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Dissipative Systems with Constraints

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1. INTRODUCTION

Throughout the paper M will be a C^∞ compact m-dimensional connected Riemannian manifold with a C^∞ Riemannian metric $\langle\ ,\ \rangle$, $\partial M = \phi$, called the configuration space. The dimension m of M is called the degree of freedom, the tangent bundle TM of M the velocity phase space and the cotangent bundle T^*M of M the momentum phase space. The vector bundle isomorphism $\mu: TM \to T^*M$ defined by $\mu(v)\,w = \langle v, w\rangle$ is a C^∞ diffeomorphism and will be referred to as the mass operator otherwise called the Legendre transformation. The function $K: TM \to R$ defined by $K(v) = \frac{1}{2}\langle v, v\rangle$ is called the kinetic energy of the system.

The constraint Σ is a C^∞ mapping $q \mapsto \Sigma_q$ which assigns to each $q \in M$ an n-dimensional subspace $\Sigma_q \subset T_qM$. The subvector bundle ΣM of TM defined by the vectors of all Σ_q, $q \in M$, is a regular submanifold of TM, and if $\Sigma_q^\perp$ is the orthogonal complement of Σ_q, in T_qM, the same is true for the subvector bundle $\Sigma^\perp M$ defined by the distribution $q \mapsto \Sigma_q^\perp$. The bundles ΣM, $\Sigma^\perp M$ have dimensions $m+n$, $2m-n$ and there are canonically defined C^∞ bundle mappings $P: TM \to \Sigma M$, $P^\perp: TM \to \Sigma^\perp M$ defined by $Pv_q =: v_q''$, $P^\perp v_q =: v_q^\perp$ where v_q'', $v_q^\perp$ are the orthogonal projections of v_q on Σ_q, $\Sigma_q^\perp$. By means of the mass operator one can translate ΣM, $\Sigma^\perp M$, P,

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$P^\perp$ to corresponding objects Σ^*M, $\Sigma^{\perp *}M$, P^*, $P^{\perp *}$. In fact, $\Sigma^*M = \mu(\Sigma M)$, $\Sigma^{\perp *}M = \mu(\Sigma^\perp M)$, $P^* = \mu P \mu^{-1}$, and $P^{\perp *} = \mu P^\perp \mu^{-1}$.

By a motion compatible with Σ we mean a C^2 function $t \to q(t)$ from some open interval $I \subset R$ into M which satisfies $\dot{q}(t) \in \Sigma_{q(t)}$ for all $t \in I$. The derivative $\dot{q}$ is called the velocity of the motion and $\mu(\dot{q})$ its momentum. We define the acceleration to be the covariant derivative $\nabla_{\dot{q}} \dot{q}$ of $\dot{q}$ (see [5, 8, 14]).

The active field of force is a C^k function F: $TM \to T^*M$, $k \geqslant 1$, such that $F(v_q) \in T_q^*M$. We let $\mathscr{F}^k$ be the Frechet space of all such F endowed with the Whitney topology. We also need to consider fields of force defined only on ΣM. The space of these fields of force can be identified with the subspace $\mathscr{F}^k_\Sigma$ of $\mathscr{F}^k$ defined by the condition $F(v) = F(Pv)$. A field of force is said to be positional if it is constant on $T_q M$, $\forall q \in M$. An important special case of positional field of force is the so called conservative field of force given by $v_q \to dV(q)$ where V: $M \to R$ is a C^{k+1} function. This field of force is said to be conservative with potential V.

A mechanical system with constraints is a triplet (M, Σ, F) and we want to distinguish between holonomic $(m = n)$ and nonholonomic $(0 < n < m)$ constraints. We single out two particular cases that are of special mechanical interest, namely the semi-holonomic and the true nonholonomic case; Σ is said to be semi-holonomic if the distribution $q \mapsto \Sigma_q$ is completely integrable [6] and Σ is true nonholonomic if given any neighborhood U of any point in M, the field $q \mapsto \Sigma_q$ is not completely integrable in U.

We say that a field of force $D \in \mathscr{F}^k$ is dissipative with respect to Σ if $P^*D(v)(v) \leqslant 0$ for each $v \in \Sigma M$, strictly dissipative if $P^*D(v)(v) = 0$ implies $v = 0$, strongly dissipative if there is a continuous function c: $M \to R^+ \setminus \{0\}$ such that $P^*D(v)(v) \leqslant -c|v|^2$. We say that (M, Σ, F) is a dissipative mechanical system if the active field of force F is the sum of a conservative field of force dV and of a field of force D dissipative with respect to Σ.

If a system (M, Σ, F) is holonomic, that is, $\Sigma M = TM$, their motions have to satisfy the Newton law

$$\mu(\nabla_{\dot{q}} \dot{q}) = F(\dot{q}) \qquad \text{(see [8, 14, 20])}.$$

But, in the nonholonomic case, in order to get motions compatible with Σ, we have to introduce a reactive field of force $R \in \mathscr{F}^k_\Sigma$ depending on M, Σ and to consider a generalized Newton law

$$\mu(\nabla_{\dot{q}} \dot{q}) = (F + R)(\dot{q}).$$

In many examples $\mu^{-1}R(v_q) \in \Sigma_q^\perp$ for all $v_q \in TM$ and all external fields of force $F \in \mathscr{F}^k$; such systems are called systems with perfect constraints.

We will not deal, in the present paper, with cases in which $\mu^{-1}R(v_q)$ is not necessarily orthogonal to the constraint Σ_q; cases like constraints with friction, as well as the case of constraints obtained by means of control apparatus and also cases describing motions of systems corresponding to the displacement in an electric field of piezoelectric matter are classical examples of nonperfect constraints; see [9, 10, 11, 12, 13] and also [27], for more examples and references.

We will be restricted to the consideration of dissipative mechanical systems (M, Σ, F) in which $F = dV + D$, Σ is a C^∞ perfect constraint and M is a compact configuration space.

Let us denote by ϕ_t the dynamical system (flow) that a system of this kind defines on ΣM. We will be interested in the set $\mathscr{A}$ of all bounded (in ΣM) orbits. This is a compact set usually referred to as the attractor because it attracts all other orbits. Clearly $\mathscr{A}$ contains the set of equilibria. One of the results is that, under the hypothesis that the center manifold of each equilibrium point coincides locally with the set of equilibria, then the α and ω-limit sets of any orbit in $\mathscr{A}$ are points. From this it follows that, under the same hypothesis, $\mathscr{A}$ is the union of the unstable manifolds of all equilibrium points.

Then we turn our attention to the relations between $\mathscr{A}$ and the configuration space M. We prove that the natural projection $\tau\colon TM \to M$ always projects $\mathscr{A}$ onto M. Another result in this direction is that if $\mathscr{A}$ is a differential manifold, and provided that some genericity conditions hold, then $\mathscr{A}$ and M have the same dimension.

The configuration space M can naturally be identified with the zero section $\mathcal{O} \subset TM$. When the conservative field of force is zero, then all points in M are equilibrium configurations for the mechanical system and correspondently all points in $\mathcal{O}$ are points of equilibrium for the dynamical system ϕ_t. In this case it is also true that $\mathscr{A}$ coincides with $\mathcal{O}$ and therefore is diffeomorphic to M. By using perturbation techniques from the theory of normally hyperbolic sets [1, 2, 3] we prove that, if a certain parameter ε which is a measure of the importance of conservative versus dissipative forces is sufficiently small, then the attractor $\mathscr{A}^\varepsilon$ is diffeomorphic to $\mathcal{O}$ and thus to M and approaches $\mathcal{O}$ as $\varepsilon \to 0$.

By means of the diffeomorphism between $\mathscr{A}^\varepsilon$ and M the flow ϕ_t^ε restricted to $\mathscr{A}^\varepsilon$ can be conjugated to a flow $\bar{\phi}_t^\varepsilon$ on M. The vector field X^ε associated with this flow converges to zero as $\varepsilon \to 0$, therefore we consider the vector field $Y^\varepsilon = \varepsilon^{-1}X^\varepsilon$ which has the same orbits as X^ε and show that as $\varepsilon \to 0$ it converges to a limit vector field Y^0 which can be explicitly computed in terms of the given conservative and dissipative fields of force. This fact has a special importance when Y^0 is a structurally stable vector field; in fact when this is the case all relevant information on the dynamics of X^ε, $\varepsilon > 0$ small, can be recovered by studying a vector field on the configuration space M instead of a vector field on the submanifold ΣM of the phase space.

2. General Theory

As for Σ we can describe $\Sigma^\perp$ in a neighbourhood of any point in M by giving $r = m - n$ C^∞ vectorfields Y_i such that for each $q \in U$ the vectors $Y_i(q)$ form a basis of the vector space $\Sigma_q^\perp$. Then the condition of being compatible with Σ for a motion $t \to q(t)$, can be expressed in the form

$$\langle \dot{q}, Y_i \rangle = 0 \qquad (1 \leqslant i \leqslant r). \tag{2.1}$$

We shall assume, as we can, that the vector fields Y_i are orthonormal. The following theorem justifies the definition of a mechanical system with constraints:

Theorem 2.1. *Given a mechanical system with perfect constraints* (M, Σ, F) *there is a unique field of force* $R \in \mathscr{F}_\Sigma^k$, *the reactive field of force, such that*

(i) $P^*R = 0$.

