ATTRACTORS AND ASYMPTOTIC STABILITY FOR FUZZY DYNAMICAL SYSTEMS

LAÉCIO C. DE BARROS*, RODNEY C. BASSANEZI*, AND PEDRO A. TONELLI†

ABSTRACT. In this work we study the assymptotic properties of maps on fuzzy spaces which are extensions of maps on $\mathbb{R}^n$. The main results are in section 4 (see Theorem 9) and we give an illustrative example in the last section.

1. PRELIMINARIES

The interest in studying Fuzzy Dynamical Systems appears at least in three different contexts: Images processing,where the functions are contractions (see for instance Cabrelli et Alli [2]); Fuzzy Game Theory, see the work of Klement and Butnariu [1] and Modeling Biological Population Dynamics, see Barros, [3]. Here we give some basic results relating attractors and stable fixed points of the Zadeh's extension of a continuous function in $\mathbb{R}^n$. Our approach is theoretical and is derived from the necessity to have a Fuzzy theory for population dynamics, but has aspects in common with the other branches cited above.

Here we fix some notations and recall known results. The family of all compact nonempty subsets of $\mathbb{R}^n$ will be denoted as $\mathcal{Q}(\mathbb{R}^n)$, while $\mathcal{Q}_c(\mathbb{R}^n)$ is for the subset of $\mathcal{Q}(\mathbb{R}^n)$, whose elements are convex set in $\mathbb{R}^n$.

We also set $\mathcal{F}(\mathbb{R}^n)$ for the family of fuzzy sets $u : \mathbb{R}^n \to [0,1]$ whose α-level:

$$[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\} \quad 0 < \alpha \leq 1 \text{ and } [u]^0 = \text{cl}\{x \in \mathbb{R}^n : u(x) > 0\}$$

are in $\mathcal{Q}(\mathbb{R}^n)$. Finally $\mathcal{E}^n$denotes the family of fuzzy sets whose α-level are in $\mathcal{Q}_c(\mathbb{R}^n)$.

It is known that the metric

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} h([u]^\alpha, [v]^\alpha)$$

where h is the Hausdorff metric in $\mathcal{Q}(\mathbb{R}^n)$, makes the spaces ($\mathcal{F}(\mathbb{R}^n)$,D) and ($\mathcal{E}^n$, D) into complete metric spaces [15].

We have also the endograph metric

$$H(u,v) = h(\text{send}(u), \text{send}(v))$$

1

where

$$\mathrm{send}(u) = ([u]^0 \times [0,1]) \cap \mathrm{end}(u)$$

with

$$\mathrm{end}(u) = \{(x,\alpha) \in \mathbb{R}^n \times [0,1] : u(x) \geq \alpha\}$$

this set is called the endograph [12] and here h means the Hausdorff metric in the corresponding space.
We say that a sequence $A_p \in \mathcal{Q}(\mathbb{R}^n)$ converges to A in the sense of Kuratowski if

$$A = \liminf_{p\to+\infty} A_p = \limsup_{p\to+\infty} A_p$$

where

$$\liminf_{p\to+\infty} A_p = \{x \in \mathbb{R}^n : x = \lim_{p\to\infty} x_p, x_p \in A_p\}$$

and

$$\limsup_{p\to+\infty} A_p = \{x \in \mathbb{R}^n : x = \lim_{p_j\to\infty} x_{p_j}, x_{p_j} \in A_{p_j}\} = \bigcap_{p=1}^{\infty} \overline{\bigcup_{n\geq p} A_p}$$

We quote the following theorem from Hausdorff [10]

Theorem 1. *Let $A_p, A \in \mathcal{Q}(\mathbb{R}^n)$, then the following are equivalents:*
(1) A_p converges to A in the Hausdorff metric h.
(2) A and A_p are contained in a compact set K, and A_p converges to A in the Kuratowski sense.

It is well known that the space$(\mathcal{F}(\mathbb{R}^n), D)$ is complete but not separable [15] whereas $(\mathcal{F}(\mathbb{R}^n), H)$ is separable but not complete [12].
In the following we will need a proposition which can be found in [14]

Proposition 1. *Let u_p be a sequence in $\mathcal{F}(\mathbb{R}^n)$and $u \in \mathcal{F}(\mathbb{R}^n)$. Then the sequence u_p converges in the endograph metric to u if and only if*

$$\begin{aligned} \{u > \alpha\} &\subset \lim_{p\to\infty} \inf\{u_p \geq \alpha\} \\ &\subset \lim_{p\to\infty} \sup\{u_p \geq \alpha\} \\ &\subset \{u \geq \alpha\}, \forall \alpha \in [0,1] \end{aligned} \tag{1}$$

and

$$\lim_{p\to\infty} h([u_p]^0, [u]^0) = 0 \tag{2}$$

In the remainder of this section T is the interval $[a,b] \subset \mathbb{R}$.

Definition 1. *Let $f : T \times \mathbb{R}^n \to \mathbb{R}^n$ be a mapping then for each fixed t we define the Zadeh extension as:*

$$\hat{f}(t,u) = \begin{cases} \sup_{\tau \in f^{-1}(t,x)} u(\tau) & \text{if} \quad f^{-1}(t,x) \neq \emptyset \\ 0 & \text{if} \quad f^{-1}(t,x) = \emptyset \end{cases}$$

for all fuzzy set u.

The proof of the following results can be found in [4]

Theorem 2. *If $f : T \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous then $\hat{f} : T \times \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is well defined and for all $t \in T$ and $\alpha \in [0,1]$ we have*

$$[\hat{f}(t,u)]^\alpha = f(t,[u]^\alpha)$$

Theorem 3. *If $f : T \times \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz with constant K so is the Zadeh extension $\hat{f} : T \times \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$, with the same Lipschitz constant with respect to the metric D.*

Note that if $x \in \mathbb{R}^n$ is a fixed point of a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$, the the fuzzy set χ_x which is one for x and zero for all other points, is a fixed point of $\hat{f}$, hence we have

Corollary 1. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction, then $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is also a contraction. And if x_0 is the only fixed point of f given by the Banach theorem, then χ_{x_0} is the only fixed point of $\hat{f}$.*

Next we recall some definitions for a typical function $F : (\mathcal{F}(\mathbb{R}^n), D) \to (\mathcal{F}(\mathbb{R}^n), D)$ that the reader can find in Hale [8].

