_f([a,b],E)$ we mean the space of all functions $\alpha:[a,b]\to E$ which are Riemann integrable with respect to $f:[a,b]\to F$.

Analogously we say that $f:[a,b]\to F$ is Riemann integrable with respect to $\alpha:[a,b]\to E$ if there exists an $I\in G$ such that for every $\varepsilon>0$, there is a constant gauge δ of $[a,b]$ such that for every δ-fine $d=(\xi_i,t_i)\in TD_{[a,b]}$,

$$\left\|\sum_{i=1}^{|d|}\alpha(\xi_i)\left[f(t_i)-f(t_{i-1})\right]-I\right\|<\varepsilon.$$

Then $R^{\alpha}([a,b],F)$ denotes the space of all functions $f:[a,b]\to F$ which are Riemann integrable with respect to a given $\alpha:[a,b]\to E$ with integral $I=\int_a^b\alpha(t)\,df(t)$.

The integrals $\int_a^b d\alpha(t)\,f(t)$ and $\int_a^b\alpha(t)\,df(t)$ defined above are known as Riemann-Stieltjes integrals.

Consider the BT (E,F,G) with $E=Y'=L(Y,\mathbb{R})$, $F=L(X,Y)$, $G=X'=L(X,\mathbb{R})$ and $\mathcal{B}(y',u)=y'\circ u$, for all $y'\in Y'$ and $u\in L(X,Y)$. We will use the identification

$$\int_c^d dy(t)\circ K(t,s)=\int_c^d K(t,s)^*\,dy(t),$$

where $y:[a,b]\to Y'$, $K:[c,d]\times[a,b]\to L(X,Y)$ and $K(t,s)^*$ denotes the adjoint of $K(t,s)\in L(X,Y)$.

2.3. Some Properties

Let E be a normed space. By $C([a,b],E)$ we mean the space of all continuous functions from $[a,b]$ to E endowed with the usual supremum norm, $\|\cdot\|_\infty$. We define

$$C_a([a,b],E)=\{f\in C([a,b],E);\, f(a)=0\}.$$

The next result is well-known. It gives the Integration by Parts Formula for the Riemann-Stieltjes integrals. For a proof of it, see for instance [13] or [7], Theorem 2.5.

Theorem 2.3 (Integration by Parts). *Let (E,F,G) be a BT. If either $\alpha\in SB([a,b],E)$ and $f\in C([a,b],F)$, or $\alpha\in C([a,b],E)$ and $f\in BV([a,b],F)$, then $\alpha\in R_f([a,b],E)$, $f\in R^{\alpha}([a,b],F)$ and the Integration by Parts Formula*

$$\int_a^b d\alpha(t)\,f(t)=\alpha(b)\,f(b)-\alpha(a)\,f(a)-\int_a^b\alpha(t)\,df(t)$$

holds.

The assertions in the next remark follow by the Integration by Parts Formula and some easy computation.

Remark 2.2. *Suppose (E,F,G) is a BT and $\alpha\in SB([a,b],E)$. If we define*

$$F_\alpha(f)=\int_a^b d\alpha(t)f(t),\qquad f\in C([a,b],F),$$

then $F_\alpha\in L(C([a,b],F),G)$ and $\|F_\alpha\|\le SB(\alpha)$. In particular we have

- *If $E = F'$ and $G = \mathbb{C}$ or $\mathbb{R}$ as in Remark 2.1, then given $\alpha \in BV([a,b],F')$, there exists*

$$F_\alpha(f) = \int_a^b \langle f(t), d\alpha(t)\rangle = \lim_{\Delta d \to 0} \sum_{i=1}^{|d|} \langle f(\xi_i), \alpha(t_i) - \alpha(t_{i-1})\rangle, \qquad f \in C([a,b],F),$$

where $d = (\xi_i, t_i) \in TD_{[a,b]}$ and $\Delta d = \max\{t_i - t_{i-1}; i = 1,2,\ldots,|d|\}$. Also $F_\alpha \in C([a,b],F')$ and $\|F_\alpha\| \leq V(\alpha)$.

- *If $\alpha \in SV([a,b],L(X,Y))$, then for every $f \in C([a,b],X)$ there exists the Riemann-Stieltjes integral $\int_a^b d\alpha(t) f(t)$. Furthermore $F_\alpha(f) = \int_a^b d\alpha(t) f(t)$, $f \in C([a,b],X)$, is such that $F_\alpha \in L(C([a,b],X),Y)$ and $\|F_\alpha\| \leq SV(\alpha)$.*

The next theorem says that all operators in $L(C([a,b],X),Y))$ can be represented by functions of bounded semi-variation. The version we present here is a special case of [13], Theorem I.5.1. In particular, it will be shown later that if $Y = Z'$, then $L(C([a,b],X),Y))$ can be represented by functions of bounded semi-variation which are right continuous.

Theorem 2.4. *The mapping*

$$\alpha \in SV_a([a,b],L(X,Y)) \mapsto F_\alpha \in L(C([a,b],X),Y)$$

where $F_\alpha(f) = \int_a^b d\alpha(t) f(t)$, for $f \in C([a,b],Y)$, is an isometry (i.e., $\|F_\alpha\| = SV(\alpha)$) of the first Banach space onto the second. We also have $\alpha(t)x = F_\alpha(\chi_{]a,t]}x)$, $x \in X$, where $\chi_{]a,t]}$ stands for the characteristic function of $]a,t]$.

We proceed with the presentation of results borrowed from [14] with their proofs.

In the sequel, we assume that c is some point in the interval $[a,b]$.

Given $\alpha \in BV([a,b],Y')$, let us define an auxiliary function $\overline{\alpha} : [a,b] \to Y'$ by

$$\overline{\alpha}(t) = \begin{cases} 0, & t = a, \\ \alpha(t+) - \alpha(a), & t \in]a,b[, \\ \alpha(b) - \alpha(a), & t = b. \end{cases} \tag{3}$$

The next result, which can be found in [14], Theorem I.2.12, will be useful to prove that the operators of $C([a,b],Y)'$ can be represented by elements of $BV_c([a,b],Y')$.

Lemma 2.3. *Let $\alpha \in BV([a,b],Y')$. Then*

(i) $\overline{\alpha} \in BV_c^+([a,b],Y')$ and $V(\overline{\alpha}) \leq V(\alpha)$;

(ii) For every $f \in C([a,b],Y)$, $F_{\overline{\alpha}}(f) = F_\alpha(f)$.

Proof. Let us prove (i). By the definition of $\overline{\alpha}$, $\overline{\alpha}(a) = 0$ and $\overline{\alpha}$ is right continuous at $t \in]a,b[$. Hence $\overline{\alpha} \in BV_c^+([a,b],Y')$. It remains to prove that $V(\overline{\alpha}) \leq V(\alpha)$.

We can suppose, without loss of generality, that $\alpha(a) = 0$. Then given $\varepsilon > 0$ and $d = (t_i) \in D_{[a,b]}$, there exist $s_i \in]t_{i-1}, t_i[$, $i = 1,2,\ldots,|d|-1$, such that $\alpha(s_i+) = \alpha(s_i)$ (by Proposition 2.1) and $\|\alpha(t_i+) - \alpha(s_i)\| \leq \varepsilon$. Therefore

$$\begin{aligned} \|\overline{\alpha}(t_i+) - \overline{\alpha}(t_{i-1})\| &\leq \|\alpha(t_i+) - \alpha(s_i)\| + \|\alpha(s_i) - \alpha(s_{i-1})\| + \|\alpha(t_{i-1}+) - \alpha(s_{i-1})\| \\ &\leq \|\alpha(s_i) - \alpha(s_{i-1})\| + 2\varepsilon. \end{aligned}$$

If we consider $d' = (s_i)$ with $a = s_0 < s_1 < \ldots < s_{|d|-1} < s_{|d|} = b$, then

$$\sum_{i=1}^{|d|} \|\overline{\alpha}(t_i+) - \overline{\alpha}(t_{i-1})\| \leq \sum_{i=1}^{|d|} \|\alpha(s_i) - \alpha(s_{i-1})\| + \sum_{i=1}^{|d|} 2\varepsilon$$

and hence $V_d(\overline{\alpha}) \leq V_{d'}(\alpha) + 2|d|\varepsilon \leq V(\alpha) + 2|d|\varepsilon$. Thus $V_d(\overline{\alpha}) \leq V(\alpha)$ (since ε is independent of the choice of d) and then $V(\overline{\alpha}) \leq V(\alpha)$.

Now we will prove (ii). By definition, we have

$$F_{\overline{\alpha}}(f) = \int_a^b \langle f(t), d\overline{\alpha}(t)\rangle = \lim_{\Delta d \to 0} \sum_{i=1}^{|d|} \langle f(\xi_i), \overline{\alpha}(t_i) - \overline{\alpha}(t_{i-1})\rangle,$$

where $\xi_i \in [t_{i-1}, t_i]$. We may suppose, without loss of generality, that $\alpha(a) = 0$ and then Proposition 2.1 implies $\overline{\alpha}(t) = \alpha(t)$, for all $t \in [a,b]$ but a countable subset. Then if we take the points t_i of the division $d = (t_i) \in D_{[a,b]}$ in the complement of that countable subset, we obtain (ii). □

The next representation theorem can be found in [14], Theorem I.2.13.

Theorem 2.5. *The mapping*

$$\alpha \in BV_c([a,b], Y') \mapsto F_\alpha \in C([a,b], Y)'$$

is an isometry (i.e., $\|F_\alpha\| = V(\alpha)$*) of the first Banach space onto the second.*

Proof. It is clear that the mapping is linear and $\|F_\alpha\| \leq V(\alpha)$. We will prove that the mapping is one-to-one, that is, $\alpha \neq 0$ implies $F_\alpha \neq 0$. It is enough to show that there exists $f \in C([a,b], Y)$ such that $F_\alpha(f) \neq 0$.

If $\alpha(b) \neq 0$, then there exists $y_0 \in Y$ such that $\langle y_0, \alpha(b)\rangle = 1$. If we take $f(t) \equiv y_0$, then

$$F_\alpha(f) = \int_a^b \langle y_0, d\alpha(t)\rangle = \langle y_0, \alpha(b)\rangle = 1.$$

If $\alpha(b) \neq 0$, let $t_0 \in]a,b[$ be such that $\alpha(t_0) \neq 0$ and consider $y_0 \in Y$ such that $\langle y_0, \alpha(t_0)\rangle = 1$. Define

$$f_n(t) = \begin{cases} 0, & t \leq t_0 \\ n(t-t_0)y_0, & t_0 \leq t \leq t_0 + \frac{1}{n} \\ y_0, & t_0 + \frac{1}{n} \leq t \leq b. \end{cases}$$

Then $f_n \in C([a,b], Y)$ and $\|f_n\| = \|y_0\|$, $n \in \mathbb{N}$, which implies

$$F_\alpha(f_n) = \int_{t_0}^{t_0+\frac{1}{n}} n(t-t_0)\langle y_0, d\alpha(t)\rangle + \int_{t_0+\frac{1}{n}}^b \langle y_0, d\alpha(t)\rangle,$$

where the second integral equals $\langle y_0, \alpha(t_0 + \frac{1}{n})\rangle$ which converges to $\langle y_0, \alpha(t_0+)\rangle = 1$ and the norm of the first integral is bounded by $\|y_0\| V_{[t_0, t_0+\frac{1}{n}]}(\alpha)$ which converges to 0.