(ii) *For each* $v_q \in \Sigma M$ *the (maximal) motion* $t \to q(t)$ *that satisfies the generalized Newton law*

$$\mu(\nabla_{\dot{q}} \dot{q}) = (F + R)(\dot{q}), \tag{2.2}$$

and the initial condition $\dot{q}(0) = v_q$, *is compatible with* Σ. *Moreover the following is true*:

(iii) *the motion in* (ii) *is of class* C^{k+2} *and is uniquely determined by* $v_q \in \Sigma M$.

(iv) *there is a field of force* $Q \in \mathscr{F}_\Sigma^\infty$ *depending only on* Σ *such that*

$$R(v_q) + p^{\perp *} F(v_q) = Q(v_q), \qquad \forall v_q \in \Sigma M. \tag{2.3}$$

Q is given locally by

$$Q(v_q) = -\mu(\langle v_q, \nabla_{v_q} Y_i \rangle \, Y_i(q)). \tag{2.4}$$

Proof. If we let $\tilde{F} = \mu^{-1} \circ F$ then Eq. (2.2) is equivalent to

$$\nabla_{\dot{q}} \dot{q} = (\tilde{F} + \tilde{R})(\dot{q}). \tag{2.5}$$

Therefore, if $t \to q(t)$, $\dot{q}(0) = v_q$, is a motion that satisfies (2.2), by applying $P^{\perp}$ to (2.5) and by setting $t = 0$ we get

$$P^{\perp} \nabla_{v_q} \dot{q} = P^{\perp} \tilde{F}(v_q) + \tilde{R}(v_q). \tag{2.6}$$

Since we have assumed that the Y_i are orthonormal, then we can write locally

$$P^{\perp} \nabla_{\dot{q}} \dot{q} = \langle \nabla_{\dot{q}} \dot{q}, Y_i \rangle \, Y_i; \tag{2.7}$$

on the other hand, if $t \to q(t)$ is compatible with Σ, differentiating (2.1) yields

$$\langle \nabla_{\dot{q}} \dot{q}, Y_i \rangle = -\langle \dot{q}, \nabla_{\dot{q}} Y_i \rangle. \tag{2.8}$$

Therefore we can write

$$P^{\perp} \nabla_{v_q} \dot{q} = -\langle v_q, \nabla_{v_q} Y_i \rangle \, Y_i(q). \tag{2.9}$$

If we introduce this equation into Eq. (2.6) and apply μ, we get Eq. (2.3) with Q defined by (2.4). Thus we have proved that if (i), (ii) hold then R is uniquely determined.

To complete the proof we need to prove that if $R \in \mathscr{F}^k_{\Sigma}$ is the field of force defined by (i) and (2.3), then the motion $t \to q(t)$ that satisfies (2.2) and the initial condition $\dot{q}(0) = v_q$ is compatible with Σ. Let us start by introducing the lifting operator [8], namely the operator $C_{v_q}\colon T_q M \to T_{v_q}(TM)$ defined by

$$C_{v_q}(w_q) =: \frac{d}{ds} (v_q + s w_q) \Big|_{s=0}. \tag{2.10}$$

The operator C_{v_q} is a linear map which is an isomorphism onto its range. Moreover, it is easily seen that if $(q, v) = (q_1, ..., q_m, v_1, ..., v_m)$ are local natural coordinates in TM (by abuse of notation we use the same symbol q both for a point in M and for the vector of its coordinates in a generic chart), then C_{v_q} has the local expression

$$(q, w) \to ((q, v), (0, w)). \tag{2.11}$$

From this and the local expression of the covariant derivative [8]

$$\nabla_{\dot{q}} \dot{q} = (\ddot{q}_k + \Gamma^k_{ij} \dot{q}_i \dot{q}_j) \frac{\partial}{\partial q_k}, \tag{2.12}$$

we get

$$C_{\dot{q}}(\nabla_{\dot{q}} \dot{q}) = \ddot{q} - S(\dot{q}), \tag{2.13}$$

where we have indicated by S the geodesic spray, that is, the vector field on TM locally expressed by $(q, v) \to ((q, v), (v, \gamma))$ with $\gamma_k =: -\Gamma^k_{ij} v_i v_j$.

Since $C_{\dot{q}}$ is injective, Eq. (2.2) is equivalent to the equation obtained by applying $C_{\dot{q}}$ to (2.2). This and Eq. (2.13) imply that (2.2) is equivalent to

$$\ddot{q} = E(\dot{q}) =: S(\dot{q}) + C_{\dot{q}}((\tilde{F} + \tilde{R})(\dot{q})), \tag{2.14}$$

therefore we only need to prove that if $R \in \mathscr{F}^k_\Sigma$ is the field of force defined by (i) and (2.3), then $v_q \in \Sigma_q$ implies $E(v_q) \in T_{v_q}(\Sigma M)$. To see this we note that for $v_q \in \Sigma M$ we have

$$\begin{aligned} C_{v_q}(Pw_q) &= \frac{d}{ds}(v_q + sPw_q)\bigg|_{s=0} = \frac{d}{ds} P(v_q + sw_q)\bigg|_{s=0} \\ &= TP \cdot \frac{d}{ds}(v_q + sw_q)\bigg|_{s=0} = TP \cdot C_{v_q}(w_q), \end{aligned} \tag{2.15}$$

where TP is the tangent linear map to P. From (2.15) it follows that we can write

$$\begin{aligned} E(v_q) &= S(v_q) + C_{v_q}(P(\tilde{F} + \tilde{R})(v_q)) + C_{v_q}(P^\perp(\tilde{F} + \tilde{R})(v_q)) \\ &= TP \cdot [S(v_q) + C_{v_q}((\tilde{F} + \tilde{R})(v_q))] \\ &\quad + S(v_q) - TP \cdot S(v_q) + C_{v_q}(P^\perp(\tilde{F} + \tilde{R})(v_q)). \end{aligned} \tag{2.16}$$

By (2.3) and the previous part of the proof we have

$$C_{v_q}(P^\perp(\tilde{F} + \tilde{R})(v_q)) = C_{v_q}(P^\perp \nabla_{v_q} \dot{p}) \tag{2.17}$$

for any motion $t \to p(t)$ satisfying $\dot{p}(0) = v_q$ and compatible with Σ. On the order hand by (2.13), (2.15) one has

$$\begin{aligned} C_{v_q}(P^\perp \nabla_{v_q} \dot{p}) &= C_{v_q}(\nabla_{v_q} \dot{p}) - C_{v_q}(P \nabla_{v_q} \dot{p}) \\ &= \ddot{p} - S(v_q) - TP \cdot (\ddot{p} - S(v_q)). \end{aligned} \tag{2.18}$$

Since $t \to p(t)$ is compatible with Σ it results $\ddot{p} = TP \cdot \ddot{p}$. Therefore from (2.16), (2.17), (2.18) it follows that $E(v_q)$ belongs to $T_{v_q}(\Sigma M)$ for $v_q \in \Sigma M$. ∎

DEFINITION 2.1. $t \to q(t)$ is called a motion of the mechanical system with constraints (M, Σ, F) if it is compatible with Σ and satisfies (2.2) with R defined by (i), (2.3).

On the basis of Theorem 2.1, finding the motions of a mechanical system with constraints amounts to the determination of the flow defined on ΣM by the vector field

$$E(v_q) = S(v_q) + C_{v_q}((P\tilde{F} + \tilde{Q})(v_q)), \qquad v_q \in \Sigma M. \tag{2.19}$$

We shall call this vector field on ΣM the GMA vector field.[1] Since S and Q do not change if F is changed, two different choices F_i, $i = 1, 2$, of F give rise to the same GMA vector field if and only if

$$C_{v_q}((P\tilde{F}_1)(v_q)) = C_{v_q}((P\tilde{F}_2)(v_q)). \tag{2.20}$$

From this and the fact that C_{v_q} and μ are injective it follows

THEOREM 2.2. *Two mechanical systems (M, Σ, F_1) and (M, Σ, F_2) have the same motions if and only if*

$$P^*F_1 = P^*F_2. \tag{2.21}$$

The determination of the flow of the GMA vector field is locally the same as to find the solutions of a system of differential equations that we now derive for later reference.

The condition of being in ΣM for a vector $v_q \in TM$ can be expressed in terms of the local coordinates (q, v) by saying that $r = m - n$ of the components of v, for instance

$$v^{\perp} =: (v_1, ..., v_r),$$

are linear combinations of the remaining

$$v'' =: (v_{r+1}, ..., v_m).$$

That is, $v^{\perp} = Av''$ for some $r \times n$ matrix A. It follows that if $t \to v_q(t) \in \Sigma M$ is a curve on ΣM then

$$\frac{dv_q}{dt}$$

has the local representation

$$\frac{dv_q}{dt} \simeq \left(\begin{pmatrix} q, & \begin{matrix} Av'' \\ v'' \end{matrix} \end{pmatrix}, \begin{pmatrix} \dot{q}, & \begin{matrix} A\dot{v}'' + \dfrac{\partial A}{\partial q_i} \dot{q}_i v'' \\ \dot{v}'' \end{matrix} \end{pmatrix} \right). \tag{2.22}$$

[1] GMA stand for Gibbs, Maggi, Appell, who first derived the equations for mechanical systems with nonholonomic constraints [9, 10].