2. Discrete Fuzzy Dynamical Systems (Fuzzy Cascades)

A *Discrete Fuzzy Dynamical System* is an iterative system of the form

$$u_{n+1} = F(u_n) \tag{3}$$

where, $F : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is a function.
Given $u_0 \in \mathcal{F}(\mathbb{R}^n)$, the sequence of elements

$$u_0,\ F(u_0),\ F(F(u_0)), \ldots$$

is called the *Positive Orbit of equation (3) from* u_0 and $F^n(u_0)$ denotes the n times composition of F.
Let $\hat{f}$ be a Zadeh's extension of a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ and take the system

$$u_{n+1} = \hat{f}(u_n). \tag{4}$$

We call (4) the *associated* fuzzy system to the deterministic system

$$x_{n+1} = f(x_n). \tag{5}$$

The following theorem gives a result about the solutions of the both equations (4) and (5). In some sense we can say that the deterministic solutions are the preferred solutions of the associated fuzzy systems

Theorem 4. *Suppose that u_n and x_n are the solutions of (4) and (5) through u_0 and x_0 respectively, and assume that $u_0(x_0) = 1$. Then $u_n(x_n) = 1$ for all $n \geq 0$.*

Proof. We have

$$u_{n+1}(x_{n+1}) = \hat{f}(u_n)(x_{n+1}) = \sup_{x_{n+1}=f(\tau)} u_n(\tau) \geq u_n(x_n) \tag{6}$$

since u_n and x_n solutions, hence

$$u_{n+1}(x_{n+1}) \geq u_n(x_n) \geq ... \geq u_0(x_0) = 1.$$

□

Note that for this result f needs not be continuous. What will be important in the continuity of f is that this implies the monotonicity of $\hat{f}$, see our work [4], that is, we have $\hat{f}^{n+1}(u_0) \geq \hat{f}^n(u_0)$ for all $n \geq m$ if $\hat{f}^{m+1}(u_0) \geq \hat{f}^m(u_0)$.

Definition 1. Let $F : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ be a map. A point $\overline{u} \in \mathcal{F}(\mathbb{R}^n)$ is called a *fixed point* of F if $F(\overline{u}) = \overline{u}$.

Observe that $F(\overline{u}) = \overline{u}$ if and only if

$$[F(\overline{u})]^\alpha = [\overline{u}]^\alpha \tag{7}$$

for all $\alpha \in [0, 1]$.
It is clear that if $F : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is a contraction then F has only one fixed point since $(\mathcal{F}(\mathbb{R}^n), D)$ is a complete metric space.
Observe also that if $\overline{u}$ is the fixed point in this case, then

$$D(F^n(u), \overline{u}) \leq \frac{k^n}{1-k} D(F(u), \overline{u}) \tag{8}$$

give us a upper bound for n-th iteration as an approximation of $\overline{u}$. Here k is the contraction constant.
The next theorem is a consequence of Tychonoff fixed point theorem, whose version in the fuzzy space was proved by Kaleva [11].

Theorem 5. *Let K be a compact convex subset of $(\mathcal{E}^n, D)$. Every continuous map $K \to K$ has a fixed point*

Note that if $F : (\mathcal{F}(\mathbb{R}^n), D) \to (\mathcal{F}(\mathbb{R}^n), D)$ is a contraction with a fixed point different from a characteristic function of a point $x \in \mathbb{R}^n$, then F isn't a Zadeh's extension.

Proposition 1. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $A \subset \mathbb{R}^n$ is a compact subset, then A is invariant for f (i.e. $f(A) = A$, see Hale [8]) if and only if, the Characteristic function χ_A is a fixed point of the Zadeh's extension $\hat{f}$.*

Proof. Using the properties of the extension of a continuous function [4] we have

$$[\hat{f}(\chi_A)]^\alpha = f([\chi_A]^\alpha) = f(A) = A = [\chi_A]^\alpha \ \forall \alpha \in [0,1]. \tag{9}$$

From this follows the result immediately. □

In the next section we study the stability of the fixed points using the properties of the space $(\mathcal{F}(\mathbb{R}^n), D)$.

3. Stability of the Fixed Points

Definition 2. A fixed point $\overline{u}$ of $F : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is *stable* if for all $\varepsilon > 0$ there is a $\delta > 0$ such that, for all u with $D(\overline{u}, u) < \delta$, we have $D(F^n(u), \overline{u}) < \varepsilon$ for all $n \geq 0$. A fixed point $\overline{u}$ is *unstable* if it isn't stable, and $\overline{u}$ is *asymptotically stable* if it is stable and there exists $r > 0$ such that for all u satisfying $D(\overline{u}, u) < r$, then $\lim\limits_{n \to +\infty} D(F^n(u), \overline{u}) = 0$.

If $\overline{u}$ is asymptotically stable then $F^n(u) \xrightarrow{H} \overline{u}$ for all u such that $D(\overline{u}, u) < r$ (cf. Kaleva [11]).
Taking into account the definition of the space $(\mathcal{F}(\mathbb{R}^n), D)$ we can rewrite the above definition as:

Definition 3. The fixed point $\overline{u}$ of $F : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is stable if and only if for all $\varepsilon > 0$ there is $\delta > 0$ such that, for all u with $\sup\limits_{0 \leq \alpha \leq 1} h([u]^\alpha, [\overline{u}]^\alpha) < \delta$, then $\sup\limits_{0 \leq \alpha \leq 1} h([F^n(u)]^\alpha, [\overline{u}]^\alpha) < \varepsilon$, for all $n \geq 0$ and $\overline{u}$ is asymptotically stable if it is stable and $\lim\limits_{n \to +\infty} \sup\limits_{0 \leq \alpha \leq 1} h([F^n(u)]^\alpha, [\overline{u}]^\alpha) = 0$ if $\sup\limits_{0 \leq \alpha \leq 1} h([u]^\alpha, [\overline{u}]^\alpha) < r$.

Then if $\overline{u}$ is stable $h([F^n(u)]^\alpha, [\overline{u}]^\alpha) < \varepsilon$ for all $\alpha \in [0,1]$ and $n \geq 0$, which means that

$$\inf\{s : [F^n(u)]^\alpha \subset B([\overline{u}]^\alpha, s) \text{ e } [\overline{u}]^\alpha \subset B([F^n(u)]^\alpha, s)\} < \varepsilon. \tag{10}$$

As a consequence we have that (see Kaleva [11])

$[F^n(u)]^\alpha \subset B([\overline{u}]^\alpha, \varepsilon)$ and $[\overline{u}]^\alpha \subset B([F^n(u)]^\alpha, \varepsilon)$ for all $\alpha \in [0,1]$ and $n \geq 0$.

If $\lim\limits_{n \to +\infty} \sup\limits_{0 \leq \alpha \leq 1} h([F^n(u)]^\alpha, [\overline{u}]^\alpha) = 0$ we have for all $\alpha \in [0,1]$

$$\lim_{n \to +\infty} h([F^n(u)]^\alpha, [\overline{u}]^\alpha) = 0 \tag{11}$$

that implies

$$\lim_{n\to+\infty} \inf([F^n(u)]^\alpha) = \lim_{n\to+\infty} \sup[F^n(u)]^\alpha = [\overline{u}]^\alpha, \tag{12}$$

according Theorem 1. Then, if $\lim_{n\to+\infty} D(F^n(u),\overline{u}) = 0$ we get

$$\begin{aligned}
[\overline{u}]^\alpha &= \{y \in \mathbb{R}^n : y = \lim_{n\to+\infty} y_n,\ y_n \in [F^n(u)]^\alpha\} \\
&= \{y \in \mathbb{R}^n : y = \lim_{j\to+\infty} y_{n_j},\ y_{n_j} \in [F^{n_j}(u)]^\alpha\} \\
&= \bigcap_{j\geq 0} \overline{\bigcup_{n\geq j}[F^n(u)]^\alpha}
\end{aligned}$$

for all $\alpha \in [0,1]$.

