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Colombeau's theory and shock
waves in a problem of hydrodynamics

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COLOMBEAU'S THEORY AND SHOCK WAVES

IN A PROBLEM OF HYDRODYNAMICS

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Introduction

The   aim   of this paper is to study the existence of discontinuous

(shocks) solutions for the system of hydrodynamics with   viscosity

in a one dimensional space

$$
\begin{cases}
\rho_t + (\rho u)_x \approx 0 \\
(\rho u)_t + (p + \rho u^2)_x \approx [(\nu \cdot \rho) u_x]_x \\
E_t + [(E + p)u]_x \approx [(\nu \cdot \rho) u u_x]_x \\
E \approx \lambda p + \frac{1}{2} \rho u^2
\end{cases}
$$

where $\nu \in C^\infty(R_+^*, R_+^*)$ is a   strictly increasing function   satisfying

some others conditions oportunely stated , $\lambda \neq 0$ is a real number

and the symbol $\approx$ denotes the association relation (see DEF. 1.1

and [C], sections 2 and 3).

In §1   we   state   some   basic   definitions and results about the

simplified algebra $\varsigma_s(\Omega)$. In   §2   we introduce the exponential and

the   logarithm of a generalized function and this is used to solve

a class of ordinary differential equations of first order in $\varsigma_s(R)$

In §2 we introduce also the composition in a very particular case.

The first two sections present the basic tools for the study of the above problem in the third section.

1 THE SIMPLIFIED ALGEBRA $\varsigma_s(\Omega)$

Our main reference for Colombeau's theory, containing an organized exposition of its basic kernel, is [A-B]. Here we summarize some special concepts which will be needed to state and solve the above problem. For the general notation, see [A-B],1.1. Here Ω denotes a non void open subset of R^n and the notation $K \subset\subset \Omega$ means that K is a compact subset of Ω. It is known that we can construct a differential algebra $\varsigma(\Omega) = \varsigma(\Omega,K)$ (in what follows the notation K denotes indistinctly R or C) which contains, as a vector subspace, the set $\mathcal{D}'(\Omega) = \mathcal{D}'(\Omega,K)$ of all distributions on Ω in such way that the partial derivation in $\varsigma(\Omega)$ generalizes exactly the partial derivation in $\mathcal{D}'(\Omega)$ (see [A-B], 2 and 5). The simplified algebra $\varsigma_s(\Omega)$ is a subalgebra of $\varsigma(\Omega)$ whose definition we shall give briefly (see [A-B], 8.1.3 (d)).

In what follows we denote by I the interval $]0,1] \subset R$ and we denote by Γ the set of all strictly increasing functions $\gamma : N \longrightarrow R_+^*$ such that $\lim_{q \to +\infty} \gamma(q) = +\infty$. The set

$$\mathcal{E}_s[\Omega] = \mathcal{E}_s[\Omega,K]: = \{u \in K^{I \times \Omega} \mid u(\epsilon, \cdot) \in C^\infty(\Omega,K) \text{ for every } \epsilon \in I\}$$

endowed with the pointwise operations is a K-algebra. The set

$\mathcal{E}_{M,s}[\Omega] = \mathcal{E}_{M,s}[\Omega,K] := \{u \in \mathcal{E}_s[\Omega,K] | \forall \; K \subset\subset \Omega \text{ and } \forall \; \alpha \in \mathbb{N}^n \; \exists \; N \in \mathbb{N},$
$c > 0 \text{ and } \eta \in I \text{ such that } |\partial^\alpha u(\epsilon,x)| \leq c\epsilon^{-N} \text{ whenever } x \in K \text{ and}$
$\epsilon \in]0,\eta] \}$

is a subalgebra of $\mathcal{E}_s[\Omega,K]$ and the set

$\mathcal{N}_s[\Omega] = \mathcal{N}_s[\Omega,K] := \{u \in \mathcal{E}_{M,s}[\Omega] | \forall \; K \subset\subset \Omega \text{ and } \forall \; \alpha \in \mathbb{N}^n \; \exists \; N \in \mathbb{N}$

and $\forall \in \Gamma$ such that $\forall \; q \geq N \; \exists \; c > 0 \text{ and } \eta \in I \text{ such that}$
$|\partial^\alpha u(\epsilon,x)| \leq c\epsilon^{\gamma(q)-N} \text{ whenever } x \in K \text{ and } \epsilon \in]0,\eta]\}$

is an ideal of $\mathcal{E}_{M,s}[\Omega]$. The elements of $\mathcal{E}_{M,s}[\Omega]$ (resp. $\mathcal{N}_s[\Omega]$) are
called moderate (resp. null) functions on Ω. We define the
algebra of (simplified) generalized functions on Ω by

$$\mathcal{G}_s(\Omega) = \mathcal{G}_s(\Omega,K) := \frac{\mathcal{E}_{M,s}[\Omega]}{\mathcal{N}_s[\Omega]}$$

The map $f \in C^\infty(\Omega) \longrightarrow \text{class}(\hat{f}) \in \mathcal{G}_s(\Omega)$, where $\hat{f}(\epsilon,x) := f(x)$ for
all $(\epsilon,x) \in I \times \Omega$, is an injective homomorphism of algebras by means
of which we identify $C^\infty(\Omega)$ with a subalgebra of $\mathcal{G}_s(\Omega)$. We have no
natural inclusion map either from $C(\Omega)$ or $\mathcal{D}'(\Omega)$ into $\mathcal{G}_s(\Omega)$ but this
is not relevant for our purposes. If Ω' and Ω are open subsets of
$\mathbb{R}^n$ such that $\emptyset \neq \Omega' \subset \Omega$ and $f \in \mathcal{G}_s(\Omega)$, the restriction of f to Ω',
denoted by $f|\Omega'$, is defined as the class of $\hat{f}|I \times \Omega'$ in $\mathcal{G}_s(\Omega')$, where
$\hat{f} \in \mathcal{E}_{M,s}[\Omega]$ is any representative of f. If $\alpha \in \mathbb{N}^n$ and $f \in \mathcal{G}_s(\Omega)$,
the partial derivative of order α of f, denoted by $\partial^\alpha f$, is defined
as the class of $\partial^\alpha \hat{f}$ in $\mathcal{G}_s(\Omega)$, where $\hat{f} \in \mathcal{E}_{M,s}[\Omega]$ is an arbitrary

representative of f and $\partial^\alpha \hat{f}(\varepsilon,x) := [\partial^\alpha \hat{f}(\varepsilon,\cdot)](x)$ for every $(\varepsilon,x) \in I \times \Omega$. It can be easily verified that the definitions of $f|\Omega'$ and $\partial^\alpha f$ are independent of the representative $\hat{f}$ of f. From [A-B], 8.1.3(f) we know that $\mathcal{G}_s$ is a fine sheaf of K-algebras on R^n. If $f \in \mathcal{G}_s(\Omega)$ and $(V_\lambda)_{\lambda \in \Lambda}$ is the family of all open subsets of Ω such that $f|V_\lambda = \emptyset$ for every $\lambda \in \Lambda$, it follows that $f|V = \emptyset$ where $V = \bigcup_{\lambda \in \Lambda} V_\lambda$ is the largest open subset of Ω where f vanishes. The set $\Omega \cap \complement V$ is called the support of f and is denoted by $\text{supp}(f)$.