If $v = (v_1, \ldots, v_m)$ are the components of $\tilde{F} + \tilde{R}$ with respect to the local basis $\partial/\partial q_i$ $(1 \leqslant i \leqslant m)$, then from (2.11), (2.14) and the definitions of S we get that E has the local expression

$$(q, v) \to ((q, v), (v, \gamma + v)).$$

From Theorem 2.1 we know that $v_q \in \Sigma M$ implies

$$E(v_q) \in T_{v_q}(\Sigma M).$$

This is the same as to say that for $v_q \in \Sigma M$ the local representation of $E(v_q)$ must be of the type (2.22), say,

$$E(v_q) \simeq \left(\left(q, \begin{matrix} Av'' \\ v'' \end{matrix} \right), \left(\begin{matrix} Av'' \\ v'' \end{matrix}, \begin{matrix} A(\gamma + v)'' + \dfrac{\partial A}{\partial q_i} \begin{pmatrix} Av'' \\ v'' \end{pmatrix}_i v'' \\ (\gamma + v)'' \end{matrix} \right) \right), \tag{2.23}$$

where $\binom{Av''}{v''}_i$ stands for the ith component of the vector $\binom{Av''}{v''}$. It follows from (2.22), (2.23), that (2.14) restricted to ΣM is locally equivalent to the following system of $m + n$ first-order differential equations

$$\begin{aligned} \dot{q}^{\perp} &= Av'', \\ \dot{q}'' &= v'', \\ \dot{v}'' &= (\gamma + v)''. \end{aligned} \tag{2.24}$$

EXAMPLE 2.1. A rigid body B which beside having a fixed point Ω is constrained to move in such a way that the angular velocity is always orthogonal to a straight line l fixed in B. A possible realization of this constraint is sketched in Fig. 1a.

In this case $M = SO(3)$, $n = 2$. Let xyz with $z = l$ be a positively oriented frame fixed in B, then as local coordinates in a neighbourhood of any given position of B one can take the angles q_1, q_2, q_3 corresponding to three suc-

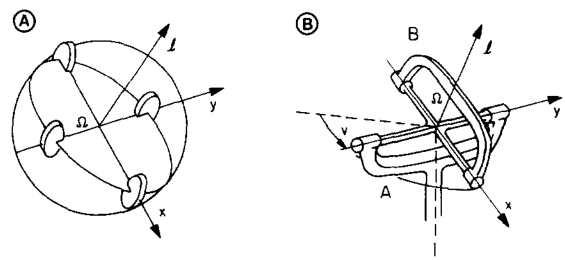

FIGURE 1

cessive rotations R_1 around y, R_2 around $R_1 x$, and R_3 around $R_2 R_1 z$. A local basis for Σ is then given by

$$X_1 = \frac{\partial}{\partial q_1}, \qquad X_2 = \frac{\partial}{\partial q_2} + \sin q_1 \frac{\partial}{\partial q_3}.$$

It follows $[X_1, X_2] = \cos q_1\, \partial/\partial q_3$ which is not a linear combination of X_1, X_2. Σ is then true nonholonomic (see [6]).

If the constraints are changed to a Cardanic suspension as in Fig. 1b then the constraints are semi-holonomic and the leaves are parametrized by angle v. We note for later reference that for the positions of B with l orthogonal to the x, y plane, the subspace Σq is the same as for the corresponding positions of the true nonholonomic system considered in Fig. 1a. Let us assume now that B is a gyroscope around l and any potential function V. If c is the moment of inertia around l and a the moment of inertia around any axis through Ω orthogonal to l, then the Riemannian metric is defined by the kinetic energy

$$K = \tfrac{1}{2}(a(\dot{q}_1^2 + \dot{q}_2^2 \cos^2 q_1) + c(-\dot{q}_2 \sin q_1 + \dot{q}_3)^2),$$

and it is easily seen that $Y = c^{-1/2}\, \partial/\partial q_3$ is a unit vector orthogonal to X_1, X_2. To compute Q we note that, as a direct computation shows, it is a general fact that the 1-form α with local expression

$$\left(\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j}\right) dq_j$$

satisfies $\alpha(v) = \langle \nabla_{\dot{q}} \dot{q}, v\rangle$, $v \in TM$. Therefore we have

$$\begin{aligned} -\langle \dot{q}, \nabla_{\dot{q}} Y\rangle &= \langle P^{\perp} \nabla_{\dot{q}} \dot{q}, Y\rangle = \langle \nabla_{\dot{q}} \dot{q}, Y\rangle \\ &= \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_3} - \frac{\partial K}{\partial q_3} = c^{1/2}(-\dot{q}_2 \sin q_1 + \dot{q}_3)^{\cdot}. \end{aligned} \tag{2.25}$$

On the other hand, in the present case Eq. (2.1) becomes

$$-\dot{q}_2 \sin q_1 + \dot{q}_3 = 0,$$

that together with (2.25) implies $Q = 0$.

For many other interesting examples see [7].

Remark. The previous example, with V the gravity potential, can be also used to show that if q is an equilibrium position for a conservative mechanical system with perfect semi-holonomic constraints and q is Liapounov stable, then, if one changes the constraints making them true nonholonomic in the class of Σ that leave the subspace Σ_q unchanged, the position q may become an unstable equilibrium. To see this we note that all positions of B such that l is vertical and the center of mass G of B is under Ω are stable equilibria when B is constrained by a Cardanic suspension as described in the last example 2.1. On the other hand it is well known that among the motions of a gyroscope that moves under the action of gravity there exist precessions around the vertical with the property that the angular velocity always stays orthogonal to the gyroscopic axis. Moreover, the angular speed that realizes this kind of motion depends on the angle v between the vertical and the gyroscopic axis and goes to zero with v. It follows that, in the case of true nonholonomic constraints the positions of B with G on the vertical under Ω are unstable equilibria.

THEOREM 2.3. *If* (M, Σ, F) *is a mechanical system with (perfect) constraints and* X_F *is the vector field on TM corresponding to Newton's law* $\mu(\nabla_{\dot q}\dot q) = F(\dot q)$, *then the* GMA *vector field* E *on* ΣM *associated with* (M, Σ, F) *is given by*

$$E = TP \cdot X_F. \tag{2.26}$$

Proof. Formula (2.19) gives for all $v_q \in \Sigma M$:

$$E(v_q) = S(v_q) + C_{v_q}(P\tilde{F}(v_q)) + C_{v_q}(\tilde{Q}(v_q)), \tag{2.27}$$

$$X_F(v_q) = S(v_q) + C_{v_q}(P\tilde{F}(v_q)), \tag{2.28}$$

and then

$$\begin{aligned} TP \cdot X_F(v_q) &= TP \cdot S(v_q) + C_{v_q}(P\tilde{F}(v_q)) \\ &= TP \cdot S(v_q) + E(v_q) - C_{v_q}(\tilde{Q}(v_q)) - S(v_q). \end{aligned} \tag{2.29}$$

But formula (2.27) applied to $\tilde{F} = 0$ shows that

$$TP \cdot S(v_q) = S(v_q) + C_{v_q}(\tilde{Q}(v_q)) \tag{2.30}$$

and then formula (2.26) is proved. ∎

3. THE ATTRACTOR AND THE SET OF EQUILIBRIA OF A DISSIPATIVE SYSTEM

In the discussion of the qualitative behavior of the flow $\phi: \mathcal{D} \subset \Sigma M \times \mathbb{R} \to \Sigma M$ defined by the GMA vector field corresponding to a dissipative system, we shall focus our attention on the set $\mathcal{A} = \{x \in \Sigma M \mid \phi(x, t)$ *is defined for* $t \in (-\infty, \infty)$ *and is bounded*$\}$. The reason for this is that globally defined and bounded orbits, for instance critical points, are usually the most interesting from a mechanical point of view. Moreover, as we shall see, strictly dissipativeness implies that $\mathscr{A}$ is a global attractor. Therefore we may conclude that $\mathscr{A}$ contains most of the relevant information about the dynamical system ϕ. We shall usually consider strictly dissipative systems because otherwise $\mathscr{A}$ may coincide with the all ΣM as for Hamiltonian systems and the study of $\mathscr{A}$ is then of no interest.

Before giving a characterization of $\mathscr{A}$ we present some results on critical points of GMA and give related results on the asymptotic behavior of trajectories.