Example 1: We consider the system

$$u_{n+1} = F(u_n) \tag{13}$$

where $F : \mathcal{E}^1 \to \mathcal{E}^1$ is given by $F(u) = \lambda u,\ \lambda \in \mathcal{E}^1$. It's clear that $\hat{0} = \chi_{\{0\}}$ is fixed point. Suppose that $[\lambda]^\alpha = [\lambda_1^\alpha, \lambda_2^\alpha]$, $[u]^\alpha = [u_1^\alpha, u_2^\alpha]$ with λ_i^α and u_i^α positive, then

$$[F(u)]^\alpha = [\lambda_1^\alpha u_1^\alpha,\ \lambda_2^\alpha u_2^\alpha],\ \alpha \in [0,1] \tag{14}$$

using the multiplication of fuzzy numbers (see for instance Negoita and Ralescu [13]) .

Given the initial condition u_0 with $[u_0]^\alpha = [u_{01}^\alpha, u_{02}^\alpha]$ and u_{01}^α positive, we have

$$[F^n(u_0)]^\alpha = [\lambda^n u_0]^\alpha = [(\lambda_1^\alpha)^n u_{01}^\alpha, (\lambda_2^\alpha)^n u_{02}^\alpha],\ \alpha \in [0,1]. \tag{15}$$

Then,

$$D(F^n(u_0), \hat{0}) = \sup_{x\in[F^n(u_0)]^0} |x| = (\lambda_2^0)^n u_{02}^0 \text{ for all } u_0 \in \mathcal{E}^1. \tag{16}$$

Hence, as $D(u_0, \hat{0}) = u_{02}^0$, we have asymptotic stability at $\hat{0}$ if $[\lambda]^0 \subset [0,1)$ and instability if $[\lambda]^0 \not\subset [0,1]$.

We remark that the length of the α-level of the solution is

$$(\lambda_2^\alpha)^n\ u_{02}^\alpha - (\lambda_1^\alpha)^n\ u_{01}^\alpha \tag{17}$$

that decreases with n when $[\lambda]^\alpha \subset [0,1)$. What is a different result from the differential fuzzy equations [5]

Let us also comment that, although F is not the extension of any function, we have that all crisp linear system with coefficient a and initial condition x_0 satisfying $\lambda_1^\alpha \leq a \leq \lambda_1^\alpha$, $u_{01}^\alpha \leq x_0 \leq u_{02}^\alpha$ for all $x \in [0,1]$, leads to

$$(\lambda_1^\alpha)^n\ u_{01}^\alpha \leq a^n\ x_0 \leq (\lambda_2^\alpha)^n\ u_{02}^\alpha,\ \forall\alpha \in [0,1]. \tag{18}$$

In other words, the solution of the deterministic system

$$x_{n+1} = ax_n,\quad x_0 \tag{19}$$

has always membership degree one. (Is a preferred solution for the fuzzy system). In fact, this example is extremely useful if one consider systems where the fuzziness appears just in a multiplicative parameter.

4. Stability of the fixed point of the Zadeh's

In this section the map $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is the Zadeh's extension of a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ and we know that $\overline{u}$ is a fixed point of $\hat{f}$ if and only if, the α-levels $[\overline{u}]^\alpha$ are compact invariant sets of f, since

$$[\overline{u}]^\alpha = [\hat{f}(\overline{u})]^\alpha = f([\overline{u}]^\alpha),\ \forall \alpha \in [0,1]. \tag{20}$$

On the other hand, as $[\hat{f}^n(u)]^\alpha = f^n([u]^\alpha)$, then $\lim_{n\to\infty} h([\hat{f}^n(u)]^\alpha,\ [\overline{u}]^\alpha) = 0$, and we have

$$\begin{aligned} [\overline{u}]^\alpha &= \{\lim_{n\to\infty} f^n(x_n),\ x_n \in [u]^\alpha\} \\ &= \{\lim_{j\to+\infty} f^{n_j}(x_j),\ x_j \in [u]^\alpha\} \\ &= \bigcap_{j\geq 0} \overline{\bigcup_{n\geq j} f^n([u]^\alpha)} \end{aligned}$$

this means $[\overline{u}]^\alpha = \omega([u]^\alpha)$ where $\omega(B)$ is the ω-limit set of B. Recall the definition of the ω-limit according Hale [8]

$$\omega(B) = \bigcap_{j\geq 0} \overline{\bigcup_{n\geq j} f^n(B)} \text{ (see Hale [8], p. 8).} \tag{21}$$

In particular, $\omega(x) \subset [\overline{u}]^\alpha$ for all $x \in [u]^\alpha$.
The following Lemma is very important for the rest of this section.

Lemma 1. *Let $\hat{X} \in \mathcal{F}(\mathbb{R}^n)$ the characteristic function of the compact set $X \subset \mathbb{R}^n$. Then the open ball is characterized by*

$$\begin{aligned} B(\hat{X},r) &= \{u \in \mathcal{F}(\mathbb{R}^n) : D(\hat{X},u) < r\} \\ &= \{u \in \mathcal{F}(\mathbb{R}^n) : [u]^0 \subset B(X,r) \quad \text{and } X \subset B([u]^1,r)\} = F \end{aligned}$$

Proof. If $u \in B(\hat{X},r)$, then $\sup_{0\leq\alpha\leq 1} h(X,[u]^\alpha) < r$
or

$$\sup_{0\leq\alpha\leq 1} \inf\{\varepsilon : [u]^\alpha \subset B(X,\varepsilon) \text{ e } X \subset B([u]^\alpha,\varepsilon)\} < r \tag{22}$$

which implies

$$i(\alpha) = \inf\{\varepsilon : [u]^\alpha \subset B(X,\varepsilon) \text{ e } X \subset B([u]^\alpha,\varepsilon)\} < r \tag{23}$$

for all $\alpha \in [0,1]$. Hence $[u]^\alpha \subset B(X,r)$ and $X \subset B([u]^\alpha,r)$ for all $\alpha \in [0,1]$. In particular we have $[u]^0 \subset B(X,r)$ and $X \subset B([u]^1,r)$ then $u \in F$.

