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Self-adjoint extensions and spectral analysis in the generalized Kratzer problem

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Abstract

We present a mathematically rigorous quantum-mechanical treatment of a one-dimensional non-relativistic motion of a particle in the potential field $V(x) = g_1x^{-1} + g_2x^{-2}$, $x \in \mathbb{R}_+ = [0, \infty)$. For $g_2 > 0$ and $g_1 < 0$, the potential is known as the *Kratzer potential* $V_K(x)$ and is usually used to describe molecular energy and structure, interactions between different molecules and interactions between non-bonded atoms. We construct all self-adjoint Schrödinger operators with the potential $V(x)$ and represent rigorous solutions of the corresponding spectral problems. Solving the first part of the problem, we use a method of specifying self-adjoint extensions by (asymptotic) self-adjoint boundary conditions. Solving spectral problems, we follow Krein's method of guiding functionals. This work is a continuation of our previous works devoted to the Coulomb, Calogero and Aharonov–Bohm potentials.

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1. Introduction

In this paper, we present a mathematically rigorous quantum-mechanical (QM) treatment of a one-dimensional (1D) non-relativistic motion on a semiaxis of a spinless particle of mass m in the potential field

$$V(x) = g_1x^{-1} + g_2x^{-2}, \quad x \in \mathbb{R}_+ = [0, \infty). \quad (1)$$

At the physical level of rigor, the Schrödinger equation with potential (1) was studied for a long time in connection with different physical problems; see, for example, [1, 2] and the books [3–5]. In particular, this potential enters the stationary radial Schrödinger equation

$$\left[\frac{d^2}{dr^2} + \frac{2m}{\hbar^2} \left(E_{nl} - U(r) - \frac{l(l+1)\hbar^2}{2mr^2} \right) \right] \psi_{nl}(r) = 0, \quad (2)$$

where n and l are radial and angular quantum numbers, after separating spherical variables in 3D spherically symmetric QM problems; see, for example, [4]. The potential (1) is singular at the origin; it is repulsive at this point for $g_2 > 0$, and has a minimum at a point $x_0 > 0$ for $g_2 > 0$ and $g_1 < 0$. The potential with g_1, g_2 in the latter range is known as the *Kratzer potential* [6]. The Kratzer potential is conventionally used to describe the molecular energy spectrum, interactions between different molecules [7] and interactions between nonbonded atoms [8]. For $g_2 < 0$ and $g_1 > 0$, we have the *inverse Kratzer potential*, which is conventionally used to describe tunnel effects, scattering of charged particles [9] and decays, in particular, molecule ionization and fluorescence [10]. In addition, valence electrons in a hydrogen-like atom are described in terms of such a potential [11]. When modeling some physical systems, a constant is usually added to the angular momentum term, $l(l+1) \rightarrow \beta + l(l+1)$, in order to take some effective potential energy into account. For example, in the model of a

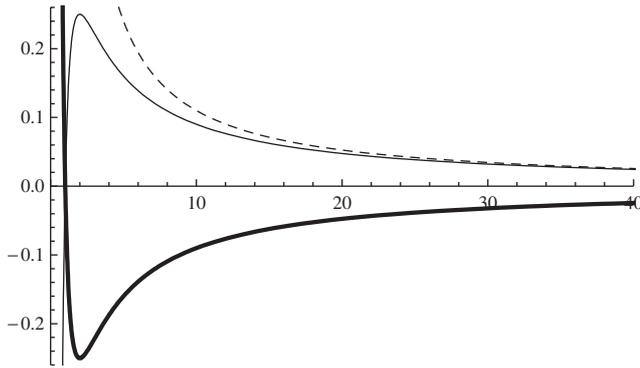


Figure 1. Potential $V(x) = g_1x^{-1} + g_2x^{-2}$, with $g_1 = g_2 = 1$ (dashed), $g_1 = -g_2 = 1$ (solid) and $g_1 = -g_2 = -1$ (thick).

molecule interaction, β can represent the dissociation energy of a diatomic molecule [7]; in the scattering problem, this parameter represents attractive ($\beta < 0$) or repulsive ($\beta > 0$) interactions between charged particles [9].

In figure 1, we show the shape of the potential under consideration for different values of the parameters.

Even though a number of works are devoted to the QM problem with the potential (1), a rigorous mathematical analysis of this problem is lacking in the literature. The aim of such an analysis (which is, in fact, the aim of the present paper) is to construct all self-adjoint (referred to as s.a. in what follows) Schrödinger operators (Hamiltonians) with the potential (1) and present rigorous solutions of the corresponding spectral problems.

When solving the first part of the problem, we use a method to specify s.a. differential operators by (asymptotic) s.a. boundary conditions (the so-called alternative method; see [12]). When solving spectral problems, we follow Krein’s method of guiding functionals; see [13] and the books [14]. This work is a continuation of our previous works [15, 16] devoted to the Coulomb, Calogero and Aharonov–Bohm potentials; using the given references, the reader can become acquainted with the necessary basic notions and constructions, such as the guiding functional and Green’s function.

As in the above-mentioned works, we start with an s.a. differential operation \check{H} on \mathbb{R}_+ ,

$$\check{H} = -d_x^2 + g_1x^{-1} + g_2x^{-2}, \quad (3)$$

examining solutions of the corresponding homogeneous differential equation $(\check{H} - W)\psi = 0$, or

$$\begin{aligned} \psi'' - (g_1x^{-1} + g_2x^{-2} - W)\psi &= 0, \quad W = |W|e^{i\varphi}, \\ 0 \leq \varphi < 2\pi, \end{aligned} \quad (4)$$

which is the Schrödinger equation (with omitted factor $2m/\hbar^2$) with a complex energy W ; for $\text{Im } W = 0$, we write $W = E$ in what follows.

The basic operator \hat{H}^+ in $L^2(\mathbb{R}_+)$ associated with \check{H} is defined on the natural domain³ $D_{\hat{H}}^*(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$,

$$D_{\hat{H}}^*(\mathbb{R}_+) = \{\psi_*(x) : \psi_*, \psi_*' \text{ are a.c. in } \mathbb{R}_+; \psi_*, \check{H}\psi_* \in L^2(\mathbb{R}_+)\}, \quad (5)$$

and it is the adjoint of the so-called initial symmetric operator \hat{H} associated with \check{H} and defined on the dense domain $D_H = \mathcal{D}(\mathbb{R}_+)$, the space of smooth functions with a compact support,

$$\mathcal{D}(\mathbb{R}_+) = \{\psi(x) : \psi \in C^\infty(\mathbb{R}_+), \text{supp } \psi \subseteq [\alpha, \beta] \subset (0, \infty)\}, \quad (6)$$

where \hat{H} acts as \check{H} . It is evident that $\mathcal{D}(\mathbb{R}_+) \subset D_{\hat{H}}^*(\mathbb{R}_+)$ and $\hat{H} \subset \hat{H}^+$. The operator \hat{H}^+ is generally not s.a.. Its quadratic asymmetry form is denoted by Δ_{H^+} . All possible s.a. Hamiltonians associated with \check{H} are defined as s.a. restrictions of \hat{H}^+ (which are simultaneously s.a. extensions of the symmetric \hat{H}) to some subspaces (domains) belonging to $D_{\hat{H}}^*(\mathbb{R}_+)$ and specified by some additional (asymptotic) s.a. boundary conditions on functions belonging to $D_{\hat{H}}^*(\mathbb{R}_+)$ under which the asymmetry form Δ_{H^+} becomes trivial (vanishes). These domains are maximum subspaces in $D_{\hat{H}}^*(\mathbb{R}_+)$ where the operator \hat{H}^+ is symmetric⁴ (see [12]). Our first aim is to describe all these Hamiltonians. The special case of $g_1 = 0$ corresponds to the Calogero potential and was already considered in [15]; we therefore keep $g_1 \neq 0$ in what follows.

This paper is organized as follows. In section 2, we present and discuss some exact solutions of equation (4) and their asymptotics. In the following five sections, we construct all s.a. extensions of \hat{H} and perform the corresponding spectral analysis of the Hamiltonians for different ranges of the parameter g_2 . In sections (3.1)–(3.4), we consider the case of $g_2 \neq 0$. The special case of $g_2 = 0$ is considered in section 3.5. In section 4, we make some remarks and highlight possible applications of the obtained results.

2. Exact solutions and asymptotics

We first consider the Schrödinger equation (4). Introducing a new variable z and new functions $\phi_\pm(z)$ instead of the respective x and $\psi(x)$,

$$\begin{aligned} z &= \lambda x, \quad \lambda = 2\sqrt{-W} = 2\sqrt{|W|}e^{i(\varphi-\pi)/2}, \\ \psi(x) &= x^{1/2\pm\mu} e^{-z/2} \phi_\pm(z), \\ \mu &= \begin{cases} \sqrt{g_2 + 1/4}, & g_2 \geq -1/4, \\ i\kappa, & \kappa = \sqrt{|g_2| - 1/4}, \quad g_2 < -1/4, \end{cases} \end{aligned} \quad (7)$$

³ a.c. means absolutely continuous.

⁴ Although the notions ‘s.a. extension of \hat{H} ’ and ‘s.a. restriction of \hat{H}^+ ’ are equivalent, it is more customary to speak of s.a. extensions; we use one or the other of the equivalent notions where appropriate.

we reduce equation (4) to the confluent hypergeometric equations for $\phi_{\pm}(z)$,

$$z d_z^2 \phi_{\pm}(z) + (\beta_{\pm} - z) d_z \phi_{\pm}(z) - \alpha_{\pm} \phi_{\pm}(z) = 0, \tag{8}$$

$$\alpha_{\pm} = 1/2 \pm \mu + g_1/\lambda, \quad \beta_{\pm} = 1 \pm 2\mu,$$

whose solutions are the known confluent hypergeometric functions $\Phi(\alpha_{\pm}, \beta_{\pm}; z)$ and $\Psi(\alpha_{\pm}, \beta_{\pm}; z)$; see [17, 18].

Solutions $\psi(x)$ of equation (4) are restored from solutions of equations (8) by the transformation (7). In what follows, we use $u_1(x; W)$, $u_2(x; W)$ and $v_1(x; W)$ defined by

$$u_1(x; W) = x^{1/2+\mu} e^{-z/2} \Phi(\alpha_+, \beta_+; z) = u_1(x; W)|_{\lambda \rightarrow -\lambda},$$

$$u_2(x; W) = x^{1/2-\mu} e^{-z/2} \Phi(\alpha_-, \beta_-; z) = u_2(x; W)|_{\lambda \rightarrow -\lambda}$$

$$= u_1(x; W)|_{\mu \rightarrow -\mu}, \tag{9}$$

$$v_1(x; W) = \lambda^{2\mu} x^{1/2+\mu} e^{-z/2} \Psi(\alpha_+, \beta_+; z)$$

$$= \lambda^{2\mu} \frac{\Gamma(-2\mu)}{\Gamma(\alpha_-)} u_1 + \frac{\Gamma(2\mu)}{\Gamma(\alpha_+)} u_2.$$

The function u_2 is not defined for $\beta_- = -n$, or $\mu = (n+1)/2, n \in \mathbb{Z}_+$, in particular, for $\mu = 1/2$. For such μ , we replace u_2 by other solutions of equation (4); they are considered in the subsequent sections.

The coefficients of the Taylor expansion of the functions $u_1(x; W)/x^{1/2+\mu}$ and $u_2(x; W)/x^{1/2-\mu}$ with respect to x are polynomials in λ . Because these functions are even in λ , the coefficients are polynomials in W , whence it follows that $u_1(x; W)$ and $u_2(x; W)$ are entire functions in W at any point x except $x = 0$ for u_2 with $\mu > 1/2$.

If $g_2 \geq -1/4$ ($\mu \geq 0$), then $u_1(x; W)$ and $u_2(x; W)$ are real-entire functions of W . If $g_2 < -1/4$ ($\mu = i\kappa$), then $u_2(x; E) = u_1(x; E)$.

The pairs u_1, u_2 with $\mu \neq 0$ and u_1, v_1 for $\text{Im } W \neq 0$ are the fundamental systems of solutions of equation (4), because the respective Wronskians are

$$\text{Wr}(u_1, u_2) = -2\mu, \tag{10}$$

$$\text{Wr}(u_1, v_1) = -\Gamma(\beta_+)/\Gamma(\alpha_+) \equiv -\omega(W).$$

The well-known asymptotics of the special functions Φ and Ψ , see e.g. [17], entering solutions (9) allows us to simply estimate the asymptotic behavior of the solutions at the origin, as $x \rightarrow 0$, and at infinity, as $x \rightarrow \infty$.