The generalized (real and complex) numbers are the elements of a K - algebra $\bar{K}_s$ defined as follows (see [A-B], 8.2 and 3.1). The set

$$\mathcal{G}_{M,s}(K) := \{ u \in K^I | \exists\, N \in \mathbb{N},\, c > 0 \text{ and } \eta \in I \text{ such that } |u(\varepsilon)| \leq c\varepsilon^{-N}$$

whenever $0 < \varepsilon < \eta \}$,

endowed with the pointwise operations is a K-algebra and the set

$$\mathcal{R}_s(K) := \{ u \in \mathcal{G}_{M,s}(K) | \exists\, N \in \mathbb{N} \text{ and } \exists\, \gamma \in \Gamma \text{ such that } \forall\, q \geq N\, \exists\, c > 0$$

and $\eta \in I$ such that $|u(\varepsilon)| \leq c\varepsilon^{\gamma(q)-N}$ whenever $0 < \varepsilon < \eta \}$

is an ideal of $\mathcal{G}_{M,s}(K)$. We define the algebra of (simplified) generalized numbers by

$$\bar{K}_s := \frac{\mathcal{G}_{M,s}(K)}{\mathcal{R}_s(K)} .$$

The natural map $z \in K \longrightarrow \text{class}(\varepsilon \longrightarrow z) \in \bar{K}_s$ and, for a given Ω, the natural map induced by

$$u \in \mathcal{G}_{M,s}(K) \longrightarrow [(\varepsilon,x) \longrightarrow u(x)] \in \mathcal{G}_{M,s}[\Omega] ,$$

are injetive and hence we can identify K (resp. $\bar{K}_s$) with a

subalgebra of $\bar{K}_s$ (resp. $\varsigma_s(\Omega)$). The elements of the image of $\bar{K}_s$ in

$\varsigma_s(\Omega)$ are called generalized constants.

The concepts of integral and association defined in the sequel are

derived from the same concepts in $\varsigma(\Omega)$. Since here we introduce

$\varsigma_s(\Omega)$ directly, they appear in this paper as definitions. If M is

a Lebesgue-measurable subset of $\mathbb{R}^n$ such that $\bar{M} \subset\subset \Omega$ and $f \in \varsigma_s(\Omega)$,

we define the integral of f on M (see [A-B], 8.2.2 (b)) by

$$\int_M f = \int_M f(x)dx : = \text{class}(\epsilon \longrightarrow \int_M \hat{f}(x)dx) \in \bar{K}_s$$

where $\hat{f} \in \varsigma_{M,s}[\Omega]$ is any representative of f. Clearly the

integral of f on M indep, ends on the representative $\hat{f}$ of f.

If $f \in \varsigma_s(\Omega)$ and $\text{supp}(f) \subset\subset \Omega$, it is easy to see that if K and L

are two compact subsets of Ω containing $\text{supp}(f)$ in its interior

then the integrals of f on K and L are equal. Then, we define (see

[A-B], 4.2.1) the integral of f on Ω as

$$\int_\Omega f = \int_\Omega f(x)dx : = \int_K f$$

where K is any compact subset of Ω containing $\text{supp}(f)$ in its

interior. The integral of a generalized function has the usual

linear properties.

We say that a generalized number $z \in \bar{K}_s$ is associated with Θ,

denoted by $z \approx \Theta$, if there exists a (or equivalently, for each)

representative $\hat{z} \in \mathcal{E}_{M,s}(K)$ of z such that $\lim_{\varepsilon \to 0} \hat{z}(\varepsilon) = 0$. Two elements $z_1, z_2 \in \bar{K}_s$ are associated if $z_1 - z_2 \tilde{\approx} 0$ and we denote this fact by $z_1 \tilde{\approx} z_2$.

DEFINITION 1.1 We say that $f, g \in \mathcal{E}_s(\Omega)$ are associated and we denote by $f \tilde{\approx} g$, if for every $\psi \in \mathcal{D}(\Omega)$ we have

$$\int_\Omega (f - g)\psi \tilde{\approx} 0 \quad \text{in} \quad \bar{K}_s .$$

Given $f, g \in \mathcal{E}_s(\Omega)$, the definition of association in $\bar{K}_s$ shows that $f \tilde{\approx} g$ if and only if, for any representatives $\hat{f}$ and $\hat{g}$ of f and g respectively and for any $\psi \in \mathcal{D}(\Omega)$, we have

$$\lim_{\varepsilon \to 0} \int_\Omega [\hat{f}(\varepsilon,x) - \hat{g}(\varepsilon,x)]\psi(x)dx = 0 .$$

The association relation has the following properties (see [A-B], 6.3.1).

PROPOSITION 1.2 (a) If $f_i, g_i \in \mathcal{E}_s(\Omega)$ and $f_i \tilde{\approx} g_i$ $(i = 1,2)$ then $f_1 + f_2 \tilde{\approx} g_1 + g_2$.

(b) If $\psi \in C^\infty(\Omega)$, $f, g \in \mathcal{E}_s(\Omega)$ and $f \tilde{\approx} g$, then $\psi f \tilde{\approx} \psi g$.

(c) If $f, g \in \mathcal{E}_s(\Omega)$, $f \tilde{\approx} g$ and $\alpha \in \mathbb{N}^n$, then $\partial^\alpha f \tilde{\approx} \partial^\alpha g$.

(d) If $f, g \in \mathcal{E}_s(\mathbb{R})$ and $\frac{\partial f}{\partial x_1} \tilde{\approx} \frac{\partial g}{\partial x_1}$, then there is $\phi \in \mathcal{E}_s(\mathbb{R})$ such that $\frac{\partial \phi}{\partial x_1} = 0$ and $f - g \tilde{\approx} \phi$. ∎

DEFINITION 1.3 Let $T \in \mathcal{D}'(\Omega)$. We say that $f \in \mathcal{E}_s(\Omega)$ has the distribution T as macroscopic aspect if there is a representative $\hat{f}$ of f such that, for every $\psi \in \mathcal{D}(\Omega)$, we have

(1.3.1) $\quad \lim\limits_{\varepsilon \longrightarrow 0} \int_{\Omega} \hat{f}(\varepsilon,x)\psi(x)dx = \langle T,\psi \rangle$.

Clearly, any other representative of f satisfies (1.3.1) for every $\psi \in \mathcal{D}(\Omega)$.

NOTATION From now on we will work only with real (classical and generalized) functions, so we can simplify the above notations $\varsigma_s(R,R)$, $\mathcal{D}'(R,R)$, $C^\infty(R,R)$, etc. to $\varsigma_s(R)$, $\mathcal{D}'(R)$, $C^\infty(R)$, etc.

DEFINITION 1.4 (a) An element $f \in \varsigma_s(R)$ is said to be a Heaviside g.f. (Heaviside generalized function) if it has the Heaviside function as macroscopic aspect, that is, if there is a representative $\hat{f}$ of f such that, for each $\psi \in \mathcal{D}(R)$, we have

(1.4.1) $\quad \lim\limits_{\varepsilon \longrightarrow 0} \int_{-\infty}^{+\infty} \hat{f}(\varepsilon,x)\psi(x)dx = \int_{0}^{+\infty} \psi(x)dx$.