If $u \in F$, then $[u]^0 \subset B(X,r)$ and $X \subset B([u]^1, r)$, which implies $[u]^\alpha \subset [u]^0 \subset B(X,r)$ and $X \subset B([u]^1,r) \subset B([u]^\alpha, r)$ for all $\alpha \in [0,1]$.
In this way, as X and $[u]^\alpha$ are compact sets, $B(X,r)$ and $B([u]^\alpha, r)$ are open sets and we have $i(\alpha) < r$ for all $\alpha \in [0,1]$. Now,

$$i(\alpha) \leq \max\{i(0), i(1)\} < r \tag{24}$$

for all $\alpha \in [0,1]$, using properties of the Hausdorff metric, since

$$i(\alpha) = h([u]^\alpha, X) \text{ and } [u]^1 \subset [u]^\alpha \subset [u]^0. \tag{25}$$

Finally,

$$\sup_{0 \leq \alpha \leq 1} i(\alpha) < r \text{ isto é, } u \in B(\hat{X}, r).$$

□

Analogously we can prove for the closed balls that

$$B[\hat{X}, r] = \{u \in \mathcal{F}(\mathbb{R}^n) : [u]^0 \subset B[X,r] \text{ e } X \subset B([u]^1, r]\} \tag{26}$$

and since $\mathcal{F}(\mathbb{R}^n, D)$ is a metric space we have

$$\overline{B(\hat{X}, r)} = B[\hat{X}, r]. \tag{27}$$

Corollary 2. *If X is a set with one element, $X = \{\overline{x}\}$, then*

$$B(\hat{X}, r) = \{u \in \mathcal{F}(\mathbb{R}^n) : [u]^0 \subset B(\overline{x}, r)\} \tag{28}$$

and from the definition of the metric D,

$$D(\chi_{\{\overline{x}\}}, u) = \sup_{x \in [u]^0} ||x - \overline{x}||.$$

With the above Corollary we can establish a result relating the stability of the fixed point $\overline{x}$ and the stability of the fuzzy set $\chi_{\{\overline{x}\}}$. We remark also that this Corollary assure the continuity of $\hat{f}$ in every point of the form $\chi_{\{\overline{x}\}}$, $x \in \mathbb{R}^n$.

Theorem 6. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous with $f(\overline{x}) = \overline{x}$ and $\hat{f}$ its Zadeh's extension. Then*
a) $\chi_{\{\overline{x}\}}$ is stable for the system (4) if and only if, $\overline{x}$ is stable for the system (5)
b) $\chi_{\{\overline{x}\}}$ is asymptotic stable for the system (4) if and only if, $\overline{x}$ is asymptotic stable for the system (5).

Proof. a) Given $\varepsilon > 0$ there is $\delta > 0$ such that if $||x - \overline{x}|| < \delta$ then $||f^n(x) - \overline{x}|| < \varepsilon$ for all $n \geq 0$. If $D(u, \chi_{\{x\}}) < \delta$ then $\sup_{x \in [u]^0} ||x - \overline{x}|| < \delta$ which implies $||f^n(x) - \overline{x}|| < \varepsilon$ for all $x \in [u]^0 \subset B(\overline{x}, \delta)$. Hence $\sup_{x \in [u]^0} ||f^n(x) - \overline{x}|| < \varepsilon$ or $\sup_{y \in f^n([u]^0)} ||y - \overline{x}|| < \varepsilon$ or yet $\sup_{y \in [\hat{f}^n(u)]^0} ||y - \overline{x}|| < \varepsilon$ since f is continuous.

Then, by Corollary 2,

$$D(\hat{f}^n(u), \chi_{\{\overline{x}\}}) < \varepsilon. \tag{29}$$

To show the stability of $\overline{x}$, given the stability of $\chi_{\{\overline{x}\}}$, it is enough to see that

$$||x - \overline{x}|| = D(\chi_{\{x\}}, \chi_{\{\overline{x}\}}) \tag{30}$$

and

$$\hat{f}^n(\chi_{\{x\}}) = \chi_{f^n(x)}. \tag{31}$$

b) There is $\delta > 0$ such that if $||x-\overline{x}|| < \delta$ then $\lim_{n\to\infty} ||f^n(x)-\overline{x}|| = 0$. Suppose that $D(\chi_{\{\overline{x}\}}, u) < \delta$ which is equivalent to $\sup_{x\in([u]^0)} ||x - \overline{x}|| < \delta$ following that for all $x \in [u]^0$, $\lim_{n\to\infty} ||f^n(x) - \overline{x}|| = 0$ or

$$\begin{aligned} 0 &= \lim_{n\to\infty} \sup_{x\in[u]^0} ||f^n(x) - \overline{x}|| = \lim_{n\to\infty} \sup_{y\in f^n([u]^0)} ||y - \overline{x}|| = \\ &= \lim_{n\to\infty} \sup_{y\in[\hat{f}^n(u)]^0)} ||y - \overline{x}|| = \lim_{n\to\infty} D(\hat{f}^n(u), \chi_{\{\overline{x}\}}). \end{aligned}$$

what proves the assertion. □

Corollary 3. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, C^1 and $f(\overline{x}) = \overline{x}$. If λ_i are the eigenvalues of $f'(\overline{x})$, then*
a) If $|\lambda_i| < 1$, for all i then $\chi_{\{\overline{x}\}}$ is asymptotic stable for (4).
b) If $|\lambda_i| > 1$, for some i, $\chi_{\{\overline{x}\}}$ is unstable for (4).

Proof. It follows immediately from the above theorem. □

Again we recall some definitions from Hale [8]
An invariant set J is said to be an *isolated* invariant set, if there is a neighborhood of J such that if K is an invariant set in this neighborhood then $K \subseteq J$.
A set $A \subset \mathbb{R}^n$ *attracts* a set $B \subset \mathbb{R}^n$ under $T : \mathbb{R}^n \to \mathbb{R}^n$ if for all $\varepsilon > 0$ there is $n_0 = n_0(\varepsilon, A, B)$ such that $T^n(B)$ is contained in $B(A, \varepsilon)$ for $n \geq n_0$ where $B(A, \varepsilon)$ is the open ball centered at A and ratio ε.
A is *global attractor* if it is compact maximal invariant set and attract every bounded subset of $\mathbb{R}^n$.
A subset A is a *local attractor* if A is compact invariant and there is a bounded neighborhood B of A such that A attracts B.

Corollary 4. *Let $u \in \mathcal{F}(\mathbb{R}^n)$ and $\hat{u}^\beta$ be characteristic functions of the levels $[u]^\beta$. Then $\hat{u}^\beta \in B(\hat{u}^\alpha, r)$ if and only if, $[u]^\beta \subset B([u]^\alpha, r)$.*

Proof. This is a consequence of the above lemma and the fact that $[u]^\alpha \subset [u]^\beta$ if $\alpha > \beta$. □

We observe that the step functions $u : \mathbb{R}^n \to [0,1]$ are of this kind.