As $x \rightarrow 0$, we have

$$u_1(x; W) = \kappa_0^{-1/2-\mu} u_{1as}(x) + O(x^{3/2+\mu}),$$

$$u_2(x; W) = \kappa_0^{-1/2+\mu} u_{2as}(x)$$

$$+ \begin{cases} O(x^{5/2-\mu}), & -1/4 < g_2 < 3/4, \quad g_2 \neq 0, \\ (0 < \mu < 1, \mu \neq 1/2), \\ O(x^{3/2}), & g_2 < -1/4 \quad (\mu = i\kappa), \end{cases} \tag{11}$$

and, if $\alpha_+ \neq -n, \alpha_- \neq -m, n, m \in \mathbb{Z}_+$,

$$v_1(x; W) = \begin{cases} \frac{\Gamma(2\mu)}{\Gamma(\alpha_+)} x^{1/2-\mu} \tilde{O}(x), \\ g_2 \geq 3/4 \quad (\mu \geq 1) \\ \frac{\lambda^{2\mu} \Gamma(-2\mu)}{\Gamma(\alpha_-)} \kappa_0^{-1/2-\mu} u_{1as}(x) \\ + \frac{\Gamma(2\mu)}{\Gamma(\alpha_+)} \kappa_0^{-1/2+\mu} u_{2as}(x) \\ + O(x^{3/2}), \quad -1/4 < g_2 < 3/4, \\ g_2 \neq 0 \quad (0 < \mu < 1, \mu \neq 1/2), \\ \frac{\lambda^{2i\kappa} \Gamma(-2i\kappa)}{\Gamma(\alpha_-)} \kappa_0^{-1/2-i\kappa} u_{1as}(x) \\ + \frac{\Gamma(2i\kappa)}{\Gamma(\alpha_+)} \kappa_0^{-1/2+i\kappa} u_{2as}(x) \\ + O(x^{3/2}), \quad g_2 < -1/4 \quad (\mu = i\kappa), \end{cases} \tag{12}$$

where $\tilde{O}(x) = 1 + O(x)$,

$$u_{1as}(x) = (\kappa_0 x)^{1/2+\mu},$$

$$u_{2as}(x) = \begin{cases} (\kappa_0 x)^{1/2-\mu} - \frac{g_1/\kappa_0}{2\mu-1} (\kappa_0 x)^{3/2-\mu}, \\ -1/4 < g_2 < 3/4, \quad g_2 \neq 0 \\ (0 < \mu < 1, \mu \neq 1/2), \\ (\kappa_0 x)^{1/2-i\kappa}, \quad g_2 < -1/4 \quad (\mu = i\kappa), \end{cases} \tag{13a}$$

and κ_0 is an arbitrary, but fixed, parameter of dimension of inverse length.

As $x \rightarrow \infty, \text{Im } W > 0$, we have

$$u_1(x; W) = \frac{\Gamma(\beta_+)}{\Gamma(\alpha_+)} \lambda^{\alpha_+ - \beta_+} x^{g_1/\lambda} e^{z/2} \tilde{O}(x^{-1})$$

$$= O\left(x^a e^{|W|^{1/2} \sin(\varphi/2)}\right),$$

$$v_1(x; W) = \lambda^{-\alpha_-} x^{-g_1/\lambda} e^{-z/2} \tilde{O}(x^{-1})$$

$$= O\left(x^{-a} e^{-|W|^{1/2} \sin(\varphi/2)}\right),$$

$$a = 2^{-1} |W|^{-1/2} g_1 \sin(\varphi/2).$$

The obtained asymptotics are sufficient to allow definite conclusions on the deficiency indices of the initial symmetric operator \hat{H} as functions of the parameters g_1, g_2 and thereby on a possible variety of its s.a. extensions. It is evident that for $\text{Im } W > 0$, the function $u_1(x; W)$ exponentially increases at infinity and is not square-integrable. The function $v_1(x; W)$ exponentially decreasing at infinity is not square-integrable at the origin for $g_2 \geq 3/4$ ($\mu \geq 1$), whereas for $g_2 < 3/4$, it is (moreover, for $g_2 < 3/4$, any solution of equation (4) is square-integrable at the origin). Because for $\text{Im } W > 0$, the functions u_1, v_1 form a fundamental system of equation (4), this equation with $\text{Im } W > 0$ has no square-integrable solutions for $g_2 \geq 3/4$, whereas for $g_2 < 3/4$, there exists one square-integrable solution, $v_1(x; W)$. This means that the deficiency indices of the initial symmetric operator \hat{H} are

equal to zero, $m_{\pm} = 0$, for $g_2 \geq 3/4$ and are equal to unity, $m_{\pm} = 1$, for $g_2 < 3/4$.

Correspondingly for $g_2 \geq 3/4$, there is a unique s.a. extension of \hat{H} , whereas for $g_2 < 3/4$, there exists a one-parameter family of s.a. extensions of \hat{H} . The structure of these extensions, in particular, the appearance of their specifying asymptotic boundary conditions, depends crucially on a specific range of values of the parameter g_2 . In what follows, we identify five such regions and consider them separately.

3. Self-adjoint extensions and spectral analysis

3.1. The first range $g_2 \geq 3/4$ ($\mu \geq 1$)

As was mentioned above, the deficiency indices of the initial symmetric operator \hat{H} with g_2 in this range are zero. This implies that for $g_2 \geq 3/4$, the operator \hat{H}^+ is s.a. and $\hat{H}_1 = \hat{H}^+$ is a unique s.a. extension of \hat{H} with the domain $D_{H_1} = D_{\hat{H}}^*(\mathbb{R}_+)$ (5).

A spectral analysis of the s.a. operator $\hat{H}_1 = \hat{H}^+$ begins with an evaluation of its Green's function $G(x, y; W)$ that is the kernel of the integral representation of the solution $\psi_*(x)$ of the inhomogeneous differential equation

$$(\check{H} - W)\psi_*(x) = \eta(x), \quad \eta(x) \in L^2(\mathbb{R}_+), \quad (14)$$

with $\text{Im } W \neq 0$ under the condition that $\psi_* \in D_{\hat{H}}^*(\mathbb{R}_+)$, i.e. under the condition that ψ_* is square-integrable⁵, $\psi_*(x) \in L^2(\mathbb{R}_+)$ (see [15, 16]). The general solution of this equation without the condition of square integrability can be represented as

$$\begin{aligned} \psi_*(x) &= a_1 u_1(x; W) + a_2 v_1(x; W) + I(x; W), \\ \psi'_*(x) &= a_1 u'_1(x; W) + a_2 v'_1(x; W) + I'(x; W), \end{aligned} \quad (15)$$

where

$$\begin{aligned} I(x; W) &= \int_0^x G^{(+)}(x, y; W) \eta(y) dy \\ &\quad + \int_x^\infty G^{(-)}(x, y; W) \eta(y) dy, \\ I'(x; W) &= \int_0^x d_x G^{(+)}(x, y; W) \eta(y) dy \\ &\quad + \int_x^\infty d_x G^{(-)}(x, y; W) \eta(y) dy, \end{aligned}$$

$$G^{(+)}(x, y; W) = \omega^{-1}(W) v_1(x; W) u_1(y; W),$$

$$G^{(-)}(x, y; W) = \omega^{-1}(W) u_1(x; W) v_1(y; W),$$

with ω given in (10). Using the Cauchy–Bunyakovskii inequality, it is easy to show that $I(x; W)$ is bounded as $x \rightarrow \infty$. The condition $\psi_*(x) \in L^2(\mathbb{R}_+)$ then implies that $a_1 = 0$, because $u_1(x; W)$ exponentially grows while $v_1(x; W)$ exponentially decreases at infinity. As $x \rightarrow 0$, we have $I(x) \sim O(x^{3/2})$, $I'(x) \sim O(x^{1/2})$ (up to the logarithmic accuracy at $g_2 = 3/4$), whereas $v_1(x; W)$

is not square-integrable at the origin. The condition $\psi_*(x) \in L^2(\mathbb{R}_+)$ then implies that $a_2 = 0$. In addition, we see that the asymptotic behavior of functions $\psi_*(x)$ belonging to $D_{\hat{H}}^*(\mathbb{R}_+)$ at the origin, as $x \rightarrow 0$, is estimated by

$$\psi_*(x) = O(x^{3/2}), \quad \psi'_*(x) = O(x^{1/2}). \quad (16)$$

Together with the fact that the functions ψ_* vanish at infinity (see below), this implies that the asymmetry form Δ_{H^+} is trivial, which confirms that in the first range the operator \hat{H}^+ is symmetric and therefore s.a. (in contrast to the ranges considered in the subsequent sections).

It follows that Green's function is given by

$$G(x, y; W) = \begin{cases} G^{(+)}(x, y; W), & x > y, \\ G^{(-)}(x, y; W), & x < y. \end{cases}$$

The representation (9) of the function v_1 in terms of the functions u_1 and u_2 is inconvenient sometimes, because the individual summands do not exist for some μ although v_1 does. For our purposes, other representations are convenient. For $m - 1 < 2\mu < m + 1$, $m \geq 2$, the function $v_1(x; W)$ can be represented as

$$\begin{aligned} v_1(x; W) &= A_m(W) u_1(x; W) + \frac{\omega(W)}{2\mu} v_{(m)}(x; W), \\ A_m(W) &= \lambda^{2\mu} \frac{\Gamma(-2\mu)}{\Gamma(\alpha_-)} + a_m(W) \frac{\Gamma(2\mu)\Gamma(\beta_-)}{\Gamma(\alpha_+)}, \\ v_{(m)}(x; W) &= u_2(x; W) - a_m(W) \Gamma(\beta_-) u_1(x; W), \\ a_m(W) &= \lambda^m \frac{\Gamma(\alpha_{\pm m})}{m! \Gamma(\alpha_{-m})}, \quad \alpha_{\pm m} = \frac{1 \pm m}{2} + g_1/\lambda. \end{aligned}$$

It is easy to see that all the coefficients $a_m(W)$ are polynomials in W which are real for $\text{Im } W = 0$ ($W = E$). In view of the relation

$$\lim_{\beta \rightarrow -n} \Gamma^{-1}(\beta) \Phi(\alpha, \beta; x) = \frac{x^{n+1} \Gamma(\alpha + n + 1)}{(n + 1)! \Gamma(\alpha)} \times \Phi(\alpha + n + 1, n + 2; x)$$

(see [17, 18]), the functions $v_{(m)}(x; W)$ and $A_m(W)$ exist for $m - 1 < 2\mu < m + 1$ and for any W . In fact, $v_{(m)}(x; W)$ are particular solutions of equation (4), which are real-entire in W and have the properties (for $m - 1 < 2\mu < m + 1$)

$$\text{Wr}(u_1, v_{(m)}) = -2\mu, \quad v_{(m)}(x; W) = x^{1/2-\mu} \tilde{O}(x), \quad x \rightarrow 0.$$

As a guiding functional, we take

$$\Phi(\xi; W) = \int_0^\infty U(x; W) \xi(x) dx, \quad \xi \in \mathbb{D} = D_r(\mathbb{R}_+) \cap D_{H_1}, \quad (17)$$

where $U(x; W) = u_1(x; W)$ and $D_r(\mathbb{R}_+)$ is the space of arbitrary functions with a support bounded from the right: $\varphi(x) \in D_r(\mathbb{R}_+) \implies \text{supp } \varphi \subseteq [0, \beta]$, $\beta < \infty$; the domain \mathbb{D} is dense in $L^2(\mathbb{R}_+)$. The functional $\Phi(\xi; W)$ (17) is a simple guiding functional, i.e. it satisfies the properties (1) for a fixed ξ , the functional $\Phi(\xi; W)$ is an entire function of W ; (2) if $\Phi(\xi_0; E_0) = 0$, $\xi_0 \in \mathbb{D}$, then the inhomogeneous equation $(\check{H} - E_0)\psi = \xi_0$ has a solution $\psi \in \mathbb{D}$; (3) $\Phi(\check{H}\xi; W) = W\Phi(\xi; W)$. It is easy to verify

⁵ We note that $D_{\hat{H}}^*(\mathbb{R}_+)$ can be considered as the space of unique square-integrable solutions of equation (14) with $\text{Im } W \neq 0$ and any $\eta(x) \in L^2(\mathbb{R}_+)$.

properties (1) and (3), and it remains to be verified that the property (2) also holds. Let

$$\begin{aligned} \Phi(\xi_0; E_0) &= \int_0^b u_1(x; E_0)\xi_0(x) dx = 0, \\ \xi_0 \in \mathbb{D}, \quad \text{supp } \xi_0 &\in [0, b]. \end{aligned} \tag{18}$$

We consider the function $\psi(x)$ defined by

$$\begin{aligned} \psi(x) &= \frac{1}{2\mu} \left[u_1(x; E_0) \int_x^b v_{(m)}(y; E_0)\xi_0(y) dy \right. \\ &\quad \left. + v_{(m)}(x; E_0) \int_0^x u_1(y; E_0)\xi_0(y) dy \right], \end{aligned} \tag{19}$$

which evidently satisfies the equation $(\hat{H} - E_0)\psi(x) = \xi_0(x)$. Using the condition (18), we find that $\text{supp } \psi \in [0, b]$, i.e. $\psi \in D_r(\mathbb{R}_+)$, and therefore, $\psi \in L^2(c, b)$ for any $c > 0$. Taking the asymptotic behavior of the functions $u_1(x; E_0)$, $v_{(m)}(x; E_0)$ and $\xi_0(x)$ at the origin into account, a simple evaluation of the integrals in representation (19) gives

$$\psi(x) = \begin{cases} O(x^{1/2+\mu}), & 1 \leq \mu < 3, \\ O(x^{7/2} \ln \delta), & \mu = 3, \\ O(x^{7/2}), & \mu > 3 \end{cases} \quad x \rightarrow 0,$$

i.e. $\psi \in D_{H_e}$, and therefore, $\psi \in \mathbb{D}$.