Since GMA is the restriction to ΣM of a second order equation, the critical points lie on the zero section $\mathcal{O}: M \to \Sigma M$ and can be identified with the equilibria of the underlying mechanical system. When this is a dissipative system, since

$$P^*D(0)(v) = \lim_{\lambda \to 0} \lambda^{-1} P^*D(\lambda v)(\lambda v) \leqslant 0, \qquad \lambda > 0, v \in \Sigma M, \tag{3.1}$$

implies $P^*D(0)(v) = 0$ or equivalently $P\tilde{D}(0)(v) = 0$ for each $v \in \Sigma M$, and on the other hand S and Q vanish on the zero section, we see from (2.19) that, if we let $X_V = P\tilde{\partial} V = P \operatorname{grad} V$, the set of critical points of GMA coincides with the intersection $X_V \cap \mathcal{O}$ of the section $X_V: M \to \Sigma M$ with the zero section. We may therefore quote from [15, 18, 19] the following result. As usual, if A, B are submanifolds of a manifold C, by $A \pitchfork B$ we mean transversal intersection that is: at each point $x \in A \cap B$, $T_x A$ and $T_x B$ span $T_x C$.

THEOREM 3.1. (i) *The set G^{k+1} of potential functions $V \in C^{k+1}(M, \mathbb{R})$ $(k \geqslant 1)$ such that $X_V \pitchfork \mathcal{O}$ is open and dense in $C^{k+1}(M, \mathbb{R})$;*

(ii) *If $V \in G^{k+1}$, then the set C_V of the critical points of* GMA *or equivalently the set ε_V of the equilibria of the underlying dissipative system is a C^k compact submanifold of dimension $r = m - n$;*

(iii) *C_V, ε_V depend C^k continuously on $V \in G^{k+1}$.*

From this theorem if follows that generically for a holonomic mechanical system the set of equilibria is made of a finite number of points; when $r = 1$ as in the case of the rigid body in Example 2.1, the set of equilibria is generically the union of a finite number of circles.

For fixed M and Σ the number of connected components in C_V, ε_V has a minimum $m(M, \Sigma)$ for V in G^{k+1}; for instance in the holonomic case if $M = S^2$, this minimum is 2; if $M = T^2$ this minimum is 4. It may be interesting to find relations between $m(M, \Sigma)$ and M, Σ, for general M and Σ. We now turn to the asymptotic behavior of the trajectories $t \to v(t)$ of GMA.

THEOREM 3.2. *The trajectories* $t \to v(t)$ *of the* GMA *vector field associated with a dissipative system are globally defined in the future and bounded. If the system is strictly dissipative all trajectories approach the set* C_V *of the critical points as* $t \to \infty$. *Moreover, if* $t \to v(t)$ *is defined also for negative time and bounded, then* $v(t)$ *approaches* C_V *as* $t \to -\infty$.

Proof. From (2.2) it follows

$$\frac{d}{dt} K(v(t)) = \langle v(t), \nabla_{v(t)} v(t) \rangle = \langle v(t), F(v(t)) \rangle$$

$$= \langle v(t), \text{grad } V(\tau v(t)) \rangle + \langle v(t), \tilde{D}(v(t)) \rangle. \tag{3.2}$$

Since GMA is the restriction to ΣM of a second-order equation, $v(t)$ coincides with the derivative with respect to t of $\tau v(t)$, therefore we have

$$\frac{dV}{dt}(\tau v(t)) = \langle v(t), \text{grad } V(\tau v(t)) \rangle. \tag{3.3}$$

Moreover, we can write

$$\langle v(t), \tilde{D}(v(t)) \rangle = \langle v(t), P\tilde{D}v(t)) \rangle = P^* D(v(t))(v(t)). \tag{3.4}$$

From (3.2), (3.3), (3.4) it follows that

$$\frac{dH}{dt}(v(t)) = P^* D(v(t))(v(t)) \leqslant 0, \tag{3.5}$$

where $H(v) =: K(v) - V(\tau v)$.

Since M is compact and K is nonnegative the open set $\mathscr{B}_a = \{v \in \Sigma M \mid H(v) < a\}$ is also bounded. Thus, if H_0 is the value of $H(v(t))$ at some initial time t_0, $v(t)$ remains in the bounded set $\mathscr{B}_{H_0}$ in its maximal interval of existence (t_0, t_1). This and the continuation theorem imply $t_1 = \infty$.

Since M is compact H is bounded below therefore from (3.5) we get that $H(v(t))$ approaches some constant H_∞ as $t \to \infty$. Therefore if v_∞ is a point in the ω-limit set of $t \to v(t)$ and $t \to w(t)$ is the trajectory through v_∞ we have $H(w(t)) = H_\infty$. Equation (3.5) implies then $P^* D(w(t))(w(t)) = 0$ and therefore if D is strictly dissipative $w(t)$ is in the zero section. From this and the fact that $w(t)$ is the derivative of $\tau w(t)$ it follows that $w(t)$ must be constant and therefore equal to v_∞ which is then a critical point of GMA. The last part of the theorem is proved in a similar way. ∎

Strictly dissipativeness implies that all trajectories of GMA approach the set of critical points but it is not a sufficient condition in order that the

ω-limit set of any orbit contains just one point. For instance, when M is a circle $C \subset \mathbb{R}^3$, s the curvilinear abscissa along C, T the unit vector tangent to C at s, $V=0$, $D(vT)(wT) = -v^2 w$, $\forall v$, $w \in \mathbb{R}$, Eqs. (2.24) take the form $\dot{s} = v$, $\dot{v} = -v^3$. From these equations it follows that $v \to 0$ as $t \to \infty$ while s grows unboundedly if the initial value of v is not zero. Therefore the ω-limit set of any orbit through any point in $TC \setminus \mathcal{O}$ is the all $\mathcal{O}$.

The main point in this example is the nongenericity of V; in fact we know from Theorem 3.1 that for $r=0$ and $V \in G^{k+1}$ the critical points of GMA are isolated and then the ω-limit set of any orbit must be a single point if the system is strictly dissipative. In the case $r=1$ even for $V \in G^{k+1}$ the critical points are not isolated.

Using a general theorem in transversality theory we will prove the next result for $r=1$. For this, call $J^2(M, R)$ the manifold of 2-jets of the C^{k+1} real-valued functions on M, $k \geqslant 2$. For every $V \in C^{k+1}(M, R)$ let $j^2 V$: $M \to J^2(M, R)$ be the C^{k-1} map such $j^2 V(p) = j_p^2 V$ is the 2-jet of V at p. Recall that $V \in C^{k+1}(M, R)$ is a Morse function if $dV(p) = 0$ for $p \in M$ implies $d^2 V(p)$ is a nondegenerate quadratic form on $T_p M$. Call $\tilde{G}^{k+1} = \{V \in G^{k+1} \mid V \text{ is a Morse function}\}$. It is well known that $\tilde{G}^{k+1}$ is open and dense and this fact will follow also in the sequel of the next arguments.

Define now the sets

$$W_1 = \{j_p^2 V \in J^2(M, R): dV_p = 0\}$$

and

$$W_2 = \{j_p^2 V \in J^2(M, R): V \in G^{k+1}; dV_p \neq 0; dV_p \mid \Sigma_p = 0; dV_p \mid T_p \varepsilon_V = 0\}.$$

Remark that W_1 is a closed embedded submanifold of $J^2(M, R)$ of codimension m. In the definition of W_2, since $V \in G^{k+1}$, the condition $dV_p \mid \Sigma_p = 0$ is equivalent to say that $p \in \varepsilon_V$, the codimension $(m-1)$ submanifold of M of all critical points of P grad V; dV_p and $T_p \varepsilon_V$ depend on Σ, $\langle \ , \ \rangle$ and $j_p^2 V$, thus W_2 is well defined. Let us show now that W_2 is a closed embedded codimension m submanifold of $J^2(M, R)$. For a given $V \in G^{k+1}$, assume $s \to q(s)$ be the arc length parametrization of C_V, $q(s_0) = p$. If $u_{v_j} (\partial/\partial q_j)$, $v = 1, 2, ..., m$, is a local basis for Σ, the definition of W_2 is equivalent to

$$f_v(p) = \left[u_{v_j} \frac{\partial V}{\partial q_j} \right] (p) = 0, \tag{3.6}$$

and

$$q_j(s_0) \frac{\partial V}{\partial q_j} (p) = 0. \tag{3.7}$$

Since $[u_{v_j}(\partial V/\partial q_j)](q(s))=0$ one obtains at $s=s_0$

$$\left[u_{v_j}\frac{\partial V}{\partial q_j\,\partial q_k}+\frac{\partial V}{\partial q_j}\cdot\frac{\partial u_{v_j}}{\partial q_k}\right]\dot{q}_k(s_0)=M_{v_k}\dot{q}_k(s_0)=0 \tag{3.8}$$

the matrix M_{v_k} with maximum rank. Besides, (3.7) and (3.8) imply

$$\phi(p)=\det\begin{bmatrix} M_{v_k} \\ \dfrac{\partial V}{\partial q_k}\end{bmatrix}(p)=0. \tag{3.9}$$

Conversely, if $V\in G^{k+1}$, (3.6) and (3.9) imply (3.6) and (3.7). It is enough now to remark that df_v and $d\phi$ are linearly independent at each point of $J^2(M, R)$ which is not too difficult but tedious. The map ev_ρ,

$$(V, p)\in C^{k+1}\times M\xrightarrow{ev_\rho} j^2_pV\in J^2(M, R)$$

is a C^{k-1} map which is onto with onto derivative at each point; this means that

$$\rho\colon\quad V\in G^{k+1}\to j^2V\in C^{k-1}(M, J^2(M, R))$$

is a C^{k-1}-representation (see [8, p. 46]). We are now able to apply the transversality theory (openness and density theorems) to show that the sets

$$A_1=\{V\in G^{k+1}; j^2V \pitchfork W_1\}\qquad\text{and}\qquad A_2=\{V\in G^{k+1}; j^2V\pitchfork W_2\}$$

are open and dense in G^{k+1}, then in $C^{k+1}(M, R)$.