Now we will prove that the mapping is onto and it is an isometry. It suffices to show that for every $F \in C([a,b], Y)'$, there exists an $\alpha \in BV([a,b], Y')$ such that $F = F_\alpha$ and $V(\alpha) \leq \|F\|$. Then by Lemma 2.3, the same applies to $\overline{\alpha}$ given by (3).

Since $C([a,b],Y)$ is a subspace of the space $B([a,b],Y)$ of bounded functions from $[a,b]$ to Y, then the Hahn-Banach Theorem implies $F \in C([a,b],Y)'$ admits a linear continuous extension $\overline{F} \in B([a,b],Y)'$ such that $\|\overline{F}\| = \|F\|$.

If for all $s \in [a,b]$, $s \neq c$, we define $\psi_s(t) = 1$ for $a \le t \le s$, $\psi_s(t) = 0$ for $s < t \le b$, and $\psi_c = 0$, then given $y \in Y$ and $t \in [a,b]$, $y\psi_t \in B([a,b],Y)$ and this function takes only two values: 0 and y. The mapping $y \in Y \mapsto \overline{F}(y\psi_t) \in \mathbb{R}$ is linear and continuous because $|\overline{F}(y\psi_t)| \le \|F\|\,\|y\|$. Therefore there exists one and only one element $\alpha(t) \in Y'$ such that $\overline{F}(y\psi_t) = \langle y, \alpha(t)\rangle$, $y \in Y$.

We assert that $\alpha \in BV([a,b],Y')$ and $V(\alpha) \le \|F\|$. Indeed, given $d = (t_i) \in D_{[a,b]}$, we have

$$\begin{aligned} V_d(\alpha) &= \sum_{i=1}^{|d|} \|\alpha(t_i) - \alpha(t_{i-1})\| \\ &= \sup\left\{ \left| \sum_{i=1}^{|d|} \langle y_i, \alpha(t_i) - \alpha(t_{i-1})\rangle \right| ; y_i \in Y,\ \|y_i\| \le 1 \right\} \\ &= \sup_{\|y_i\|\le 1} \left| \overline{F}\left(\sum_{i=1}^{|d|} y_i(\psi_{t_i} - \psi_{t_{i-1}}) \right) \right| \le \|F\|, \end{aligned}$$

since $\|\sum_{i=1}^{|d|} y_i(\psi_{t_i} - \psi_{t_{i-1}})\| \le 1$. Also, given $f \in C([a,b],Y)$, we have $F(f) = \int_a^b \langle f(t), d\alpha(t)\rangle$, that is, $F_\alpha = F$. Indeed because for $\xi_i \in [t_{i-1}, t_i]$, we have

$$\begin{aligned} \int_a^b \langle f(t), d\alpha(t)\rangle &= \lim_{\Delta d \to 0} \sum_{i=1}^{|d|} \langle f(\xi_i), \alpha(t_i) - \alpha(t_{i-1})\rangle \\ &= \lim_{\Delta d \to 0} \overline{F}\left(\sum_{i=1}^{|d|} f(\xi_i)(\psi_{t_i} - \psi_{t_{i-1}}) \right) = F(f), \end{aligned}$$

since $\lim_{\Delta d \to 0} \sum_{i=1}^{|d|} f(\xi_i)(\psi_{t_i} - \psi_{t_{i-1}}) = f$, where the limit is taken in the space $B([a,b],Y)$ of bounded functions from $[a,b]$ to Y. Also

$$\left\| f - \sum_{i=1}^{|d|} f(\xi_i)(\psi_{t_i} - \psi_{t_{i-1}}) \right\| = \sup_{a \le s \le b} \left| f(s) - \sum_{i=1}^{|d|} f(\xi_i)[\psi_{t_i}(s) - \psi_{t_{i-1}}(s)] \right| \le \omega_d(f),$$

where $\omega_d(f)$ is the oscillation of f with respect to d and converges to 0 as $\Delta d \to 0$. □

Remark 2.3. *Theorem 2.5 also holds for $BV_c([a,b],Y')$, $c \in [a,b]$, instead of $BV_a([a,b],Y')$.*

The next two results are also borrowed from [14] (see respectively Theorem I.3.8 and Corollary I.3.9 there).

Theorem 2.6. *The mapping*

$$\alpha \in SV_c^+([a,b], L(X,Z')) \mapsto F_\alpha \in L(C([a,b],X),Z')$$

is a linear isometry (i.e., $\|F_\alpha\| = SV(\alpha)$) of the first Banach space onto the second.

Proof. The mapping is clearly linear. Also $\|F_\alpha\| \le SV(\alpha)$ by Remark 2.2.

Let us prove that the mapping is injective. If $\alpha \neq 0$, there exists $t_0 \in]a,b]$ such that $\alpha(t_0) \neq 0$. Hence there exists $z \in Z$ such that $z \circ \alpha(t_0) \neq 0$, where $z \circ \alpha(t) \in X'$. Therefore $z \circ \alpha \in BV_c^+([a,b],X')$ and $z \circ \alpha \neq 0$. By Theorem 2.5, $F_{z\circ\alpha} \neq 0$, where $F_{z\circ\alpha}$ is the element of $C([a,b],X)'$ defined by $z \circ \alpha$. Thus there exists $f \in C([a,b],X)$ such that $F_{z\circ\alpha}(f) \neq 0$. On the other hand,

$$F_{z\circ\alpha}(f) = \int_a^b \langle f(t), d(z\circ\alpha)(t)\rangle = \langle z, \int_a^b d\alpha(t) f(t)\rangle = \langle z, F_\alpha(f)\rangle$$

and hence $F_\alpha(f) \neq 0$, that is, $F_\alpha \neq 0$.

Now we will show that given $F \in L(C([a,b],X),Z')$, there exists $\alpha \in SV_c^+([a,b],L(X,Z'))$ such that $F = F_\alpha$ and $SV(\alpha) \le \|F\|$.

For every $z \in Z$, we have $z \circ F \in C([a,b],X)'$ and $\|z\circ F\| \le \|z\|\,\|F\|$. Then Theorem 2.5 implies that there is one and only one element $\alpha_z \in BV_c^+([a,b],X')$ such that $z \circ F = F_{\alpha_z}$, that is, for every $f \in C([a,b],X)'$, we have

$$(z\circ F)(f) = \int_a^b \langle f(t), d\alpha_z(t)\rangle$$

and $V(\alpha_z) = \|z \circ F\|$.

We assert that $\alpha_{z_1+z_2}(t) = \alpha_{z_1}(t) + \alpha_{z_2}(t)$, $t \in [a,b]$. Indeed, we have $(z_1+z_2)\circ F = z_1 \circ F + z_2 \circ F$ and hence for every $f \in C([a,b],X)'$, we have

$$\int_a^b \langle f(t), d\alpha_{z_1+z_2}(t)\rangle = \int_a^b \langle f(t), d(\alpha_{z_1} + \alpha_{z_2})(t)\rangle$$

and then the uniqueness of the representation in Theorem 2.5 implies (i). In a similar way, one proves that $\alpha_{\lambda z}(t) = \lambda\alpha_z(t)$. Then for $t \in [a,b]$ and $x \in X$, we define $\alpha(t)x \in \mathbb{R}^Z$ by $(\alpha(t)x)z = \langle x, \alpha_z(t)\rangle$ and hence the mapping $z \in Z \mapsto (\alpha(t)x)z = \langle x, \alpha_z(t)\rangle \in \mathbb{R}$ is linear. It is also continuous, since $\|\alpha(t)x\| = \sup\{|\langle x, \alpha_z(t)\rangle|; z \in Z, \|z\| \le 1\}$ and $|\langle x, \alpha_z(t)\rangle| \le \|x\|\,\|\alpha_z(t)\| \le \|x\| V(\alpha_z) = \|x\|\,\|z\circ F\| \le \|x\|\,\|F\|\,\|z\|$ and hence $\|\alpha(t)x\| \le \|x\|\,\|F\|$.

We have $\alpha \in L(X,Z')$, since the mapping $x \in X \mapsto \alpha(t)x \in Z'$ is linear and $\|\alpha(t)\| \le \|F\|$ (by the last inequality).

We also assert that $\alpha \in SV_c^+([a,b],L(X,Z'))$ and $SV(\alpha) \le \|F\|$. Indeed, by the definition of α, we have $z \circ \alpha = \alpha_z \in BV_c^+([a,b],X')$ for every $z \in Z$. Hence by Proposition 2.2 (i),

$$SV(\alpha) = \sup\{V(z\circ\alpha); z \in Z, \|z\| \le 1\} = \sup\{V(\alpha_z); z \in Z, \|z\| \le 1\} \le \|F\|,$$

since $V(\alpha_z) = \|z\circ F\| \le \|z\|\,\|F\|$.

Finally $F = F_\alpha$, since for every $z \in Z$, we have $z \circ F = F_{\alpha_z} = F_{z\circ\alpha} = z \circ F_\alpha$ and then $z \circ F = z \circ F_\alpha$. □

The next result is a consequence of Theorem 2.6 with $Z = Y'$.

Corollary 2.2. *For every $F_\alpha \in L(C([a,b],X),Y)$, there exists one and only one $\alpha \in SV_c^+([a,b],L(X,Y''))$ such that $F = F_\alpha$.*

Remark 2.4. *Under the hypotheses of Corollary 2.2, we write $\alpha_F = \alpha$. Note that the mapping $F \mapsto \alpha_F$ may not be onto if $Y'' \neq Y$.*

By Corollary 2.1, if $\alpha \in SV([a,b],L(X,Z'))$, then for every $t \in [a,b[$, there exists one and only one element $\alpha(t+) \in L(X,Z')$ such that for every $x \in X$ and every $z \in Z$, we have

$$\lim_{\varepsilon \to 0} \langle \alpha(t+\varepsilon)x, z\rangle = \langle \alpha(t+)x, z\rangle.$$

If we define $\alpha^+(t) = \alpha(t+)$, $a < t < b$, and $\alpha^+(a) = \alpha(a)$, then α^+ is a function of bounded semi-variation and we write $\alpha^+ \in SV^+(]a,b],L(X,Z'))$. Moreover for every $f \in C([a,b],X)$, we have $\int_a^b d\alpha^+(t)f(t) = \int_a^b d\alpha(t)f(t)$ and $\|F_\alpha\| = SV(\alpha^+)$.

The next result follows from Theorem 2.6 and [1], Satz 10.

Theorem 2.7. *The mapping*

$$\alpha \in SV_c^+(]a,b],L(X,Z')) \mapsto F_\alpha \in L(C([a,b],X),Z')$$

is a linear isometry (i.e., $\|F_\alpha\| = SV(\alpha)$) of the first Banach space onto the second.

Similarly as in Corollary 2.2, we have

Corollary 2.3. *For every $F_\alpha \in L(C([a,b],X),Y)$, there exists one and only one $\alpha \in SV_c^+(]a,b],L(X,Y''))$ such that $F = F_\alpha$.*

Let $SV_b^{+\int}(]a,b],L(X,\dot{Y}))$ denote the space of functions $\alpha \in SV_b^{+\int}(]a,b],L(X,Y''))$ such that $\alpha(a) \in L(X,Y)$ and $\int_a^t \alpha(s)xds \in Y$, for all $t \in [a,b]$ and $x \in X$. Let χ_A be the characteristic function of a set $A \subset [a,b]$. The next theorem completes Corollary 2.3 and characterizes the image of the mapping $F \mapsto \alpha_F$. It is borrowed from [11] (Theorem 1.4 there).