Theorem 7. *Let $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ be a Zadeh's extension of the continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$. If $\overline{u}$ is a fixed point of $\hat{f}$ and $\lim_{n\to\infty} D(\hat{f}^n(u), \overline{u}) = 0$ for $D(\overline{u}, u) < r$, then the levels $[\overline{u}]^\alpha$ attract the levels $[u]^\alpha$ by f.*

Proof. If $u \in B(\overline{u}, r)$, then we have for all $\varepsilon > 0$ that there is $n_0 = n_0(\varepsilon)$ such that if $n \geq n_0$, $D(\hat{f}^n(u), \overline{u}) < \varepsilon$ what ensures $h(f^n([u]^\alpha), [\overline{u}]^\alpha) < \varepsilon \ \forall\alpha$
or
(32)
$h(f^n([u]^\alpha), [\overline{u}]^\alpha) = \inf\{s : f^n([u]^\alpha) \subset B([\overline{u}]^\alpha, s) \text{ e } [\overline{u}]^\alpha \subset B(f^n([u]^\alpha), s)\} < \varepsilon \ \forall\alpha$
leading to

$$f^n([u]^\alpha) \subset B([\overline{u}]^\alpha, \varepsilon) \tag{33}$$

□

Note that in this metric D, there is a kind of uniformity in α in the sense that $n = n(\varepsilon)$ for all α.
As $f^n([u]^\alpha) \subset B([\overline{u}]^\alpha, \varepsilon)$ for $n \geq n_0$ allow us to conclude that for $x \in [u]^\alpha$

$$\{f^n(x)\} \subset B([\overline{u}]^\alpha, \varepsilon) \subset \overline{B([\overline{u}]^\alpha, \varepsilon)}. \tag{34}$$

As $\overline{B([\overline{u}]^\alpha, \varepsilon)}$ is compact, since $[\overline{u}]^\alpha$ is compact, then $f^n(x)$ has a convergent subsequence. Hence $\omega(x) \neq \phi$ for all $x \in [u]^\alpha$.

Corollary 5. *If f and $\hat{f}$ are as in Theorem 7, then*

1. *$\lim_{n\to+\infty} D(\hat{f}^n(u), \overline{u}) = 0$ for all $u \in B(\overline{u}, r), r > 0$, and $X \subset \mathbb{R}^n$ is compact invariant for f with $[\overline{u}]^0 \subset B(X, r)$ and $X \subset B([\overline{u}]^1, r)$, then $\overline{u} = \hat{X}$.*
2. *$\overline{u}$ is a fixed point of $\hat{f}$ with $[\overline{u}]^\alpha \subset B([\overline{u}]^1, r)$ and $[\overline{u}]^0 \subset B([\overline{u}]^\alpha, r)$, for some $\alpha \in [0, 1]$, then $[\overline{u}]^0 = [\overline{u}]^1$.*
3. *If $[\overline{u}]^0 \subset B([\overline{u}]^1, r)$ then $[\overline{u}]^0 = [\overline{u}]^1$.*
4. *the fixed point $\hat{X}$ of $\hat{f}$ is stable if and only if, for all $\varepsilon > 0$ there is $\delta > 0$ such that if $u \in \mathcal{F}(\mathbb{R}^n)$ with $[u]^0 \subset B(X, \delta)$ and $X \subset B([u]^1, \delta)$ then $f^n([u]^0) \subset B(X, \varepsilon)$ and $X \subset B(f^n([u]^1, \varepsilon)$ for all $n \geq 0$.*
5. *If f, $\hat{f}$ and $\hat{X}$ are as above then $\omega(x) \subset X$ for all x near X.*

Proof. 1. It follows from lemma 1 $D(\overline{u}, \hat{X}) < r$, then $\lim_{n\to+\infty} D(\hat{f}^n(\hat{X}), \overline{u}) = 0$ or $\lim_{n\to+\infty} D(\hat{X}, \overline{u}) = 0$ since $\hat{X}$ is a fixed point for $\hat{f}$. Then

$$D(\hat{X}, \overline{u}) = 0 \ \text{ or } \ \overline{u} = \hat{X}.$$

2. In the first item we make $[\overline{u}]^\alpha = X$.
3. Immediate.
4. Consequence of Lemma 1.

5. Let $[u]^\alpha = B[X,\delta]$ with $0 < \delta < r$. By lemma 1 $u \in B(\hat{X}, r)$ then $X = \omega(B[X,\delta])$, and finally $\omega(x) \subset X$.

□

Remark: It follows from Corollaries 5 that the only fixed points of $\hat{f}$, asymptotic stable, with $h([u]^0, [u]^1) < r$, are the characteristic functions of some compact subset X of $\mathbb{R}^n$. Observe that $B[X,\delta] \subset \mathbb{R}^n$ is compact if X is also compact.
The case where the fixed points are isolated plays an important role as in classical dynamical systems. Corollary 4 suggests that we concentrate the study for the fixed points $u : \mathbb{R}^n \to [0,1]$ that are step functions, moreover by Corollary 5 we know that the fixed point $\hat{X}$ of $\hat{f}$ is the characteristic function of the compact set $X \subset \mathbb{R}^n$.
We list some results on the fixed point $\hat{X}$ of $\hat{f}$,

Theorem 8. *Let $\hat{X}$ be the characteristic function of the compact set $X \subset \mathbb{R}^n$ such that $\hat{f}(\hat{X}) = \hat{X}$ and $\hat{X}$ is asymptotic stable for $\hat{f}$. Under these conditions we have*

$$X = \bigcap_{n \geq 0} f^n(B[X,\delta]). \tag{35}$$

for all $\delta \in [0, r[$ with r given by the Definition 2

Proof. Let $u \in \mathcal{F}(\mathbb{R}^n)$ such that $[u]^\alpha = B[X,\delta]$ where $\delta \in [0, r[$.
By lemma 1 $u \in B(\hat{X}, r)$. Hence X attracts $B[X,\delta]$. As $X \subset B[X,\delta]$ and $w(B[X,\delta]) = X$, it follows from Lemma 2.1.1, p. 9, Hale [8], that

$$X = \bigcap_{n \geq 0} f^n(B[X,\delta]).$$

□

Note that in particular X is a local attractor since it attracts the neighborhood $B[X,\delta]$, as seen in Theorem 7, although it is not sufficient for $\hat{X}$ to be asymptotic stable.
Example 2 Consider $x_{n+1} = f(x_n)$, with

$$f(x) = \sqrt[3]{x-3} + 3 - \frac{2\sqrt{3}}{9}. \tag{36}$$

The fixed points of f are $x_1 = 3 - \frac{8}{3\sqrt{3}}$ and $x_2 = 3 + \frac{1}{3\sqrt{3}}$, with $f'(x_2) = 1$. The interval $X = [x_1, x_2]$ is a local attractor for f. We assert that the characteristic function $\hat{X}$, of X, is not asymptotic stable for $\hat{f}$, since that for any $u_0 \in \mathcal{F}(\mathbb{R}^n)$ close to $\hat{X}$, of the form

$$[u_0]^\alpha = [u_0]^0 = [x_1 + \delta,\ x_2 - \delta] \tag{37}$$

for all $\alpha \in [0,1]$, and $\delta > 0$, appropriated

$$[\hat{f}^n(u_0)]^\alpha = f^n([u_0]^0) \tag{38}$$

and

$$\lim_{n\to+\infty} h(f^n([u_0]^0), \{x_1\}) = 0, \tag{39}$$

which implies

$$\lim_{n\to+\infty} h(f^n([u_0]^0), [u_o]^0) \neq 0. \tag{40}$$

Then by Proposition 1, $\hat{f}^n(u_0) \overset{H}{\not\to} \hat{X}$ and it follows $\hat{f}^n(u_0) \overset{D}{\not\to} \hat{X}$, that shows the assertion. In fact, we can see in this example that $X \not\subset B(f^n([u]^1, \varepsilon)$ for

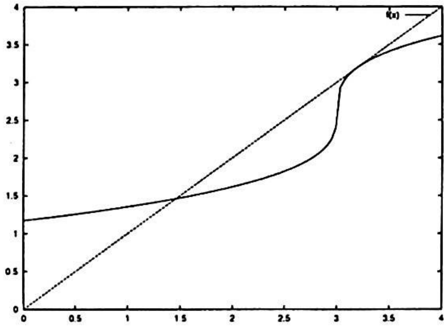

FIGURE 1. Fixed Points of $f(x)$

infinity n, for any interval $[u]^1$ contained in X and by lemma 1, $\hat{X}$ cannot be asymptotic stable.