(b) An element $f \in \varsigma_s(R)$ is said to be a Dirac g.f. (Dirac generalized function) if it has the Dirac measure as macroscopic aspect, that is, if there is a representative $\hat{f}$ of f such that, for every $\psi \in \mathcal{D}(R)$, we have

(1.4.2) $\quad \lim\limits_{\varepsilon \longrightarrow 0} \int_{-\infty}^{+\infty} \hat{f}(\varepsilon,x)\psi(x)dx = \psi(0)$.

REMARK The above concepts of Heaviside generalized function and Dirac generalized function are called respectively Heaviside s.g.f and Dirac s.g.f., in [A-B],8.3.3, in order to distinguish them from the generalized functions $H \in \mathcal{D}'(R) \subset \varsigma(R)$ and $\delta \in \mathcal{D}'(R) \subset \varsigma(R)$, see [A-B], 2.5.3 (a),(b) and 5.2.3.

EXAMPLE 1.5 (a) Let $\hat{f} \in \mathcal{G}_{M,s}[R]$ and suppose that there is a function $A : I \longrightarrow R_+^*$ such that :

(I) $\lim_{\varepsilon \longrightarrow 0} A(\varepsilon) = 0$; (II) $\hat{f}(\varepsilon,x) = 0$ if $x < -A(\varepsilon)$ and $\varepsilon \in I$;

(III) $\hat{f}(\varepsilon,x) = 1$ if $x > A(\varepsilon)$ and $\varepsilon \in I$; (IV) $\lim_{\varepsilon \longrightarrow 0} \int_{-A(\varepsilon)}^{+A(\varepsilon)} |\hat{f}(\varepsilon,x)| dx = 0$

Then the class $f \in \mathcal{G}_s(R)$ of $\hat{f}$ is a Heaviside g.f.. The condition (IV) can be replaced by the stronger one :

$$(IV') \sup_{(\varepsilon,x)\in I\times R} |\hat{f}(\varepsilon,x)| < \infty .$$

(b) Let $\hat{f} \in \mathcal{G}_{M,s}[R]$ and suppose that there is a function $A : I \longrightarrow R_+^*$ such that :

(I) $\lim_{\varepsilon \longrightarrow 0} A(\varepsilon) = 0$; (II) $\hat{f}(\varepsilon,x) = 0$ if $|x| \geq A(\varepsilon)$ and $\varepsilon \in I$;

(III) $\int_{-\infty}^{+\infty} \hat{f}(\varepsilon,x) dx = 1$ $(\varepsilon \in I)$; (IV) $\sup_{\varepsilon \in I} \int_{-\infty}^{+\infty} |\hat{f}(\varepsilon,x)| dx < \infty .$

Then the class $f \in \mathcal{G}_s(R)$ of $\hat{f}$ is a Dirac g.f.

(c) Let $\rho \in \mathcal{D}(R)$ be a function verifying the following conditions:

$$\rho \geq 0 , \rho(0) > 0 , supp(\rho) = [-1,1] \text{ and } \int_{-\infty}^{+\infty} \rho(s) ds = 1 .$$

Then the function $\hat{f}_\rho$ defined by $\hat{f}_\rho(\varepsilon,x) := \varepsilon^{-1} \rho(\varepsilon^{-1} x)$ if $(\varepsilon,x)\in I\times R$ belongs to $\mathcal{G}_{M,s}[R]$ and clearly satisfies the above conditions (I) to (IV) of (b) (with $A(\varepsilon) = \varepsilon$ for every $\varepsilon \in I$), hence the class f_ρ of $\hat{f}_\rho$ is a Dirac g.f.. The class of the function

$$\hat{g}_\rho : (\varepsilon,x) \longrightarrow \int_{-\infty}^{x} \hat{f}_\rho(\varepsilon,t) dt \in R$$

belongs to $\sigma_{M,s}[R]$ and satisfies the conditions (I) to (IV) (or (IV´)) of (a) (with $A(\varepsilon) = \varepsilon$ for every $\varepsilon \in I$) which shows that the class g_p of $\hat{g}_p$ is Heaviside g.f.. Note that we have $\hat{g}'_p = \hat{f}_p$. ,

The most elementary properties of Heaviside g.f. and Dirac g.f. appear in the proposition below, whose proof follows easily from DEF.1.1 and DEF.1.3.

PROPOSITION 1.6 (a) Any representative of a Heaviside g.f. (resp. Dirac g.f.) satisfies the condition (1.4.1) (resp. (1.4.2)).

(b) If f and g are both Heaviside g.f. or both Dirac g.f. then $f \overset{\sim}{} g$.

(c) If f is a Heaviside g.f. then f' is a Dirac g.f.

(d) If f is a Heaviside g.f. then $f|R_-^* \overset{\sim}{} 0$ in $\sigma_s(R_-^*)$ and $f|R_+^* \overset{\sim}{} 1$ in $\sigma_s(R_+^*)$.

(d´) If f is a Dirac g.f. then $f|R^* \overset{\sim}{} 0$ in $\sigma_s(R^*)$.

(e) If f is a Heaviside g.f. or a Dirac g.f., $\alpha \in R$ and $\alpha f \overset{\sim}{} 0$ in $\sigma_s(R)$ then $\alpha = 0$. ∎

NOTATION From now on we denote the Heaviside g.f. by H,K,L,M,Y,etc and the Dirac g.f. by δ, δ_1, etc.

We will distinguish now an important class of Heaviside g.f.

DEFINITION 1.7 We say that $H \in \sigma_s(R)$ is a proper Heaviside g.f. if H has a representative $\hat{H}$ verifying the following conditions :

(PH1) For every K ⊂⊂ R there are c > 0 and $\eta \in I$ such that

$$\sup_{(\varepsilon,x) \in]0,\eta]\times K} |\hat{H}(\varepsilon,x)| \leq c .$$

(PH2)
$$\lim_{\varepsilon \longrightarrow 0} \hat{H}(\varepsilon,x) = \begin{cases} 0 \text{ if } x < 0 \\ 1 \text{ if } x > 0 . \end{cases}$$

PROPOSITION 1.8 (a) Every representative of a proper Heaviside g.f satisfies conditions (PH1) and (PH2).

(b) Every proper Heaviside g.f. is a Heaviside g.f..

(c) If $(H_i)_{1\leq i \leq m}$ is a non void finite sequence of proper Heaviside g.f., $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_m) \in N^m$ and $|\alpha| > 0$, then $H_1^{\alpha_1}H_2^{\alpha_2}\ldots H_m^{\alpha_m}$ is a proper Heaviside g.f. and therefore $H_1^{\alpha_1}H_2^{\alpha_2}\ldots H_m^{\alpha_m} \approx H$ in $\mathcal{G}_s(R)$ for every Heaviside g.f. H.

Proof. The statement (a) follows immediately from the definition of $\hat{n}_s[R]$, (b) and (c) are easy consequences of the dominated convergence theorem. ∎

2 SOME SPECIAL GENERALIZED FUNCTIONS

The content of this section is a fitting of concepts and results developed in [V] (§ 3). The relation $u \in \mathcal{G}_{M,s}[R]$ does not imply that $e^u \in \mathcal{G}_{M,s}[R]$, so we shall define a vector subspace $\mathcal{G}_{M,s}^{log}[R]$ of $\mathcal{G}_{M,s}[R]$ such that e^u is moderated whenever u belongs to this subspace. More generaly, we will give a sufficient condition for the validity of the relation $\psi \cdot f \in \mathcal{G}_s(R)$ when $\psi \in C^{\infty}(R)$ and

$f \in \mathcal{G}_s(R)$ adequate to the requirements of this work.