The derivative of the spectral function is given by

$$\sigma'(E) = \pi^{-1} \text{Im}[\omega^{-1}(E + i0)A_m(E + i0)]. \tag{20}$$

Because $\omega^{-1}(W)A_m(W)$ is an analytic function of μ , its value at $\mu = m/2$ is a limit as $\mu \rightarrow m/2$. For $\mu \neq m/2$, the representation (20) can be simplified to

$$\sigma'(E) = \text{Im } \Omega^{-1}(E + i0), \quad \Omega(W) = \frac{\pi \Gamma(\alpha_-)\Gamma(\beta_+)}{\lambda^{2\mu}\Gamma(-2\mu)\Gamma(\alpha_+)}.$$

For $E = p^2 \geq 0$, $p \geq 0$, $\lambda = 2p e^{-i\pi/2}$, we find that

$$\sigma'(E) = \left(\frac{|\Gamma(\alpha_+)|}{\Gamma(\beta_+)} \right)^2 \frac{(2p)^{2\mu} e^{-\pi g_1/2p}}{2\pi} > 0. \tag{21}$$

We see that $\sigma'(E)$ is a nonsingular function for $E \geq 0$. It follows that the spectrum of the s.a. Hamiltonian \hat{H}_1 is continuous for all such values of E . For $E = -\tau^2 < 0$, $\tau > 0$, $\lambda = 2\tau$, the function $\Omega^{-1}(E)$ is real for all values of E where $\Omega^{-1}(E)$ is finite, which implies that $\text{Im } \Omega^{-1}(E + i0)$ can differ from zero only at the discrete points E_n where $\Omega(E_n) = 0$. It is easy to see that the latter equation is reduced to the equations $\alpha_+(E_n) = -n$, $n \in \mathbb{Z}_+$, which have solutions only if $g_1 < 0$, and the solutions E_n are then given by

$$E_n = -g_1^2(1 + 2\mu + 2n)^{-2}, \quad \tau_n = |g_1| (1 + 2\mu + 2n)^{-1}. \tag{22}$$

We thus find that for $E < 0$, the function $\sigma'(E)$ is equal to zero if $g_1 > 0$, whereas if $g_1 < 0$, this function is given by

$$\sigma'(E) = \sum_{n=0}^{\infty} Q_n^2 \delta(E - E_n), \quad Q_n = \frac{(2\tau_n)^{\mu+1}}{\Gamma(\beta_+)} \sqrt{\frac{\Gamma(1 + 2\mu + n)}{(1 + 2\mu + 2n)n!}}.$$

The final result of this section is as follows.

For $g_2 > 3/4$ ($\mu > 1$), the spectrum of a unique s.a. operator (Hamiltonian) \hat{H}_1 is simple and is given by

$$\text{spec } \hat{H}_1 = \begin{cases} \mathbb{R}_+, & g_1 > 0, \\ \mathbb{R}_+ \cup \{E_n\}, & g_1 < 0. \end{cases}$$

For $g_1 > 0$, the generalized eigenfunctions $U_E(x) = \sqrt{\sigma'(E)}u_1(x; E)$, $E \geq 0$, of \hat{H}_1 form a complete orthonormalized system in $L^2(\mathbb{R}_+)$. For $g_1 < 0$, the generalized eigenfunctions $U_E(x) = \sqrt{\sigma'(E)}u_1(x; E)$, $E \geq 0$, of \hat{H}_1 together with the eigenfunctions $U_n(x) = Q_n u_1(x; E_n)$, $n \in \mathbb{Z}_+$, form a complete orthonormalized system in $L^2(\mathbb{R}_+)$.

3.2. The second range $3/4 > g_2 > -1/4$, $g_2 \neq 0$ ($1 > \mu > 0$, $\mu \neq 1/2$)

We note that in this section, we consider the range $3/4 > g_2 > -1/4$ excluding the point $g_2 = 0$ ($\mu = 1/2$); the reason is that the function u_2 we use here is not defined for $\mu = 1/2$. The case $g_2 = 0$ ($\mu = 1/2$) is considered separately in the last subsection.

The operator \hat{H}^+ with g_2 in the second range is not s.a., and we must construct its s.a. reductions. In accordance with the general procedure of the alternative method (see, e.g., [12] and also [15, 16]), we begin by evaluating the quadratic asymmetry form Δ_{H^+} in terms of quadratic boundary forms, which are determined by the asymptotics of functions $\psi_*(x)$ belonging to the natural domain $D_{\hat{H}}^*(\mathbb{R}_+)$ at the origin (the left boundary form) and at infinity (the right boundary form). Because the potential vanishes at infinity, the right boundary form is trivial (zero)⁶ (see [12]), and the asymmetry form Δ_{H^+} is reduced to (minus) the left boundary form. To determine the asymptotic behavior of functions ψ_* at the origin, we consider these functions as solutions of the inhomogeneous equation (14) with $W = 0$. Because, in the range under consideration, any solution of the homogeneous equation (4) is square-integrable at the origin, the general solution of equation (14) with $W = 0$ can be represented as

$$\begin{aligned} \psi_*(x) &= a_1 u_1(x; 0) + a_2 u_2(x; 0) - \frac{1}{2\mu} \int_0^x [u_1(x; 0)u_2(y; 0) \\ &\quad - u_2(x; 0)u_1(y; 0)]\eta(y) dy. \end{aligned} \tag{23}$$

The asymptotic behavior of the functions u_1 and u_2 in the representation (23) as $x \rightarrow 0$ is given by (11) and (13a); the asymptotic behavior of the integral terms is estimated using the Cauchy–Bunyakovskii inequality, and we find that

$$\begin{aligned} \psi_*(x) &= a_1 u_{1as}(x) + a_2 u_{2as}(x) + O(x^{3/2}), \\ \psi'_*(x) &= a_1 u'_{1as}(x) + a_2 u'_{2as}(x) + O(x^{1/2}). \end{aligned} \tag{24}$$

With these asymptotics, we calculate the left boundary form $[\psi_*, \psi'_*](0) = \lim_{x \rightarrow 0} (-\bar{\psi}_*(x)\psi'_*(x) + \bar{\psi}'_*(x)\psi_*(x))$ and obtain a representation of the quadratic asymmetry form as a quadratic form in the coefficients a_1 and a_2 in (24):

$$\Delta_{H^+}(\psi_*) = -2\mu k_0 (\bar{a}_1 a_2 - \bar{a}_2 a_1).$$

⁶ Moreover, we can prove that ψ_* vanishes at infinity together with its derivative, $\psi_*(x), \psi'_*(x) \xrightarrow{x \rightarrow \infty} 0$.

The coefficients a_1, a_2 are called the (left) asymptotic boundary (a.b.) coefficients⁷. The requirement on the a.b. coefficients that Δ_{H^+} vanish results in the relation⁸

$$a_2 \sin \nu = a_1 \cos \nu, \quad \nu \in \mathbb{S}(-\pi/2, \pi/2), \quad (25)$$

between these coefficients. It follows that the quadratic asymmetry form Δ_{H^+} becomes trivial on the subspaces of $D_{\hat{H}}^*$ such that the a.b. coefficients of functions $\psi_*(x)$ belonging to $D_{\hat{H}}^*$ satisfy relation (25) with a fixed ν . These subspaces are just the domains of s.a. restrictions of \hat{H}^+ , and relation (25), with a fixed ν , defines the asymptotic boundary conditions specifying these s.a. operators.

We thus find that for each g_2 in the second range, there exists a family of s.a. Hamiltonians $\hat{H}_{2,\nu}$ parametrized by the parameter ν on a circle with the domains $D_{H_{2\nu}}$ that are the subspaces of functions belonging to $D_{\hat{H}}^*(\mathbb{R}_+)$ and having the following asymptotic behavior as $x \rightarrow 0$:

$$\begin{aligned} \psi(x) &= C\psi^{\text{as}}(x) + O(x^{3/2}), \\ \psi'(x) &= C\psi^{\text{as}'}(x) + O(x^{1/2}), \\ \psi^{\text{as}}(x) &= u_{1\text{as}}(k_0x) \sin \nu + u_{2\text{as}}(x, k_0) \cos \nu. \end{aligned} \quad (26)$$

The spectral analysis of $\hat{H}_{2,\nu}$ is similar to that for \hat{H}_1 in the previous section, the difference being that the function $v_1(x; W)$ is now square-integrable at the origin and we must take into account asymptotic boundary conditions (26). To evaluate the Green's function for $\hat{H}_{2,\nu}$, we take the representation (15) with $a_1 = 0$ for $\psi_*(x)$ belonging to $D_{H_{2\nu}}$; boundary conditions (26) and asymptotics (11), (13a) then yield

$$\begin{aligned} a_2 &= k_0^{-2\mu} \omega^{-1}(W) \left[\frac{\Gamma(2\mu)}{\Gamma(\alpha_+)} \sin \nu - \frac{\Gamma(-2\mu)(\lambda/k_0)^{2\mu}}{\Gamma(\alpha_-)} \cos \nu \right]^{-1} \\ &\times \cos \nu \int_0^\infty v_1(x; W) \eta(x) dx. \end{aligned}$$

Representing the function $v_1(x; W)$ in the form

$$\begin{aligned} v_1(x; W) &= (2\mu)^{-1} k_0^{-1/2+\mu} \omega(W) [\tilde{\omega}_{2,\nu}(W) u_{2,\nu}(x; W) \\ &\quad + \omega_{2,\nu}(W) \tilde{u}_{2,\nu}(x; W)], \\ u_{2,\nu}(x; W) &= k_0^{1/2+\mu} u_1(x; W) \sin \nu + k_0^{1/2-\mu} u_2(x; W) \cos \nu, \\ \tilde{u}_{2,\nu}(x; W) &= -k_0^{1/2+\mu} u_1(x; W) \cos \nu + k_0^{1/2-\mu} u_2(x; W) \sin \nu, \\ \omega_{2,\nu}(W) &= \sin \nu + f(W) \cos \nu, \\ \tilde{\omega}_{2,\nu}(W) &= \cos \nu - f(W) \sin \nu, \\ f(W) &= (\lambda/k_0)^{2\mu} \frac{\Gamma(\alpha_+) \Gamma(\beta_-)}{\Gamma(\alpha_-) \Gamma(\beta_+)}, \end{aligned}$$

where ω is given in (10), the functions $u_{2,\nu}(x; W)$ and $\tilde{u}_{2,\nu}(x; W)$ are real-entire in W solutions of equation (4) and $u_{2,\nu}(x; W)$ satisfies boundary condition (26), we obtain the

⁷ The inertia indices of the quadratic form $(1/2i\mu\kappa_0)\Delta_+$ are 1, 1, which confirms the previous assertion in section 2 that the deficiency indices of \hat{H} are $m_\pm = 1$, see [12].

⁸ Here and in what follows, we use the notation $\mathbb{S}(a, b) = [a, b]$, $a \sim b$.

Green's function

$$\begin{aligned} G(x, y; W) &= \Omega^{-1}(W) u_{2,\nu}(x; W) u_{2,\nu}(y; W) \\ &\quad + \frac{1}{2\mu k_0} \begin{cases} \tilde{u}_{2,\nu}(x; W) u_{2,\nu}(y; W), & x > y, \\ u_{2,\nu}(x; W) \tilde{u}_{2,\nu}(y; W), & x < y, \end{cases} \end{aligned} \quad (27)$$

where

$$\Omega(W) = 2\mu k_0 \omega_2(W) \tilde{\omega}_2^{-1}(W). \quad (28)$$

We note that the second summand in (27) is real for real $W = E$.

As a guiding functional we take the functional $\Phi(\xi; W)$ given by (17) with $U(x; W) = u_{2,\nu}(x; W)$ and $\xi \in \mathbb{D} = D_r(\mathbb{R}_+) \cap D_{H_{2\nu}}$. The domain \mathbb{D} is dense in $L^2(\mathbb{R}_+)$, $\overline{\mathbb{D}} = L^2(\mathbb{R}_+)$. Following the procedure of the previous section, we show that $\Phi(\xi; z)$ is a simple guiding functional, i.e. satisfies the properties (1)–(3) cited in subsection 3.1. It is easy to verify the properties (1) and (3). We prove that the property (2) also holds. Let

$$\begin{aligned} \Phi(\xi_0; E_0) &= \int_0^b u_{2,\nu}(x; E_0) \xi_0(x) dx = 0, \\ \xi_0 &\in \mathbb{D}, \quad \text{supp } \xi_0 \in [0, b]. \end{aligned} \quad (29)$$

We consider the function

$$\begin{aligned} \psi(x) &= \frac{1}{2\mu\kappa_0} \left[u_{2,\nu}(x; E_0) \int_x^b \tilde{u}_{2,\nu}(y; E_0) \xi_0(y) dy \right. \\ &\quad \left. + \tilde{u}_{2,\nu}(x; E_0) \int_0^x u_{2,\nu}(y; E_0) \xi_0(y) dy \right], \end{aligned}$$

which is a solution of equation

$$(\check{H} - E_0)\psi(x) = \xi_0(x).$$

Using condition (29), we find that $\text{supp } \psi \in [0, b]$, i.e. $\psi \in D_r(\mathbb{R}_+)$, and therefore $\psi \in L^2(c, b)$ for any $c > 0$.

The function $\psi(x)$ allows the representation

$$\begin{aligned} \psi(x) &= c u_{2,\nu}(x; E_0) + \tilde{u}_{2,\nu}(x; E_0) \int_0^x u_{2,\nu}(y; E_0) \xi_0(y) dy \\ &\quad - u_{2,\nu}(x; E_0) \int_0^x \tilde{u}_{2,\nu}(y; E_0) \xi_0(y) dy, \\ c &= \frac{1}{2\mu\kappa_0} \int_0^b \tilde{u}_{2,\nu}(y; E_0) \xi_0(y) dy. \end{aligned} \quad (30)$$

Using the asymptotics of functions $u_{2,\nu}(x; E_0)$, $\tilde{u}_{2,\nu}(x; E_0)$, and $\xi_0(x)$ and simple estimates of the asymptotic behavior of the integral terms at the origin, we find that the asymptotic of $\psi(x)$ at the origin is given by

$$\psi(x) = c u_{2,\nu}(x; E_0) + O(x^{5/2-\mu}), \quad x \rightarrow 0,$$

which implies that $\psi \in D_{H_{2\nu}}$ and therefore $\psi \in \mathbb{D}$.