Lemma 2. *Let $\hat{f}$ be the Zadeh's extension of the continuous function f : $\mathbb{R}^n \to \mathbb{R}^n$. If*
$\lim_{n\to+\infty} D(\hat{f}^n(\hat{x}), u) = 0$ *for any $x \in \mathbb{R}^n$, then u is the characteristic function of some point in $\mathbb{R}^n$.*

Proof. It is enough to observe that

$$\lim_{n\to+\infty} D(\hat{f}^n(\hat{x}), u) = 0 \Rightarrow \lim_{n\to+\infty} \sup_{y\in[u]^0} ||f^n(x) - y|| = 0 \tag{41}$$

which leads to the result. □

Theorem 9. *If f and $\hat{f}$ are as above, $\overline{u}$ is a fixed point of $\hat{f}$, asymptotic stable with diameter $diam[\overline{u}]^0 < r$, when r is from definition 2 then $\overline{u}$ is the characteristic function of some point in R^n, moreover if $\overline{u}$ is globally asymptotic stable, then $\overline{u}$ is the characteristic function of some point in $\mathbb{R}^n$.*

Proof. If $diam[\overline{u}]^0 < r$, then by lemma 1 for all $x \in [\overline{u}]^0$, we have $D(\overline{u}, \hat{x}) < r$. Hence $\lim_{n \to +\infty} D(\hat{f}^n(\hat{x}), \hat{u}) = 0$ and from Lemma 2 we have the first part.

For the second part just note that in this case $\lim_{n \to +\infty} D(\hat{f}^n(\hat{x}), \hat{u}) = 0$ for all $x \in \mathbb{R}^n$. □

Next we give some results about the periodic orbits.

Definition 4. A point $u^* \in \mathcal{F}(\mathbb{R}^n)$ is *periodic point with period p* of $F : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ if p is the least positive integer such that $F^p(u^*) = u^*$, where F^p stands for composition. The set of all iterated of a periodic point is called *p-periodic orbit* or *p-cycle.*

A p-periodic point u^* is a fixed point of F^p. Consequently, the notion of stability of u^* is the same of a fixed point.

Definition 5. A p-periodic point u^* is *stable, asymptotic stable*, or *unstable* if u^* is a point stable, asymptotic stable or unstable respectively of F^p.

If $\hat{f}$ is the Zadeh extension of the continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$, then x^* is p-periodic point of f if and only if, the characteristic function $\hat{x}^*$ of x^* is a p-periodic point of $\hat{f}$, since $[\hat{f}^n(\hat{x})]^\alpha = f^n(x)$ for all $n \geq 0$ and $\alpha \in [0, 1]$, where $\hat{x}$ is the characteristic function $x \in \mathbb{R}^n$. We recall also that $\hat{f}$ is continuous at $\hat{x}$.

Corollary 6. *If $\hat{f}$ is the Zadeh extension of the continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$, then x^* is a stable p-periodic point (asymptotic stable, unstable) for f if and only if, $\hat{x}^*$ is p-periodic stable (asymptotic stable, unstable) for $\hat{f}$.*

Proof. Just apply Theorem 6 for f^p. □

Corollary 7. *Let x^* a 2-periodic point of the function of class $C^1, f : \mathbb{R} \to \mathbb{R}$. If $|f'(f(x^*))f'(x^*)| < 1$, then the orbit by $\hat{x}^*$ is stable.*

Proof. Use the fact that $|f'(f(x^*)).f'(x^*)| < 1$ implies the stability of the orbit by x^*. See Edelstein [7]. □

5. An Example

We consider the normalized logistic function

$$f(x) = ax(1 - x), \tag{42}$$

with $1 \le a \le 4$, as we want just $x \ge 0$. Let us research the fixed points of the Zadeh's extension $\hat{f}$, different from the characteristic functions $\hat{0}$ and $\hat{x}_a, x_a = 1 - \frac{1}{a}$.
According we have seen until here, we have to solve the following equation $f([u]^\alpha) = [u]^\alpha = [u_1^\alpha, u_2^\alpha]$, for all $\alpha \in [0,1]$, i. e.

$$(43) \qquad \begin{cases} u_1^\alpha = \min\limits_{u_1^\alpha \le x \le u_2^\alpha} f(x) \\ u_2^\alpha = \max\limits_{u_1^\alpha \le x \le u_2^\alpha} f(x) \end{cases}$$

If $1 \le a \le 2$, then $x_a = 1 - \frac{1}{a} \le \frac{1}{2}$
Let $u \in \mathcal{E}^1$ be such that $\hat{f}(u) = u$. Then

$$(44) \qquad u_2^0 = \max_{u_1^0 \le x \le u_2^0} f(x) \le \max_{x \in \mathbb{R}} f(x) = f(\frac{1}{2}) = \frac{a}{4} \le \frac{1}{2}.$$

Then, $u_2^\alpha \le \frac{1}{2}$ for all $\alpha \in [0,1]$. Hence, as f is increasing in $(-\infty, \frac{1}{2}]$, 43 is given by

$$(45) \qquad \begin{cases} u_1^\alpha = f(u_1^\alpha) = au_1^\alpha(1 - u_1^\alpha) \\ u_2^\alpha = f(u_2^\alpha) = au_2^\alpha(1 - u_2^\alpha) \end{cases}$$

Then the only solution of equation 45 excluding $\hat{0}$ and $\hat{x}_a$, is the fuzzy set $u \in \mathcal{E}^1$ such that $[u]^\alpha = [0, x_a]$, for all $\alpha \in [0,1]$.
If $a > 2$, then $x_a = 1 - \frac{1}{a} > \frac{1}{2}$. In this case we consider all the possibilities of fixed points using u_1^0 and u_2^0.