Theorem 2.8. *The mapping*

$$\alpha \in SV_b^{+\int}(]a,b],L(X,\dot{Y})) \mapsto F_\alpha \in L(C([a,b],X),Y),$$

where $F_\alpha(f) = \int_a^b d\alpha(t)f(t)$, is an isometry (i.e., $\|F_\alpha\| = SV(\alpha)$) of the first Banach space onto the second. Furthermore $\int_a^t \alpha(s)xds = -F_\alpha(g_{t,x})$ and $\alpha(a)x = -F_\alpha(\chi_{[a,b]}x)$, where for $t \in [a,b]$ and $x \in X$, we define $g_{t,x}(s) = (s-a)x$, if $a \le s \le t$, and $g_{t,x}(s) = (t-a)x$, if $t \le s \le b$.

Proof. Given $F \in L(C([a,b],X),Y)$, let α be the corresponding element by Corollary 2.3. We assert that $\alpha \in SV_b^{+\int}([a,b],L(X,\dot{Y}))$. Indeed. Since $g_{t,x} \in C([a,b],X)$, then $F(g_{t,x}) \in Y$. But

$$F(g_{t,x}) = \int_a^b d\alpha(s)g_{t,x}(s) = -\int_a^b \alpha(s)dg_{t,x}(s) = -\int_a^t \alpha(s)xds,$$

where we applied the Integration by Parts Formula (Theorem 2.10) to get the second equality (with $\alpha(b) = 0$ and $g_{t,x}(a) = 0$). Similarly, one can prove that $F(\chi_{[a,b]}x) = -\alpha(a)x \in Y$.

The functions $g_{t,x}$ and $\chi_{[a,b]}x$ form a total subset of $C([a,b],X)$. By the fact that $SV_b^{+\int}(]a,b],L(X,\dot{Y}))$ is a closed subset of $SV_b^+(]a,b],L(X,Y''))$, it follows that the isometry is onto. □

Now, let us denote by $SV_b^{+\int}([a,b],L(X,\dot{Y}))$ be the set of all $\alpha \in SV_b^{+\int}(]a,b],L(X,\dot{Y}))$ such that $\alpha(a+) = \alpha(a)$ instead of $\alpha(a) \in L(X,Y)$. The next two results are respectively Theorems 1.5 and 1.6 from [11].

Theorem 2.9. *The mapping*

$$\alpha \in SV_b^{+\int}([a,b],L(X,\dot{Y})) \mapsto F_\alpha \in L(C_a([a,b],X),Y)$$

is an isometry of the first Banach space onto the second.

Proof. At first, we will prove that the mapping is one-to-one. Let α be such that $F_\alpha f = 0$, for every $f \in C_a([a,b],X)$. Then for every $t \in [a,b]$, $x \in X$ and $y \in Y$, we have

$$0 = \langle F_\alpha(g_{t,x}), y\rangle = -\int_a^t \langle \alpha(s)x, y\rangle ds.$$

Hence $\alpha = 0$, since by hypothesis $\alpha(s+) = \alpha(s)$, for every $s \in [a,b[$.

Now we will prove that the mapping is onto. Given $\alpha \in SV_b^{+\int}(]a,b],L(X,\dot{Y}))$, let us define $\alpha_a(a) = \alpha(a+)$ and $\alpha_a(t) = \alpha(t)$, if $a < t \leq b$. Then $\alpha_a \in SV_b^{+\int}([a,b],L(X,\dot{Y}))$, since $\alpha_a(a+) = \alpha_a(a)$. Also, for every $f \in C_a([a,b],X)$, we have $F_{\alpha_a}(f) = F_\alpha(f)$.

The isometry follows from Theorem 2.8. □

The notation below is going to be used in the next theorem.

Given a function $\alpha : [c,d] \times [a,b] \to L(X,Y'')$, we write $\alpha^t(s) = \alpha_s(t) = \alpha(t,s)$ and we consider the following properties:

(C^σ): For $a \leq s \leq b$ and $x \in X$, the function $t \in [c,d] \mapsto \alpha_s(t)x \in Y$ is continuous,

($\widetilde{C}^\sigma$): For $a < s \leq b$ and $x \in X$, the function $t \in [c,d] \mapsto \int_a^s \alpha(t,\sigma)x d\sigma \in Y$ is continuous, and for $x \in X$ (and $s = a$), the function $t \in [c,d] \mapsto \alpha_a(t)x \in Y$ is continuous,

(SV^u): For $t \in [c,d]$, $\alpha^t \in SV([a,b],L(X,Y))$ and $SV^u(\alpha^t) := \sup_{c \leq t \leq d} SV(\alpha^t) < \infty$. If moreover $\alpha(t,b) = 0$ for all $t \in [c,d]$, then we write (SV_b^u) instead of (SV^u).

($SV_b^{+\int u}$): For $t \in [c,d]$, $\alpha^t \in SV_b^{+\int}([a,b],L(X,\dot{Y}))$ and $SV^u(\alpha) := \sup_{c \leq t \leq d} SV(\alpha^t) < \infty$.

We write $\alpha \in C^\sigma SV^u([c,d] \times [a,b],L(X,Y))$ if α satisfies (C^σ) and (SV^u). Analogously, $\alpha \in \widetilde{C}^\sigma SV_b^u([c,d] \times [a,b],L(X,Y))$ if α satisfies ($\widetilde{C}^\sigma$) and (SV_b^u). We write $\alpha \in \widetilde{C}^\sigma SV_b^{+\int u}([c,d] \times [a,b],L(X,\dot{Y}))$ if α satisfies ($\widetilde{C}^\sigma$) and ($SV_b^{+\int u}$).

The next theorem is borrowed from [11], Theorem 1.6.

Theorem 2.10. *The mapping*

$$\alpha \in \widetilde{C}^\sigma SV_b^{+\int u}([c,d] \times [a,b],L(X,\dot{Y})) \mapsto F_\alpha \in L(C([a,b],X),C([c,d],Y)),$$

where $(F_\alpha f)(t) = \int_a^b d_s\alpha(t,s)f(s)$, $c \leq t \leq d$, *is an isometry (i.e.,* $\|F_\alpha\| = SV^u(\alpha)$*) of the first Banach space onto the second. Besides* $\int_a^s \alpha(t,\sigma)x d\sigma = -F_\alpha(g_{s,x})(t)$, $a \leq s \leq b$ *and* $\alpha(t,a)x = -F_\alpha(\chi_{[a,b]}x)(t)$.

Proof. By [13], Theorem I.5.10 and the remark that follows it ([13], p. 49-52), specialized for the Riemann-Stieltjes integral, the mapping

$$\alpha \in \widetilde{C}^{\sigma} SV_b^u([c,d]\times[a,b], L(X,Y)) \mapsto F_\alpha \in L(C_a([a,b],X), C([c,d],Y))$$

is an isometry. Reciprocally, given $F \in L(C_a([a,b],X), C([c,d],Y))$ and $t \in [c,d]$, Theorem 2.9 implies the continuous mapping $f \in C([a,b],X) \mapsto (Ff)(t) \in Y$ can be represented by an $\alpha^t \in SV_b^{+\int}([a,b], L(X,\dot{Y}))$. The proof is complete. □

We call any subset $A \subset X$ *relatively compact* if the closure of A in X is compact. We denote by $\mathcal{K}(X,Y)$ the subspace of compact linear operators in $L(X,Y)$. In particular, we write $\mathcal{K}(X) = \mathcal{K}(X,X)$. We conclude this section of auxiliary results mentioning the Fredholm Alternative for the Riemann-Siteltjes integral. For a proof of it, see [12] or [3], Theorems 2.4 and 2.5.

Theorem 2.11. *Suppose $K \in C^{\sigma}(SV_a)^u([a,b]\times[a,b], L(X))$. Given $t \in [a,b]$, let*

$$K(t,s_0)^* x' = \lim_{s \downarrow s_0} K(t,s)^* x'$$

for every $s_0 \in\,]a,b[$ and every $x' \in X'$. Suppose the mapping

$$K^{\diamondsuit} : t \in [a,b] \mapsto K^{\diamondsuit}(t) = K^t \in SV_a([a,b], L(X))$$

belongs to $C([a,b], SV_a([a,b], \mathcal{K}(X)))$. Given $\lambda \in \mathbb{R}$, $\lambda \neq 0$, consider the integral equations

$$\lambda x(t) - \int_a^b d_s K(t,s)\, x(s) = f(t), \quad t \in [a,b], \tag{4}$$

$$\lambda u(t) - \int_a^b d_s K(t,s)\, u(s) = 0, \quad t \in [a,b], \tag{5}$$

$$\lambda y(s) - \int_a^b K(t,s)^* \, dy(t) = g(s), \quad s \in [a,b], \tag{6}$$

$$\lambda z(s) - \int_a^b K(t,s)^* \, dz(t) = 0, \quad s \in [a,b]. \tag{7}$$

Then the Fredholm Alternative holds for these equations, that is,

(i) either for every $f \in C([a,b],X)$, equation (4) has exactly one solution and the same applies to equation (6),

(ii) or equation (5) has non-trivial solutions and the same applies to equation (7).

If (ii) holds, then equation (4) (respectively equation (6)*) admits a solution if and only if $\int_a^b f(t)\, dz(t) = 0$ for every solution z of equation (7) (respectively $\int_a^b u(t)\, dg(t) = 0$ for every solution u of equation (5)) and the space of solutions of* (5) *has finite dimension equal to that of the space of solutions of* (7) *which equals the codimension of $(\lambda I - F_K)C([a,b],X)$ in $C([a,b],X)$ and the codimension of $(\lambda I - (F_K)^*)BV_b^+([a,b],X')$ in $BV_b^+([a,b],X')$.*

3. Gauge Integrals in Banach Spaces

3.1. Definitions and Terminology

In this section, we consider functions $\alpha : [a,b] \to L(X,Y)$ and $f : [a,b] \to X$.

We say that α is *Kurzweil f-integrable* (or *Kurzweil integrable with respect to f*), if there exists $I \in Y$ such that for every $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$\left\| \sum_{i=1}^{|d|} \alpha(\xi_i)\left[f(t_i) - f(t_{i-1})\right] - I \right\| < \varepsilon.$$

In this case, we write $I = (K)\int_a^b \alpha(t)\, df(t)$ and $\alpha \in K_f([a,b], L(X,Y))$.

Analogously, we say that f is *Kurzweil $\alpha-$integrable* (or *Kurzweil integrable with respect to α*), if there exists $I \in Y$ such that given $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that

$$\left\| \sum_{i=1}^{|d|} \left[\alpha(t_i) - \alpha(t_{i-1})\right] f(\xi_i) - I \right\| < \varepsilon,$$

whenever $d = (\xi_i, t_i) \in TD_{[a,b]}$ is δ-fine. In this case, we write $I = (K)\int_a^b d\alpha(t) f(t)$ and $f \in K^{\alpha}([a,b], X)$.