The meaning of these transversalities is the following: if $V\in A_1$ and $j^2_pV\in W_1$, the condition $j^2V\pitchfork W_1$, $p\in M$, implies that V is a Morse function ($A_1=\tilde{G}^{k+1}$) and there exist just a finite number of points $p\in M$ with that property for a given V; if $V\in A_2$ and $j^2_pV\in W_2$ (also for at most a finite number of points), the transversality condition $j^2V\pitchfork_p W_2$ means that ε_V is tangent at p to Σ_p with a generic kind of tangency. For each V in the open and dense set $A_1\cap A_2$ there is at most a finite number of points of C_V for which

$$dV(\dot{q}(s))=\frac{d}{ds}V(q(s))=0.$$

THEOREM 3.2. *Let $r=m-n=1$, then there is an open and dense set $A_1\cap A_2\subset G^{k+1}$, $k\geqslant 2$, such that if $V\in A_1\cap A_2$, V is a Morse function and there are at most a finite number of points of C_V for which $V|C_V$ is not strictly monotonic. Moreover, if the system is strictly dissipative, then the ω-limit set of any orbit of the* GMA *contains just one point. The same is true for the α-limit set of any negatively bounded orbit.*

The next theorem concerns the case of a generic value of r and gives conditions in order that the ω-limit set of any orbit contains just one point.

We state the theorem without specific reference to the GMA vector field because the result can be applied to any evolutionary equation (e.g., gradient systems) that satisfies the property that the ω-limit set of any bounded orbit contains only critical points. The result we present includes as special cases theorems of Malkin [26], Hale and Massatt [22] concerning the situation where the set of critical points is a one-dimensional manifold. With standard notation, if x is a critical point of a vector field $X \in C^1$ $(\Omega, \mathbb{R}^n)$, then W_x^{ss}, W_x^{cs}, W_x^{c}, W_x^{cu}, W_x^{uu} we mean local stable, centerstable, center, center unstable, unstable manifold of x. For a definition of these concepts as well as standard theory see, for instance, [1, 23, 24].

THEOREM 3.4. *Suppose that the ω-limit set $\omega(\gamma)$ of a bounded orbit γ of a vector field $X \in C^1$ $(\Omega, \mathbb{R}^n)$ contains only critical points. Then a sufficient condition in order that $\omega(\gamma)$ contains just one point is that the local center manifold W_x^c at each critical point coincides locally with the set of critical points. A similar result holds true for the α-limit set of a negatively bounded orbit.*

The proof of this theorem is a plain consequence of the following fundamental result stated in Henry [23].

LEMMA 3.1. *Let x be a critical point of $X \in C^1$ $(\Omega, \mathbb{R}^n)$ and $\phi: \mathscr{D} \subset \Omega \times \mathbb{R} \to \Omega$ be the local dynamical system defined by X. Then there exist C^0 fibrations $\pi: U \to W_x^{cu} \cap U$, U an open neighborhood of x in Ω, $\pi_0: U_0 \to W_x^c \cap U_0$, U_0 an open neighborhood of x in W_x^{cu}, such that $\pi^{-1}(x) = W_x^{ss} \cap U$; $\pi_0^{-1}(x) = W_x^{uu} \cap U_0$ are C^1 manifolds and π commutes with ϕ in the sense that for each $y \in U$ there is an interval $(\alpha_1, \alpha_2) \subset \mathbb{R}$ such that $\pi(\phi(y, t)) = \phi(\pi y, t)$, $t \in (\alpha_1, \alpha_2)$ and α_1 is either $-\infty$ (α_2 is either $+\infty$) or $\phi(y, \alpha_1) \in \partial U$ (or $\phi(y, \alpha_2) \in \partial U$) and similarly for π_0.*

Proof of Theorem 3.4. Suppose $z \in \Omega$ is such that the orbit $\gamma(z)$ through z is bounded and $\omega(\gamma)$ is contained in the set of critical points C. If $x \in \omega(\gamma)$ then there are fibrations π, π_0 as in Lemma 3.1. We may assume that $U_0 = W_x^{cu} \cap U$ and define a new fibration $\tilde{\pi}: U \to W_x^c \cap U$ by letting $\tilde{\pi} = \pi_0 \circ \pi$. By Lemma 3.1, as long as all expressions are meaningfull, we have

$$\tilde{\pi}(\phi(y, t)) = \pi_0(\phi(\pi(y), t)) = \phi(\tilde{\pi}(y), t); \tag{3.10}$$

on the other hand, since by assumption W_x^c coincides locally with C we have $\phi(\tilde{\pi}(y), t) = \tilde{\pi}(y)$. Therefore from (3.10) it follows that

$$\tilde{\pi}(\phi(y, t)) = \tilde{\pi}(y); \tag{3.11}$$

that is, the fibers of $\tilde{\pi}$ are locally invariant manifolds. Given $y \in W_x^c \cap U$ let $\overline{\tilde{\pi}^{-1}(y)}$ be the closure of the fiber $\tilde{\pi}^{-1}(y)$ over y and let d_y be the distance between $\overline{\tilde{\pi}^{-1}}(y) \cap \partial U$ and $C \cap \bar{U}$, the set of critical points in the closure of U. Since $\tilde{\pi}$ is a fibration and W_x^c and C are locally coincident, by reducing the sets U_0, U if necessary, we may assume that $d_y \geqslant \delta > 0$ with δ independent of y.

Since $x \in \omega(\gamma)$ there is a divergent sequence t_k such that $x_k =: \phi(z, t_k)$ converges to x as $k \to \infty$. Let $y_k =: \tilde{\pi}(x_k)$ and suppose that for each k there is a $t'_k > t_k$ such that

$$x'_k =: \phi(z, t'_k) \in \partial U.$$

Let x' be the limit of a convergent subsequence of x'_k, then $x' \in \partial U$ and $d(x', C \cap \bar{U}) \geqslant \delta$. Thus x' is a point in $\omega(\gamma)$ which is not a critical point in contradiction with the hypothesis $\omega(\gamma) \subset C$. It follows that there is a value $\bar{k}$ of k such that $\phi(z, t)$ stays in $\overline{\tilde{\pi}^{-1}(y_{\bar{k}})}$ for all $t \geqslant t_{\bar{k}}$. Since $\omega(\gamma) \subset C$ and $x_{\bar{k}}$ is the only critical point in $\overline{\tilde{\pi}^{-1}}(y_{\bar{k}})$ it must be $\omega(\gamma) = \{y_{\bar{k}}\} = \{x\}$. ∎

We now begin the study of $\mathscr{A}$ by giving a characterization of the attractor and some of its properties. We need the following lemma that was originally proved by La Salle and Billoti [21]. The version we quote is more or less the one in [16] where a complete proof is also given.

LEMMA 3.2. *Let (S, d) be a complete metric space and Ω a bounded open set in S. Suppose $\psi: S \to S$ is completely continuous and such that*

(i) *given $x \in S$, $\psi^n(x) \in \Omega$ for all sufficiently large n;*
(ii) $\overline{\psi(\Omega)} \subset \Omega$;

then $\mathscr{J}_\psi = \bigcap_{i \geqslant 0} \psi^i(\Omega)$ is a nonempty compact ψ-invariant set, it is connected if Ω is connected and contains every compact ψ-invariant set. Given $\varepsilon > 0$ there is an integer n_ε such that for all $n \geqslant n_\varepsilon$ $\psi^n(\Omega)$ is contained in the ε-neighborhood of $\mathscr{J}_\psi$.