Let $\mathcal{G}^{\log}[R] = \mathcal{G}^{\log}[R,R]$ be the vector space of all functions $u \in \mathcal{G}_s[R]$ verifying the following condition :

[Log] For all $K \subset\subset R$, there are $c \geq 1$, $N \in \mathbb{N}$ and $\eta \in I$ such that $|u(\epsilon,x)| \leq \log(c\epsilon^{-N})$ for each $(\epsilon,x) \in]0,\eta[\times K$.

Next we define

$$\mathcal{G}^{\log}_{M,s}[R] = \mathcal{G}^{\log}_{M,s}[R,R] := \mathcal{G}^{\log}[R] \cap \mathcal{G}_{M,s}[R]$$

$$\mathfrak{n}^{\log}_{s}[R] = \mathfrak{n}^{\log}_{s}[R,R] := \mathcal{G}^{\log}[R] \cap \mathfrak{n}_{s}[R]$$

We say that an element $f \in \mathcal{G}_s(R)$ has the property (LLG) (locally logarithmic growth) if there is a representative $\hat{f}$ of f which belongs to $\mathcal{G}^{\log}_{M,s}[R]$. Clearly, if f has the property (LLG), then each representative of f belongs to $\mathcal{G}^{\log}_{M,s}[R]$. Since $\mathfrak{n}^{\log}_{s}[R]$ is the kernel of the restriction to $\mathcal{G}^{\log}_{M,s}[R]$ of the natural map of $\mathcal{G}_{M,s}[R]$ onto $\mathcal{G}_s(R)$, it follows that the quotient vector subspace of all generalized functions which has property (LLG) :

$$\mathcal{G}^{\log}_{s}(R) := \frac{\mathcal{G}^{\log}_{M,s}[R]}{\mathfrak{n}^{\log}_{s}[R]}$$

can be identified to a vector subspace of $\mathcal{G}_s(R)$.

PROPOSITION 2.1 (a) If $u \in \mathcal{G}^{\log}_{M,s}[R]$ then $e^u \in \mathcal{G}_{M,s}[R]$.

(b) If $u_1,u_2 \in \mathcal{G}^{\log}_{M,s}[R]$ and $u_1 - u_2 \in \mathfrak{n}^{\log}_{s}[R]$ then $e^{u_1} - e^{u_2} \in \mathfrak{n}_{s}[R]$

Proof. (a) From the chain rule, for all $(\epsilon,x) \in I\times R$ and $p \in \mathbb{N}^*$, it follows that (see [F]) :

(2.1.1) $(e^u)^{(p)}(\varepsilon,x) = e^{u(\varepsilon,x)} \sum_{m=1}^{p} \sum_{\alpha} c_p(\alpha,m) u^{(\alpha_1}(\varepsilon,x) \ldots u^{(\alpha_m}(\varepsilon,x)$

where α runs over the set $I(p,m) := (\alpha=(\alpha_1,\ldots,\alpha_m) \in N^{*m} | \ |\alpha| = p)$

and $c_p(\alpha,m) := p!(m!\alpha!)^{-1}$. Therefore, the moderation of e^u results

from (2.1.1) and the condition [Log].

(b) We fix K CC R and $p \in N$ and prove that there exist $N \in N$ and

$\gamma \in \Gamma$ such that for each $q \geq N$ there are $c > 0$ and $\eta \in I$ such that

(2.1.2) $|(e^{u_1} - e^{u_2})^{(p)}(\varepsilon,x)| \leq c\varepsilon^{\gamma(q) - N}$ for all $(\varepsilon,x) \in]0,\eta[\times K$.

Indeed, since u_i satisfies the condition [Log] ($i = 1,2$), there are

$c_1 \geq 1$, $N_1 \in N$ and $\eta_1 \in I$ such that

(2.1.3) $|u_i(\varepsilon,x)| \leq \log(c_1\varepsilon^{-N_1})$ for all $(\varepsilon,x) \in]0,\eta_1[\times K$ ($i=1,2$).

It follows that

(2.1.4) $e^{u_i(\varepsilon,x)} \leq c_1\varepsilon^{-N_1}$ for every $(\varepsilon,x) \in]0,\eta_1[\times K$ ($i = 1,2$)

and

(2.1.5) $|(u_1 - u_2)(\varepsilon,x)| \leq \log(c_1^2\varepsilon^{-2N_1})$ for all $(\varepsilon,x) \in]0,\eta_1[\times K$.

From (2.1.3), (2.1.5) and the mean value theorem we get

(2.1.6) $|(e^{u_1} - e^{u_2})(\varepsilon,x)| \leq 2|(u_1-u_2)(\varepsilon,x)|c_1^3\varepsilon^{-3N_1}$ if $(\varepsilon,x) \in]0,\eta_1[\times K$

Since $u_1 - u_2 \in R_s^{log}[R]$, there are $N_2 \in N$ and $\gamma \in \Gamma$ such that for

every $q \geq N_2$ we can find $c_2 > 0$ and $\eta_2 \in I$ such that

(2.1.7) $|(u_1 - u_2)(\varepsilon,x)| \leq c_2\varepsilon^{\gamma(q) - N_2}$ if $(\varepsilon,x) \in]0,\eta_2[\times K$.

Let $N := 3N_1 + N_2$. Given $q \geq N$, fix $c := 2c_1^3c_2$ and $\eta := \min(\eta_1,\eta_2)$

then, from (2.1.6) and (2.1.7) we get (2.1.2) in the case $p = 0$.

Suppose now p ≥ 1. By applying (2.1.1) with u_1 instead of u, we get

$$(e^{u_1} - e^{u_2})^{(p)}(\epsilon,x) = T_1(\epsilon,x) + T_2(\epsilon,x) \quad \text{for all } (\epsilon,x) \in I \times R$$

where T_1 and T_2 are the moderate functions defined by

$$T_1(\epsilon,x) := (e^{u_1} - e^{u_2})(\epsilon,x) \sum_{m=1}^{p} \sum_{\alpha} c_p(\alpha,m) \prod_{i=1}^{m} u_1^{(\alpha_i)}(\epsilon,x)$$

$$T_2(\epsilon,x) := e^{u_2(\epsilon,x)} \sum_{m=1}^{p} \sum_{\alpha} c_p(\alpha,m) \sum_{i=1}^{m} [u_1^{(\alpha_1)} \ldots (u_1 - u_2)^{(\alpha_i)} \ldots u_2^{(\alpha_m)}](\epsilon,x)$$

Clearly, it is enough to find $\gamma_J \in \Gamma$ and $N_J' \in N$ such that for each

q ≥ N there are $c_J > 0$ and $\eta_J \in I$ verifying

(2.1.8) $|T_j(\epsilon,x)| \le c_J \epsilon^{\gamma_J(q) - N_J'}$ for all $(\epsilon,x) \in]0,\eta_J[\times K (j=1,2)$

In the case j = 1, (2.1.8) follows from (2.1.2) for p = 0 and from the moderation of u_1. In the case j = 2, it is a consequence of (2.1.4) and the relation $u_1 - u_2 \in \mathfrak{n}_s^{log}[R]$. ∎

Therefore, the following definition is meaningful.