If the gauge δ in the definition of $\alpha \in K_f([a,b], L(X,Y))$ is a constant function, then we obtain the Riemann-Stieltjes integral $\int_a^b \alpha(t)\, df(t)$ and we write $\alpha \in R_f([a,b], L(X,Y))$. Similarly, when we consider only constant gauges δ in the definition of $f \in K^{\alpha}([a,b], X)$, we obtain the Riemann-Stieltjes integral $\int_a^b d\alpha(t) f(t)$ and we write $f \in R^{\alpha}([a,b], X)$.

The vector integral of Henstock is more restrictive than that of Kurzweil in a general Banach space context. We define it in the sequel.

We say that α is *Henstock f-integrable* (or *Henstock variationally integrable with respect to f*), if there exists a function $A_f : [a,b] \to Y$ (called the *associate function* of α) such that for every $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$\sum_{i=1}^{|d|} \left\| \alpha(\xi_i)\left[f(t_i) - f(t_{i-1})\right] - \left[A_f(t_i) - A_f(t_{i-1})\right] \right\| < \varepsilon.$$

We write $\alpha \in H_f([a,b], L(X,Y))$ in this case.

In an analogous way we define the Henstock α-integrability of $f : [a,b] \to X$ and we write $f \in H^{\alpha}([a,b], X)$ in this case (see [6]).

Clearly $H_f([a,b], L(X,Y)) \subset K_f([a,b], L(X,Y))$ and $H^{\alpha}([a,b], X) \subset K^{\alpha}([a,b], X)$. If we identify the isomorphic spaces $L(\mathbb{R},\mathbb{R})$ and $\mathbb{R}$, then all the spaces $K_f([a,b], L(\mathbb{R}))$, $K_f([a,b], \mathbb{R})$, $H_f([a,b], L(\mathbb{R}))$ and $H_f([a,b], \mathbb{R})$ can also be identified, since $K_f([a,b], \mathbb{R}) = H_f([a,b], \mathbb{R})$ (see, for instance, [16], for a proof of this fact).

Given $f : [a,b] \to X$ and $\alpha \in K_f([a,b], L(X,Y))$, we define the indefinite integral $\widetilde{\alpha}_f : [a,b] \to Y$ of α with respect to f by

$$\widetilde{\alpha}_f(t) = (K)\int_a^t \alpha(s)\, df(s), \quad t \in [a,b].$$

If in addition $\alpha \in H_f([a,b], L(X,Y))$, then $\widetilde{\alpha}_f(t) = A_f(t) - A_f(a)$ for every $t \in [a,b]$.

In an analogous way, given $\alpha : [a,b] \to L(X,Y)$, we define the indefinite integral $\widetilde{f}^{\alpha} : [a,b] \to Y$ of f with respect to α by

$$\widetilde{f}^{\alpha}(t) = (K)\int_a^t d\alpha(s) f(s), \quad t \in [a,b],$$

for every $f \in K^{\alpha}([a,b], X)$.

In particular, when $\alpha(t) = t$, then instead of $K^{\alpha}([a,b],X)$, $R^{\alpha}([a,b],X)$, $H^{\alpha}([a,b],X)$ and $\widetilde{f}^{\alpha}$ we write, respectively, $K([a,b],X)$, $R([a,b],X)$, $H([a,b],X)$ and $\widetilde{f}$, that is, $\widetilde{f}(t) = (K)\int_a^t f(s)\,ds$, for every $t \in [a,b]$.

We proceed so as to define the equivalence classes of Kurzweil and of Henstock integrable functions.

Let m denote the Lebesgue measure. A function $f : [a,b] \to X$ satisfies the *Strong Lusin Condition* and we write $f \in SL([a,b],X)$ if given $\varepsilon > 0$ and $B \subset [a,b]$ with $m(B) = 0$, there is a gauge δ of B such that for every δ-fine $d = (\xi_i, t_i) \in TPD_{[a,b]}$ with $\xi_i \in B$ for all i, we have

$$\sum_{i=1}^{|d|} \| f(t_i) - f(t_{i-1}) \| < \varepsilon.$$

If we denote by $AC([a,b],X)$ the space of all absolutely continuous functions from $[a,b]$ to X, then we have

$$AC([a,b],X) \subset SL([a,b],X) \subset C([a,b],X).$$

In $SL([a,b],X)$, we consider the usual supremum norm, $\|\cdot\|_\infty$, induced from $C([a,b],X)$.

Given $f \in SL([a,b],X)$ and $\alpha \in H_f([a,b],L(X,Y))$, let $\beta : [a,b] \to L(X,Y)$ be such that $\beta = \alpha$ m-almost everywhere. Then $\beta \in H_f([a,b],L(X,Y))$ and $\widetilde{\beta}_f(t) = \widetilde{\alpha}_f(t)$, for every $t \in [a,b]$. See [6] (the corollary after Theorem 5 there) for a proof of this fact. An analogous result holds when we replace $H_f([a,b],L(X,Y))$ by $K_f([a,b],L(X,Y))$.

Suppose $f \in SL([a,b],X)$. Two functions $\beta, \alpha \in K_f([a,b],L(X,Y))$ are called *equivalent* if and only if $\widetilde{\beta}_f = \widetilde{\alpha}_f$. We denote by $\mathbf{K}_f([a,b],L(X,Y))$ and $\mathbf{H}_f([a,b],L(X,Y))$ respectively the spaces of all equivalence classes of functions of $K_f([a,b],L(X,Y))$ and of $H_f([a,b],L(X,Y))$ and we endow these spaces with the *Alexiewicz norm*

$$\|\alpha\|_{A,f} = \sup\left\{ \left\| (K)\int_a^t \alpha(s)\,df(s) \right\| ; t \in [a,b] \right\} = \|\widetilde{\alpha}_f\|_\infty,$$

where we recall that $\|\cdot\|_\infty$ is the usual supremum norm.

3.2. Some Properties

In this section, me mention several properties of the gauge integrals of Kurzweil and of Henstock. As it should be expected, both Kurzweil and Henstock vector integrals are linear, additive over non-overlapping intervals and invariant with respect to changes on sets of Lebesgue measure zero.

The result that follows is known as the Saks-Henstock Lemma and it is useful in many situations. For a proof of it, see [17], Proposition 16, for instance. A similar lemma also holds if we replace $K_f([a,b],L(X,Y))$ by $R_f([a,b],L(X,Y))$.

Lemma 3.1 (Saks-Henstock Lemma). *Given* $f : [a,b] \to X$, *let* $\alpha \in K_f([a,b], L(X,Y))$ *that is, for every* $\varepsilon > 0$, *there is a gauge* δ *of* $[a,b]$ *such that*

$$\left\| \sum_{i=1}^{|d|} \alpha(\xi_i)[f(t_i) - f(t_{i-1})] - (K)\int_a^b \alpha(t)\,df(t) \right\| < \varepsilon,$$

whenever $d = (\xi_i, t_i) \in TD_{[a,b]}$ *is* δ*-fine. Then for every* δ*-fine* $d' = (\zeta_j, s_j) \in TPD_{[a,b]}$,

$$\left\| \sum_{j=1}^{|d'|} \left\{ (K)\int_{s_{j-1}}^{s_j} \alpha(t)\,df(t) - \alpha(\zeta_j)[f(s_j) - f(s_{j-1})] \right\} \right\| < \varepsilon.$$

The next result is the Fundamental Theorem of Calculus for the Henstock integral. The proof follows standard steps (see [16], p. 43, for instance) adapted to Banach space-valued functions.

Theorem 3.1 (Fundamental Theorem of Calculus). *If* $F \in C([a,b], X)$ *and there exists the derivative* $F'(t) = f(t)$, *for every* $t \in [a,b]$, *then* $f \in H([a,b], X)$ *and*

$$(K)\int_a^t f(s)\,ds = F(t) - F(a), \qquad t \in [a,b].$$

The next two versions of the Fundamental Theorem of Calculus for Henstock vector integrals can be found in [6], respectively Theorems 1 and 2 there.

Theorem 3.2. *If* $f \in SL([a,b], X)$ *and* $A \in SL([a,b], Y)$ *are both differentiable and* $\alpha : [a,b] \to L(X,Y)$ *is such that* $A'(t) = \alpha(t) f'(t)$ *for m-almost every* $t \in [a,b]$, *then* $\alpha \in H_f([a,b], L(X,Y))$ *and* $A = \widetilde{\alpha}_f$.

Theorem 3.3. *If* $f \in SL([a,b], X)$ *is differentiable and* $\alpha \in H_f([a,b], L(X,Y))$ *is bounded, then* $\widetilde{\alpha}_f \in SL([a,b], Y)$ *and there exists the derivative* $(\widetilde{\alpha}_f)'(t) = \alpha(t) f'(t)$ *for m-almost every* $t \in [a,b]$.

Corollary 3.1. *Suppose* $f \in SL([a,b], X)$ *is differentiable and non-constant on any non-degenerate subinterval of* $[a,b]$ *and* $\alpha \in H_f([a,b], L(X,Y))$ *is bounded and such that* $\widetilde{\alpha}_f = 0$. *Then* $\alpha = 0$ *m-almost everywhere.*

The next result is a particular case of [7], Theorem 2.2.

Theorem 3.4. *If* $f \in C([a,b], X)$ *and* $\alpha \in K_f([a,b], L(X,Y))$, *then* $\widetilde{\alpha}_f \in C([a,b], Y)$.

For the Henstock vector integral we have the following analogue of Theorem 3.4. A proof of this result can be found in [6], Theorem 7.

Theorem 3.5. *If* $f \in SL([a,b], X)$ *and* $\alpha \in H_f([a,b], L(X,Y))$, *then* $\widetilde{\alpha}_f \in SL([a,b], Y)$.

The next result follows from Theorem 3.5. A proof of it can be found in [5], Theorem 5.

Theorem 3.6. *Suppose $f \in SL([a,b],X)$ is non-constant on any non-degenerate subinterval of $[a,b]$. Then the mapping*

$$\alpha \in \mathbf{H}_f([a,b],L(X,Y)) \mapsto \widetilde{\alpha}_f \in C_a([a,b],X)$$

is an isometry (i.e., $\left\|\widetilde{\alpha}_f\right\|_\infty = \|\alpha\|_{A,f}$) onto a dense subspace of $C_a([a,b],X)$.

The next result gives us a substitution formula for Kurzweil vector integrals. It also holds for the Riemann-Stieltjes integral instead. For a proof of it, see [4], Theorem 11.

Theorem 3.7. *Let $\alpha \in SV([a,b],L(X,Y))$, $f:[a,b] \to Z$, $\beta \in K_f([a,b],L(Z,X))$ and $g(t) = \tilde{\beta}_f(t) = \int_a^t \beta(s)df(s)$, $t \in [a,b]$. Then $\alpha \in K_g([a,b],L(X,Y)$ if and only if $\alpha\beta \in K_f([a,b],L(Z,Y))$. In this case, we have*

$$(K)\int_a^b \alpha(t)\beta(t)\,df(t) = \int_a^b \alpha(t)\,dg(t) = \int_a^b \alpha(t)\,d\left[\int_a^t \beta(s)df(s)\right] \tag{8}$$

and

$$\left\|(K)\int_a^b \alpha(t)\beta(t)\,df(t)\right\| \le [SV(\alpha) + \|\alpha(a)\|]\,\|\beta\|_{A,f}. \tag{9}$$

Using Theorems 3.4 and 2.3, we have the next corollary whose proof can be found in [4], Corollary 8.