Moreover, if ψ is uniformly continuous on an η-neighborhood N_η of Ω for some $\eta > 0$, then given $\varepsilon > 0$ there is a $\delta > 0$ such that for any $\bar{\psi}: S \to S$ that satisfies (i), (ii), *and $d(\bar{\psi}(x), \psi(x)) < \delta$ on N_η the set $\mathscr{J}_{\bar{\psi}}$ is contained in a ε-neighborhood of $\mathscr{J}_\psi$.*

To apply Lemma 3.2 to our situation we identify ψ with the time one map ϕ_1 associated with ϕ. The existence of ϕ_1 is assured by Theorem 3.2. Furthermore we note that, since $H(0_q) = V(q)$ for $0_q \in \mathcal{O}$, compactness of M implies that H is bounded on the zero section and therefore that, if a is sufficiently large, the zero section is contained in the set $\mathscr{B}_a$. Let Ω_a be the connected component of $\mathscr{B}_a$ containing $\mathcal{O}$. If the system is strictly dissipative, then Eq. (3.5) implies $\overline{\phi_1(\Omega_a)} \subset \Omega_a$. Since by Theorem 3.2 the ω-limit set of any orbit is in $\mathcal{O} \subset \Omega_a$, ϕ_1 and Ω_a satisfy also condition (i). Therefore we can apply Lemma 3.2 and obtain a set $\mathcal{J}_{\phi_1}$ with the property stated in the lemma. It is easy to check $\mathcal{J}_{\phi_1} = \mathcal{A}$. In fact it is quite obvious that $\mathcal{A} \subset \mathcal{J}_{\phi_1}$. On the other hand if x is in $\mathcal{J}_{\phi_1}$ then $\phi(x, \cdot)$ is globally defined because $\mathcal{J}_{\phi_1}$ is invariant; moreover, it is also bounded because $\phi(x, -n) \in \mathcal{J}_{\phi_1} \subset \Omega_a$ and Ω_a is a positively invariant bounded set. From Lemma 3.2 it also follows that $\mathcal{A}$ is uniformly asymptotically stable. Remark that $\Omega_a = \mathcal{B}_a$ since M is connected. We have, then, the following

THEOREM 3.5. *If $\phi: \mathcal{D} \subset \Sigma M \times \mathbb{R} \to \Sigma M$ is the dynamical system associated with a strictly dissipative mechanical system and $\mathcal{A} = \{x \in \Sigma M \mid \phi(x, t)$ is defined for $t \in (-\infty, \infty)$ and bounded$\}$, then*

(i) *$\mathcal{A}$ is compact, connected, invariant, and maximal.*

(ii) *$\mathcal{A}$ is a uniformly asymptotically stable set for the flow ϕ.*

(iii) *$\mathcal{A}$ is an upper semicontinuous function of the potential V and of the dissipative field of force D.*

(iv) *If ϕ_1 is the time one map associated with ϕ and $\mathcal{B} = \{x \in \Sigma M \mid H(x) < a\}$ with a sufficiently large $a > 0$, then*

$$\mathcal{A} = \bigcap_{n \geq 0} \phi_1^n(\mathcal{B}).$$

It is interesting to remark that, if the α-limit set of any negatively bounded orbit contains just a point, as for instance in the cases described in Theorems 3.2, 3.3, 3.4, then

$$\mathcal{A} = \bigcup_{x \in C} W_x^u.$$

EXAMPLE 3.1. For a pendulum that beside gravity is subjected also to a dissipative force of viscous-type equations (2.24) take the form

$$\begin{aligned} \dot{v} &= \eta, \\ \dot{\eta} &= -g/l \sin v - c\eta, \end{aligned} \tag{3.12}$$

where l is the length of the pendulum, g the gravity acceleration, $c > 0$ a constant and v the anomaly from the stable equilibrium. Since there are only two equilibria $(0, 0)$ and $(\pi, 0)$ we have $\mathcal{A} = W^u_{(0,0)} \cup W^u_{(\pi,0)}$. For any c, $W^u_{(0,0)}$ contains only the point $(0, 0)$; $W^u_{(\pi,0)}$ contains the unstable equilibrium $(\pi, 0)$ and two orbits connecting $(\pi, 0)$ with $(0, 0)$ therefore $\mathcal{A}$ is a cicle homeomorphic to the zero section (see Fig. 2a). The influence of c is on the smoothness of $\mathcal{A}$. In fact for $c \geq 2\sqrt{g/l}$, $\mathcal{A}$ is a C^∞-differential

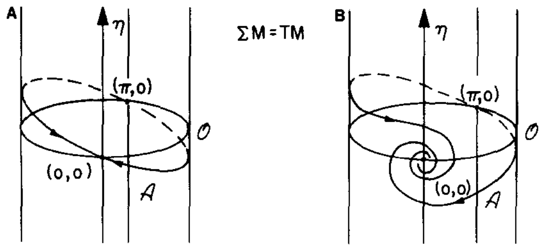

FIGURE 2

manifold diffeomorphic to $\mathcal{O}$, for $c < 2\sqrt{g/l}$, (0, 0) is a stable focus and $\mathscr{A}$ looses its differential structure at (0, 0) (see Fig. 2b).

4. RELATIONSHIPS BETWEEN THE ATTRACTOR AND THE CONFIGURATION SPACE

One of the basic questions in the description of the structure of the attractor which is a subset of ΣM is to see how this structure is related to the basic manifold M, the configuration space of the mechanical system under study. The following theorem says that $\mathscr{A}$ is at least as large as M.

THEOREM 4.1. *Let $\mathscr{A}$ be the attractor of a strictly dissipative system; then the image of $\mathscr{A}$ under the natural projection $\tau: TM \to M$ is the all configuration space.*

Proof. Since $\mathscr{A}$ is compact and τ is continuous $\tau(\mathscr{A})$ is also compact, therefore to prove that $\tau(\mathscr{A}) = M$ it suffices to show that $\tau(\mathscr{A})$ is dense in M. We can regard M and ΣM as metric spaces with distance d_M, d_Σ, then given $\varepsilon > 0$, by Theorem 3.5(ii) there is a t_ε such that the image $\phi_{t_\varepsilon}(\mathcal{O})$ of the zero section under ϕ_{t_ε} is contained in the ε-neighborhood U_ε of $\mathscr{A}$ defined by

$$U_\varepsilon =: \{X_q \mid X_q \in \Sigma M, d_\Sigma(X_q, \mathscr{A}) < \varepsilon\}.$$

Since τ is continuous it is uniformly continuous on U_ε and therefore given $\varepsilon' > 0$ we can choose ε so small that $d_\Sigma(X_p, X_q) < \varepsilon$ implies $d_M(q, p) < \varepsilon'$. Let $\psi: M \to M$ be the map defined by $\psi(q) =: \tau\phi_{t_\varepsilon}(0_q)$, $0_q \in \mathcal{O}$. Then ψ is onto M. This follows from the observation that, as a consequence of the general theorem on continuous dependence on initial data, the map ψ is homotopic to $\psi_0 = \tau\phi_0 = \mathrm{id}_M$. In fact, since M is a manifold without boundary, the homotopy between ψ and id_M implies that ψ and id_M have the same topological degree, that is $d(\psi, M, q) = d(\mathrm{id}, M, q)$ for any given

$q \in M$. Since $d(\mathrm{id}, M, q) = 1$ this implies that $\psi^{-1}(q)$ is nonempty and therefore onto-ness of ψ. From the definition of ψ we have then $M = \psi(M) = \tau\phi_{t_\varepsilon}(\mathcal{O})$ and therefore for any given $q \in M$ there is an $X_q \in \phi_{t_\varepsilon}(\mathcal{O})$. Since $\phi_{t_\varepsilon}(\mathcal{O})$ is in U_ε there is an $X_p \in \mathscr{A}$ such that $d_\Sigma(X_q, X_p) < \varepsilon$ thus by the remark about uniform continuity of τ we have $d_M(q, p) < \varepsilon'$. This concludes the proof because it means that given $q \in M$ and $\varepsilon' > 0$ there is a point $p \in \tau(\mathscr{A})$ the distance of which from q is less than ε'. ∎

It may be interesting to remark that Theorem 4.1 implies that given any point $q \in M$ there is a $v_q \in \Sigma_q$ such that the orbit of GMA through v_q is globally defined and bounded.

The following theorem gives conditions in order that the attractor and the configuration space have the same dimension.

THEOREM 4.2. *If V is a Morse function, D is strongly dissipative and $\mathscr{A}$ is a differential manifold, then*

$$\dim \mathscr{A} = \dim M. \tag{4.1}$$

To prove this theorem we use two lemmas.

LEMMA 4.1. *Given a critical point $O_{\bar{q}} \in C_V$ there are local natural coordinates such that the $(m+n)\times(m+n)$ matrix L corresponding to the linearization of the* GMA *vector field at $O_{\bar{q}}$ is given by*

$$L = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{0} & 0 \\ \cline{3-3} \multicolumn{2}{c|}{} & I \\ \hline F & G & H \end{array}\right] \tag{4.2}$$

where F is some $n \times r$ matrix and the $n \times n$ matrices G, H are defined in terms *of the potential V, of the components D_i $(1 \leqslant i \leqslant m)$ of D and of certain constants σ_k^{jh} $(1 \leqslant j \leqslant m;\ 1 \leqslant h, k \leqslant n)$ by*

$$G_{hk} = \frac{\partial^2 V}{\partial q_{r+h}\,\partial q_{r+k}} + \sigma_k^{jh} \frac{\partial V}{\partial q_j},$$
$$H_{nk} = \frac{\partial D_{r+h}}{\partial v_{r+k}}. \tag{4.3}$$

Proof. In deriving Eqs. (2.24) we used for $E(v_q)$ the expression (2.16). We could as well use the equivalent expression (2.19); therefore we may also regard the v_i as the components of $P\tilde{F} + \tilde{Q}$; moreover $\tilde{Q}$ contributes to v_i with a term that depends quadratically on v. Since also γ_i depends quadratically on v, it follows that, in computing the linear part of the right-hand side of the third of the Eqs. (2.24) the only term we need to consider is the contribution of $P\tilde{F}$ to v_i. The result follows in a standard way. ∎

LEMMA 4.2. *Suppose V is a Morse function and D is strongly dissipative. Then if $\bar{q}$ is a point of minimum of V the matrix L corresponding to $O_{\bar{q}}$ has n eigenvalues with negative real part and n eigenvalues with positive real part.*

Proof. The eigenvalues of L are 0 with multiplicity r and the $2n$ eigenvalues of the $2n \times 2n$ matrix

$$L' = \left[\begin{array}{c|c} O & I \\ \hline G & H \end{array}\right].$$

When $\bar{q}$ is a point of minimum, G reduces to the positive definite symmetric matrix

$$G_{hk} = \frac{\partial^2 V}{\partial q_{r+h}\,\partial q_{r+k}}.$$

Moreover, it is easy to check that strongly dissipativeness implies $v''^{\mathrm{T}} H v'' < 0$, $\forall v'' \in \mathbb{R}^n \setminus \{0\}$, that is, the symmetric part of H is a negative definite matrix.