DEFINITION 2.2 Let $f \in \varsigma_s^{log}(R)$. The exponential of f, denoted e^f or exp(f), is the class in $\varsigma_s(R)$ of the moderate function $e^{\hat{f}}$ where $\hat{f}$ is any representative of f.

If I is an open interval of R then for each $f \in \varsigma_s(I)$ there exists $F \in \varsigma_s(I)$ such that $F' = f$ (F is called a primitive of f, see [A-B] 4.3.2, 8.1.3(a) and (d)).

THEOREM 2.3 Let $f,g \in \varsigma_s(R)$ and suppose that f has a primitive $F \in \varsigma_s^{log}(R)$. Then, every solution $y \in \varsigma_s(R)$ of the differential

equation

$(2.3.1)$ $$y' + fy = g$$

has a representative $\hat{g}$ given by

$$\hat{y}(\epsilon,x) = [\hat{\lambda}(\epsilon) + \int_a^x \hat{g}(\epsilon,t)e^{\hat{F}(\epsilon,t)}dt]e^{-\hat{F}(\epsilon,x)} \quad \text{for all } (\epsilon,x) \in I \times R$$

where $\hat{g}$ and $\hat{F}$ are any representatives of g and F respectively, $a \in R$ and $\hat{\lambda} \in \mathcal{G}_{M,s}(R)$.

Proof. Fix any solution y of (2.3.1). Since $F' = f$ it follows that $(ye^F)' = ge^F$ hence the generalized function $h := ye^F$ is a solution of the differential equation $z' = ge^F$. By consequence, for every $\lambda \in \mathcal{G}_{M,s}(R)$ the moderate function defined by

$$\hat{h}(\epsilon,x) = \hat{\lambda}(\epsilon) + \int_a^x \hat{g}(\epsilon,t)e^{\hat{F}(\epsilon,t)}dt \quad \text{for every } (\epsilon,x) \in I \times R$$

is a representative of h. ∎

DEFINITION 2.4 Let $\beta \in]0,+\infty]$, $\psi \in C^\infty(]0,\beta[)$ and $u \in \mathcal{G}_s[R]$ satisfying the condition:

$[V_\beta]$ $\qquad u(\epsilon,x) \in]0,\beta[$ for all $(\epsilon,x) \in I \times R$.

(1) For every $p \in N$ we denote by $\psi^{(p)} \cdot u$ the element of $\mathcal{G}_s[R]$ defined by

$(2.4.1)$ $\qquad (\psi^{(p)} \cdot u)(\epsilon,x) := \psi^{(p)}(u(\epsilon,x))$ for all $(\epsilon,x) \in I \times R$.

(2) We say that u satisfies the condition $[V'_\beta]$ if the following assertion holds:

$[V'_\beta]$ $\quad$ For every $K \subset\subset R$ there are $\eta \in I$ and $(a,b) \in R^2$ such that
$\qquad 0 < a < b < \beta$ and $u(\epsilon,x) \in [a\epsilon,b]$ for all $(\epsilon,x) \in]0,\eta[\times K$.

In what follows, "u satisfies $[V_\beta]$" means that there is $\beta \in]0,+\infty]$ such that u satisfies the condition $[V_\beta]$; the same abbreviation holds for $[V'_\beta]$. In the case $\beta = +\infty$ we write $[V_\infty]$ (or $[V'_\infty]$) instead of $[V_{+\infty}]$ (or $[V'_{+\infty}]$).

PROPOSITION 2.5 Let $u_1, u_2 \in \mathcal{E}_{M,s}[R]$ such that $u_1 - u_2 \in \mathcal{R}_s[R]$ and suppose that u_i (i = 1,2) satisfies $[V_\beta]$. We have :

(a) If u_1 satisfies $[V'_\beta]$ then u_2 satisfies $[V'_\beta]$.

(b) Assume that u_1 satisfies $[V'_\beta]$ and that $\varphi \in C^\infty(]0,\beta[)$ verifies the condition :

(2.5.1) $\left|\begin{array}{l} \text{For every } p \in N \text{ and every } (A,B) \in R^\bullet \text{ with } 0 < A < B < \beta \\ \text{there are } c > 0, \quad N \in N \quad \text{and} \quad \eta \in I \quad \text{such} \quad \text{that} \\ |\varphi^{(p)}(x)| \leq c\varepsilon^{-N} \text{ for all } (\varepsilon,x) \in]0,\eta[\times[A\varepsilon,B]. \end{array}\right.$

Then, for every $K \subset\subset R$ and every $p \in N$, there are $\gamma \in \Gamma$ and $N \in N$ such that, for each $q \geq N$, we can find $c > 0$ and $\eta \in I$ satisfying

(2.5.2) $|(\varphi^{(p)} \cdot u_1 - \varphi^{(p)} \cdot u_2)(\varepsilon,x)| \leq c\varepsilon^{\gamma(q)-N} \quad \forall \ (\varepsilon,x) \in]0,\eta[\times K$.

Proof Clearly we can suppose that $\beta < \infty$. Since (a) is obvious let us prove (b). Fix $K \subset\subset R$ and $p \in N$. By (a), u_1 and u_2 satisfy $[V'_\beta]$ hence we can find $\eta_1 \in I$ and $(a,b) \in R^\bullet$ such that $0 < a < b < \beta$ and

(2.5.3) $a\varepsilon \leq u_i(\varepsilon,x) \leq b$ for every $(\varepsilon,x) \in]0,\eta_1[\times K$.

The assumption $u := u_1 - u_2 \in \mathcal{R}_s[R]$ implies that there is $\eta_2 \in I$ such that $\eta_2 \leq \eta_1$ and

(2.5.4) $|u(\varepsilon,x)| \leq a'\varepsilon$ whenever $(\varepsilon,x) \in]0,\eta_2[\times K$,

where $0 < a' < \min(a,b')$ and $b' > 0$ satisfies $b + b' < \beta$. Now, we set

$A := a - a'$ and $B := b + b'$, then $0 < A < B < \beta$. We claim that

(2.5.5) $\quad \left| \begin{array}{l} \theta(t) := u_1(\varepsilon,x) + tu_2(\varepsilon,x) \in [A\varepsilon, B] \\ \text{for every } (\varepsilon,x,t) \in]0,\eta_2[\times K \times [0,1] \end{array} \right.$

Indeed, by (2.5.3) and (2.5.4) we get $\theta(t) \leq u_1(\varepsilon,x) + tu(\varepsilon,x) \leq$

$b + a'\varepsilon < b + b' < \beta$ and $A\varepsilon \leq u_1(\varepsilon,x) = \theta(t) - tu(\varepsilon,x) \leq$

$\theta(t) + |u(\varepsilon,x)| \leq \theta(t) + a'\varepsilon$, which proves (2.5.5). The assumption

$u \in \tilde{\eta}_s[R]$ implies that there exist $\gamma \in \Gamma$ and $N_1 \in N$ such that, for

every $q \geq N_1$ we can find $c_1 > 0$ and $\eta_3 \in I$ satisfying

(2.5.6) $|u(\varepsilon,x)| \leq c_1 \varepsilon^{\gamma(q) - N_1}$, for all $(\varepsilon,x) \in]0,\eta_3[\times K$.