The derivative of the spectral function reads

$$\sigma'(E) = \pi^{-1} \text{Im } \Omega^{-1}(E + i0).$$

It is convenient to consider the cases $|\nu| < \pi/2$ and $\nu = \pm\pi/2$ separately.

Finally, we consider the case $\nu = \pi/2$ where we have

$$u_{2,\pi/2}(x; W) = k_0^{1/2+\mu} u_1(x; W),$$

$$\Omega^{-1}(W) = -\frac{\Gamma(\alpha_+) \Gamma(\beta_-) (\lambda/k_0)^{2\mu}}{2\mu k_0 \Gamma(\alpha_-) \Gamma(\beta_+)}.$$

We see that all results for the spectrum and system of the normalized (generalized) eigenfunctions coincide with those of the first range ($g_2 \geq 3/4$). In particular, the expressions for discrete energy levels (we will denote them by \mathcal{E}_n) are given by equation (22):

$$\mathcal{E}_n = -\frac{g_1^2}{(1+2\mu+2n)^2}, \quad \tau_n = \sqrt{|\mathcal{E}_n|} = \frac{|g_1|}{1+2\mu+2n}.$$

We obtain the same results for the case $\nu = -\pi/2$.

Secondly, we consider the case $\nu = 0$. Here we have

$$u_{2,0}(x; W) = k_0^{1/2-\mu} u_2(x; W),$$

$$\sigma'(E) = \pi^{-1} \text{Im } \Omega^{-1}(E+i0),$$

$$\Omega^{-1}(W) = \frac{(k_0/\lambda)^{2\mu} \Gamma(\beta_+) \Gamma(\alpha_-)}{2\mu k_0 \Gamma(\beta_-) \Gamma(\alpha_+)}.$$

We let $g_1 > 0$. For $E = p^2 \geq 0$, $p \geq 0$, $\lambda = 2p e^{-i\pi/2}$, we have

$$\sigma'(E) = \left(\frac{|\Gamma(\alpha_-)|}{|\Gamma(\beta_-)|} \right)^2 \frac{(k_0/2p)^{2\mu} e^{-\pi g_1/2p}}{2\pi k_0}. \quad (31)$$

For $E = -\tau^2 < 0$, $\tau > 0$, $\lambda = 2\tau$, the function $\Omega^{-1}(E)$ is real in the case of those E for which $\Omega^{-1}(E)$ is finite, such that $\text{Im } \Omega^{-1}(E+i0)$ can differ from zero only for E that provide $\Omega^{-1}(E) = \infty$. The latter is possible only for $\alpha_- = -n$ ($\Gamma(\alpha_-) = \infty$), $n \in \mathbb{Z}_+$, or

$$1 - 2\mu + g_1/\tau = -2n, \quad n \in \mathbb{Z}_+. \quad (32)$$

Equation (32) has no solutions for $0 < \mu < 1/2$ and has one solution for $1/2 < \mu < 1$: $n = 0$, $\tau = \tau_{-1}(0) = g_1/(2\mu - 1)$, $E = E_{-1}(0) = -\tau_{-1}^2(0)$.

We now let $g_1 < 0$. For $E = p^2 \geq 0$, $p \geq 0$, $\lambda = 2p e^{-i\pi/2}$, the derivative of the spectral function is given by equation (31).

For $E = -\tau^2 < 0$, $\tau > 0$, $\lambda = 2\tau$, the function $\Omega^{-1}(E)$ is real for $E \neq E_n(0)$ ($\Omega(E_n(0)) = \infty$), such that $\sigma'(E)$ does not vanish only in the points $E = E_n(0)$. The equation $\Omega^{-1}(E_n(0)) = \infty$ implies the condition $\alpha_- = 1/2 - \mu - |g_1|/2\tau_n(0) = -n$ ($\Gamma(\alpha_-) = \infty$), which gives

$$E_n(0) = -\tau_n^2(0) = -\left(\frac{g_1}{1-2\mu+2k} \right)^2,$$

$$k = \begin{cases} n, & 0 < \mu < 1/2, \\ n+1, & 1/2 < \mu < 1, \end{cases} \quad n \in \mathbb{Z}_+.$$

Thus, for $g_1 < 0$, the simple spectrum of $\hat{H}_{2,0}$ is given by $\text{spec } H_{2,0} = \mathbb{R}_+ \cup \{E_n(0), n \in \mathbb{Z}_+\}$ and a complete

orthonormalized system in $L^2(\mathbb{R}_+)$ consists of (generalized) eigenfunctions

$$U_E(x) = \sqrt{\sigma'(E)} u_2(x; E), \quad E \geq 0,$$

$$U_n(x) = \frac{(2\tau_n)^{1-\mu}}{|\Gamma(\beta_-)|} \sqrt{\frac{\Gamma(1-2\mu+k)}{(1-2\mu+2k)k!}} u_2(x; E_n(0)), \quad n \in \mathbb{Z}_+, \quad (33)$$

of $\hat{H}_{2,0}$.

For $0 < \mu < 1/2$, $g_1 > 0$, the simple spectrum of $\hat{H}_{2,0}$ is given by $\text{spec } \hat{H}_{2,0} = \mathbb{R}_+$ and a complete orthonormalized system in $L^2(\mathbb{R}_+)$ consists of generalized eigenfunctions $U_E(x)$ (33) with the corresponding parameters and the function $\sigma'(E)$.

For $1/2 < \mu < 1$, $g_1 > 0$, the simple spectrum of $\hat{H}_{2,0}$ is given by $\text{spec } H_{2,0} = \mathbb{R}_+ \cup \{E_{-1}(0)\}$ and a complete orthonormalized system in $L^2(\mathbb{R}_+)$ consists of generalized eigenfunctions $U_E(x)$ (33) with the corresponding parameters and the function $\sigma'(E)$, and only one eigenfunction of the discrete spectrum $U_{-1}(x)$, given by (33) with $n = -1$ and $k = 0$.

Now, we turn to the general case $|\nu| < \pi/2$. In this case, we have

$$\sigma'(E) = (2\pi \mu k_0 \cos^2 \nu)^{-1} \text{Im } f_v^{-1}(E+i0),$$

$$f_v(W) = f(W) + \tan \nu, \quad f(W) = \frac{(\lambda/k_0)^{2\mu} \Gamma(\beta_-) \Gamma(\alpha_+)}{\Gamma(\beta_+) \Gamma(\alpha_-)}.$$

For $E = p^2 \geq 0$, $p \geq 0$, $\lambda = 2p e^{-i\pi/2}$, we have

$$\sigma'(E) = \frac{B(E)}{2\pi k_0 \cos^2 \nu [A^2(E) + \mu^2 B^2(E)]}, \quad (34)$$

where $A(E) = \text{Re } f_v(E)$ and $\mu B(E) = -\text{Im } f_v(E)$. A direct calculation gives

$$A(E) = \frac{\mu |\Gamma(\alpha_+)|^2 (2p/k_0)^{2\mu}}{\Gamma^2(\beta_+) \sin(2\pi \mu)} (e^{-\pi g_1/2p} \cos(2\pi \mu) + e^{\pi g_1/2p}) + \tan \nu, \quad (35)$$

$$B(E) = \frac{|\Gamma(\alpha_+)|^2 (2p/k_0)^{2\mu} e^{-\pi g_1/2p}}{\Gamma^2(\beta_+)}.$$

For $E = -\tau^2 < 0$, $\tau > 0$, $\lambda = 2\tau$, the function $f_v(E)$ is real; therefore, $\sigma'(E)$ can differ from zero only at the discrete points $E_n(\nu)$ such that $f_v(E_n(\nu)) = 0$, or $f(E_n(\nu)) = -\tan \nu$, and we find that (derivatives with respect to E are denoted by primes in equation (36))

$$\sigma'(E) = \sum_n [-2\mu k_0 f'_v(E_n(\nu)) \cos^2 \nu]^{-1} \delta(E - E_n(\nu)),$$

$$f'_v(E_n(\nu)) = f'(E_n(\nu)) < 0, \quad \partial_\nu E_n(\nu) = -\cos^{-2} \nu [f'(E_n(\nu))]^{-1} > 0. \quad (36)$$

I. Let $g_1 > 0$.

For $E = p^2 > 0$, $p > 0$, the function $\sigma'(E)$ (34) is a finite positive function. At $E = 0$, we have $B(0) = 0$ and

$$A(0)|_{\nu=\nu_{-1}} = 0, \quad \tan \nu_{-1} = -\frac{(g_1/k_0)^{2\mu}\Gamma(\beta_-)}{\Gamma^{-1}(\beta_+)},$$

$$\begin{cases} \nu_{-1} > 0, & 1/2 < \mu < 1, \\ \nu_{-1} < 0, & 0 < \mu < 1/2. \end{cases}$$

It is easy to see that

$$f_\nu(W) = \tan \nu - \tan \nu_{-1} - (2\mu k_0 \cos^2 \nu_{-1})^{-1} \times \Psi^{-2}W + O(W), \quad W \rightarrow 0,$$

$$\Psi = \frac{g_1(g_1/k_0)^{-\mu}}{\mu \cos \nu_{-1}} \sqrt{\frac{3\Gamma(1+2\mu)}{2k_0(1+2\mu)\Gamma(2-2\mu)}}.$$

It follows that for $\nu \neq \nu_{-1}$, the function $\sigma'(E)$ is finite at $E = 0$. But for $\nu = \nu_{-1}$ and for small E , we have

$$\sigma'(E) = -\frac{1}{\pi}\Psi^2 \operatorname{Im}(E+i0)^{-1} + O(1) = \Psi^2\delta(E) + O(1),$$

which means that there is the eigenvalue $E = 0$ in the spectrum of the s.a. Hamiltonian $\hat{H}_{2,\nu_{-1}}$.

For $E = -\tau^2 < 0$, $\lambda = 2\tau$, the function $f(E)$,

$$f(E) = \frac{\Gamma(\beta_-)}{\Gamma(\beta_+)} \frac{\Gamma(1/2 + \mu + g_1/2\tau)(2\tau/k_0)^{2\mu}}{\Gamma(1/2 - \mu + g_1/2\tau)},$$

has the properties that it is a smooth function for $E \in (-\infty, 0)$, $f(E) \rightarrow \infty$ as $E \rightarrow -\infty$,

$$f(0) = -\tan \nu_{-1} \begin{cases} < 0, & 1/2 < \mu < 1, \\ > 0, & 0 < \mu < 1/2. \end{cases}$$

Because $f'(E_n(\nu)) < 0$ (see equation (36)), the straight line $\tilde{f}(E) = \tan \nu$, $E \in (-\infty, 0]$, can intersect the plot of the function $f(E)$ not more than once. That is why the equation $f_\nu(E) = 0$ has no solutions for $\nu \in (\nu_{-1}, \pi/2)$, whereas for any fixed $\nu \in (-\pi/2, \nu_{-1}]$, this equation has only one solution $E_{-1}(\nu) \in (-\infty, 0]$, which increases monotonically from $-\infty$ to 0 as ν changes from $-\pi/2 + 0$ to ν_{-1} .

We thus find that the spectrum of $\hat{H}_{2,\nu}$, $|\nu| < \pi/2$, with $g_1 > 0$, is simple and is given by

$$\operatorname{spec} \hat{H}_{2,\nu} = \begin{cases} \mathbb{R}_+ \cup \{E_{-1}(\nu)\}, & \nu \in (-\pi/2, \nu_{-1}], \\ \mathbb{R}_+, & \nu \in (\nu_{-1}, \pi/2). \end{cases}$$

The generalized eigenfunctions

$$U_E(x) = \sqrt{\sigma'(E)}u_{2,\nu}(x; E), \quad E \geq 0,$$

and (for $\nu \in (-\pi/2, \nu_{-1}]$) the eigenfunction

$$U_{-1}(x) = [-2\mu k_0 f'(E_{-1}(\nu)) \cos^2 \nu]^{-1/2} u_{2,\nu}(x; E_{-1}(\nu))$$

of $\hat{H}_{2,\nu}$ form a complete orthonormalized systems in $L^2(\mathbb{R}_+)$.

II. Let $g_1 < 0$.

Then for $E = p^2 \geq 0$, $p \geq 0$, $\lambda = 2p e^{-i\pi/2}$, formulae (34) and (35) hold true. Because the functions $A(E)$ and $B(E)$ are finite at $E = 0$ ($B(0) \neq 0$), the function $\sigma'(E)$ (34) is a finite positive function for $E \geq 0$. This means that for $E \geq 0$, the spectra of s.a. Hamiltonians $\hat{H}_{2,\nu}$ are simple, purely continuous, and given by $\operatorname{spec} \hat{H}_{2,\nu} = \mathbb{R}_+$.

For $E = -\tau^2 < 0$, $\tau > 0$, $\lambda = 2\tau$, we have

$$f(E) = \frac{\Gamma(\beta_-)}{\Gamma(\beta_+)} \frac{\Gamma(1/2 + \mu - |g_1|/2\tau)(2\tau/k_0)^{2\mu}}{\Gamma(1/2 - \mu - |g_1|/2\tau)}.$$

It is easy to see that for fixed ν , the spectrum is bounded from below and the equation $f_\nu(E_n(\nu)) = 0$ has an infinite number of solutions

$$E_n(\nu) = -g_1^2/4n^2 + O(n^{-3}), \quad (37)$$

asymptotically coinciding with (22) as $n \rightarrow \infty$.