- For $u_2^0 \le \frac{1}{2}$, the solutions of 45, for all $\alpha \in [0,1]$, are $u_1^\alpha = 0$ and $u_2^\alpha = x_a > \frac{1}{2}$ what contradicts $u_2^\alpha \le u_2^0 \le \frac{1}{2}$. Then, there are no new fixed points if $u_2^0 \le \frac{1}{2}$.
- For $u_2^0 \ge \frac{1}{2}$, 45 is given by

$$(46) \qquad \begin{cases} u_1^\alpha = f(u_2^\alpha) = au_2^\alpha(1 - u_2^\alpha) \\ u_2^\alpha = f(u_1^\alpha) = au_1^\alpha(1 - u_1^\alpha) \end{cases}$$

Since f is decreasing in $[\frac{1}{2}, +\infty)$.
Changing variables we transform 46 in

$$(47) \qquad \begin{cases} x = ay(1 - y) \\ y = ax(1 - x) \end{cases}$$

from this e have also ,

$$(48) \qquad x = a^2x(1 - x)[1 - ax(1 - x)] = f^2(x),$$

Explaining: the coordinates of the solutions of the above system are exactly the fixed points of $f^2(x)$ which are given by $x = 0$, $x_a = 1 - \frac{1}{a}$ and x_1, $x_2 = \dfrac{a + 1 \pm \sqrt{(a-3)(a+1)}}{2a}$, (See Edelstein [[7]]). So, if $2 < a < 3$, x_1 and x_2 are not reals, and the only solution of 46 is $u_1^\alpha = u_2^\alpha = x_a$, $\forall \alpha \in [0,1]$. On the

other hand if $3 \leq a < 4$, the solutions of 46 are $u_1^\alpha = u_2^\alpha = x_a$ or $u_1^\alpha = x_1$ and $u_2^\alpha = x_2$, $\forall \alpha \in [0,1]$. But in this last case we must observe that $u_1^0 = x_1 \geq \frac{1}{2}$ if and only if, $3 \leq a \leq 1+\sqrt{5}$. In this way, apart $\hat{0}$ and $\hat{x}_a$, we have a new fixed point $\overline{u}$ given by $[\overline{u}]^\alpha = [x_1, x_2] = [x_1, \; f(x_1)]$ for all $\alpha \in [0,1]$.
Finally, if $a = 4$, we have again $u_1^\alpha = u_2^\alpha = x_a$ as unique solution of 46.
• Now, if $u_1^\alpha < \frac{1}{2}$ and $u_2^\alpha > \frac{1}{2}$ for some α, then 43 is given by

$$\begin{cases} u_1^\alpha = \min\{f(u_1^\alpha), f(u_2^\alpha)\} \\ \quad u_2^\alpha = f(\frac{1}{2}) = \frac{a}{4} \end{cases} \tag{49}$$

Hence, if $u_1^\alpha = f(u_1^\alpha)$ then $u_1^\alpha = 0$. If $u_1^\alpha = f(u_2^\alpha)$, then $u_1^\alpha = f(\frac{a}{4}) = \frac{a^2}{16}(4-a)$, we compute that $f(\frac{a}{4}) \leq \frac{1}{2}$, That is,

$$\frac{a^2}{16}(4-a) \leq \frac{1}{2} \iff a \geq 1+\sqrt{5} \tag{50}$$

As a resume we have
• If $1 \leq a \leq 2$, the only fixed points ares: $\hat{0}; \hat{x}_a$ and $\overline{u}_1$ given by $[\overline{u}_1]^\alpha = [0, x_a]$, $\forall \alpha \in [0,1]$.
• If $2 < a \leq 3$, apart $\hat{0}$ and $\hat{x}_a$, we have fixed point $\overline{u}_2$ given by $[\overline{u}_2]^\alpha = [0, \frac{a}{4}]$.
• If $3 < a \leq 1+\sqrt{5}$, excluding $\hat{0}$, $\hat{x}_a$ and $\overline{u}_2$, we have also the fixed point $\overline{u}_3$ with $[\overline{u}_3]^\alpha = [x_1, x_2]$, $\forall \alpha \in [0,1]$.
• If $1+\sqrt{5} < a < 4$, the fixed points are $\hat{0}, \hat{x}_a, \overline{u}_2, \overline{u}_3, \overline{u}_4$, with $[\overline{u}_4]^\alpha = [f(\frac{a}{4}), \frac{a}{4}]$ e $\overline{u}_5$ given by:

$$[\overline{u}_5] = \begin{cases} [0, f(\frac{a}{4})] & \text{If} \quad \alpha \leq \overline{\alpha} \\ [f(\frac{a}{4}), \frac{a}{4}] & \text{If} \quad \alpha > \overline{\alpha} \end{cases} \tag{51}$$

for all $\alpha \in [0,1]$ and some $\overline{\alpha}$.
• If $a = 4$ the only fixed points are $\hat{0}, \hat{x}_a$ and $\overline{u}_6$ with $[\overline{u}_6]^\alpha = [0,1]$ for all $\alpha \in [0,1]$:
We are going to study the stability of the new fixed points for the function $\hat{f}$ since the old one have the same behavior of the deterministic counterpart.
For $1 < a \leq 2$, the fixed point $\overline{u}_1$ is unstable (cf. Corollary 5) since $D(u, \overline{u}_1) < \delta$ if $[u]^\alpha = [\delta, x_a]$ and $f^n([\delta, x_a]) = [f^n(\delta), \; x_a]$ for $\delta > 0$, small, since that f is increasing $[0, \frac{1}{2}]$. Then $[0, x_a] \not\subset B(f^n([u]^1, \varepsilon)$ for infinitely many $n \in \mathbb{N}$.
For $2 < a \leq 4$ we have:
• $\overline{u}_2, \overline{u}_3$ and $\overline{u}_6$ aren't stable according Corollary 5.
• $\overline{u}_5$ is also unstable since given $0 < \delta < f(\frac{a}{4})$, the fuzzy set u defined by its levels

$$[u]^\alpha = \begin{cases} [\frac{\delta}{2}, \frac{a}{4}] & \text{If} \quad \alpha \leq \overline{\alpha} \\ [f(\frac{a}{4}), \frac{a}{4}] & \text{If} \quad \alpha > \overline{\alpha} \end{cases} \tag{52}$$

is such that $D(u,\overline{u}_5) = \frac{\delta}{2} < \delta$ and $D(\hat{f}^n(u),\overline{u}_5) > \varepsilon$ for infinitely many $n \in \mathbb{N}$ taking $0 < \varepsilon < \delta$, since that

$$(53) \qquad D(\hat{f}^n(u),\overline{u}_5) = \sup_{0\leq\alpha\leq1} h(f^n([u]^\alpha),\ [\overline{u}_5]^\alpha) = h(f^n([\frac{\delta}{2},\frac{a}{4}]),[0,\frac{a}{4}])$$

and $f^n([\frac{\delta}{2},\frac{a}{4}]) \subset [\varepsilon,\frac{a}{4}]$ for n big, since f is increasing in $(0,\frac{1}{2}]$.

