CLOSED HYPERSURFACES OF S^4 WITH CONSTANT MEAN CURVATURE AND CONSTANT SCALAR CURVATURE

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§ 0 Introduction

S. S. Chern proposed the following problem (see [C, D, K] and [Y_2]):

"Let Σ be the set of closed minimal immersed hypersurfaces of S^{n+1} with constant scalar curvature. Let $\kappa: \Sigma \to \mathbb{R}$ be the function given by: $\kappa(M)$ = scalar curvature of M.

Question: Is $\kappa(\Sigma)$ a discrete set of real numbers?"

Probably the main motivation for this conjecture is Simon's formula ([S]) according to which, if $M \in \Sigma$ and $\kappa(M) < 1$ then $\kappa(M) \leq 1 - \frac{1}{n-1}$. It comes immediately from the definitions of minimality and scalar curvature that $\kappa(M) \leq 1$ for $M \in \Sigma$. It is clear that if $M \in \Sigma$ and $\kappa(M) = 1$ then M is a great round hypersphere of S^{n+1}. Chern, Do Carmo and Kobayachi ([C, D, K]) proved that if $M \in \Sigma$ and $\kappa(M) = 1 - \frac{1}{n-1}$ then M is the riemannian product of two round spheres.

Peng and Terng ([P, T]) made a breakthrough on this problem proving that if $M \in \Sigma$ and $\kappa(M) < 1 - \frac{1}{n-1}$ then $\kappa(M) < 1 - \frac{1}{n-1} - \frac{1}{12n^2(n-1)}$. Unfortunately, we don't know examples realizing this last value for the scalar curvature of M. However, for the special case where n = 3, the same authors got a stronger and sharp result that reads:

If dim M = 3, $M \in \Sigma$ and $\kappa(M) < \frac{1}{2}$ then $\kappa(M) \leq 0$.

Notice that $\kappa(M) = 0$ is realized by Cartan's minimal isoparametric hypersurface of S^4 (see [C] and also [N]).

In this paper, we extend Peng and Terng's result for hypersurfaces of S^4 having constant mean curvature. Moreover we prove uniqueness of Cartan's isoparametric hypersurfaces of S^4 in the sense that they are the only closed hypersurfaces of S^4 having constant mean curvature and vanishing scalar curvature. Precisely, we prove:

Theorem:

"Let $M^3 \subset W^4$ be a 3-dimensional closed immersed hypersurface in a 4-dimensional space of constant curvature W. If M^3 has constant mean curvature and constant non negative scalar curvature then M is isoparametric."

Corollary 1:

"If $M^3 \subset S^4$ is a closed 3-dimensional immersed hypersurface of S^4 with constant mean curvature H and constant non negative scalar curvature κ, then M^3 is isoparametric and consequently one of the following three possibilities holds:

1) M^3 is a round sphere

2) M^3 is a Clifford torus $S^2(r) \times S^1(t)$ where $S^2(r)$ and $S^1(t)$ are a sphere and a circle of radius r and t respectively.

3) M^3 is the unique isoparametric hypersurface of S^4 with vanishing scalar curvature and given mean curvature H."

Corollary 2:

"The only closed hypersurfaces of S^4 with constant mean curvature and every where vanishing scalar curvature are Cartan's isoparametric hypersurfaces of S^4."

Constancy of both κ and H is crucial for the characterization theorem stated above. Indeed, Hsiang ([Hs_1], [Hs_2]) gave many (embedded!) examples of non isoparametric hyperspheres of S^{n+1} (of S^4, in particular) having constant mean curvature. We should mention that hypersurfaces of S^{n+1} with constant mean curvature and restrictions on the sectional curvatures or on the Ricci curvature had already been treated by Yau [Y1] and Nomizu and Smyth [N, S].

The specific case of closed hypersurfaces of S^{n+1} with constant mean curvature and constant scalar curvature was studied by Okumura [O]. It was proved that $\kappa > 1 - \frac{2}{n(n-1)} + \frac{n(n-2)}{(n-1)^2} H^2$ implies the hypersurface is a round sphere. Our Corollary 1 provides a complete and sharp answer for this pinching problem in the case $n = 3$ and $\kappa > 0$. We should finally remark that the existence of closed hypersurfaces of S^{n+1} with negative scalar curvature is, as far as we know, an open problem.