We thus find that the spectrum of $\hat{H}_{2,\nu}$, $|\nu| < \pi/2$, with $g_1 < 0$, is simple and given by $\operatorname{spec} \hat{H}_{2,\nu} = \mathbb{R}_+ \cup \{E_n(\nu)\}$. The corresponding generalized eigenfunctions of the continuous spectrum

$$U_E(x) = \sqrt{\sigma'(E)}u_{2,\nu}(x; E), \quad E \geq 0,$$

and eigenfunctions of the discrete spectrum

$$U_n(x) = [-2\mu k_0 f'(E_n(\nu)) \cos^2 \nu]^{-1/2} \times u_{2,\nu}(x; E_n(\nu)), \quad E_n(\nu) < 0,$$

of $\hat{H}_{2,\nu}$ form a complete orthonormalized system in $L^2(\mathbb{R}_+)$.

It is possible to present a description of the discrete spectrum of the Hamiltonians $\hat{H}_{2,\nu}$, $|\nu| < \pi/2$, $g_1 < 0$ in more detail.

The function $f(E)$ has the properties $f(E) \rightarrow \infty$ as $E \rightarrow -\infty$; $f(\mathcal{E}_n \pm 0) = \pm\infty$, $n \in \mathbb{Z}_+$ and we have

$$E_n(0) < \mathcal{E}_n < E_{n+1}(0) < \mathcal{E}_{n+1}, \quad n \in \mathbb{Z}_+.$$

Taking into account the third equality in (36), we can see that in each energy interval $(\mathcal{E}_{n-1}, \mathcal{E}_n)$, $n \in \mathbb{Z}_+$, for a fixed $\nu \in (-\pi/2, \pi/2)$, there is one discrete level $E_n(\nu)$ that increases monotonically from $\mathcal{E}_{n-1} + 0$ to $\mathcal{E}_n - 0$ when ν changes from $\pi/2 - 0$ to $-\pi/2 + 0$ (we set $\mathcal{E}_{-1} = -\infty$). We note that the relations

$$\lim_{\nu \rightarrow \pi/2} E_n(\nu) = \lim_{\nu \rightarrow -\pi/2} E_{n+1}(\nu) = \mathcal{E}_n, \quad n \in \mathbb{Z}_+$$

confirm the equivalence of s.a. extensions with parameters $\nu = -\pi/2$ and $\nu = \pi/2$.

It should also be pointed out that bound states exist even for the repulsive potential, $g_2, g_1 > 0$; see the dashed line in figure 1.

3.3. The third range $g_2 = -1/4$ ($\mu = 0$)

The analysis in this section is similar to that in the previous one; a peculiarity is that $\alpha_+ = \alpha_- = \alpha = 1/2 + g_1/\lambda$, $\beta_+ = \beta_- = 1$, $u_1(x; W) = u_2(x; W)$, and the representation (9) of $v_1(x; W)$ in terms of u_1 and u_2 does not hold. As the solutions of equation (4) with $g_2 = -1/4$, we therefore use

the functions $u_1(x; W)$, $u_3(x; W)$ and $v_1(x; W)$, respectively defined by

$$\begin{aligned} u_1(x; W) &= x^{1/2} e^{-z/2} \Phi(\alpha, 1; z) = u_1(x; W)|_{\lambda \rightarrow -\lambda}, \\ u_3(x; W) &= \frac{\partial}{\partial \mu} [u_1(x; W)|_{\mu \neq 0}]_{\mu=0} + \ln k_0 u_1(x; W), \\ v_1(x; W) &= x^{1/2} e^{-z/2} \Psi(\alpha, 1; z) \\ &= \Gamma^{-1}(\alpha) [\omega_0(W) u_1(x; W) - u_3(x; W)], \\ \omega_0(W) &= 2\psi(1) - \psi(\alpha) - \ln(\lambda/k_0), \end{aligned}$$

where $\psi(\alpha) = \Gamma'(\alpha)/\Gamma(\alpha)$ and k_0 is a constant. The functions $u_1(x; W)$ and $u_3(x; W)$ are real-entire in W .

The asymptotic behavior of these functions at the origin and at infinity is, respectively, as follows.

As $x \rightarrow 0$, $z = \lambda x \rightarrow 0$, we have

$$\begin{aligned} u_1(x; W) &= k_0^{-1/2} u_{1as}(x) + O(x^{3/2}), \\ u_{1as}(x) &= (k_0 x)^{1/2}, \\ u_3(x; W) &= k_0^{-1/2} u_{3as}(x) + O(x^{3/2} \ln x), \\ u_{3as}(x) &= (k_0 x)^{1/2} \ln(k_0 x), \\ v_1(x; W) &= k_0^{-1/2} \Gamma^{-1}(\alpha) [\omega_0(W) u_{1as}(x) \\ &\quad - u_{3as}(x)] + O(x^{3/2} \ln x). \end{aligned} \tag{38}$$

As $x \rightarrow \infty$, $\text{Im } W > 0$, we have

$$\begin{aligned} u_1(x; W) &= \Gamma^{-1}(\alpha) \lambda^{\alpha-1} x^{g_1/\lambda} e^{z/2} \tilde{O}(x^{-1}) \rightarrow \infty, \\ v_1(x; W) &= \lambda^{-\alpha} x^{-g_1/\lambda} e^{-z/2} \tilde{O}(x^{-1}) \rightarrow 0. \end{aligned} \tag{39}$$

Both sets u_1, u_3 and u_1, v_1 are linearly independent,

$$\text{Wr}(u_1, u_3) = 1, \quad \text{Wr}(u_1, v_1) = -\Gamma^{-1}(\alpha),$$

and in particular, u_1 and v_1 form a fundamental system of solutions of equation (4) for $\text{Im } W \neq 0$ and $W = 0$; see section 2,

We recall that, for $g_2 = -1/4$, the deficiency indices of the initial symmetric operator \hat{H} are $m_{\pm} = 1$, and therefore there exists a one-parameter family of s.a. extensions of \hat{H} with $g_2 = -1/4$; see section 2.

To evaluate the asymmetry form in terms of a.b. coefficients, we need to determine the asymptotics of functions ψ_* belonging to the natural domain $D_{\hat{H}}^*(\mathbb{R}_+)$ at the origin. To this end, we use the general solution (23) of equation (14), performing there the substitutions $a_2 u_2 \rightarrow a_2 u_3$ and $u_2/2\mu \rightarrow -u_3$.

Taking into account that the potential vanishes as $x \rightarrow \infty$, we have $\psi_*(x), \psi'_*(x) \xrightarrow{x \rightarrow \infty} 0$. Using the Cauchy–Bunyakovskii inequality to estimate the integral terms, we find that the desired asymptotic as $x \rightarrow 0$ is given by

$$\begin{aligned} \psi_*(x) &= a_1 u_{1as}(x) + a_2 u_{3as}(x) + O(x^{3/2} \ln x), \\ \psi'_*(x) &= a_1 u'_{1as}(x) + a_2 u'_{3as}(x) + O(x^{1/2} \ln x). \end{aligned}$$

Then we find that $\Delta_{H^+}(\psi_*) = k_0(\bar{a}_1 a_2 - \bar{a}_2 a_1)$. Therefore, the deficiency indices of \hat{H} are $m_{\pm} = 1$. The requirement that

Δ_{H^+} vanishes results in the relation

$$a_1 \cos \vartheta = a_2 \sin \vartheta, \quad \vartheta \in \mathbb{S}(-\pi/2, \pi/2).$$

Thus, there exists a family of s.a. Hamiltonians $\hat{H}_{3,\vartheta}$ with the domains $D_{H_{3,\vartheta}}$ that consist of functions from $D_{\hat{H}}^*(\mathbb{R}_+)$ with the following asymptotic behavior as $x \rightarrow 0$:

$$\begin{aligned} \psi &= C \psi_{3,\vartheta as}(x) + O(x^{3/2} \ln x), \\ \psi' &= C \psi'_{3,\vartheta as}(x) + O(x^{1/2} \ln x), \end{aligned} \tag{40}$$

$$\psi_{3,\vartheta as}(x) = u_{1as}(x) \sin \vartheta + u_{3as}(x) \cos \vartheta.$$

Therefore,

$$D_{H_{3,\vartheta}} = \left\{ \psi : \psi \in D_{\hat{H}}^*(\mathbb{R}_+), \psi \text{ obey (40)} \right\}.$$

Imposing s.a. boundary condition (40) on the functions (15) (with $a_1 = 0$), and using asymptotics (38), we find the Green's functions of the operators $\hat{H}_{3,\vartheta}$:

$$\begin{aligned} G(x, y; W) &= \Omega^{-1}(W) u_{3,\vartheta}(x; W) u_{3,\vartheta}(y; W) \\ &\quad + \begin{cases} \tilde{u}_{3,\vartheta}(x; W) u_{3,\vartheta}(y; W), & x > y, \\ u_{3,\vartheta}(x; W) \tilde{u}_{3,\vartheta}(y; W), & x < y, \end{cases} \end{aligned}$$

where

$$\begin{aligned} \Omega(W) &= (\omega_0 \cos \vartheta + \sin \vartheta)(\omega_0 \sin \vartheta - \cos \vartheta)^{-1}, \\ u_{3,\vartheta}(x; W) &= u_1(x; W) \sin \vartheta + u_3(x; W) \cos \vartheta, \\ \tilde{u}_{3,\vartheta}(x; W) &= u_1(x; W) \cos \vartheta - u_3(x; W) \sin \vartheta, \\ \Gamma(\alpha) v_1 &= (\omega_0 \sin \vartheta - \cos \vartheta) u_{3,\vartheta} + (\omega_0 \cos \vartheta + \sin \vartheta) \tilde{u}_{3,\vartheta}. \end{aligned}$$

We note that $u_{3,\vartheta}$ and $\tilde{u}_{3,\vartheta}$ are solutions of equation (4) real-entire in W , the solution $u_{3,\vartheta}$ satisfies the boundary condition (40) and the second summand in $G(x, y; W)$ is real for real $W = E$.

It is easy to verify that the guiding functional given by (17) with $U = u_{3,\vartheta}$ satisfies the properties (1) and (3) cited in section 3.1. The proof that it satisfies the property (2) is identical to that presented in section 3.2 for the second range $1 > \mu > 0$. It follows that the spectra of $\hat{H}_{3,\vartheta}$ are simple.

The derivative of the spectral function is given by $\sigma'(E) = \pi^{-1} \text{Im } \Omega^{-1}(E + i0)$.

We first consider the case $\vartheta = \pi/2$, where we have

$$u_{3,\pi/2}(x; W) = u_1(x; W), \quad \Omega(W) = -[\psi(\alpha) + \ln(\lambda/k_0)]^{-1}.$$

For $E = p^2 \geq 0$, $p \geq 0$, $\lambda = 2p e^{-i\pi/2}$, we find that

$$\sigma'(E) = \frac{1}{2} \left(1 - \tanh \frac{\pi g_1}{2p} \right) \geq 0.$$

For $E = -\tau^2 < 0$, $\tau > 0$, $\lambda = 2\tau$ and $g_1 > 0$, the function $\Omega(E)$ is of the form

$$\Omega(E) = -[\psi(1/2 + g_1/2\tau) + \ln(2\tau/k_0)]^{-1},$$

which implies that for $g_1 > 0$, there is no negative part of the spectrum.

For $E = -\tau^2 < 0$, $\tau > 0$, $\lambda = 2\tau$ and $g_1 < 0$, we have

$$\Omega(E) = -[\psi(1/2 - |g_1|/2\tau) + \ln(2\tau/k_0)]^{-1},$$

which implies that there are discrete negative energy levels \mathcal{E}_n in the spectrum,

$$\mathcal{E}_n = -g_1^2(1+2n)^{-2}, \quad \tau_n = |g_1|(1+2n)^{-1}, \quad n \in \mathbb{Z}_+$$

$$\sigma'(E) = \sum_{n \in \mathbb{Z}_+} Q_n^2 \delta(E - \mathcal{E}_n), \quad Q_n = 2|g_1|(1+2n)^{-3/2}.$$

It is easy to see that for the case of $\vartheta = -\pi/2$, we obtain the same results for the spectrum and eigenfunctions as it must be.

We thus find that for $g_1 > 0$, the spectrum of $\hat{H}_{3,\pm\pi/2}$ is simple, given by $\text{spec } \hat{H}_{3,\pm\pi/2} = \mathbb{R}_+$, and a complete orthonormalized system in $L^2(\mathbb{R}_+)$ of its generalized eigenfunctions is

$$U_E(x) = \sqrt{\sigma'(E)} u_1(x; E), \quad E \geq 0.$$

For $g_1 < 0$, the spectrum of $\hat{H}_{3,\pm\pi/2}$ is simple and is given by $\text{spec } \hat{H}_{3,\pm\pi/2} = \mathbb{R}_+ \cup \{\mathcal{E}_n, n \in \mathbb{Z}_+\}$, and a complete orthonormalized system in $L^2(\mathbb{R}_+)$ of its (generalized) eigenfunctions consists of the functions

$$U_E(x) = \sqrt{\sigma'(E)} u_1(x; E), \quad E \geq 0, \\ U_n(x) = 2|g_1|(1+2n)^{-3/2} u_1(x; \mathcal{E}_n), \quad \mathcal{E}_n < 0.$$

We note that the spectrum and eigenfunctions for $\hat{H}_{3,\pi/2}$ coincide with those for \hat{H}_1 with $g_2 \geq 3/4$ if we set $\mu = 0$ in the respective formulae in section 3.1.