Since φ satisfies (2.5.1) there are $c_2 > 0$, $N_2 \in N$ and $\eta_4 \in I$ such

that

(2.5.7) $|\varphi^{(p)}(y)| \leq c_2 \varepsilon^{-N_2}$ for all $(\varepsilon,y) \in]0,\eta_3[\times [A\varepsilon, B]$.

Let $N := N_1 + N_2$. Fix $q \geq N$, let $c := c_1 c_2$ and $\eta := \min_{2 \leq i \leq 4} (\eta_i)$,

then, for each $(\varepsilon,x) \in]0,\eta[\times K$, by applying the mean value theorem

and (2.5.5) we have

$|(\varphi^{(p)} \cdot u_1 - \varphi^{(p)} \cdot u_2)(\varepsilon,x)| \leq |u(\varepsilon,x)| \cdot \sup_{y \in [A\varepsilon, B]} |\varphi^{(p+1)}(y)|$

which implies (2.5.2) by (2.5.6) and (2.5.7). ∎

THEOREM 2.6 Let $\beta \in]0,+\infty]$ and $\varphi \in C^\infty(]0,\beta[)$ verifying (2.5.1).

We have :

(a) If $u \in \mathcal{G}_{M,s}[R]$ satisfies the conditions $[V_\beta]$ and $[V'_\beta]$, then

$\varphi \cdot u \in \mathcal{G}_{M,s}[R]$;

(b) If $u_1, u_2 \in G_{M,s}[R]$ and $u_1 - u_2 \in R_s[R]$, $u_i (i = 1,2)$ satisfies $[V_\beta]$ and u_1 satisfies $[V_\beta']$ then $\phi \cdot u_1 - \phi \cdot u_2 \in R_s[R]$.

Proof. (a) Since u satisfies $[V_\beta]$, the result follows from the assumption (2.5.1) and

(2.6.1) $\quad (\phi \cdot u)^{(p)}(\epsilon,x) = \sum_{m=1}^{p} \sum_{\alpha \in I(p,m)} c_p(\alpha,m)\phi^{(m)}(u(\epsilon,x))\prod_{i=1}^{m} u^{(\alpha_i)}(\epsilon,x)$,

where $c_p(\alpha,m)$ and $I(p,m)$ are as in the proof of PROP. 2.1 .

(b) Fixed $K \subset\subset R$ and $p \in N$ we will show that there are $N \in N$ and $\gamma \in \Gamma$ such that, for each $q \ge N$, we can find $c > 0$ and $\eta \in I$ such that

(2.6.2) $\quad |(\phi \cdot u_1 - \phi \cdot u_2)^{(p)}(\epsilon,x)| \le c\epsilon^{\gamma(q)-N}$ whenever $(\epsilon,x)\in]0,\eta[\times K$.

Since the case $p = 0$ follows directly from (2.5.2), we can assume that $p \ge 1$. From (2.6.1) we get

$\quad (\phi \cdot u_1 - \phi \cdot u_2)^{(p)}(\epsilon,x) = T_1(\epsilon,x) + T_2(\epsilon,x) \quad ((\epsilon,x) \in I\times R)$,

where T_1 and T_2 are the moderate functions :

$T_1(\epsilon,x) := \sum_{m=1}^{p} \sum_{\alpha} c_p(\alpha,m)(\phi^{(m)} u_1 - \phi^{(m)} u_2)(\epsilon,x)\prod_{i=1}^{m} u_1^{(\alpha_i)}(\epsilon,x)$,

$T_2(\epsilon,x) := \sum_{m=1}^{p} \sum_{\alpha} c_p(\alpha,m)(\phi^{(m)} u_2)(\epsilon,x).\sum_{i=1}^{m} P_i(\epsilon,x)$, where

$P_i(\epsilon,x) := [u_1^{(\alpha_1)}....u_1^{(\alpha_{i-1})} (u_1-u_2)^{(\alpha_i)}u_2^{(\alpha_{i+1})}...u_2^{(\alpha_m)}](\epsilon,x)$.

Now, for having (2.6.2) it is enough that T_j $(j = 1,2)$ satisfies (2.1.8) (see the proof of PROP. 2.1). We start from the following remark : if $\phi \in C^\infty(]0,\beta[)$ satisfies (2.5.1) then, for every $m \in N$

$\psi^{(m)}$ satisfies (2.5.1). Hence, (a) shows that $\psi^{(m)} \cdot u \in \mathcal{G}_{M,s}[R]$ if $u \in \mathcal{G}_{M,s}[R]$, thus the inequalities (2.1.8) follow from (2.5.2), the moderation of u_1 and u_2 and the assumption $u_1 - u_2 \in \mathcal{R}_s[R]$.∎

DEFINITION 2.7 Let $\beta \in]0,+\infty]$, $\psi \in C^{\infty}(]0,\beta[)$ satisfying (2.5.1) and assume that $f \in \mathcal{G}_s(R)$ has a representative verifying $[V_\beta]$ and $[V'_\beta]$. The composition $\psi \cdot f \in \mathcal{G}_s(R)$ is the class of the moderate function $\psi \cdot \hat{f}$, where $\hat{f}$ is any representative of f satisfying $[V_\beta]$.

PROPOSITION 2.8 Let $\beta \in]0,+\infty]$ and assume that $f \in \mathcal{G}_s(R)$ has a representative $\hat{f}$ satisfying $[V_\beta]$ and $[V'_\beta]$, then :

(a) $\psi \cdot \hat{f}$ is a representative of $\psi \cdot f$ for every $\psi \in C^{\infty}(]0,\beta[)$ satisfying (2.5.1),

(b) f is an invertible element in $\mathcal{G}_s(R)$ and the function $\hat{f}^{-1}$ given by $\hat{f}^{-1}(\varepsilon,x) := [\hat{f}(\varepsilon,x)]^{-1}$ for all $(\varepsilon,x) \in I \times R$, is a representative of f^{-1};

(c) The generalized function e^f is well defined and the function $e^{\hat{f}} : (\varepsilon,x) \in I \times R \longrightarrow \exp[\hat{f}(\varepsilon,x)] \in R$ is a representative of e^f,

(d) The generalized functions $\log f := \log \cdot f$ and $e^{\log f} := (\exp \cdot \log) \cdot f$ are well defined and the functions $\log \hat{f} := \log \cdot \hat{f}$ and $\hat{f}$ are representatives of $\log f$ and $e^{\log f}$, respectively (hence, $e^{\log f} = f$, see REMARK 2.9, (d) below).

Proof. (a) Follows from DEF. 2.7 and TH. 2.6 . (b) Clearly, the

function $\psi : y \in R_+^* \longrightarrow y^{-1} \in R$ satisfies (2.5.1), hence (a) shows

that $\hat{f}^{-1} := \psi \cdot \hat{f}$ is a representative of $g := \psi \cdot f$. By applying

(2.4.1) we get $\hat{f}^{-1}(\varepsilon, x) = [\hat{f}(\varepsilon, x)]^{-1}$ for every $(\varepsilon, x) \in I \times R$ and

therefore $fg = class(\hat{f}\hat{f}^{-1}) = 1$, which shows that f is invertible

in $\mathcal{G}_s(R)$ and that $\hat{f}^{-1}$ is a representative of $f^{-1} = g$. (c) The

function $\psi : y \in R_+^* \longrightarrow e^y \in R$ satisfies (2.5.1) and $e^f = \psi \cdot f$.