§ 1 Proof of the theorem

It is clearly sufficient to prove the theorem for oriented M^3 and the proof will be divided into two parts:

1st part: Suppose M^3 has three distinct principal curvatures at each point. Let $\{e_1, e_2, e_3\}$ be a local orthonormal frame belonging to the orientation of M^3 and dyagonalizing the second fundamental form of the isometric immersion $i: M^3 \to W$. Let $\{\omega_1, \omega_2, \omega_3\}$ be the associated local coframe. We are, locally, in the following situation:

a) $Ae_i = \lambda_i e_i \quad 1 \leq i \leq 3$ on U, $\lambda_1 < \lambda_2 < \lambda_3$ on M.

b) $\omega_i(e_j) = \delta_{ij}$, $\omega_i \in \Lambda^1(U, \mathbb{R})$

where A denotes the matriz of the second fundamental form of the immersion, λ_i is a principal curvature, U is an open set of M and δ_{ij} is the classical Kroenecker symbol.

Define the 2-form $\Psi \in \Lambda^2(U, \mathbb{R})$ by

$$\Psi = \omega_{12} \wedge \omega_3 - \omega_{13} \wedge \omega_2 - \omega_1 \wedge \omega_{23}, \qquad (1)$$

ω_{ij}, $1 \leq i, j \leq 3$ being local Cartan connection forms associated to the given local orthonormal frame in such a way that

$$d\omega_i = -\sum_{i=1}^{3} \omega_{ij} \wedge \omega_j \quad , \quad \omega_{ij}(u) = \langle \nabla_u e_j, e_i \rangle,$$

Fact 1: Ψ is globally definable:

To see this let us consider another local frame $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ defined on an open set $V \subset M$. Suppose $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ satisfies a) and b). If, moreover, $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ belongs to the orientation of M then one of the four possibilities holds in $U \cap V$.

i) $\bar{e}_i = e_i$, $1 \leq i \leq 3$; ii) $\bar{e}_1 = \bar{e}_1$, $\bar{e}_2 = -\bar{e}_2$, $\bar{e}_3 = -\bar{e}_3$; ; iii) $\bar{e}_1 = -\bar{e}_1$, $\bar{e}_2 = e_2$, $\bar{e}_3 = -e_3$; iv) $\bar{e}_1 = -e_1$, $\bar{e}_2 = -e_2$, $\bar{e}_3 = e_3$

Now, defining $\bar{\Psi}$ in the same way as Ψ one can easily check that $\Psi_{U\cap V} = \bar{\Psi}_{U\cap V}$ □

Fact 2: $d\Psi = \left| \dfrac{|\nabla f|^2}{9(\lambda_1-\lambda_2)^2(\lambda_2-\lambda_3)^2(\lambda_3-\lambda_1)^2} + 3\kappa \right| \nu$

where:

$f = \lambda_1^3 + \lambda_2^3 + \lambda_3^3$ = trace A^3 = trace $(A \circ A \circ A)$

κ is the scalar curvature function of M

and ν is the volume element of M

Proof:

Using Cartan structure equations,

$$d\omega_{ij} = -\sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj} + \Omega_{ij} , \quad d\omega_i = -\sum_{j=1}^{3} \omega_{ij} \wedge \omega_j \ ; \ 1 \leq i, j \leq 3$$

and (1) yields:

$$d\Psi = -\omega_{13} \wedge \omega_{23} \wedge \omega_3 - \omega_{21} \wedge \omega_{31} \wedge \omega_1 + \omega_{12} \wedge \omega_{32} \wedge \omega_2 +$$
$$+ \Omega_{12} \wedge \omega_3 - \Omega_{13} \wedge \omega_2 + \omega_1 \wedge \Omega_{23} \qquad (2)$$

Notice that $\Omega_{ij} = R_{ij}\, \omega_i \wedge \omega_j$, $R_{ij} = c + \lambda_i \lambda_j$ being the sectional curvature of M in the direction of the plane generated by e_i and e_j and c being the sectional curvature of the constantly curved ambient space W.