We now turn to the case $|\vartheta| < \pi/2$. In this case, $\sigma'(E)$ can be represented as

$$\sigma'(E) = (\pi \cos^2 \vartheta)^{-1} \text{Im}[\omega_3(E + i0)]^{-1},$$

$$\omega_3(W) = \psi(\alpha) + \ln(\lambda/k_0) - 2\psi(1) - \tan \vartheta.$$

For $E = p^2 \geq 0$, $p \geq 0$, $\lambda = 2p e^{-i\pi/2}$ and $g_1 < 0$, we have

$$\sigma'(E) = \frac{B(E)}{\pi \cos^2 \vartheta [A^2(E) + B^2(E)]}, \quad (41)$$

where $\omega_3(E) = A(E) - iB(E)$. The function $B(E)$ can be explicitly calculated:

$$B(E) = \frac{\pi}{2} \left(1 - \tanh \frac{\pi g_1}{2\sqrt{E}} \right) > 0, \quad \forall E \geq 0, \quad (42)$$

whence it follows that for all $E \geq 0$, the spectrum of $\hat{H}_{3,\vartheta}$ is purely continuous.

For $E = p^2 > 0$, $p > 0$, $\lambda = 2p e^{-i\pi/2}$ and $g_1 > 0$, the spectral function is given by the same equations (41) and (42). But in this case, $B(0) = 0$ and the limit $\lim_{W \rightarrow 0} \omega_3(W)$ must be carefully examined.

At small W , we have

$$\omega_3(W) = (\tan \vartheta_{(-)} - \tan \vartheta) - (6g_1^2)^{-1} W + O(W^2),$$

$$\tan \vartheta_{(-)} = \ln(g_1/k_0) - 2\psi(1).$$

For $\vartheta \neq \vartheta_{(-)}$, the function $\sigma'(E)$ is finite at $E = 0$. But for $\vartheta = \vartheta_{(-)}$ and small E , we have

$$\sigma'(E) = -\frac{6g_1^2}{\pi \cos^2 \vartheta_{(-)}} \text{Im}(E + i0)^{-1} + O(1) \\ = \frac{6g_1^2}{\cos^2 \vartheta_{(-)}} \delta(E) + O(1),$$

which means that the spectrum of the Hamiltonian $\hat{H}_{3,\vartheta_{(-)}}$ contains an eigenvalue $E = 0$.

For $E = -\tau^2 < 0$, $\tau > 0$, $\lambda = 2\tau$, the function $\omega_3(E)$ is real; therefore, $\sigma'(E)$ can differ from zero only at zero-points $E_n = E_n(\vartheta)$ of $\omega_3(E)$, which yields

$$\sigma'(E) = \sum_n [-k_0 \omega'_3(E_n(\vartheta)) \cos^2 \vartheta]^{-1} \delta(E - E_n(\vartheta)),$$

$$\omega_3(E_n(\vartheta)) = 0, \quad \omega'_3(E_n(\vartheta)) < 0,$$

and

$$\partial_\vartheta E_n(\vartheta) = [\cos^2 \vartheta \omega'_3(E_n(\vartheta))]^{-1} < 0. \quad (43)$$

For $g_1 > 0$, we have

$$\omega_3(E) = \psi(1/2 + g_1/2\tau) + \ln(2\tau/g_1) + \tan \vartheta_{(-)} - \tan \vartheta, \\ \omega_3(E) = (1/2) \ln |E| - \tan \vartheta + O(1), \quad E \rightarrow -\infty, \\ \omega_3(0) = \tan \vartheta_{(-)} - \tan \vartheta.$$

For $\vartheta < \vartheta_{(-)}$, the equation $\omega_3(E) = 0$ has no solution, whereas for $\vartheta \geq \vartheta_{(-)}$, it has only one solution $E^{(-)}(\vartheta)$. Because equation (43) holds for $\partial_\vartheta E^{(-)}(\vartheta)$, $E^{(-)}(\vartheta)$ increases from $-\infty$ to 0 when ϑ changes from $\pi/2 - 0$ to $\vartheta_{(-)}$.

For $g_1 < 0$, we have

$$\omega_3(E) = \psi(1/2 - |g_1|/2\tau) + \ln(2\tau/k_0) - 2\psi(1) - \tan \vartheta, \\ \omega_3(E) = (1/2) \ln |E| - \tan \vartheta + O(1), \quad E \rightarrow -\infty.$$

It is easy to verify that the equation $\omega_3(E) = 0$ has an infinite number of solutions $E_n, n \in \mathbb{Z}_+$, bounded from below and asymptotically coinciding with (22) as $n \rightarrow \infty$, $E_n = -g_1^2/4n^2 + O(n^{-3})$.

We thus find that for $g_1 > 0$, the spectrum of $\hat{H}_{3,\vartheta}$ is simple and given by $\text{spec } \hat{H}_{3,\vartheta} = \mathbb{R}_+ \cup \{E^{(-)}(\vartheta)\}$ and a complete orthonormalized system in $L^2(\mathbb{R}_+)$ of its (generalized) eigenfunctions consists of functions

$$U_E(x) = \sqrt{\sigma'(E)} u_{3,\vartheta}(x; E), \quad E \geq 0, \\ U(x) = [-k_0 \cos^2 \vartheta \omega'_3(E^{(-)}(\vartheta))]^{-1/2} u_{3,\vartheta}(x; E^{(-)}(\vartheta))$$

(the eigenvalue $E^{(-)}(\vartheta)$ exists and therefore $E^{(-)}(\vartheta)$ and the corresponding eigenfunction $U(x)$ enter the inversion formulae only if $\vartheta \geq \vartheta_{(-)}$); for $g_1 < 0$, the spectrum of $\hat{H}_{3,\vartheta}$ is simple and is given by $\text{spec } \hat{H}_{3,\vartheta} = \mathbb{R}_+ \cup \{E_n\}$ and a complete orthonormalized system in $L^2(\mathbb{R}_+)$ of its (generalized) eigenfunctions consists of the functions

$$U_E(x) = \sqrt{\sigma'(E)} u_{3,\vartheta}(x; E), \quad E \geq 0, \\ U_n(x) = [-k_0 \cos^2 \vartheta \omega'_3(E_n(\vartheta))]^{-1/2} \\ \times u_{3,\vartheta}(x; E_n(\vartheta)), \quad E_n(\vartheta) < 0.$$

It is possible to describe the discrete spectrum for $|\vartheta| < \pi/2$ and $g_1 < 0$ in more detail. To this end, we represent the equation $\omega_3(E(\vartheta)) = 0$ in the equivalent form

$$f(E) = \tan \vartheta, \quad f(E) = \psi(1/2 - |g_1|/2\tau) + \ln(2\tau/k_0) - 2\psi(1).$$

Then we have

$$f(-\infty) = \infty, \quad f(\mathcal{E}_n \pm 0) = \pm\infty, \quad n \in \mathbb{Z}_+.$$

Because equation (43) holds, we can see that in each interval $(\mathcal{E}_n, \mathcal{E}_{n+1})$, $n \in \{-1\} \cup \mathbb{Z}_+$, there is one discrete eigenvalue $E_n(\vartheta)$ and $E_n(\vartheta)$ increases monotonically from $\mathcal{E}_n + 0$ to $\mathcal{E}_{n+1} - 0$ when ϑ changes from $\pi/2 - 0$ to $-\pi/2 + 0$ (we set $\mathcal{E}_{-1} = -\infty$). We note the relations

$$\lim_{\vartheta \rightarrow -\pi/2} E_{n-1}(\vartheta) = \lim_{\vartheta \rightarrow \pi/2} E_n(\vartheta) = \mathcal{E}_n.$$

3.4. The fourth range $g_2 < -1/4$ ($\mu = i\kappa$, $\kappa > 0$)

The analysis in this section is similar to that in section 3.2 (although the results for the spectrum differ drastically). We therefore briefly outline the basic points.

According to section 2, the deficiency indices of the initial symmetric operator \hat{H} with $g_2 < -1/4$ are $m_{\pm} = 1$ and therefore there exists a one-parameter family of its s.a. extensions.

To evaluate the asymmetry form Δ_{H^+} , we determine the asymptotics of functions ψ_* belonging to $D_{\hat{H}}^*(\mathbb{R}_+)$ at the origin using representation (23) with $\mu = i\kappa$ of the general solution of equation (14) with $W = 0$ and estimating the integral terms by means of the Cauchy–Bunyakovskii inequality; we find that, as $x \rightarrow 0$,

$$\begin{aligned} \psi_*(x) &= a_1 u_{1as}(x) + a_2 u_{2as}(x) + O(x^{3/2}), \\ \psi'_*(x) &= a_1 u'_{1as}(x) + a_2 u'_{2as}(x) + O(x^{1/2}), \\ u_{1as}(x) &= (k_0 x)^{1/2+i\kappa}, \quad u_{2as}(x) = (k_0 x)^{1/2-i\kappa} = \overline{u_{1as}(x)}. \end{aligned}$$

Thus, we find that $\Delta_{H^+}(\psi_*) = -2i\kappa(\overline{a_1}a_1 - \overline{a_2}a_2)$, which means that the deficiency indices of \hat{H} are $m_{\pm} = 1$. The condition $\Delta_{H^+}(\psi_*) = 0$ yields $a_1 = e^{2i\theta}a_2$, $\theta \in \mathbb{S}(0, \pi)$. Therefore, there exists a family of s.a. Hamiltonians $\hat{H}_{4,\theta}$ with the domains $D_{H_{4,\theta}}$ that consist of functions from $D_{\hat{H}}^*(\mathbb{R}_+)$ with the following asymptotic behavior as $x \rightarrow 0$:

$$\begin{aligned} \psi &= C\psi_{4as}(x) + O(x^{3/2}), \quad \psi' = C\psi'_{4as}(x) + O(x^{1/2}), \\ \psi_{4as}(x) &= e^{i\theta}u_{1as}(x) + e^{-i\theta}u_{2as}(x) = \overline{\psi_{4as}(x)}. \end{aligned} \quad (44)$$

Therefore,

$$D_{H_{4,\theta}} = \left\{ \psi : \psi \in D_{\hat{H}}^*(\mathbb{R}_+), \psi \text{ obey (44)} \right\}.$$

Imposing the s.a. boundary condition (44) on the functions (15) (with $a_1 = 0$), and using asymptotics (11), we obtain the Green's function of the operators $\hat{H}_{4,\theta}$,

$$G(x, y; W) = \Omega^{-1}(W)u_{4,\theta}(x; W)u_{4,\theta}(y; W) - \frac{1}{4\kappa k_0} \begin{cases} \tilde{u}_{4,\theta}(x; W)u_{4,\theta}(y; W), & x > y, \\ u_{4,\theta}(x; W)\tilde{u}_{4,\theta}(y; W), & x < y, \end{cases}$$

where

$$\begin{aligned} \Omega(W) &= \frac{4i\kappa k_0 \omega_{4,\theta}(W)}{\tilde{\omega}_{4,\theta}(W)}, \quad \omega_{4,\theta}(W) = a(W) + b(W), \\ \tilde{\omega}_{4,\theta}(W) &= a(W) - b(W), \quad a(W) = e^{i\theta} \frac{\Gamma(\beta)(\lambda/k_0)^{-i\kappa}}{\Gamma(\alpha)}, \\ b(W) &= e^{-i\theta} \frac{\Gamma(\beta_-)(\lambda/k_0)^{i\kappa}}{\Gamma(\alpha_-)}, \\ u_{4,\theta}(x; W) &= e^{i\theta} k_0^{1/2+i\kappa} u_1(x; W) + e^{-i\theta} k_0^{1/2-i\kappa} u_2(x; W), \\ \tilde{u}_{4,\theta}(x; W) &= i[e^{-i\theta} k_0^{1/2-i\kappa} u_2(x; W) - e^{i\theta} k_0^{1/2+i\kappa} u_1(x; W)], \\ 4\kappa V_1(x; W) &= -(\lambda/k_0)^{i\kappa} k_0^{-1/2+i\kappa} [i\tilde{\omega}_{4,\theta}(W)u_{4,\theta}(x; W) \\ &\quad + \omega_{4,\theta}(W)V_{\theta}(x; W)], \end{aligned}$$

where $u_{4,\theta}$ and $\tilde{u}_{4,\theta}$ are real-entire in W solutions of equation (4), the solution $u_{4,\theta}$ satisfies boundary conditions (44), and the second term in $G(x, y; W)$ is real for real $W = E$.

It is easy to verify that the guiding functional given by (17) with $U = u_{4,\theta}$ satisfies the properties (1)–(3) cited in section 3.1, whence it follows that the spectra of $\hat{H}_{4,\theta}$ are simple.

The derivative of the spectral function has the form $\sigma'(E) = \pi^{-1} \text{Im} \Omega^{-1}(E + i0)$.

For $E = p^2 \geq 0$, $p \geq 0$, $\lambda = 2p e^{-i\pi/2}$, and $g_1 < 0$, we have

$$\begin{aligned} \sigma'(E) &= \pi^{-1} \text{Im} \Omega^{-1}(E) = \frac{(4\pi\kappa k_0)^{-1} (1 - |D(E)|^2)}{(1 + D(E))(1 + \bar{D}(E))}, \\ D(E) &= \frac{a(E)}{b(E)} = \frac{e^{-2i\theta} \Gamma(\beta) \Gamma(\alpha_-) e^{2i\kappa \ln(k_0/2p)} e^{-\pi\kappa}}{\Gamma(\beta_-) \Gamma(\alpha)}. \end{aligned} \quad (45)$$

Because

$$|D(E)|^2 = \frac{1 + e^{-2\pi\kappa} e^{-\pi g_1/p}}{1 + e^{2\pi\kappa} e^{-\pi g_1/p}} < 1, \quad \forall p \geq 0, \quad (46)$$

we have $\text{spec} \hat{H}_{4,\theta} = \mathbb{R}_+$.