Since

$$\det\left[\begin{array}{c|c} -\lambda I & I \\ \hline G & H - \lambda I \end{array}\right] = \det\left[\begin{array}{c|c} 0 & \\ \hline G + \lambda(H - \lambda I) & H - \lambda I \end{array}\right],$$

for an eigenvalue λ of L' we have

$$\det(L' - \lambda I) = (-1)^n \det(-\lambda^2 I + \lambda H + G) = 0.$$

Thus λ is an eigenvalue of L' if and only if there is a nonzero $w \in \mathbb{C}^n$ such that

$$\lambda^2 w - \lambda H w - G w = 0. \tag{4.4}$$

If w^* is the conjugate transpose of w, then $w^* G w$ is real and positive because G is a symmetric positive matrix. On the other hand since the symmetric part of H is negative definite the real part of $w^* H w$ is negative. Therefore (4.4) implies the existence of $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha, \gamma > 0$ such that

$$\lambda^2 + (\alpha + i\beta)\,\lambda - \gamma = 0. \tag{4.5}$$

and $\alpha, \gamma > 0$ implies that (4.5) has no roots in the imaginary axis; the same is true for εH instead of H, $0 \leqslant \varepsilon \leqslant 1$. The result follows just computing the eigenvalues of L' when $H = 0$. ∎

Proof of Theorem 4.2. Since $\mathscr{A}$ is compact there is a point $v_{\bar{q}} \in \mathscr{A}$ such that the energy function $H = K - V$ satisfies $H(v_q) \leqslant H(v_{\bar{q}})$, $v_q \in \mathscr{A}$. Clearly $v_{\bar{q}} \in \mathcal{O}$ because otherwise from (3.5) and strictly dissipativeness would follow $\dot{H}(v_{\bar{q}}) < 0$ and therefore the orbit $\gamma(v_{\bar{q}}) \subset \mathscr{A}$ though $v_{\bar{q}}$ would contain a point v_q such that $H(v_q) > H(v_{\bar{q}})$ in contradiction with the maximality of $H(v_{\bar{q}})$. Thus $v_{\bar{q}} = O_{\bar{q}}$ and $H(O_{\bar{q}}) = -V(\bar{q})$. It follows that V restricted to $\tau(\mathscr{A})$ attains its minimum at $\bar{q}$. Since for each critical point p of V the point O_p is a critical point of GMA and therefore belongs to $\mathscr{A}$, $V(\bar{q})$ is the absolute minimum of V (this can also be derived from the fact that Theorem 4.1 imply $\tau(\mathscr{A}) = M$). Therefore $O_{\bar{q}}$ is a critical point of GMA and the hypothesis and Lemma 4.2 imply

$$\dim W^s(O_{\bar{q}}) = n. \tag{4.6}$$

The maximality of $H(O_{\bar{q}})$ and (3.5) yield $W^s(O_{\bar{q}}) \cap \mathscr{A} = \{O_{\bar{q}}\}$. This, the hypothesis on $\mathscr{A}$ and (4.6) imply

$$\dim \mathscr{A} \leqslant \dim \Sigma M - \dim W^s(O_{\bar{q}}) \leqslant m. \tag{4.7}$$

On the other hand by Theorem 4.1 we have $\tau(\mathscr{A}) = M$ and therefore

$$\dim \mathscr{A} \geqslant m. \tag{4.8}$$

In fact $\tau|_{\mathscr{A}}$ is a C^1 mapping onto M, thus by Sard's theorem there is a regular value q of τ. It follows that if $v_q \in \tau^{-1}(q) \cap \mathscr{A}$ the linear map $T\tau|_{\mathscr{A}}(v_q)\colon T_{v_q}\mathscr{A} \to T_q M$ has maximal rank. This implies

$$\dim T_{v_q}\mathscr{A} \geqslant \dim T_q M. \tag{4.9}$$

and therefore (4.8). Equations (4.7), (4.8) imply (4.1). ∎

Remark. In order that Theorem 4.2 holds it sufficies to require that V is nondegenerate at $\bar{q}$ and that D is strongly dissipative at $O_{\bar{q}}$.

In the remaining part of the section we shall discuss some aspects of the dependence of attractor on the potential and on the dissipative field of force. First, let us note that $V=0$ implies that $\mathscr{A}$ is coincident with $\mathcal{O}$ and thus diffeomorphic to M. In fact, every point of M is then a point of equilibrium; vice versa, since the orbit $\gamma(v_q)$ though a point $v_q \in \mathscr{A}$ has by Theorem 3.2 its α and ω limit sets in $\mathcal{O}$ and the energy H is constant on $\mathcal{O}$ because $V=0$, Eq. (3.5) implies $\dot{H}=0$ and therefore $v_q \in \mathcal{O}$ by strict dissipativeness. One main result is the following perturbation theorem that roughly speaking says $\mathscr{A}$ is a smooth object close to $\mathcal{O}$ for V close to 0.

THEOREM 4.3. *Given a strongly dissipative field of force $D \in \mathscr{J}^k$ there is a neighborhood $\mathscr{N}$ of $o \in C^{k+1}(M)$ such that, if $\mathscr{A}^V$ is the attractor corresponding to $V \in \mathscr{N}$ and the given D, then*

(i) *$\mathscr{A}^V$ is a C^k differential manifold and $\tau|_{\mathscr{A}^V}$ is a C^k diffeomorphism of $\mathscr{A}^V$ onto M.*

(ii) *$\mathscr{A}^V$ depends C^k continuously on $V \in \mathscr{N}$ and $\mathscr{A}^0 = \mathcal{O}$.*

From Theorem 4.3 we can derive the following result that applies to situations where V is not necessarely small.

THEOREM 4.4. *Given $\bar{V} \in C^{k+1}(M)$ and a strongly dissipative field of force $\bar{D} \in \mathscr{J}^k$ that satisfies $\bar{D}(\lambda v_q) = \lambda \bar{D}(v_q)$, $\lambda > 0$, $v_q \in \Sigma M$, let $\mathscr{A}^{\alpha,\beta}$ be the attractor corresponding to $V = \alpha\bar{V}$, $D = \beta\bar{D}$, $\alpha \geqslant 0$, $\beta > 0$. Then there is $\varepsilon > 0$ such that for $\alpha/\beta^2 < \varepsilon$*

(i) *$\mathscr{A}^{\alpha,\beta}$ is a C^k differential manifold and $\tau|_{\mathscr{A}^{\alpha,\beta}}$ is a diffeomorphism of $\mathscr{A}^{\alpha,\beta}$ onto M.*

(ii) *$\mathscr{A}^{\alpha,\beta}$ depends C^k continuously on α, β and $\mathscr{A}^{0,\beta} = \mathcal{O}$.*

Theorem 4.4. generalizes to a large class of dissipative mechanical systems the situation occuring for the pendulum in Example 3.1. In fact for $c > 2\sqrt{g/l}$ the attractor $\mathscr{A}^c$ is a circle that approaches the zero section for $c \to \infty$.

To see that Theorem 4.4 follows from Theorem 4.3 let $v_{\bar{V}}$, $v_{\bar{D}}$, v_Q be the contributions to v in Eq. (2.24) due to the conservative field of force corresponding to V, to the dissipative field of force $\bar{D}$ and of the field force Q in Theorem 2.1, respectively. Then Eq. (2.24) corresponding to $V = \alpha\bar{V}$, $D = \beta\bar{D}$ are

$$\begin{aligned} \dot{q}^{\perp} &= Av'', \\ \dot{q}'' &= v'', \\ \dot{v}'' &= \hat{\gamma}''(q, v'') + \alpha v''_{\bar{V}}(q) + \beta v''_{\bar{D}}(q, v''), \end{aligned} \tag{4.10}$$

where we have set $\hat{\gamma} = \gamma + v_Q$. Since $\hat{\gamma}$ is quadratic in v'' and $v_{\bar{D}}$ is homogeneous of degree 1 with respect to v'' because $\bar{D}$ is, the transformation $v'' = \beta w''_q$, $t = s/\beta$ applied to (4.10) yields

$$\frac{dq^{\perp}}{ds} = Aw'',$$
$$\frac{dq''}{ds} = w'', \tag{4.11}$$
$$\frac{dw''}{ds} = \hat{\gamma}(q, w'') + \frac{\alpha}{B^2} v''_{\bar{V}}(q) + v''_{\bar{D}}(q, w'').$$

These equations may be interpreted as Eqs. (2.24) for $V = (\alpha/\beta^2)\,\bar{V}$, $D = \bar{D}$. It follows that the diffeomorphism $\sigma: \Sigma M \to \Sigma M$ defined by $\sigma(v_q) = \beta v_q$, transform orbits of the GMA vector field corresponding to $V = \alpha/\beta^2 \bar{V}$, $D = \bar{D}$ into orbits of the GMA vector field corresponding to $V = \alpha\bar{V}$, $D = \beta\bar{D}$. This implies

$$\mathscr{A}^{\alpha,\beta} = \sigma(\mathscr{A}^{(\alpha/\beta^2)\bar{V}})$$

and therefore our claim.