Let us write

$\omega_{ij} = \sum_{k=1}^{3} \Gamma_{ij}^{k}\, \omega_k$ and use Codazzi equations to get:

$$\Gamma_{ij}^{i} = -\Gamma_{ji}^{i} = (\lambda_j - \lambda_i)\; d\lambda_i\,(e_j) \tag{3}$$

$$\frac{\Gamma_{12}^{3}}{\lambda_2-\lambda_1} = \frac{\Gamma_{13}^{2}}{\lambda_3-\lambda_1} = \frac{\Gamma_{23}^{1}}{\lambda_3-\lambda_2} \tag{4}$$

Let us now denote

$\lambda_{ij} = d\lambda_i\,(e_j)$.

Using (2), (3) and (4) one gets:

$$d\Psi = [3\kappa - (\frac{\lambda_{13}\,\lambda_{23}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)} + \frac{\lambda_{21}\,\lambda_{31}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} + \frac{\lambda_{12}\,\lambda_{32}}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)})\Big|\; \nu$$

where $\kappa = c + \frac{1}{3}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) = \frac{1}{3}(R_{12} + R_{13} + R_{23})$

The relation between κ and H is given by

$$\kappa = \frac{1}{6}(9H^2 - S) + c$$

where:

$H = \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3)$ is the mean curvature of M

$S = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ is the square of the norm of the second fundamental form of M.

So, if K and H are constant so is S. Deriving S and H gives

$$dS\,(e_k) = 2\sum_{i=1}^{3} \lambda_i\, \lambda_{ik} = 0 \;,\; 1 \leq k \leq 3 \tag{5}$$

$$dH\,(e_k) = \sum_{i=1}^{3} \lambda_{ik} = 0 \quad,\; 1 \leq k \leq 3 \tag{6}$$

Now, using (5) and (6), we may write

$$d\tau = \left| \kappa + \frac{\lambda_{11}^2}{(\lambda_2 - \lambda_1)^2} + \frac{\lambda_{22}^2}{(\lambda_3-\lambda_1)^2} + \frac{\lambda_{33}^2}{(\lambda_2-\lambda_3)^2} \right| \nu \qquad (7)$$

A straightforward computation using (5) and (6) gives:

$$df(e_1) = f_1 = 3(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\,\lambda_{11},$$
$$df(e_2) = f_2 = 3(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)\,\lambda_{22}; \qquad (8)$$
$$df(e_3) = f_3 = 3(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)\,\lambda_{33}$$

Observe that $\nabla f = f_1 e_1 + f_2 e_2 + f_3 e_3$ and this ends the proof of fact 2. □

Applying Stokes theorem yields the desired result for hypersurfaces having three distinct principal curvatures at each point.

2nd part: General case, i.e., there are points $x \in M$ such that $\lambda_i(x) = \lambda_j(x)$, $i \neq j$:

We proceed by steps and general strategy is: M non isoparametric implyes $\kappa < 0$

Step 1) If there is a point $x \in M$ such that $\lambda_1(x) = \lambda_2(x) = \lambda_3(x)$: then M is isoparametric.

Proof:

Suppose $\lambda_1 = \lambda_2 = \lambda_3$ on a point x. So $S = 3H^2$ on x and because S and H are constant, $S = 3H^2$ all over M.

Now, $3H^2 = \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3)^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = S$ all over M and so $0 = \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3)^2 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 = -\frac{1}{3}[(\lambda_1 - \lambda_2)^2 + (\lambda_2-\lambda_3)^2 + (\lambda_3-\lambda_1)^2]$ all over M.

Therefore $\lambda_1 = \lambda_2 = \lambda_3 = H$ are constant on M □

Let us now suppose that there are no points $x \in M$ such that $\lambda_1(x) = \lambda_2(x) = \lambda_3(x)$ and that $\lambda_1 \leq \lambda_2 \leq \lambda_3$ on M

Step 2: Maximum and minimum of f:

Proposition

$f = \lambda_1^3 + \lambda_2^3 + \lambda_3^3$ attains its maximum when $\lambda_1 = \lambda_2$ and its minimum when $\lambda_2 = \lambda_3$ whenever points of this type exist.