Corollary 3.2. *If $\alpha \in SV([a,b],L(X,Y))$, $f \in C([a,b],W)$, $\beta \in K_f([a,b],L(W,X))$ and $g(t) = \tilde{\beta}_f(t) = \int_a^t \beta(s)df(s)$, $t \in [a,b]$, then $\alpha\beta \in K_f([a,b],L(W,Y))$ and* (8) *and* (9) *hold.*

By $E([a,b],L(X,Y))$ we mean the space of all step functions from $[a,b]$ to $L(X,Y)$, that is, $\alpha[a,b] \to L(X,Y)$ belongs to $E([a,b],L(X,Y))$ if and only if there is a division $d = (t_i) \in D_{[a,b]}$ and there are numbers $\alpha_1, \alpha_2, \ldots, \alpha_{|d|}$ such that $\alpha(t) = \sum_{i=1}^{|d|} \alpha_i \chi_{[t_{i-1},t_i[}(t)$, for every $t \in [a,b[$.

For a proof of the next proposition, see [5], Theorem 8.

Proposition 3.1. *Let $f \in SL([a,b],X)$ be differentiable and non-constant on any non-degenerate subinterval of $[a,b]$. Then the spaces $C([a,b],L(X,Y))$ and $E([a,b],L(X,Y))$ are dense in $\mathbf{K}_f([a,b],L(X,Y))$ in the Alexiewicz norm $\|\cdot\|_{A,f}$.*

4. Auxiliary Results

In this section we prove auxiliary results concerning vector gauge integrals which will be useful in the next section. We start by giving a representation theorem which says that the elements of $\mathbf{K}_g([a,b],L(\mathbb{R},X))'$ can be represented by functions of bounded variation which are continuous to the right.

Theorem 4.1. *Given $g \in SL([a,b],\mathbb{R})$ differentiable and non-constant in any non-degenerate subinterval of $[a,b]$, then the mapping*

$$\alpha \in BV_b^+([a,b],X') \mapsto H_{\alpha,g} \in \mathbf{K}_g([a,b],L(\mathbb{R},X))',$$

where $H_{\alpha,g}(f) = (K)\int_a^b \alpha(s)f(s)dg(s)$, is an isometry (i.e., $\|H_{\alpha,g}\| = V(\alpha)$) onto and, for every $t \in [a,b]$ and every $x \in L(\mathbb{R},X)$, we have $\int_a^t \alpha(s)xdg(s) = H_{\alpha,g}(\chi_{[a,t]}x)$.

Proof. The mapping is clearly linear.

We assert that the mapping is one-to-one. Indeed. Given $\rho \in [a,b]$ and $x \in L(\mathbb{R},X)$, we define a function $f_{\rho,x} : [a,b] \to L(\mathbb{R},X)$ by $f_{\rho,x}(s) = x$ if $s \in]a,\rho]$, and by $f_{\rho,x}(s) = 0$ otherwise. Then Theorem 2.3 implies $f_{\rho,x} \in R_g([a,b],L(\mathbb{R},X))$. Also $H_{\alpha,g}(f_{\rho,x}) = \int_a^\rho \alpha(s)xdg(s)$. Since for each $x \in L(\mathbb{R},X)$, the function $\alpha(\cdot)x : [a,b] \to L(\mathbb{R})$ is such that $\alpha(\cdot)x \in R_g([a,b],L(\mathbb{R})) \subset K_g([a,b],L(\mathbb{R})) = H_g([a,b],L(\mathbb{R}))$, then by the Fundamental Theorem of Calculus (Theorem 3.2), there exists $d_\rho\left((K)\int_a^\rho \alpha(s)xdg(s)\right) = \alpha(\rho)xg'(\rho)$ for m-almost every $\rho \in [a,b]$.

If $\alpha \neq 0$, then there exist $\overline{\rho} \in [a,b]$ and $\overline{x} \in L(\mathbb{R},X)$ such that $\alpha(\overline{\rho})\overline{x} \neq 0$. Besides, we can suppose without loss of generality that $g'(\overline{\rho}) \neq 0$ by the invariance of the integral over sets of Lebesgue measure zero. Indeed. From [6], we have $(K)\int_a^\rho \alpha(s)xdg(s) = (K)\int_a^\rho \alpha(s)xg'(s)$, for every $\rho \in [a,b]$. In fact the integral above is in the Riemann-Stieltjes sense. Therefore we have, in particular, $\int_a^{\overline{\rho}} \alpha(s)\overline{x}dg(s) = \int_a^{\overline{\rho}} \alpha(s)\overline{x}g'(s)$ and hence $d_\rho\left(\int_a^{\overline{\rho}} \alpha(s)\overline{x}dg(s)\right) = \alpha(\overline{\rho})xg'(\overline{\rho}) \neq 0$. Thus $H_{\alpha,g}(f_{\overline{\rho},x}) = \int_a^{\overline{\rho}} \alpha(s)\overline{x}dg(s)$ is non-constant and hence $H_{\alpha,g}(f_{\overline{\rho},x}) \neq 0$ and the mapping is one-to-one.

Since $SV([a,b],L(X,\mathbb{R})) = BV([a,b],X')$, it follows from Corollary 3.2 that $\|H_{\alpha,g}\| \leq V(\alpha)$.

Let $f \in \mathbb{K}_g([a,b],L(\mathbb{R},X))$ and $H \in \mathbb{K}_g([a,b],L(\mathbb{R},X))'$ and define $\widehat{H}(\widetilde{f}_g) = -H(f)$. By Theorem 3.6, there is a unique continuous extension of $\widehat{H}$ to $C_a([a,b],X)$ which we still denote by $\widehat{H}$. This new operator, $\widehat{H}$, belongs to $C_a([a,b],X)'$. If α represents $\widehat{H}$, then Theorem 2.9 implies $\widehat{H}(\widetilde{f}_g) = \int_a^b d\alpha(s)\widetilde{f}_g(s)$ and $\|\widehat{H}\| = V(\alpha)$. Moreover

$$H(f) = -\widehat{H}(\widetilde{f}_g) = -\int_a^b \alpha(s)d\widetilde{f}_g(s) = \int_a^b d\alpha(s)\widetilde{f}_g(s) = (K)\int_a^b \alpha(s)f(s)dg(s),$$

where we applied Theorem 2.3 and Corollary 3.2 to obtain respectively the last two equalities. Since by definition $\|f\|_g = \|\widetilde{f}_g\|$, then $\|H\| = \|\widehat{H}\| = V(\alpha)$ and the result follows. □

Let $g \in SL([a,b],Z)$. Given a function $\alpha : [c,d] \times [a,b] \to L(X,Y'')$, if α satisfies the properties $(\widetilde{C}_g^\sigma)$ and $(SV_b^{+\int u})$, where

> **$(\widetilde{C}_g^\sigma)$:** For $a < s \leq b$ and $x \in L(Z,X)$, the function $t \in [c,d] \mapsto \int_a^s \alpha(t,\sigma)xdg(\sigma) \in Y$ is continuous, and for $x \in L(Z,X)$ (and $s = a$), the function $t \in [c,d] \mapsto \alpha(t,a)xg'(a) \in Y$ is continuous,

and, as before,

> **$(SV_b^{+\int u})$:** For $t \in [c,d]$, $\alpha^t \in SV_b^{+\int}([a,b],L(X,\dot{Y}))$ and $SV^u(\alpha) := \sup_{c \leq t \leq d} SV(\alpha^t) < \infty$,

then we write $\alpha \in \widetilde{C}_g^\sigma SV_b^{+\int u}([c,d] \times [a,b],L(X,\dot{Y}))$. If, in addition, we consider only functions of bounded variation in property $(SV_b^{+\int u})$, then we write $\alpha \in \widetilde{C}_g^\sigma BV_b^{+\int u}([c,d] \times [a,b],L(X,\dot{Y}))$ instead of $\alpha \in \widetilde{C}_g^\sigma SV_b^{+\int u}([c,d] \times [a,b],L(X,\dot{Y}))$.

Notice that in the particular case when $Y = \mathbb{R}$, the spaces $\widetilde{C}_g^\sigma BV_b^{+\int u}([c,d] \times [a,b],L(X,\dot{Y}))$ and $\widetilde{C}_g^\sigma BV_b^u([c,d] \times [a,b],L(X,Y''))$ can be identified and then we write simply $\widetilde{C}_g^\sigma BV_b^u([c,d] \times [a,b],X')$.

The proof of the next result follows the steps of the proof of Theorem 4.1. However instead of Theorem 2.9, one should apply Theorem 2.10.

Theorem 4.2. *Let $g \in SL([a,b],\mathbb{R})$ be differentiable and non-constant in any non-degenerate subinterval of $[a,b]$. Then the mapping*

$$\alpha \in \widetilde{C}_g^{\sigma} BV_b^{+u}([c,d]\times[a,b],X') \mapsto H_{\alpha,g} \in L(\mathbf{K}_g([a,b],L(\mathbb{R},X)),C([c,d],\mathbb{R})),$$

where $(H_{\alpha,g}(\beta))(t) = (K)\int_a^b \alpha(t,s)\beta(s)dg(s)$ for each $t \in [c,d]$, is an isometry (i.e., $\|H_{\alpha,g}\| = V(\alpha)$) onto and, for every $s \in [a,b]$, every $t \in [a,b]$ and every $x \in L(\mathbb{R},X)$, we have $\int_a^s \alpha(t,\sigma)xdg(\sigma) = (H_{\alpha,g}(\chi_{[a,s]}x))(t)$.

Using Proposition 3.1, the next result follows easily.

Corollary 4.1. *Under the hypotheses of Theorem 4.2, for every $s \in [a,b]$, every $t \in [a,b]$ and every $\beta \in \mathbf{K}_g([a,b],L(\mathbb{R},X))$, we have $\int_a^s \alpha(t,\sigma)\beta(\sigma)dg(\sigma) = (H_{\alpha,g}(\chi_{[a,s]}\beta))(t)$.*

Let $g \in SL(a,b],\mathbb{R})$. We say that an operator $H \in L(\mathbf{K}_g([a,b],L(\mathbb{R})),C([a,b],\mathbb{R}))$ is *causal* if given $f \in \mathbf{K}_g([a,b],L(\mathbb{R})) \cong \mathbf{K}_g([a,b],\mathbb{R})$ and $t \in [a,b]$, then $f|_{[a,t]} = 0$ implies $H(f)|_{[a,t]} = 0$, where $h|_A$ denotes the restriction of a function h to a subset A of its domain.

We proceed as to show that in fact the isometry in Theorem 4.2 is onto over the space of causal operators, provided $\alpha(t,s) = 0$ for $s > t$. We need the next lemma.

Lemma 4.1. *Let $\alpha \in \widetilde{C}_g^{\sigma} SV_b^{+\int u}([a,b]\times[a,b],L(X,\dot{Y}))$, $g \in SL([a,b],Z)$ be differentiable and non-constant in any non-degenerate subinterval of $[a,b]$, and $\beta \in \mathbf{K}_g([a,b],L(Z,X))$. Then the mapping*

$$t \in [a,b] \mapsto (K)\int_a^t \alpha(t,s)\beta(s)dg(s) \in Y$$

is continuous.