The proof of Theorem 4.3 may be based on general theorems on normally hyperbolic sets such as Theorem 4.1 in [1] (see also [17, 23]).

5. The GMA Vector Field Restricted to the Attractor

Since all orbits of GMA approach the attractor as $t \to \infty$, once $\mathscr{A}$ is known an important step towards understanding the flow of GMA is the description of the flow on the attractor. When, as in the situations described in Theorems 4.3, 4.4, $\mathscr{A}$ is diffeomorphic to M, to study the flow on $\mathscr{A}$ is the same as to study a first-order equation on M. To see how this equation is related to the potential and dissipative field of force we consider a potential of the type εV with $V \in C^2(M)$, $\varepsilon \geqslant 0$ a strongly dissipative field of force $D \in \mathscr{J}^1$, then, for ε sufficiently small, Theorem 4.3 implies the attractor $\mathscr{A}^{\varepsilon}$ is a C^1 differential manifold diffeomorphic to $\mathscr{O}$ and approaches $\mathscr{O}$ in the C^1 sense as $\varepsilon \to 0$. This implies that the restriction $\tau|_{\mathscr{A}^{\varepsilon}}$ of the natural projection to $\mathscr{A}^{\varepsilon}$ is then a diffeomorphism of $\mathscr{A}^{\varepsilon}$ onto M if ε is sufficiently small. It follows that given a point $q \in M$ there is a unique point

$$(\tau|_{\mathscr{A}^{\varepsilon}})^{-1}(q)$$

in $\Sigma_q \cap \mathscr{A}^\varepsilon$. Therefore $t \to \dot{q}_\varepsilon(t)$ is an orbit of GMA in $\mathscr{A}^\varepsilon$ if and only if

$$\dot{q}_\varepsilon(t) = (\tau|_{\mathscr{A}^\varepsilon})^{-1}\,(q_\varepsilon(t)),$$

that is, if and only if the corresponding motion $t \to q_\varepsilon(t)$ is a solution of the first-order equation

$$\dot{q}_\varepsilon = X^\varepsilon(q) =: (\tau|_{\mathscr{A}^\varepsilon})^{-1}\,(q). \tag{5.1}$$

The vector field X^ε depends on $\mathscr{A}^\varepsilon$ and cannot be computed explicitly unless one knows $\mathscr{A}^\varepsilon$ which is not the case in general. Therefore one may try to get informations on the orbits of (5.1) by studying the behavior of $X^\varepsilon(q)$ for $\varepsilon \to 0$. Since $\mathscr{A}^\varepsilon$ approaches $\mathcal{O}$ as $\varepsilon \to 0$ we have

$$\lim_{\varepsilon \to 0} X^\varepsilon(q) = 0$$

thus we consider the vector field $Y^\varepsilon =: \varepsilon^{-1} X^\varepsilon$ which has the same orbits as (5.1) and study the limit of Y^ε for $\varepsilon \to 0$. The importance of this study stays in the fact that, if this limit exists and if the limit Y^0 is a structurally stable vector field, then provid ε sufficiently small, the flow of (5.1) and the flow of $\dot{q} = Y^0(q)$ are topologically equivalent, that is there is a homeomorphism $h^\varepsilon: M \to M$ that approaches the identity for $\varepsilon \to 0$ and takes orbits of (5.1) into orbits of $\dot{q} = Y^0(q)$ preserving orientation.

If (q, v) are local natural coordinates on TM, then the function $\sigma^\varepsilon: \mathcal{O} \to \Sigma M$ describing the attractor $\mathscr{A}^\varepsilon$ has a local representation of the type $q \to (\bar{q}(\varepsilon, q),\ \bar{v}(\varepsilon, q))$ and $\bar{q}(\varepsilon, \cdot)$, $\bar{v}(\varepsilon, \cdot)$ are C^1 functions such that $\bar{q}(\varepsilon, \cdot) \to id$; $\bar{v}(\varepsilon, \cdot) \to 0$ with respect to the C^1 topology as $\varepsilon \to 0$. Moreover $\bar{q}(\varepsilon, \cdot)$ has a C^1 inverse because $\tau|_{\mathscr{A}^\varepsilon}$ is a C^1 diffeomorphism and the same is true for σ^ε.

THEOREM 5.1. *If $\mathscr{A}^\varepsilon$ is a smooth function of ε in the sense that, $\bar{q}, \bar{v}$ and their first derivatives with respect to q are continuously differentiable with respect to ε, then as $\varepsilon \to 0$ Y^ε converges in C^1 to the C^1 vector field Y^0 defined by*

$$Y^0 = -(PF\tilde{D})^{-1}\, P \operatorname{grad} V, \tag{5.2}$$

where P, $\tilde{D}$ are defined as in part 2 and $F\tilde{D}$ is the fiber derivative [8] of $\tilde{D}$.

Proof. From (5.1) it follows that X^ε has the local expression

$$q \to (q, \bar{v}(\varepsilon, \bar{q}^{-1}(\varepsilon, q))), \tag{5.3}$$

therefore the local expression of Y^ε is

$$q \to (q, \bar{\varepsilon}^1 \bar{v}(\varepsilon, \bar{q}^{-1}(\varepsilon, q))). \tag{5.4}$$

The hypothesis on $\bar{q}$, $\bar{v}$ imply that the function

$$w(\varepsilon, q) =: \bar{v}(\varepsilon, \bar{q}^1(\varepsilon, q))$$

is continuously differentiable and moreover $w(0, q) = 0$ because $\bar{v}(0, q) = 0$ and $\bar{q}^{-1}(0, q) = q$. Therefore we have

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} w(\varepsilon, q) = \frac{\partial w}{\partial \varepsilon}(0, q) = \frac{\partial \bar{v}}{\partial \varepsilon}(\varepsilon, \bar{q}^{-1}(\varepsilon, q)$$

$$+ \frac{\partial \bar{v}}{\partial q}(\varepsilon, \bar{q}^{-1}(\varepsilon, q)) \frac{\partial \bar{q}^{-1}}{\partial \varepsilon}(\varepsilon, q)) \bigg|_{\varepsilon = 0} = \frac{\partial \bar{v}}{\partial \varepsilon}(0, q)$$

that shows the existence of Y^0. To show that Y^0 has the expression (5.2) we observe that any solution $t \to q_\varepsilon(t)$ of (5.1) is a motion of the mechanical system under examination and thus a solution of Eq. (2.5) this equation for a dissipative system with potential εV may be written as

$$\nabla_{\dot{q}} \dot{q} = \varepsilon P \operatorname{grad} V(q) + P\tilde{D}(\dot{q}) + \tilde{Q}(\dot{q}). \tag{5.5}$$

From Eq. (5.1) we have $\dot{q} = \varepsilon Y^\varepsilon(q)$ and therefore (5.5) implies

$$\varepsilon^2 \nabla_{Y^\varepsilon(q)} Y^\varepsilon = \varepsilon P \operatorname{grad} V(q) + P\tilde{D}(\varepsilon Y^\varepsilon(q)) + \varepsilon^2 \tilde{Q}(Y^\varepsilon(q)). \tag{5.6}$$

Dividing this equation by ε and taking the limit for $\varepsilon \to 0$ yields

$$0 = P \operatorname{grad} V + PF\tilde{D}(Y^0), \tag{5.7}$$

from which (5.2) follows because strongly dissipativeness implies that the vector bundle map $PF\tilde{D}\colon \Sigma M \to \Sigma M$ is a diffeomorphism. ∎

Remarks. As Theorem 4.3 implies Theorem 4.4, from Theorem 5.1 one may obtains corresponding conclusions for the case when the potential and dissipative field of force are of the type αV, βD.

A simple example of a situation where Theorem 5.1 applies is the pendulum in Example 3.1. In this case we have

$$Y^0(v) = -\frac{g}{lc} \sin v \tag{5.8}$$

which is a structurally stable vector field on S^1.

Finally we remark that some results on dissipative systems were obtained by Shahshahani in [20] for the holonomic case (without constraints). He proved that for a given Morse function V there is an open and dense set of strictly dissipative fields of force for which the corresponding holonomic mechanical system is Ω-stable (in fact, Ω consists of a finite number of hyperbolic zeros), TM is the union of stable manifolds and at every zero the dimension of the stable manifold is at least as large as the dimension of the unstable manifold.

After the present paper was finished I. Kupka and W. M. Oliva proved that for a given Morse function V there is an open and dense set of strictly fields of force for which the corresponding holonomic mechanical system is Morse–Smale, then structurally stable (see [16, 17], for the stability of Morse–Smale maps).

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