(d) Since $\log f = \psi \cdot f$, where ψ is given by $\psi(y) = \log y$ for each

$y \in R_+^*$, it is enough to show that ψ verifies (2.5.1). This is

clear if $p \geq 1$ and it follows easily in the case $p = 0$. Since

$\log \hat{f} = \psi \cdot \hat{f}$, the first statement is a consequence of (a). If we

set $\psi := (exp \cdot log)|R_+^*$ we have $e^{\log f} = \psi \cdot f$, since ψ satisfies

clearly (2.5.1) and $e^{\log \hat{f}} = \psi \cdot \hat{f} = \hat{f}$. ∎

REMARK 2.9 (a) In [A-B], § 7 it is defined the composition of

generalized maps which we will sketch briefly here in a particular

case. Let U and V be two open subsets of R. It can be easily

showed (see [A-B],7.3.1 and 8.1.3,(d)) that $f \in \mathcal{G}_s(U)$ is valued in

V if and only if, f has a representative $\hat{f}$ verifying the following

condition : for every $K \subset\subset R$ there are $K' \subset\subset V$ and $\eta \in I$ such that

$\{f(\varepsilon, x) | (\varepsilon, x) \in]0, \eta[\times K\} \subset K'$. If $f \in \mathcal{G}_s(R)$ is valued in V, it is

shown in [A-B] that the composition $g \cdot f$ is well defined whenever

$g \in \mathcal{G}_s(V)$. The concept of composition resulting from DEF. 2.7

is not a particular case of the this one since the condition $[V'_\beta]$ does not imply that f is valued in $]0,\beta[$. So, the concept of composition introduced in DEF 2.7 is an extension of the concept defined in [A-B], §7 specially adapted to the applications that we intend to do in this paper. The substitution of the hypothesis "f is valued in $]0,\beta[$" by the weaker condition $[V_\beta]$ is compensated by the simplicity of the generalized functions involved in these applications.

(b) If $f \in \mathcal{G}_s(R)$ has a representaive f satisfying the conditions $[V_\beta]$ and $[V'_\beta]$ (for any $\beta \in]0,+\infty]$) then $\hat{f}$ satisfies the condition $[Log]$, that is $f \in \mathcal{G}_s^{log}(R)$. Thus, the definition of exp(f) derived from PROP. 2.8(c) is a particular case of that resulting from DEF. 2.2.

(c) If $u \in \mathcal{G}_{M,s}[R]$ and $-u$ satisfies the conditions $[V_\beta]$ and $[V'_\beta]$ then $f := class(-u)$ is invertible in $\mathcal{G}_s(R)$, hence $-f = class(u)$ is also invertible in $\mathcal{G}_s(R)$. Thus, the conditions $[V_\beta]$ and $[V'_\beta]$ on a representative of $f \in \mathcal{G}_s(R)$, are only sufficient conditions for the invertibility of f.

(d) Let $f \in \mathcal{G}_s(R)$ and assume that $\hat{f}$ is a representative of f which satisfies $[V_\beta]$ and $[V'_\beta]$ for any $\beta \in]0,+\infty]$. It is easy to see that the generalized function log f (see PROP. 2.8(d)) belongs to

$\mathcal{G}_s^{\log}(R)$ and that the generalized function $e^{\log f}$ of PROP. 2.8(d) coincides with the exponential of $\log f$, obtained from DEF. 2.2 and also denoted by $e^{\log f}$.

The notations which we use now are suggested by those used in § 3.

PROPOSITION 2.10 Let $(P_r,P_1) \in R^2$ such that $0 < P_r < P_1$, $\Delta P := P_r - P_1 \dots v \in C^\infty(R_+^*,R_+^*)$ a strictly increasing function, $v_r := v(P_r)$, $v_1 := v(P_1)$, $\Delta v := v_r - v_1$ and $\hat{H} \in \mathcal{E}_{M,s}[R]$ satisfying the condition (see (PH1) in DEF. 1.7):

(2.10.1) $|\hat{H}(\varepsilon,x)| \le (\Delta P)^{-1}(P_r \varepsilon - P_1)$ for all $(\varepsilon,x) \in IxR$.

If P_* is the class in $\mathcal{G}_s(R)$ of $\hat{P}_* := \Delta P\hat{H} + P_1$, then the following statements hold:

(I) $\hat{P}_*$ satisfies the conditions $[V_\infty]$ and $[V'_\infty]$(see DEF. 2.4).

(II) If $\hat{H}$ satisfies the condition (PH2) of DEF. 1.7 and v satisfies the condition (2.5.1)(see PROP. 2.5); then the element :

(2.10.2) $V := (\Delta v)^{-1}(v \cdot P_* - v_1)$

is a proper Heaviside g.f.

(III) Conversely, assume that $Im(v) =]0,b[$ with $b \in]0,\infty]$, that v^{-1} satisfies the condition (2.5.1) and that there is $V \in \mathcal{G}_s(R)$ having a representative $\hat{V}$ such that $(v,\hat{V})$ verifies one of the two following conditions:

(2.10.3) $\begin{cases} (a) \ 0 \le \hat{V}(\varepsilon,x) \le (\Delta v)^{-1}[v_r \varepsilon - v_1] \ \forall \ (\varepsilon,x) \in IxR \\ (b) \ \forall \ s > 0 \ \exists \ \eta \in I \text{ such that } v(s\varepsilon) \le v(s)\varepsilon \ \forall \ \varepsilon \in]0,\eta[, \end{cases}$
or.

(2.10.4) $\left|\begin{array}{l}\text{(a)} \quad 0 \leq \hat{0}(\epsilon,x) \leq (\Delta\nu)^{-1} \cdot [\nu(\rho_r\epsilon) - \nu_1] \; \forall \; (\epsilon,x) \in I\times R \\ \text{(b)} \; \forall \; t \; > \; 0 \; \exists \; \eta \in I \text{ such that } \nu(t)\epsilon \leq \nu(t\epsilon) \; \forall \; \epsilon \in]0,\eta[.\end{array}\right.$

Then the generalized function

(2.10.5) $\quad H := (\Delta\rho)^{-1} \cdot [\nu^{-1} \cdot (\Delta\nu.V + \nu_1) - \rho_1]$

belongs to $\mathcal{G}_s(R)$ and has a representative $\hat{H}$ which satisfies the condition (2.10.1). Moreover, if $\hat{0}$ verifies the condition (PH2), then H is a proper Heaviside g.f. (and in this case, also V is a proper Heaviside g.f. since either of the statements (a) of (2.10.3) and (2.10.4) implies (PH1)).