Proof. Since $E([a,b],L(Z,X))$ is $\|\cdot\|_g$-dense in $\mathbf{K}_g([a,b],L(Z,X))$ by Proposition 3.1, it is enough to prove the result for every step function $\beta : [a,b] \to L(Z,X)$.

Let $x \in L(Z,X)$. We assert that the mapping

$$t \in [a,b] \mapsto (K)\int_a^t \alpha(t,s)xdg(s) \in Y$$

is continuous. Indeed. Given $\varepsilon > 0$ and $\rho > 0$, we have

$$(K)\int_a^{t+\rho} \alpha(t+\rho,s)xdg(s) = (K)\int_a^t \alpha(t+\rho,s)xdg(s) + (K)\int_t^{t+\rho} \alpha(t+\rho,s)xdg(s),$$

where $(K)\int_a^t \alpha(t+\rho,s)xdg(s)$ converges to $(K)\int_a^t \alpha(t,s)xdg(s)$ as $\rho \to 0$ by condition $(\widetilde{C}_g^{\sigma})$ for α.

Let δ be the gauge of $[a,b]$ from the definition of $(K)\int_a^b \alpha(t,s)xdg(s)$ and suppose $(\xi,[t,t+\rho]) \in TPD_{[a,b]}$ is δ-fine (that is, $\xi \in [t,t+\rho]$ and $\rho < \delta(\xi)$). Then

$$\left\|(K)\int_t^{t+\rho} \alpha(t+\rho,s)xdg(s)\right\| \leq$$

$$\leq \left\| (K)\int_t^{t+\rho} \alpha(t+\rho,s)xdg(s) - \alpha(t+\rho,\xi)x[g(t+\rho)-g(t)] \right\| +$$
$$+ \|\alpha(t+\rho,\xi)x[g(t+\rho)-g(t)]\| < \varepsilon + SV^u(\alpha)\|x\| \, \|g(t+\rho)-g(t)\|,$$

where we applied the Saks-Henstock Lemma (Lemma 3.1) to the first summand. Then from the continuity of g, the mapping

$$t \in [a,b] \mapsto (K)\int_a^t \alpha(t,s)xdg(s) \in Y$$

is right continuous and, in an analogous way, one can prove the left continuity. □

Theorem 4.3. *Let* $\alpha : [a,b]\times[a,b] \to L(\mathbb{R})$ *be such that* $\alpha(t,s) = 0$*, for all* $s > t$*, and let* $g \in SL([a,b],\mathbb{R})$ *be differentiable and non-constant in any non-degenerate subinterval of* $[a,b]$*. Then the mapping*

$$\alpha \in \widetilde{C}_g^{\sigma} BV_b^{+u}([a,b]\times[a,b], L(\mathbb{R})) \mapsto H_{\alpha,g} \in L(\mathbf{K}_g([a,b],L(\mathbb{R})), C([a,b],\mathbb{R})),$$

where $(H_{\alpha,g}(f))(t) = (K)\int_a^b \alpha(t,s)f(s)dg(s)$ *for each* $t \in [c,d]$*, is an isometry (i.e.,* $\|H_{\alpha,g}\| = V(\alpha)$*) onto the subspace of causal operators.*

Proof. By Theorem 4.2, it is enough to show that the mapping is onto.

If $\alpha : [a,b]\times[a,b] \to L(\mathbb{R})$ satisfies $(\widetilde{C}_g^{\sigma})$ and $(SV_b^{+\int u})$ and moreover $\alpha(t,s) = 0$, for all $s > t$, then Lemma 4.1 implies $H_{\alpha,g}$ is causal.

Reciprocally if $H \in L(\mathbf{K}_g([a,b],L(\mathbb{R})), C([a,b],\mathbb{R}))$ is causal, then Theorem 4.2 implies there is a unique $\alpha \in \widetilde{C}_g^{\sigma} BV_b^{+\int u}([a,b]\times[a,b], L(\mathbb{R}))$ such that $H = H_{\alpha,g}$. From the causality of $H = H_{\alpha,g}$, we have $0 = \left(H_{\alpha,g}(\chi_{[t,b]}x)\right)(t) = \int_t^b \alpha(t,\sigma)xdg(\sigma)$, for all $t \in [a,b]$ and $x \in L(\mathbb{R})$. Besides $\alpha^t(\cdot)x \in R_g([a,b],L(\mathbb{R})) \subset K_g([a,b],L(\mathbb{R})) = H_g([a,b],L(\mathbb{R}))$ and then Corollary 3.1 implies $\alpha(t,s) = 0$, for all $s > t$. □

The next lemma will be employed in the proof of the theorem following it. For a proof of the lemma, see [9] for instance.

Lemma 4.2 (Straddle Lemma). *Suppose* $f, F : [a,b] \to X$ *are such that* $F'(\xi) = f(\xi)$*, for all* $\xi \in [a,b]$*. Then given* $\varepsilon > 0$*, there exists* $\delta(\xi) > 0$ *such that*

$$\|F(t) - F(s) - f(\xi)(t-s)\| < \varepsilon(t-s),$$

whenever $\xi - \delta(\xi) < s < \xi < t < \xi + \delta(\xi)$.

When $g(s) = s$, Theorem 4.4 below is, in fact, a particular case of [3], Theorems 3.6 and 3.7. The proof we give here is merely an adaptation of those theorems in [3] so that we can obtain results concerning integral equations of Stieltjes-type characterized by the presence of a function g (in equations (1) and (2), for instance).

Theorem 4.4. *Let* $\alpha \in \widetilde{C}_g^{\sigma} BV_b^{+u}([c,d]\times[a,b], X')$*,* $g \in SL([a,b],\mathbb{R})$ *be differentiable and non-constant in any non-degenerate subinterval of* $[a,b]$*, and* $K_g : [c,d]\times[a,b] \to X'$ *be* *such that* $K_g(t,s)x = \int_a^s \alpha(t,\sigma)x dg(\sigma)$*, for each* $x \in L(\mathbb{R},X)$*. Then* $K_g \in \widetilde{C}^\sigma SV_b^{+u}([a,b]\times[a,b],X')$*. Besides, we have*

$$\int_a^b d_s K_g(t,s)\beta(s) = \int_a^b \alpha(t,s)\beta(s)dg(s)$$

for every function $\beta \in C([a,b],L(\mathbb{R},X))$ *and all* $t\in[c,d]$*. Suppose, in addition, that* $\alpha^t \in SV([a,b],\mathcal{K}(X,\mathbb{R}))$*, for every* $t\in[c,d]$*. Then the mapping*

$$K_g^\diamond : t\in[c,d] \mapsto K_g(t,\cdot) \in SV_a([a,b],X')$$

is continuous.

Proof. Since α fulfills condition $(\widetilde{C}_g^\sigma)$, then $K_g(\cdot,s)$ fulfills condition (C^σ), for each $s\in[a,b]$.

Let $x\in L(\mathbb{R},X)$ and $d=(s_i)\in D_{[a,b]}$. Then

$$\left\|\sum_{i=1}^{|d|}\left[K_g^t(s_i)-K_g^t(s_{i-1})\right]x\right\| = \left\|\sum_{i=1}^{|d|}\left[\int_{s_{i-1}}^{s_i}\alpha(t,\rho)xdg(\rho)\right]\right\| = \left\|\int_a^b \alpha(t,\rho)xdg(\rho)\right\| \leq$$

$$\leq V(\alpha^t)\|x\|\,\|g(b)-g(a)\|$$

and hence $K_g^t = K_g(t,\cdot)\in SV([a,b],X')$, for each $t\in[c,d]$.

Note that, for every $t\in[c,d]$ and every $x\in L(\mathbb{R},X)$, the function $K_g(t,\cdot)x$ is continuous on $[a,b]$, since it is an indefinite integral (see Theorem 3.4).

We assert that $K_g^\diamond \in C([c,d],SV_a([a,b],X'))$. Indeed. Let $t_0\in[c,d]$ and $d=(s_i)\in D_{[a,b]}$. Then

$$\left\|K_g^t-K_g^{t_0}\right\| = SV\left(K_g^t-K_g^{t_0}\right) = \sup_{d,\,\|x\|\leq 1}\left\|\sum_{i=1}^{|d|}\left[\left(K_g^t-K_g^{t_0}\right)(s_i)-\left(K_g^t-K_g^{t_0}\right)(s_{i-1})\right]x\right\| =$$

$$= \sup_{d,\,\|x\|\leq 1}\left\|\sum_{i=1}^{|d|}\left[\int_{s_{i-1}}^{s_i}\alpha(t,\rho)xdg(\rho)-\int_{s_{i-1}}^{s_i}\alpha(t_0,\rho)xdg(\rho)\right]\right\| =$$

$$= \sup_{d,\,\|x\|\leq 1}\left\|\int_a^b\alpha(t,\rho)xdg(\rho)-\int_a^b\alpha(t_0,\rho)xdg(\rho)\right\| = \sup_{d,\,\|x\|\leq 1}\|K_g(t,b)x-K_g(t_0,b)x\|$$

which tends to zero as $t\to t_0$, since $K_g(\cdot,s)\in C^\sigma([c,d],X')$, for each $s\in[a,b]$ and, in particular, $K_g(\cdot,b)\in C^\sigma([c,d],X')$.

Let $\beta\in C([a,b],L(\mathbb{R},X))$, $t\in[c,d]$ and $\gamma=\widetilde{\beta}_g$. In accordance with Theorem 3.1 and Corollary 3.7, $\alpha^t\beta\in K_g([a,b],L(\mathbb{R}))$ and

$$(K)\int_a^b\alpha^t(s)\beta(s)dg(s) = \int_a^b\alpha^t(s)d\gamma(s) = -\int_a^b d_s\left(\alpha^t(s)\right)\gamma(s), \tag{10}$$

where we applied Theorem 2.3 to obtain the last equality. Since $K_g(t,s)x=\int_a^s\alpha^t(\rho)xdg(\rho)$ with $K_g(t,\cdot)\in SV([a,b],X')$ for each $t\in[c,d]$, then Theorem 2.3 implies the Riemann-Stieltjes integral $\int_a^b d_sK_g(t,s)\beta(s)$ exists for each $t\in[c,d]$ and each $\beta\in C([a,b],L(\mathbb{R},X))$.