• $\overline{u}_4$ is asymptotic stable:

First we show the stability:

By continuity of f, we have that $\varepsilon > 0$, exists $0 < \delta < \varepsilon$ with $\frac{a}{4} - \delta > x_a > \frac{1}{2}$ and $f(\frac{a}{4}) - \delta > 0$ such that $f(\frac{a}{4} - \delta) < f(\frac{a}{4}) + \varepsilon < \frac{1}{2}$. Hence if $D(u,\overline{u}_4) < \delta$, that is,

$$(54) \qquad [u]^0 \subset B([f(\frac{a}{4}),\frac{a}{4}],\delta) \text{ e } [f(\frac{a}{4}),\frac{a}{4}] \subset B(f([u]^1),\delta),$$

we have since $x_a > \frac{1}{2}$, that

$$(55) \qquad f([f(\frac{a}{4})+\delta,\frac{a}{4}-\delta]) = [f(\frac{a}{4}-\delta),\frac{a}{4}]$$

and

$$(56) \qquad f^2([f(\frac{a}{4})+\delta,\frac{a}{4}-\delta]) = [f(\frac{a}{4}),\frac{a}{4}],$$

concluding that

$$(57) \qquad f^n([f(\frac{a}{4})+\delta,\frac{a}{4}-\delta]) = [f(\frac{a}{4}),\frac{a}{4}] \quad \forall n \geq 2.$$

Then $[f(\frac{a}{4}),\frac{a}{4}] \subset B(f^n([f(\frac{a}{4})+\delta,\frac{a}{4}-\delta]),\ \varepsilon) \quad \forall n \geq 0$.
from hypothesis $[f(\frac{a}{4})+\delta,\frac{a}{4}-\delta] \subset [u]^1$, it follows that

$$(58) \qquad [f(\frac{a}{4}),\frac{a}{4}] \subset B(f^n([u]^1),\varepsilon) \quad \forall n \geq 0.$$

Taking $[u]^0 = [l,m]$ and supposing that $[u]^0 \not\subset [f(\frac{a}{4}),\frac{a}{4}] = [\overline{u}_4]^0$, since in this case we have nothing to do.

$$(59) \qquad f^n([u]^0) = [\min\{f^n(l),\ f^n(m)\},\frac{a}{4}]$$

for all $n \geq 1$.

As $f^n(l)$ and $f^n(m)$ are increasing sub sequences with the same limits $f(\frac{a}{4})$, we have

$$(60) \qquad f^n([u]^0) \subset B([f(\frac{a}{4}),\frac{a}{4}],\varepsilon) \quad \forall n \geq 0$$

Using Corollary 5 we have that $\overline{u}_4$ is stable.

It is clear that $\overline{u}_4$ is asymptotic stable since $\lim_{n\to+\infty} D(\hat{f}^n(u),\overline{u}_4) = 0$ for all $u \in \mathcal{E}^1$ with $D(u,\overline{u}_4) < r$ where r is such that $f(\frac{a}{4}) - r > 0$ and $\frac{a}{4} + r < 1$.

In the classical (crisp) case $a = 3$ is a *bifurcation* value, i. e. if a is greater than 3 the fixed point x_a lose its stability and appears a 2-periodic orbit.

If a is greater than $1+\sqrt{6}$, it will appear a stable 4-periodic orbit.
From about $a = 3.89$ we have chaos. (See Hale [9], Edelstein [7], Devaney [6]).
In the fuzzy case, $a = 1$, $a = 2$, $a = 3$, $a = 1+\sqrt{5}$ are also values of bifurcations. We summarize our results in figure 2

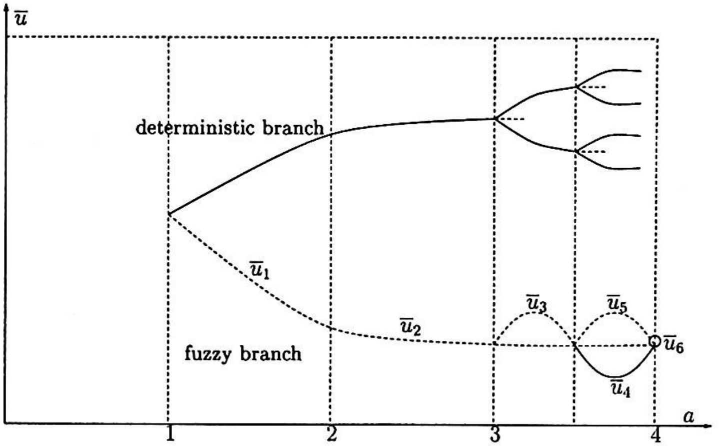

FIGURE 2. Bifurcation Diagram

REFERENCES

[1] Butanariu, D. ; Klement, E. P. - " Core, Value and Equilibria for Market Games: On a Problem of Aumann and Shapley"- *Int. Journal of Game Theory* **25**, pp 149-160 (1996).

[2] Cabrelli, C.A.; Forte, B.; Molter U. M.; Vrscay E. R. - Iterated Fuzzy Set Systems: A new Approach to the Inverse Problem for Fractals and other sets- *Journ. Math. Anal. and Appl.* **171**, pp 79-100 (1992).

[3] Barros, L. C. - *Sobre Sistemas Dinamicos Fuzzy* - Phd Thesis, Unicamp, Campinas (1997).

[4] Barros, L. C. ; Bassanezi, R. C. ; Tonelli, P. A. " On the continuity of Zadeh's extension" - Pre-print (1996).

[5] Barros, L. C. ; Bassanezi, R. C. ; Tonelli, P. A. " Remarks on Deterministic Orbits of Fuzzy Dynamical Systems " - Pre-print (1996).

[6] Devaney, R. L.- *An Introduction to Chaotic Dynamical Systems*- Benjamin/Cummings, Menlo Park (1989).

[7] Edelstein-Keshet, L.-*Mathematical Models in Biology*- Mc Graw-Hill - Mexico (1988).

[8] Hale, J. K.; - *Asymptotic Behavior of Dissipative Systems*- Math. Surveys and Monographs **25**, American Mathematical Society, Providence (1988).
[9] Hale, J. K. and Koçak, H. - *Dynamics and Bifurcations* - Springer Verlag, New York (1991).
[10] Hausdorff, F.- *Set Theory*- Chelsea Press, New York (1957)
[11] Kaleva, O. - " On the Convergence of Fuzzy Sets" - Fuzzy Sets and Systems, **17**, pp. 53-65 (1985).
[12] Kloeden, P. E. - Compact Supported Endographs and Fuzzy Sets - *Fuzzy Sets and Systems*, **4**, pp. 193-201 (1980).
[13] Negoita C. V. and Ralescu, D. A.- *Applications of Fuzzy Sets to Systems Analisis*- Birkhäuser - Basel (1975)
[14] Quelho, E. F. R. - Sobre Γ-convergencias - Phd Thesis - Unicamp - Campinas (1989).
[15] Puri, M. L. ; Ralescu, D. A. - Fuzzy Random Variables- *Journ. Math. Anal. and Appl.* **114**, pp 409-422 (1986)
[16] Rojas-Medar, M. ; Roman-Flores, H. - On the equivalence of convergence of fuzzy sets - *Fuzzy Sets and Systems*, **80**, pp. 217-224 - (1996)

* INSTITUTO DE MATEMÁTICA, UNIVERSIDADE ESTADUAL DE CAMPINAS, C. P. 6065, CEP 13081-979, CAMPINAS - SP - BRASIL

† INSTITUTO DE MATEMÁTICA E ESTATISTICA, UNIVERSIDADE DE SÃO PAULO, C. P. 66281 CEP 05315-970, SÃO PAULO - SP - BRASIL