For $E = p^2 > 0$, $p > 0$, $\lambda = 2p e^{-i\pi/2}$, and $g_1 > 0$, expressions (45) and (46) for $\sigma'(E)$ hold true. But in this case, we have $|D(0)| = 1$ and must carefully examine the limit $\lim_{W \rightarrow 0} \Omega^{-1}(W)$.

It is easy to see that for small W , we have the representation

$$\begin{aligned} \Omega^{-1}(W) &= -\frac{i}{4\kappa k_0} \frac{1 + e^{2i(\theta_0 - \theta)}}{[1 - e^{2i(\theta_0 - \theta)}] + iW/A} + O(1), \\ A &= \frac{3g_1^2}{\kappa(1 + 4\kappa^2)}, \\ \theta_0 &= \varphi - \pi[\varphi/\pi], \\ \varphi &= \kappa \ln(g_1/k_0) - \theta_{\Gamma} + \pi/2, \\ \theta_{\Gamma} &= \frac{1}{2i} \ln \frac{\Gamma(\beta)}{\Gamma(\beta_-)}, \end{aligned}$$

where $[\varphi/\pi]$ is the entire part of φ/π . For $\theta \neq \theta_0$, the function $\sigma'(E)$ is finite at $E = 0$. But for $\theta = \theta_0$, we find that

$$\begin{aligned} \sigma'(E + 0) &= -\pi^{-1} (A/2\kappa k_0) \text{Im}(E + i0)^{-1} + O(1) \\ &= (A/2\kappa k_0) \delta(E) + O(1), \end{aligned}$$

which means that the spectrum of the Hamiltonian \hat{H}_{4,θ_0} with $g_1 > 0$ contains the eigenvalue $E = 0$.

For $E = -\tau^2 < 0$, $\tau > 0$, $\lambda = 2\tau$, the function $\Omega(E)$ can be represented as

$$\begin{aligned} \Omega(E) &= [\pi \tan \Theta(E)]^{-1}, \\ \Theta(E) &= \theta + \theta_\Gamma - \theta_\Gamma(E) + \kappa \ln(k_0/2\tau), \end{aligned}$$

where

$$\begin{aligned} \theta_\Gamma(E) &= \frac{1}{2i} [\ln \Gamma(1/2 + g_1/2\tau + i\kappa) - \ln \Gamma(1/2 + g_1/2\tau - i\kappa)] \\ &= \begin{cases} -\pi |g_1|/2\tau + \kappa \ln(|g_1|/2\tau) + O(1), & g_1 < 0, \\ \kappa \ln(g_1/2\tau) + O(\tau), & g_1 > 0, \end{cases} & E \rightarrow 0, \\ \theta_\Gamma(-\infty) &= \frac{1}{2i} \ln \frac{\Gamma(1/2 + i\kappa)}{\Gamma(1/2 - i\kappa)} + O(1/\tau), & E \rightarrow -\infty. \end{aligned}$$

The asymptotic behavior of $\Theta(E)$ at the origin and at minus infinity is given by

$$\Theta(E) = \begin{cases} \begin{cases} \pi |g_1|/2\tau + O(1), & g_1 < 0, \\ \theta + \theta_\Gamma + \kappa \ln(k_0/g_1) + O(\tau), & g_1 > 0, \end{cases} & E \rightarrow 0, \\ \begin{cases} \theta + \theta_\Gamma - \theta_\Gamma(-\infty) \\ + \kappa \ln(k_0/2\tau) + O(1/\tau), \end{cases} & E \rightarrow -\infty. \end{cases}$$

Because $\Omega(E)$ is a real function for $E < 0$, $\sigma'(E)$ can differ from zero only at the points $E_n(\theta)$ where $\Theta(E_n(\theta)) = \pi/2 + \pi n$, $n \in \mathbb{Z}$, which yields

$$\begin{aligned} \sigma'(E) &= \sum_n Q_n^2 \delta(E - E_n(\theta)), \quad Q_n = [4\kappa k_0 \Theta'(E_n(\theta))]^{-1/2}, \\ \Theta'(E_n(\theta)) &> 0. \end{aligned}$$

We can obtain additional information about the discrete spectrum of $\hat{H}_{4,\theta}$. Representing the equation $\Theta(E_n(\theta)) = \pi/2 + \pi n$, $n \in \mathbb{Z}$, in an equivalent form

$$\begin{aligned} f(E_n(\theta)) &= \pi/2 + \pi(n - \theta/\pi), \\ f(E) &= \theta_\Gamma - \theta_\Gamma(E) + \kappa \ln(k_0/2\tau), \\ \partial_\theta E_n(\theta) &= -[f'(E_n(\theta))]^{-1} = -[\Theta'(E_n(\theta))]^{-1} < 0, \end{aligned}$$

we find that the following assertions hold.

- (a) The eigenvalue $E_n(\theta)$ with fixed n decreases monotonically from $E_n(0)$ to $E_n(\pi) - 0$ when θ changes from 0 to $\pi - 0$. In particular, we have $E_{n-1}(\theta) < E_n(\theta)$, $\forall n$.
- (b) For any g_1 , the spectrum is unbounded from below: $E_n(\theta) \rightarrow -\infty$ as $n \rightarrow -\infty$.
- (c) For any θ , the negative part of the spectrum is of the form $E_n(\theta) = -k_0^2 m^2 e^{2\pi|n|/\kappa} (1 + O(1/n))$ as $n \rightarrow -\infty$, where $m = m(g_1, g_2, \theta)$ is a scale factor, and asymptotically (as $n \rightarrow -\infty$) coincides with the negative part of the spectrum in the Calogero model with coupling constant g_2 under appropriate identification of scale factors.

(d) For $g_1 < 0$, the discrete part of the spectrum has an accumulation point $E = 0$. More specifically, the spectrum is of the form $E_n(\theta) = -g_1^2/4n^2 + O(1/n^3)$ as $n \rightarrow \infty$ (as in all the previous ranges of the parameter g_2) and asymptotically coincides with the spectrum for $g_2 = 0$; see below.

(e) For $g_1 > 0$, the discrete spectrum has no finite accumulation points. In particular, possible values of n are restricted from above, $n \leq n_{\max}$, where

$$n_{\max} = \begin{cases} n_0 & \text{if } \begin{cases} f(0)/\pi - 1/2 = n_0, & 0 \leq \theta < \pi \\ f(0)/\pi - 1/2 > n_0, & 0 \leq \theta < \theta_0 \end{cases} \\ n_0 + 1 & \text{if } f(0)/\pi - 1/2 > n_0 \text{ and } \theta_0 \leq \theta < \pi, \end{cases}$$

and the level $E = 0$ is present in the spectrum for $\theta = \theta_0$ only.

The final result is as follows: the spectrum of $\hat{H}_{4,\theta}$ is simple and given by

$$\text{spec } \hat{H}_{4,\theta} = \mathbb{R}_+ \cup \{E_n \leq 0, -\infty < n < n_{\max}\},$$

where $n_{\max} < \infty$ for $g_1 > 0$ and $n_{\max} = \infty$ for $g_1 < 0$, and the set of corresponding (generalized) eigenfunctions

$$\begin{aligned} U_E(x) &= \sqrt{\sigma'(E)} u_{4,\theta}(x; E), \quad E \geq 0, \\ U_n(x) &= Q_n u_{4,\theta}(x; E_n(\theta)), \quad E_n(\theta) \leq 0, \end{aligned}$$

form a complete orthonormalized system in $L^2(\mathbb{R}_+)$.

3.5. The fifth range $g_2 = 0$ ($\mu = 1/2$)

The analysis in this section is similar to that in section 3.2. A peculiarity is that the function u_2 is not defined for $\mu = 1/2$, and we therefore use the following solutions of equation (4):

$$\begin{aligned} u_1(x; W) &= x e^{-z/2} \Phi(\alpha_{1/2}, 2; z), \\ u_5(x; W) &= \tilde{u}_5(x; W) - g_1 \ln k_0 u_1(x; W), \\ v_1(x; W) &= x e^{-z/2} \Psi(\alpha_{1/2}, 2; z) \\ &= \Gamma^{-1}(\alpha_{1/2}) [\omega_{1/2}(W) u_1(x; W) + u_5(x; W)], \end{aligned}$$

where

$$\begin{aligned} \alpha_{1/2} &= 1 + g_1/\lambda, \\ \tilde{u}_5(x; W) &= e^{-z/2} x^{1/2} [x^{-\mu} \Phi(\alpha_-, \beta_-; z) \\ &\quad + g_1 \Gamma(\beta_-) x^\mu \Phi(\alpha_+, \beta_+; z)]_{\mu \rightarrow 1/2}, \\ \omega_{1/2}(W) &= g_1 \mathbf{C} + g_1 [\psi(\alpha_{1/2}) + \ln(\lambda/k_0)] - g_1 - \lambda/2, \end{aligned}$$

where \mathbf{C} is the Euler constant. The asymptotics of these functions at the origin and at infinity are, respectively, as follows.

As $x \rightarrow 0$, $z = \lambda x \rightarrow 0$, we have

$$\begin{aligned} u_1(x; W) &= k_0^{-1} u_{1as}(x) + O(x^2), \\ u_5(x; W) &= u_{5as}(x) + O(x^2 \ln x), \\ v_1(x; W) &= \Gamma^{-1}(\alpha_{1/2}) [k_0^{-1} \omega_{1/2}(W) u_{1as}(x) + u_{5as}(x)] \\ &\quad + O(x^2 \ln x), \\ u_{1as}(x) &= k_0 x, \quad u_{5as}(x) = 1 + g_1 x \ln(k_0 x) + \mathbf{C} g_1 x. \end{aligned} \tag{47}$$

As $x \rightarrow \infty$, $\text{Im } W > 0$, we have

$$u_1(x; W) = \Gamma^{-1}(\alpha_{1/2})\lambda^{-1+g_1/\lambda}x^{+g_1/\lambda}e^{z/2}\tilde{O}(x^{-1}) \rightarrow \infty,$$

$$v_1(x; W) = \lambda^{-g_1/\lambda}x^{-g_1/\lambda}e^{-z/2}\tilde{O}(x^{-1}) \rightarrow 0.$$

The functions $u_1(x; W)$ and $u_5(x; W)$ are real-entire in W . These functions form a fundamental system of solutions of equation (4); the same holds for the functions u_1, v_1 for $\text{Im } W \neq 0$, see section 3.2:

$$\text{Wr}(u_1, u_5) = -1, \quad \text{Wr}(u_1, v_1) = -1/\Gamma(\alpha_{1/2}) = -\omega(W).$$

As we know from section 3.2, at $g_2 < -1/4$, the deficiency indices of the initial symmetric operator \hat{H} are $m_{\pm} = 1$, and therefore there exists a one-parameter family of its s.a. extensions.

To evaluate the asymmetry form Δ_{H^+} , we determine the asymptotics of functions ψ_* , belonging to $D_{\hat{H}}^*(\mathbb{R}_+)$, at the origin using the representation (23) of the general solution of equation (14) with $W = 0$, where the natural substitutions $a_2u_2 \rightarrow a_2u_5$ and $u_2/2\mu \rightarrow u_5$ must be made, and estimating the integral terms by means of the Cauchy–Bunyakovskii inequality, which yields

$$\psi_*(x) = a_1u_{1as}(x) + a_2u_{5as}(x) + O(x^{3/2}),$$

$$\psi'_*(x) = a_1u'_{1as}(x) + a_2u'_{5as}(x) + O(x^{1/2}),$$

and we find that $\Delta_{H^+}(\psi_*) = -k_0(\overline{a_1}a_2 - \overline{a_2}a_1)$. This structure of Δ_{H^+} confirms that the deficiency indices of \hat{H} are $m_{\pm} = 1$. The requirement that Δ_{H^+} vanishes results in the relation $a_1 \cos \epsilon = a_2 \sin \epsilon$, $\epsilon \in \mathbb{S}(-\pi/2, \pi/2)$.

The final result is that for $g_2 = 0$, there exists a family of s.a. Hamiltonians $\hat{H}_{5,\epsilon}$ with the domains $D_{H_{5,\epsilon}}$ that consist of functions from $D_{\hat{H}}^*(\mathbb{R}_+)$ with the following asymptotic behavior as $x \rightarrow 0$:

$$\psi = C\psi_{5,\epsilon as}(x) + O(x^{3/2}),$$

$$\psi' = C\psi'_{5,\epsilon as}(x) + O(x^{1/2}), \tag{48}$$

$$\psi_{5,\epsilon as}(x) = u_{1as}(k_0x) \sin \epsilon + u_{5as}(x) \cos \epsilon.$$

Therefore,

$$D_{H_{5,\epsilon}} = \left\{ \psi : \psi \in D_{\hat{H}}^*(\mathbb{R}_+), \psi \text{ satisfy (48)} \right\}.$$

To find the Green's function $G(x, y; W)$ for $\hat{H}_{5,\epsilon}$, we use representation (15) with $a_1 = 0$ for $\psi_*(x)$ belonging to $D_{H_{4,\theta}} \subset D_{\hat{H}}^*(\mathbb{R}_+)$, boundary conditions (48) and asymptotics (47); then we find that

$$G(x, y; W) = \Omega^{-1}(W)u_{5,\epsilon}(x; W)u_{5,\epsilon}(y; W) - \frac{1}{k_0} \begin{cases} \tilde{u}_{5,\epsilon}(x; W)u_{5,\epsilon}(y; W), & x > y, \\ u_{5,\epsilon}(x; W)\tilde{u}_{5,\epsilon}(y; W), & x < y, \end{cases}$$

where

$$\Omega(W) = k_0 [k_0 \sin \epsilon - \omega_{1/2}(W) \cos \epsilon] \times [\omega_{1/2}(W) \sin \epsilon + k_0 \cos \epsilon]^{-1},$$

$$u_{5,\epsilon}(x; W) = k_0u_1(x; W) \sin \epsilon + u_5(x; W) \cos \epsilon,$$

$$\tilde{u}_{5,\epsilon}(x; W) = k_0u_1(x; W) \cos \epsilon - u_5(x; W) \sin \epsilon,$$

$$k_0\Gamma(\alpha_{1/2})v_1(x; W) = [\omega_{1/2}(W) \cos \epsilon - k_0 \sin \epsilon] \tilde{u}_{5,\epsilon}(x; W) + [\omega_{1/2}(W) \sin \epsilon + k_0 \cos \epsilon] u_{5,\epsilon}(x; W).$$

We note that $u_{5,\epsilon}(x; W)$ and $\tilde{u}_{5,\epsilon}(x; W)$ are solutions of equation (4) real-entire in W , the solution $u_{5,\epsilon}(x; W)$ satisfies boundary conditions (48), and the second summand in $G(x, y; W)$ is real for real $W = E$.