We also assert that

$$\int_a^b d_s K_g(t,s)\beta(s) = (K)\int_a^b \alpha^t(s)\beta(s)dg(s), \tag{11}$$

for every $t \in [c,d]$ and every $\beta \in C([a,b],L(\mathbb{R},X))$. Indeed. It is enough to prove that (11) holds when β is a step function. Hence we need to show that for every $t \in [c,d]$ and every $x \in L(\mathbb{R},X)$,

$$\int_a^b d_s K_g(t,s)x = (K)\int_a^b \alpha^t(s)xdg(s). \tag{12}$$

Since $\alpha^t(\cdot)x \in R_g([a,b],L(\mathbb{R}))$ by Theorem 2.3 and $R_g([a,b],L(\mathbb{R})) \subset K_g([a,b],L(\mathbb{R})) = H_g([a,b],L(\mathbb{R}))$, it follows from the Fundamental Theorem of Calculus (Theorem 3.3) that there exists $d_s\left(\int_a^s \alpha^t(\rho)xdg(\rho)\right) = \alpha^t(s)xg'(s)$ m-almost everywhere on $[a,b]$. Therefore given $t \in [c,d]$ and $x \in L(\mathbb{R},X)$, we have $d_s K_g(t,s)x = d_s(K_g(t,s)x) = d_s\left(\int_a^s \alpha^t(\rho)xdg(\rho)\right) = \alpha^t(s)xg'(s)$ for m-almost every $s \in [a,b]$. Then from the invariance of the integral over sets of m-measure zero, we obtain

$$\int_a^b d_s K_g(t,s)x = (K)\int_a^b \alpha^t(s)xg'(s)ds. \tag{13}$$

Now we will prove that $\int_a^b \alpha^t(s)xg'(s)ds = \int_a^b \alpha^t(s)xdg(s)$ which, together with (13), implies (12). Given $\varepsilon > 0$, $t \in [c,d]$ and $x \in L(\mathbb{R},X)$, let δ_1 and δ_2 be constant gauges of $[a,b]$ from the definitions of $\int_a^b \alpha^t(s)xdg(s)$ and $\int_a^b \alpha^t(s)xg'(s)ds$ respectively. Given $\xi \in [a,b]$, let $\delta_3(\xi) > 0$ be such that

$$|g(v) - g(s) - g'(\xi)(v-s)| < \varepsilon(v-s) \tag{14}$$

whenever $\xi - \delta_3(\xi) < s < \xi < v < \xi + \delta_3(\xi)$ (see Lemma 4.2) and let $\delta(\xi) = \min\{\delta_i(\xi); i = 1,2,3\}$. Then for every δ-fine $d = (\xi_i, s_i) \in TD_{[a,b]}$, we have

$$\begin{aligned}
&\left\|\int_a^b \alpha^t(s)xdg(s) - \int_a^b \alpha^t(s)xg'(s)ds\right\| \leq \\
&\leq \left\|\int_a^b \alpha^t(s)xdg(s) - \sum_{i=1}^{|d|}\alpha^t(\xi_i)x\left[g(s_i) - g(s_{i-1})\right]\right\| + \\
&+ \left\|\sum_{i=1}^{|d|}\alpha^t(\xi_i)x\left[g(s_i) - g(s_{i-1})\right] - \sum_{i=1}^{|d|}\alpha^t(\xi_i)xg'(\xi_i)(s_i - s_{i-1})\right\| + \\
&+ \left\|\sum_{i=1}^{|d|}\alpha^t(\xi_i)xg'(\xi_i)(s_i - s_{i-1}) - \int_a^b \alpha^t(s)xg'(s)ds\right\| < \\
&< \varepsilon + \sum_{i=1}^{|d|} V(\alpha^t)\|x\|\,|g(s_i) - g(s_{i-1}) - g(\xi_i)(s_i - s_{i-1})| + \varepsilon < 2\varepsilon + V(\alpha^t)\|x\|\varepsilon(b-a),
\end{aligned}$$

where we applied the integrability with respect to g of $\alpha^t(\cdot)x$, the integrability of $\alpha^t(\cdot)xg'(\cdot)$ and (14). The result follows easily. $\square$

For a proof of the next theorem, see [12], Theorems 3.8 and 3.4.

Theorem 4.5. *Let $K \in C^\sigma SV^u([a,b] \times [a,b], L(X))$. Suppose there is a division $d = (s_i) \in D_{[a,b]}$ such that*

$$\sup\left\{SV_{[s_{i-1},t]}(K^t); t \in [s_{i-1}, s_i]\right\} < 1, \qquad i = 1,2,\ldots,|d|,$$

where $SV_{[s_{i-1},t]}(K^t)$ denotes the semi-variation of the function $K(t,\cdot)$ in the interval $[s_{i-1},t]$. Then the following properties are equivalent:

(i) for every $t \in [a,b[$, the operator $I - K(t+,t)$ is invertible, where I denotes the identity in $L(X)$;

(ii) for every $h \in C([a,b],X)$, the mapping

$$y(t) - \int_a^t d_s K(t,s) y(s) = h(t), \quad t \in [a,b]$$

admits a unique solution $y_h \in C([a,b],X)$.

Let $K \in C^\sigma SV^u([a,b] \times [a,b], L(X))$. We say that a function $R \in C^\sigma SV^u([a,b] \times [a,b], L(X))$ such that $R(t,s) = I_X$, for all $s \geq t$, is a resolvent of K, whenever R satisfies the equation

$$R(t,s) - I_X + \int_s^t d_\tau K(t,\tau) \circ R(\tau,s) = 0, \quad a \leq s \leq t \leq b.$$

The next result can be found in [2] or in [12], Theorem 3.9.

Theorem 4.6. *Given $K \in C^\sigma SV^u([a,b] \times [a,b], L(X))$, it there is a division $d = (s_i) \in D_{[a,b]}$ such that*

$$\sup\left\{SV_{[s_{i-1},t]}(K^t); t \in [s_{i-1}, s_i]\right\} < 1, \qquad i = 1,2,\ldots,|d|,$$

then K has resolvent given by the Neumann series.

Theorem 4.7 below and its proof are borrowed from [11], Theorem 3.1.

Theorem 4.7. *Let E be a normed space and F be a Banach space such that $F \subset E$ with continuous immersion. Let $H \in L(E,F)$ be such that for every $f \in E$, the equation $x - Hx = f$ admits one and only one solution $x_f \in E$. Then the mapping $f \in E \mapsto x_f \in E$ is bicontinuous. If in addition the Neumann series $I + H + H^2 + H^3 + \ldots = (I-H)^{-1}$ converges in $L(F)$, then it also converges in $L(E)$.*

Proof. For every $g \in F$, the equation $y - Hy = g$ has one and only one solution $y_g \in F$ and the mapping $g \in F \mapsto y_g \in F$ is bicontinuous. Indeed. Since $F \subset E$, the equation $y - Hy = g$ with $g \in F$ has one and only one solution by hypothesis. But since $H(E) \subset F$, we have $Hy_g \in F$. Hence $y_g = Hy_g + g \in F$. On the other hand, the mapping $y \in F \mapsto g = y - H_y \in F$ is a continuous injection and therefore the closed graph theorem implies its inverse $g \in F \mapsto y_g \in F$ is continuous.

The equation $x - Hx = f$, with $x, f \in E$, is equivalent to the equation $y - Hy = g$, with $g = Hf$ and $y = x - f$. The mapping $f \in E \mapsto g = Hf \in E$ is continuous. By the previous paragraph, the mapping $Hf \in F \mapsto y_{Hf} \in E$ is also continuous. Thus the composed mapping

$f \in E \mapsto y_{Hf} \in E$ is continuous and so is the mapping $f \in E \mapsto y_{Hf} + f \in E$. The first part of the theorem follows from $y_{Hf} = x_f - f$, that is, $x_f = y_{Hf} + f$.

It remains to show the second part. Let $(I-H)^{-1} = I + H + H^2 + H^3 + \dots$ be convergent in $L(F)$. Since $H \in L(E,F)$ and the immersion $F \subset E$ is continuous, then the series is also convergent in $L(E)$. Also, if a Neumann series is convergent in some $L(Z)$ (Z not necessarily complete), then it converges to $(I-H)^{-1}$. □

5. The Fredholm Alternative for the Kurzweil-Henstock-Stieltjes Integral

In this and the next section, we write $\widetilde{C}_g^{\sigma} BV_b^{+u}([a,b] \times [a,b], \mathbb{R})$ instead of $\widetilde{C}_g^{\sigma} BV_b^{+u}([a,b] \times [a,b], L(\mathbb{R}))$.

Theorem 5.1. *Given $g \in SL([a,b],\mathbb{R})$ differentiable and non-constant in any non-degenerate subinterval of $[a,b]$ and $H \in L(\mathbf{K}_g([a,b],\mathbb{R}), C([a,b],\mathbb{R}))$, let $\alpha \in \widetilde{C}_g^{\sigma} BV_b^{+u}([a,b] \times [a,b], \mathbb{R})$ be the corresponding kernel by Theorem 4.2 (i.e., $H = H_{\alpha,g}$). Suppose H is such that for each $f \in \mathbf{K}_g([a,b],\mathbb{R})$, the linear Fredholm-Stieltjes integral equation in the sense of the Henstock-Kurzweil integral*

$$x(t) - (K)\int_a^b \alpha(t,s)\, x(s)\, dg(s) = f(t), \qquad t \in [a,b], \tag{15}$$

(i.e., the equation $x - H(x) = f$) admits a unique solution $x_f \in \mathbf{K}_g([a,b],\mathbb{R})$. Then there exists a unique kernel $\rho \in \widetilde{C}_g^{\sigma} BV_b^{+u}([a,b] \times [a,b], \mathbb{R})$ and, for each $f \in \mathbf{K}_g([a,b],\mathbb{R})$,

$$x_f(t) = f(t) - (K)\int_a^b \rho(t,s)\, f(s)\, dg(s), \qquad t \in [a,b].$$

Proof. This proof follows the steps of the proof of [11], Theorem 3.2 adapted to the Stieltjes case.

If in Theorem 4.7 we take $E = \mathbf{K}_g([a,b],\mathbb{R})$ and $F = C([a,b],\mathbb{R})$, then $(I-H)^{-1} \in L(\mathbf{K}_g([a,b],\mathbb{R}))$. If we define $I - R = (I-H)^{-1}$, then $R = H(I-H)^{-1}$ belongs to $L(\mathbf{K}_g([a,b],\mathbb{R}), C([a,b],\mathbb{R}))$ and it can be represented by a kernel $\rho \in \widetilde{C}_g^{\sigma} BV_b^{+u}([a,b] \times [a,b], \mathbb{R})$ (by Theorem 4.2). □

Theorem 5.2. *Let $g \in SL([a,b],\mathbb{R})$ be differentiable and non-constant in any non-degenerate subinterval of $[a,b]$, $f \in \mathbf{K}_g([a,b],\mathbb{R})$ and $\alpha \in \widetilde{C}_g^{\sigma} BV_b^{+u}([a,b] \times [a,b], \mathbb{R})$. Consider the linear integral Fredholm-Henstock-Kurzweil-Stieltjes integral*

$$x(t) - (K)\int_a^b \alpha(t,s)\, x(s)\, dg(s) = f(t), \qquad t \in [a,b], \tag{16}$$

and its corresponding homogeneous equation

$$u(t) - (K)\int_a^b \alpha(t,s)\, u(s)\, dg(s) = 0, \qquad t \in [a,b]. \tag{17}$$

Consider also the following integral equations

$$y(s) - \int_a^s \left[\int_a^b \alpha(t,\sigma)\, dy(t) \right] dg(\sigma) = w(s), \qquad s \in [a,b], \tag{18}$$

and

$$z(s) - \int_a^s \left[\int_a^b \alpha(t,\sigma)\, dz(t) \right] dg(\sigma) = 0, \qquad s \in [a,b]. \tag{19}$$