Proof: It is sufficient to study the function

$g: \mathbb{R}^3 \to \mathbb{R}$ given by

$g(x, y, z) = x^3 + y^3 + z^3$ subject to the restrictions

$x + y + z = H$ and $x^2 + y^2 + z^2 = S$ using Lagrange's multiphyers method. ◻

Step 3: Analyticity of f:

Let us now observe that M is a real analytic submanifold of W and $f: M \to \mathbb{R}$ is a real analytic function on M. This is a consequence of a well known theorem stating that hypersurfaces of constant mean curvature on spaces of constant curvature are real analytic submanifolds (see, for example [M]). In particular f has a finite number of critical values because M is closed.

Proposition 2:

If $\lambda_1 = \lambda_2$ on M or $\lambda_2 = \lambda_3$ on M then M is isoparametric.

Proof:

Just solve the system on the unknowns $\lambda_1, \lambda_2, \lambda_3$:

$$\lambda_1 + \lambda_2 + \lambda_3 = H$$
$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = S$$
$$\lambda_1 = \lambda_2 \quad (\text{resp } \lambda_2 = \lambda_3)$$

and write λ_i in terms of H and S. ◻

Suppose now $p \in M$ satisfies $\lambda_1(p) = \lambda_2(p)$ and take $c_2 \in \mathbb{R}$ to be a regular value of f close to max $f = f(p)$. Suppose $c_2 \neq f(p)$ (this is possible otherwise, by proposition 2, M is isoparametric). Denote

$L_{c_2} = f^{-1}(c_2)$ the closed level surface of f in M determined by c_2.

Analogously, take $q \in M$ such that $\lambda_2(q) = \lambda_1(q)$ in such a way that min $f = f(q)$ and $c_1 \neq f(q)$; c_1 being a regular value of f close to $f(q)$. Let $f^{-1}(c_1) = L_{c_1}$ be the corresponding level surface.

Setting

$N_{c_1}^{c_2} = f^{-1}([c_1, c_2])$ gives a hypersurface of W^* with boundary $L_{c_1} \cup L_{c_2}$.

We convention that $L_{c_1} = \emptyset$ if there are no points q on M such that $\lambda_2(q) = \lambda_1(q)$. In the same way, $L_{c_2} = \emptyset$ if there are no points p on M such that $\lambda_1(p) = \lambda_2(p)$.

Step 4: Applying Stokes to $N_{c_1}^{c_2}$ and Ψ:

In our case, Stokes theorem with boundary reads:

$$\int_{N_{c_1}^{c_2}} d\Psi = \int_{L_{c_2}} i_2^*\Psi + \int_{L_{c_1}} i^*\Psi$$

Here, $i_1 : L_{c_1} \to N_{c_1}^{c_2}$ and $i_2 : L_{c_2} \to N_{c_1}^{c_2}$ are natural inclusions and the boundary is "outward" oriented, i.e., the normal vector to L_{c_2} has the same sense as ∇f and the normal vector to L_{c_1} has the opposite sense of ∇f.

Step 5: Applying a theorem due to Robert M. Hardt (see[Ha]).

Theorem (R.M. Hardt)

Let f be a real analytic function of a compact sub analytic subset of a real analytic manifold M. Let k be a positive integer. Then,

$$\sup \{\mathcal{H}^k (f^{-1}(\{y\})) \text{ such that } \dim f^{-1}(\{y\}) \leq k \} < \infty$$

where $\mathcal{H}^k$ represents the k-dimensional Hausodorff measure.

Applying this theorem to the situation described in this paper we get the following:

Proposition 3:

There is a real number T such that

area $(L_c = f^{-1}(\{c\})) < T$, for all regular values c of f. □

Step 6: Computing $i_1^* \Psi$ and $i_2^* \Psi$:

$$i_2^* \Psi = \frac{1}{|\nabla f|} \left| \frac{(\lambda_1 - H)}{(\lambda_1-\lambda_2)^2 (\lambda_3-\lambda_1)^2} f_1^2 + \frac{(\lambda_2 - H)}{(\lambda_1-\lambda_2)^2 (\lambda_2-\lambda_3)^2} f_2^2 + \right.$$

$$\left. + \frac{(\lambda_3 - H)}{(\lambda_2-\lambda_3)^2 (\lambda_3-\lambda_1)^2} f_3^2 \right| \nu_2 \; , \tag{9}$$

$$i_1^* \Psi = -\frac{1}{|\nabla f|} \left| \frac{(\lambda_1 - H)}{(\lambda_1-\lambda_2)^2 (\lambda_3-\lambda_1)^2} f_1^2 + \frac{(\lambda_2 - H)}{(\lambda_1-\lambda_2)^2 (\lambda_2-\lambda_3)^2} f_2^2 + \right.$$

$$\left. + \frac{(\lambda_3 - H)}{(\lambda_2-\lambda_3)^2 (\lambda_3-\lambda_1)^2} f_3^2 \right| \nu_1 \; , \tag{10}$$

, ν_1 and ν_2 standing for volume elements of L_{c_1} and L_{c_2} respectively.