It is easy to verify that the guiding functional given by (17) with $U = u_{5,\epsilon}$ satisfies the properties (1)–(3) cited in section 3.1, whence it follows that the spectra of $\hat{H}_{5,\epsilon}$ are simple.

The derivative of the spectral function is given by $\sigma'(E) = \pi^{-1} \text{Im } \Omega^{-1}(E + i0)$.

We first consider the case of $\epsilon = \pi/2$, where we have $u_{5,\pi/2}(x; W) = k_0u_1(x; W)$ and

$$\sigma'(E) = (\pi k_0^2)^{-1} \text{Im } \Theta(E + i0),$$

$$\Theta(W) = g_1\psi(\alpha_{1/2}) + g_1 \ln(\lambda/k_0) - \lambda/2.$$

For $E = p^2 \geq 0$, $p \geq 0$, $\lambda = 2p e^{-i\pi/2}$, we have

$$\sigma'(E) = \frac{|g_1|e^{-\pi g_1/2p}}{2k_0^2 \sinh(\pi |g_1|/2p)} \geq 0.$$

For $E = -\tau^2 < 0$, $\tau > 0$, $\lambda = 2\tau$ and $g_1 > 0$, $\alpha_{1/2} = 1 + g_1/2\tau$, the function $\Theta(E)$ is finite and real, whence it follows that there are no negative spectrum points.

For $E = -\tau^2 < 0$, $\tau > 0$, $\lambda = 2\tau$ and $g_1 < 0$, $\alpha_{1/2} = 1 - |g_1|/2\tau$, we have

$$\sigma'(E) = \sum_{n \in \mathbb{Z}_+} \frac{4}{k_0^2} \left(\frac{|g_1|}{2+2n} \right)^3 \delta(E - \mathcal{E}_n),$$

$$\mathcal{E}_n = -\frac{g_1^2}{(2+2n)^2}, \quad n \in \mathbb{Z}_+.$$

It is easy to see that for the case of $\epsilon = -\pi/2$, we obtain the same results for the spectrum and eigenfunctions as it must be.

We thus find that for $g_1 > 0$, the spectrum of $\hat{H}_{5,\pi/2}$ is simple, continuous and given by $\text{spec } \hat{H}_{5,\pm\pi/2} = \mathbb{R}_+$ and the set of generalized eigenfunctions $U_E(x) = \sqrt{\sigma'(E)}u_{5,\pi/2}(x; E)$, $E \geq 0$, forms a complete orthonormalized system in $L^2(\mathbb{R}_+)$.

For $g_1 < 0$, the spectrum of $\hat{H}_{5,\pm\pi/2}$ is simple and given by $\text{spec } \hat{H}_{5,\pm\pi/2} = \mathbb{R}_+ \cup \{\mathcal{E}_n, n \in \mathbb{Z}_+\}$ and the set of (generalized) eigenfunctions

$$U_E(x) = \sqrt{\sigma'(E)}u_{5,\pi/2}(x; E), \quad E \geq 0,$$

$$U_n(x) = \frac{2}{k_0} \left(\frac{|g_1|}{2+2n} \right)^{3/2} u_{5,\pi/2}(x; \mathcal{E}_n), \quad n \in \mathbb{Z}_+,$$

forms a complete orthonormalized system in $L^2(\mathbb{R}_+)$.

We now turn to the case $|\epsilon| < \pi/2$ where we have

$$\sigma'(E) = (\pi \cos^2 \epsilon)^{-1} \text{Im } [\omega_5(E + i0)]^{-1},$$

$$\omega_5(W) = k_0 \tan \epsilon - \omega_{1/2}(W).$$

For $g_1 < 0$, $E = p^2 \geq 0$, $p \geq 0$, $\lambda = 2p e^{-i\pi/2}$, we find that

$$\sigma'(E) = \frac{B(E)}{\pi \cos^2 \epsilon [A^2(E) + B^2(E)]}, \tag{49}$$

where $\omega_5(E) = A(E) - iB(E)$. The function $B(E)$ is explicitly given by

$$B(E) = \frac{\pi}{2} \frac{|g_1|e^{-\pi g_1/2p}}{\sinh(\pi|g_1|/2p)} > 0, \quad \forall p \geq 0. \quad (50)$$

It follows that for $g_1 < 0, E \geq 0$, the spectrum of $\hat{H}_{5,\epsilon}$ is purely continuous.

For $g_1 > 0, E = p^2 > 0, p > 0, \lambda = 2p e^{-i\pi/2}$, the derivative of the spectral function is also given by equations (49) and (50). But in this case, we have $B(0) = 0$ and the limit $\lim_{W \rightarrow 0} \omega_5(W)$ has to be carefully examined. For small W , we have

$$\omega_5(W) = (\tan \epsilon - \tan \epsilon_0)k_0 - \frac{1}{3g_1}W + O(W^2),$$

$$\tan \epsilon_0 = (g_1/k_0) [\ln(g_1/k_0) + C - 1].$$

For $\epsilon \neq \epsilon_0$, the function $\sigma'(E)$ has a finite limit as $E \rightarrow 0$. But for $\epsilon = \epsilon_0$ and small E , we have

$$\sigma'(E) = \frac{3g_1}{\cos^2 \epsilon_0} \delta(E) + O(1),$$

which means that the spectrum of the Hamiltonian \hat{H}_{5,ϵ_0} has an eigenvalue $E = 0$.

For $E = -\tau^2 < 0, \tau > 0, \lambda = 2\tau$, the function $\omega_5(E)$ is real. Therefore, $\sigma'(E)$ can differ from zero only at zero points $E_n(\epsilon)$ of $\omega_5(E)$ and $\sigma'(E)$ is represented as

$$\sigma'(E) = \sum_n [-\omega'_5(E_n(\epsilon))]^{-1} \delta(E - E_n(\epsilon)),$$

$$\omega_5(E_n(\epsilon)) = 0, \quad \omega'_5(E_n(\epsilon)) < 0.$$

For $g_1 > 0$, we have

$$\omega_5(E) = -g_1 \psi(1 + g_1/2\tau) - g_1 \ln(2\tau/g_1) + \tau$$

$$+ k_0(\tan \epsilon - \tan \epsilon_0),$$

$$\omega_5(E) = \sqrt{|E|} - (g_1/2) \ln |E| + O(1), \quad E \rightarrow -\infty,$$

$$\omega_5(0) = k_0(\tan \epsilon - \tan \epsilon_0).$$

For $\epsilon > \epsilon_0$, the equation $\omega_5(E) = 0$ has no solution, while for $\epsilon \in (-\pi/2, \epsilon_0]$ it has a unique solution $E^{(-)}(\epsilon)$. It is easy to see that

$$\partial_\epsilon E^{(-)}(\epsilon) = -k_0[\omega'_5(E^{(-)}(\epsilon)) \cos^2 \epsilon]^{-1} > 0,$$

so that $E^{(-)}(\epsilon)$ increases monotonically from $-\infty$ to 0 when ϵ changes from $-\pi/2 + 0$ to ϵ_0 .

For $g_1 < 0$, we have

$$\omega_5(E) = |g_1| \psi(1/2 - |g_1|/2\tau) + |g_1| \ln(2\tau/k_0) + \tau - \tilde{\epsilon},$$

$$\tilde{\epsilon} = g_1 C - g_1 - k_0 \tan \epsilon.$$

Representing the equation $\omega_5(E_n) = 0$ in the equivalent form

$$f(E_n) = \tilde{\epsilon}, \quad f(E) = |g_1| \psi(1/2 - |g_1|/2\tau) + |g_1| \ln(2\tau/k_0) + \tau,$$

we can see that

(a)

$$f(E) \xrightarrow{E \rightarrow -\infty} \infty, \quad f(E_n \pm 0) = \pm \infty,$$

such that in each region of energy $(\mathcal{E}_n, \mathcal{E}_{n+1})$, $n \in (-1) \cup \mathbb{Z}_+$, the equation $\omega_5(E_n) = 0$ has one solution $E_n(\epsilon)$ for any fixed $\epsilon, |\epsilon| < \pi/2$, and $E_n(\epsilon)$ increases monotonically from $\mathcal{E}_n + 0$ to $\mathcal{E}_{n+1} - 0$ as ϵ changes from $-\pi/2 + 0$ to $\pi/2 - 0$ (here, by definition, $\mathcal{E}_{-1} = -\infty$).

(b) For any fixed $\epsilon, E_n(\epsilon) = -g_1^2/4n^2 + O(n^{-3})$ as $n \rightarrow \infty$, asymptotically coinciding with (22).

(c) The point $E = 0$ is an accumulation point of a discrete spectrum for $g_1 < 0$.

Note the relation

$$\lim_{\epsilon \rightarrow \pi/2} E_{n-1}(\epsilon) = \lim_{\epsilon \rightarrow -\pi/2} E_n(\epsilon) = \mathcal{E}_n, \quad n \in \mathbb{Z}_+.$$

The above results can be briefly summarized as follows.

For $g_1 < 0$, the spectrum of $\hat{H}_{5,\epsilon}$ is simple and given by

$$\text{spec } \hat{H}_{5,\epsilon} = \mathbb{R}_+ \cup \{E_n(\epsilon) < 0, \quad n \in (-1) \cup \mathbb{Z}_+\}.$$

The (generalized) eigenfunctions

$$U_E(x) = \sqrt{\sigma'(E)} u_{5,\epsilon}(x; E), \quad E \geq 0,$$

$$U_n(x) = [-\omega'_5(E_n(\epsilon))]^{-1/2} u_{5,\epsilon}(x; E_n(\epsilon)), \quad n \in (-1) \cup \mathbb{Z}_+,$$

form a complete orthonormalized system in $L^2(\mathbb{R}_+)$.

For $g_1 > 0$, the spectrum of $\hat{H}_{5,\epsilon}$ is simple and given by $\text{spec } \hat{H}_{5,\epsilon} = \mathbb{R}_+ \cup \{E^{(-)}(\epsilon) \leq 0\}$. For $\epsilon \in (-\pi/2, \epsilon_0]$ the (generalized) eigenfunctions

$$U_E(x) = \sqrt{\sigma'(E)} u_{5,\epsilon}(x; E), \quad E \geq 0,$$

$$U(x) = [-\omega'_5(E^{(-)}(\epsilon))]^{-1/2} u_{5,\epsilon}(x; E^{(-)}(\epsilon))$$

form a complete orthonormalized system in $L^2(\mathbb{R}_+)$. For $\epsilon > \epsilon_0$, the spectrum has no negative eigenvalues.

We note that the above results (for the spectrum and eigenfunctions) can be extracted from the results in section 3.2 for the case $g_2 \neq 0 (\mu \neq 1/2)$.

4. Some concluding remarks

We would like to finish our consideration with a remark on the Kratzer potential [6] mentioned in the introduction. This potential corresponds to a particular case of parameters $g_2 > 0$ and $g_1 < 0$. It is shown by the thick line in the graph of figure 1. As was already stated, the Kratzer potential is extensively used to describe the molecular structure and interactions [19]. In such cases, the Kratzer potential appears in the radial part of the Schrödinger equation (2) and has the form

$$V(x) = -2D_e \left(\frac{a}{x} - \frac{1}{2} \frac{a^2}{x^2} \right), \quad (51)$$

where D_e is the dissociation energy and a is the equilibrium inter-nuclear separation. As x goes to zero, $V(x)$ goes to

infinity, describing the internuclear repulsion, and as x goes to infinity, $V(x)$ goes to zero, describing the decomposition of molecules. Putting the potential (51) in the radial equation (2) and comparing with the Schrödinger equation (4), we have the following identification:

$$g_1 = -\frac{4m}{\hbar^2} D_e a, \quad g_2 = \frac{2m}{\hbar^2} D_e a^2 + l(l+1).$$

We can now calculate the value of g_2 for real diatomic molecules. Using the data from [20], even for $l = 0$, we have $g_2 = 4.53 \times 10^4$ for CO. The parameter g_2 is of the same order as for molecules of NO, O₂, I₂ and H₂. Thus, we see that for realistic Kratzer potentials, the corresponding radial equations (2) are always of the form (4) with $g_2 > 3/4$. Thus, the corresponding radial problem belongs to the first range described in subsection 3.1. In this case, there exists only one s.a. radial Hamiltonian defined on the natural domain (5); functions from this domain have asymptotics (16).

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