Then

(i) either for each $f \in \mathbf{K}_g([a,b],\mathbb{R})$*, equation* (16) *admits a unique solution* $x_f \in \mathbf{K}_g([a,b],\mathbb{R})$ *given by*

$$x_f(t) = f(t) - (K)\int_{[a,b]} \rho(t,s) f(s)\, dg(s), \qquad t \in [a,b],$$

where the kernel $\rho \in \widetilde{C}_g^{\sigma} BV_b^{+u}([a,b]\times[a,b],\mathbb{R})$ *is uniquely determined, and for each* $w \in C([a,b],\mathbb{R})$*, equation* (18) *admits a unique solution* $y_g \in C([a,b],\mathbb{R})$*;*

(ii) or equation (17) *admits non-trivial solutions in* $K_g([a,b],\mathbb{R})$*. In this case, equation* (16) *admits a solution if and only if for each solution* $z \in C([a,b],\mathbb{R})$ *of* (19)*,*

$$\int_a^b \left[(K)\int_a^b \alpha(t,s) f(s) dg(s) \right] dz(t) = 0$$

$$\left(resp. \quad \int_a^b \left[(K)\int_a^b \alpha(t,s) u(s) dg(s) \right] dw(t) = 0 \right).$$

Analogously, equation (18) *admits a solution if and only if for each solution* $u \in \mathbf{K}_g([a,b],\mathbb{R})$ *of* (17)*, we have*

$$\int_a^b \left[(K)\int_a^b \alpha(t,s) u(s) dg(s) \right] dw(t) = 0.$$

Proof. Let $h(t) = (K)\int_a^b \alpha(t,s) f(s) dg(s)$ and $v = x - f$. Then $v(t) = (K)\int_a^b \alpha(t,s) x(s) dg(s)$ and, by Theorem 4.2, $h, v \in C([a,b],\mathbb{R})$. Thus equation (16) is equivalent to the following equation

$$y(t) - (K)\int_a^b \alpha(t,s)\, y(s)\, dg(s) = h(t), \qquad t \in [a,b]. \tag{20}$$

Let $K_g : [a,b]\times[a,b] \to L(\mathbb{R})$ be such that $K_g(t,s)x = \int_a^s \alpha^t(\sigma) x dg(\sigma)$ for every $x \in \mathbb{R}$. According to Theorem 4.4, K_g satisfies conditions ($\widetilde{C}^{\sigma}$) and (SV_a^u). Moreover the mapping

$$K_g^{\diamond} : t \in [a,b] \mapsto K_g(t,\cdot) \in SV_a([a,b],L(\mathbb{R}))$$

is continuous and we have

$$\int_a^b d_s K_g(t,s) v(s) = (K)\int_a^b \alpha^t(s) v(s) dg(s).$$

Therefore (20) is equivalent to

$$v(t) - \int_a^b d_s K_g(t,s)v(s) = h(t), \qquad t \in [a,b], \tag{21}$$

and by the Fredholm Alternative for the Riemann-Stieltjes integral (Theorem 2.11), either $h \in C([a,b],\mathbb{R})$ and equation (21) admits one and only one solution which implies that for each $f \in \mathbf{K}_g([a,b],\mathbb{R})$, equation (16) admits a unique solution, or the homogeneous equation corresponding to (21) (respectively (16)) admits a non-trivial solution. In the second case, the adjoint $(K_g)^*$ equals K_g. Then equation (16) (respectively equation (18)) admits a solution (not necessarily in $\mathbf{K}_g([a,b],\mathbb{R})$) if and only if for each solution $z \in C([a,b],\mathbb{R})$ of (19), we have $\int_a^b \left[(K)\int_a^b \alpha(t,s)f(s)dg(s)\right] dz(t) = 0$ (respectively for each solution $u \in \mathbf{K}_g([a,b],\mathbb{R})$ of (17), we have $\int_a^b \left[(K)\int_a^b \alpha(t,s)u(s)dg(s)\right] dw(t) = 0$).

The missing part of assertion (i) follows easily from Theorem 5.1. □

6. The Linear Integral Equation of Volterra-Henstock-Kurzweil-Stieltjes

Theorem 6.1. *Let $g \in SL([a,b],\mathbb{R})$ be differentiable and non-constant in any non-degenerate subinterval of $[a,b]$, $f \in \mathbf{K}_g([a,b],\mathbb{R})$ and $\alpha \in \widetilde{C}_g^{\sigma}BV_b^{+u}([a,b]\times[a,b],\mathbb{R})$ such that $\alpha(t,s)=0$, for $s>t$. Then the linear integral equation of Volterra-Stieltjes in the sense of the Henstock-Kurzweil integral*

$$x(t) - (K)\int_{[a,t]} \alpha(t,s)x(s)\,dg(s) = f(t), \quad t \in [a,b], \tag{22}$$

admits one and only one solution $x_f \in \mathbf{K}_g([a,b],\mathbb{R})$. Furthermore, the operator $H_{\alpha,g} \in L(K_g([a,b],\mathbb{R}),C([a,b],\mathbb{R}))$ given by $(H_{\alpha,g}f)(t) = (K)\int_a^b \alpha(t,s)f(s)dg(s)$, for each $t \in [a,b]$, is causal as well as the bijection $f \mapsto x_f$ which can be written as

$$x_f(t) = f(t) - (K)\int_a^t \rho(t,s)f(s)dg(s), \qquad t \in [a,b], \tag{23}$$

where $\rho \in \widetilde{C}_g^{\sigma}BV_b^{+u}([c,d]\times[a,b],\mathbb{R})$ and $\rho(t,s)=0$ for $s>t$, and the Neumann series $I - H_{\rho,g} = I + H_{\alpha,g} + (H_{\alpha,g})^2 + (H_{\alpha,g})^3 + \ldots$ converges in $L(\mathbf{K}_g([a,b],\mathbb{R}))$.

Proof. Let $y = x - f$ and $h(t) = (K)\int_a^b \alpha(t,s)f(s)dg(s)$. Both functions y and h are continuous, since $h = H_{\alpha,g}(\chi_{[a,t]}f)$ and $y = H_{\alpha,g}(\chi_{[a,t]}x)$ (see Corollary 4.1) with $\chi_{[a,t]}f$ and $\chi_{[a,t]}x$ in $K_g([a,b],\mathbb{R})$ and $H_{\alpha,g} \in L(\mathbf{K}_g([a,b],\mathbb{R}),C([a,b],\mathbb{R}))$ (see Theorem 4.3). Then equation (22) is equivalent to the following equation

$$y(t) - (K)\int_a^t \alpha(t,s)y(s)\,dg(s) = h(t), \quad t \in [a,b]. \tag{24}$$

Let $K_g : [a,b]\times[a,b] \to \mathbb{R}$ be such that $K_g(t,s)x = \int_t^s \alpha^t(\sigma)x\,dg(\sigma)$, for every $x \in \mathbb{R}$. Then by Theorem 4.4, we have

$$\int_a^t d_s K_g(t,s)y(s) = (K)\int_a^t \alpha^t(s)y(s)dg(s).$$

Thus equation (24) is equivalent to the following equation

$$y(t) - \int_a^t d_s K_g(t,s) y(s) = h(t), \quad t \in [a,b]. \tag{25}$$

By Theorem 4.4, $K_g \in C^\sigma BV^u([a,b]\times[a,b],\mathbb{R})$ and since $\alpha(t,s)=0$ for $s>t$, then $K_g(t,s)=0$ for $s>t$. Thus, according to Theorem 4.5, in order to prove that given $h \in C([a,b],\mathbb{R})$, equation (25) has one and only one solution $y_h \in C([a,b],\mathbb{R})$ and the operator $h \mapsto y_h$ is causal, it is enough to show that

(i) there exists a division $d=(s_i) \in D_{[a,b]}$ such that $\sup\left\{SV_{[s_{i-1},t]}((K_g)^t); t \in [s_{i-1},s_i]\right\} < 1$, $i=1,2,\ldots,|d|$.

(ii) for each $t \in [a,b[$, the operator $I - K_g(t+,t)$ is invertible.

Proof of (ii). By the continuity of g, given $\xi \in [a,b]$ and $\varepsilon > 0$ with $\varepsilon < 1/2V^u(\alpha)$, there exists $\delta(\xi) > 0$ such that $|g(t)-g(\xi)| < \varepsilon$, for all $t \in [a,b]$ with $0 < |t-\xi| < \delta(\xi)/2$. Then

$$\|K_g(t+\sigma,t)x\| = \left\|\int_t^{t+\sigma} \alpha(t+\sigma,s)x dg(s)\right\| \leq V^u(\alpha)|x|\,|g(t+\sigma)-g(t)|$$

which tends to zero as σ goes to zero. It follows that $K_g(t+,t)$ is invertible.

Proof of (i). Let us consider the gauge δ of $[a,b]$ defined as above and let $d=(\xi_i,s_i) \in TD_{[a,b]}$ be δ-fine. Given $i \in \{1,2,\ldots,|d|\}$ and $t \in [s_{i-1},s_i]$, let $d_i=(r^j)$ be a division of $[s_{i-1},t]$. Then

$$\left\|\sum_j \left[(K_g)^t(r^j)-(K_g)^t(r^{j-1})\right]x_j\right\| = \left\|\sum_j \int_{r^{j-1}}^{r^j} \alpha^t(\sigma)x_j dg(\sigma)\right\| = \left\|\int_{s-i}^t \alpha^t(\sigma)x_j dg(\sigma)\right\| \leq$$

$$\leq V^u(\alpha)|x_j|\,|g(t)-g(s_{i-1})| < V^u(\alpha)|x_j|\varepsilon < \frac{|x_j|}{2}.$$

Hence $BV_{[s_{i-1},t]}((K_g)^t) \leq 1/2$ and (i) follows.

Finally, the assertion about the Neumann series for the resolvent of equation (25) in $L(C([a,b],\mathbb{R}))$ follows from Theorem 4.6. Besides, the operator F_{K_g} given by $\left(F_{K_g}y\right)(t) = \int_a^t d_s K_g(t,s)y(s)$ is causal as well as $\left(F_{K_g}\right)^n$. Therefore $\left(I-F_{K_g}\right)^{-1}$ is also causal. By Theorem 4.7, the same applies to the resolvent of equation (25) in $L(\mathbf{K}_g([a,b],\mathbb{R}))$. Thus from the fact that $F_{K_g} = H_{\alpha,g}$ (see Theorems 4.4 and 4.3), the operator $I - H_{\rho,g} = (I-H_{\alpha,g})^{-1}$ is causal. □

Remark 6.1. *Given $g \in SL([a,b],\mathbb{R})$ differentiable and non-constant in any non-degenerate subinterval of $[a,b]$, let $R_g^2([a,b],X)_A$ be the space, endowed with the Alexiewicz norm, of the equivalence classes of functions $f:[a,b]\to X$ which are improper Riemann-Stieltjes integrable with respect to g and have a finite number of singularities. Also, let $L_{1,g}([a,b],X)_A$ be the space, endowed with the Alexiewicz norm, of all equivalence classes of functions $f:[a,b]\to X$ which are Bochner integrable with respect to g (i.e., in the Stieltjes sense) with finite integral. Then Theorems 5.2 and 6.1 hold for $R_g^2([a,b],X)_A$ or $L_{1,g}([a,b],X)_A$ instead of $\mathbf{K}_g([a,b],X)$.*

Acknowledgement

We are greatful for the referee's important comments and corrections.

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