Proof:

Notice that

$$\psi = \Gamma^{1}_{12}\ \omega_1 \wedge \omega_3 + \Gamma^{2}_{12}\ \omega_2 \wedge \omega_3 - \Gamma^{1}_{13}\ \omega_1 \wedge \omega_2 - \Gamma^{3}_{13}\ \omega_3 \wedge \omega_2 -$$

$$- \Gamma^{3}_{23}\ \omega_1 \wedge \omega_3 - \Gamma^{2}_{23}\ \omega_1 \wedge \omega_2$$

Now, using fōrmulas (3) and (8) yields:

$$\Psi = \frac{(\lambda_1 - H)}{(\lambda_1-\lambda_2)^2 (\lambda_3-\lambda_1)^2}\, f_1\ \omega_2 \wedge \omega_3 + \frac{(\lambda_2 - H)}{(\lambda_1-\lambda_2)^2 (\lambda_2-\lambda_3)^2}\, f_2\ \omega_3 \wedge \omega_1 +$$

$$+ \frac{(\lambda_3 - H)}{(\lambda_2-\lambda_3)^2 (\lambda_3-\lambda_1)^2}\, f_3\ \omega_1 \wedge \omega_2$$

Now, observe that

$i_1^{*}\ (\omega_2 \wedge \omega_3) = \langle e_1, n_1 \rangle\ \nu_1$, n_1 being the "outward" normal vector to L_{c_1} on M. Also,

$i_1^{*}\ (\omega_3 \wedge \omega_1) = \langle e_2, n_1 \rangle\ \nu_1$,

$i_1^{*}\ (\omega_1 \wedge \omega_2) = \langle e_3, n_1 \rangle\ \nu_1$ and similar formulas hold for i_2 and n_2 (the "outward" normal vector to L_{c_2} on M).

It is important to observe that:

a) n_1 has the opposite sense and the same direction of ∇f because of the orientation and the fact that L_{c_1} is a level surface of f. So, $n_1 = - \dfrac{\nabla f}{|\nabla f|}$

b) By the same reasons, n_2 has the same sense and direction of ∇f and $n_2 = \dfrac{\nabla f}{|\nabla f|}$

c) $\nabla f \neq 0$ on L_{c_1} and on L_{c_2} because c_1 and c_2 are regular values of f.

Therefore,

$$i_1^* \Psi = \frac{(\lambda_1 - H)}{(\lambda_1 - \lambda_2)^2(\lambda_3 - \lambda_1)^2} f_1 \, i^*(\omega_2 \wedge \omega_3) + \frac{(\lambda_2 - H)}{(\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2} f_2 \, i^*(\omega_3 \wedge \omega_1)$$

$$+ \frac{(\lambda_3 - H)}{(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2} f_3 \, i^*(\omega_1 \wedge \omega_2) =$$

$$= -\frac{1}{|\nabla f|}\left| \frac{(\lambda_1 - H)}{(\lambda_1 - \lambda_2)^2(\lambda_3 - \lambda_1)^2} f_1^2 + \frac{(\lambda_2 - H)}{(\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2} f_2^2 + \right.$$

$$\left. + \frac{(\lambda_3 - H)}{(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2} f_3^2 \right| \nu_1$$

and the same computation is applyable to $i_2^* \Psi$. □

Step 7: Controlling integral in the boundary:

Given $\epsilon > 0$, $\exists$ c_1, c_2 sufficiently close to $f(q)$ and $f(p)$ respectively, such that

$$\left| \int_{L_{c_2}} \frac{(\lambda_3 - H)}{(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_2)^2 \, |\nabla f|} \cdot f_3^2 \right| < \frac{\epsilon}{2} \quad \text{and}$$

$$\left| \int_{L_{c_1}} \frac{(H - \lambda_1)}{(\lambda_1 - \lambda_2)^2 \, (\lambda_3 - \lambda_1)^2 \, |\nabla f|} \, f_1^2 \right| < \frac{\epsilon}{2} .$$

Proof: Clearly, $\frac{f_1^2}{|\nabla f|} \leq |\nabla f|$ and $\frac{f_2^2}{|\nabla f|} \leq |\nabla f|$

Also, $|\nabla f| \to 0$ as $c_1 \to q$ or $c_2 \to p$.

Moreover, $(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2 > L_1 > 0$ or M for some constant L_1. This is so because $\lambda_1 \leq \lambda_2 \leq \lambda_3$, M is closed and, by assumption, there are no points where $\lambda_1 = \lambda_2 = \lambda_3$.

In the same way, $(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2 > L_2 > 0$ on M for some constant L_2. Now, apply step 5 and get step 7.

Step 8: Conclusion:

First suppose $L_{c_1} \neq \emptyset$ and $L_{c_2} \neq \emptyset$.

Notice that $\lambda_1 - H < 0$ and $\lambda_3 - H > 0$ because $\lambda_1 \leq \lambda_2 \leq \lambda_3$ and $H = \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3)$.

Moreover, for a point q_1 close to q and a point p_1 close to p we have

$$\lambda_2(p_1) - H < 0 \quad \text{and} \quad \lambda_2(q_1) - H > 0$$

Therefore, by step 7,

$$\int_{L_{c_1}} i_1^* \gamma - \int_{L_{c_2}} i_2^* \gamma < c \text{ for } c_1 \text{ close to } f(q) \text{ and } c_2 \text{ close to } f(p).$$

So, by step 4,

$$\int_{N_{c_1}^{c_2}} d\gamma < c$$

Now, taking $\bar{c}_2$, $\bar{c}_1$ such that $c_2 < \bar{c}_2 < f(q)$, $f(p) > \bar{c}_1 > c_1$ we get $N^{\bar{c}_2}_{\bar{c}_1} \supset N^{c_2}_{c_1}$ and

$$\int_{N^{\bar{c}_2}_{\bar{c}_1}} d\,\Psi < \frac{\varepsilon}{n} \quad \text{for a suitable choice of } \bar{c}_1 \text{ and } \bar{c}_2 \text{ for a given } n \in N.$$

Suppose by contradiction that $\kappa > 0$.

Using formula (7) yields

$$\int_{N^{c_2}_{c_1}} \frac{\lambda_{11}^2}{(\lambda_2-\lambda_3)^2} + \frac{\lambda_{22}^2}{(\lambda_3-\lambda_1)^2} + \frac{\lambda_{33}^2}{(\lambda_1-\lambda_2)^2} < \int_{N^{\bar{c}_2}_{\bar{c}_1}} \frac{\lambda_{11}^2}{(\lambda_2-\lambda_3)^2} +$$

$$+ \frac{\lambda_{22}^2}{(\lambda_3-\lambda_1)^2} + \frac{\lambda_{33}^2}{(\lambda_1-\lambda_2)^2} < \frac{\varepsilon}{n} \quad \forall n \in N,\ \bar{c}_1\bar{c}_2 \text{ conveniently chosen.}$$

Consequently $\lambda_{ii} \equiv 0 \quad 1 \leq i \leq 3$ on $N^{c_2}_{c_1}$

The choice of c_1, c_2 being arbitrary implyes:

$\lambda_{ii} \equiv 0$ in $M - (f^{-1}(p) \cup f^{-1}(q))$

Then $\lambda_{ii} \equiv 0$ in one non empty open set of M and so, because of formulae (8), $\nabla f \equiv 0$ on this open set. Analyticity of f implyes f is constant on M. Therefore M is isoparametric (contradiction).

Finally, if $L_{c_1} = L_{c_2} = \emptyset$, the proof was given in part 1.

If $L_{c_1} \neq \emptyset$ and $L_{c_2} = \emptyset$, then.

$$\int_M d\Psi = \int_{L_{c_1}} i_1^* \Psi$$

If $L_{C_2} \neq \emptyset$ and $L_{C_1} = \emptyset$ then

$\int_M d\Psi = \int_{L_{C_2}} i_2^* \Psi$. In both cases proofs are obvious reductions of the general one. □

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