Proof. From (2.10.1) it follows that

(2.10.6) $0 < \rho_r\epsilon \leq \hat{\rho}_*(\epsilon,x) \leq 2\rho_1 - \rho_r\epsilon < 2\rho_1 < +\infty \; \forall \; (\epsilon,x) \in I\times R$

which shows that $\hat{\rho}_*$ satisfies the conditions $[V_\infty]$ and $[V'_\infty]$, hence (I) holds. Therefore, since ν verifies the condition (2.5.1), we deduce that $\nu \cdot \rho_*$ is well defined and (see PROP. 2.8(a)) $\nu \cdot \hat{\rho}_*$ is a representative of $\nu \cdot \rho_*$. Thus, the function

$\hat{0} : (\epsilon,x) \in I\times R \longrightarrow (\Delta\nu)^{-1} \cdot [(\nu \cdot \hat{\rho}_*)(\epsilon,x) - \nu_1] \in R$

is moderate and its class in $\mathcal{G}_s(R)$ is the generalized function V defined by (2.10.2). Since ν is strictly increasing, from (2.10.6), the definition of $\hat{0}$ and the inequalities $\Delta\nu < 0$ and $\nu_1 > 0$, we get

$|\hat{0}(\epsilon,x)| \leq -(\Delta\nu)^{-1} \cdot [\nu(2\rho_1) + \nu_1] \quad$ for every $(\epsilon,x) \in I\times R$

which implies that $\hat{0}$ verifies the condition (PH1). On the other hand, since $\hat{H}$ satisfies (PH2) we have

$$\lim_{\varepsilon \to 0} (v \circ \hat{P}_{\pi})(\varepsilon, x) = v_1 \text{ (resp. } v_r) \text{ if } x < 0 \text{ (resp. } x > 0)$$

hence, from the definition of $\hat{U}$ we obtain $\lim_{\varepsilon \to 0} \hat{U}(\varepsilon, x) = 0$ (resp.1)

if $x < 0$ (resp. $x > 0$), and the proof of (II) is complete.

(III) **Case 1** : The condition (2.10.3) holds. Since $\Delta v < 0$, from the

statement (a) it follows that

(2.10.7) $\quad v_r \varepsilon \le \Delta v . \hat{U}(\varepsilon, x) + v_1 \le v_1$ for every $(\varepsilon, x) \in I \times R$

and the statement (b) shows that there is $\eta \in I$ such that $v(P_r \varepsilon) \le$

$v_r \varepsilon$ for all $\varepsilon \in]0, \eta[$. Let $\hat{W} : I \times R \longrightarrow R$ be the function given by

$$\hat{W}(\varepsilon, x) := \begin{cases} \hat{U}(\varepsilon, x), & \text{if } (\varepsilon, x) \in]0, \eta[\times R \\[2mm] (\Delta v)^{-1} . [v(P_r \varepsilon) - v_1], & \text{if } (\varepsilon, x) \in [\eta, 1] \times R \end{cases}$$

Clearly, we have $\hat{W} \in \mathcal{E}_{M,s}[R]$ and $\hat{U} - \hat{W} \in \mathcal{R}_s[R]$, hence $\hat{W}$ is a

representative of U. From the definition of $\hat{W}$ and (2.10.7) we get

(2.10.8) $\quad \left| \begin{array}{l} 0 < v_r \varepsilon \le \Delta v . \hat{W}(\varepsilon, x) + v_1 \le v_1 < \beta, \text{ if } (\varepsilon, x) \in]0, \eta[\times R \\[2mm] 0 < \Delta v . \hat{W}(\varepsilon, x) + v_1 = v(P_r \varepsilon) < \beta, \text{ if } (\varepsilon, x) \in [\eta, 1] \times R \end{array} \right.$

which shows that $\Delta v . W + v_1$ satisfies the conditions $[V_\beta]$ and $[V'_\beta]$.

Since v^{-1} verifies the condition (2.5.1) it follows (see DEF. 2.7)

that the composition $v^{-1} \circ (\Delta v . V + v_1)$ is well defined and, from

PROP.2.8(a), has the function $v^{-1} \circ (\Delta v . \hat{W} + v_1)$ as a representative.

Therefore, the function

(2.10.9) $\quad \hat{H}(\varepsilon, x) := (\Delta \rho)^{-1} . [v^{-1} (\Delta v . \hat{W}(\varepsilon, x) + v_1) - \rho_1] \ \forall \ (\varepsilon, x) \in I \times$

is moderate and its class in $\mathcal{G}_s(R)$ is the generalized function

defined by (2.10.5). Since $v(P_r \varepsilon) \le v_r \varepsilon$ whenever $\varepsilon \in]0, \eta[$ and v^{-}

is strictly increasing, the definition of $\hat{H}$ and (2.10.8) imply

$0 \leq \hat{H}(\varepsilon,x) \leq (\Delta P)^{-1}.(P_r\varepsilon - P_1)$ for all $(\varepsilon,x) \in]0,\eta[\times R$ and $\hat{H}(\varepsilon,x) =$

$(\Delta P)^{-1}.(P_r\varepsilon - P_1) \geq 1$ for each $(\varepsilon,x) \in [\eta,1]\times R$, showing that $\hat{H}$

satisfies the condition (2.10.1).

Case 2 : The condition (2.10.4) holds. Since $\Delta\nu < 0$ the statement

(a) implies

(2.10.10) $\nu(P_r\varepsilon) \leq \Delta\nu.\hat{V}(\varepsilon,x) + \nu_1 \leq \nu_1$ for each $(\varepsilon,x) \in I\times R$

and the statement (b) shows that there is $\eta \in I$ such that $\nu_r\varepsilon \leq$

$\nu(P_r\varepsilon)$ if $\varepsilon \in]0,\eta[$, hence by (2.10.10) we have $\nu_r\varepsilon \leq \Delta\nu.\hat{V}(\varepsilon,x) +$

$\nu_1 \leq \nu_1$ for every $(\varepsilon,x) \in]0,\eta[\times R$ which, together with (2.10.10),

implies that $\Delta\nu.\hat{V} + \nu_1$ satisfies $[V_\beta]$ and $[V'_\beta]$. Since ν^{-1} verifies

the condition (2.5.1), it follows that $\nu^{-1}.(\Delta\nu.\hat{V} + \nu_1)$ is a

representative of $\nu^{-1}.(\Delta\nu.V + \nu_1)$ and therefore, the function $\hat{H}$

defined by

(2.10.11) $\hat{H}(\varepsilon,x) := (\Delta P)^{-1}.[\nu^{-1}(\Delta\nu.\hat{V}(\varepsilon,x) + \nu_1) - P_1] \forall (\varepsilon,x)\in I\times R$

is a representative of the generalized function H given by

(2.10.5). Now, since ν^{-1} is strictly increasing, from (2.10.10) and

(2.10.11) we get (2.10.1). For the last statement of (III), it is

enough to show that $\hat{H}$ verifies (PH2). Let $\hat{U}$ denotes either of the

moderate functions $\hat{V}$ or $\hat{Q}$ which defines $\hat{H}$ (see (2.10.9) (resp.

(2.10.11)) in the Case 1(resp. Case 2)) then , since $\hat{U}$ satisfies

(PH2) (because $\hat{U}(\varepsilon,-) = \hat{V}(\varepsilon,-)$ for small ε) we have

$$\lim_{\varepsilon\to 0} [v^{-1}\cdot(\Delta v\cdot\hat{U} + v_1)](\varepsilon,x) = \rho_1 \text{ (resp. } \rho_r) \text{ if } x < 0 \text{ (resp. } x > 0)$$

hence $\hat{A}$ satisfies (PH2) by its own definition. ∎

Given $y \in C^\infty(R)$ let us consider the function $y^* \in C^\infty(R^2)$ defined by

$$[2.1] \qquad y^*(x,t) := x - y(t) \qquad \text{for each } (x,t) \in R^2 .$$

The map $L \in C^\infty(R^2,R^2)$ defined by $L(x,t) := (y^*(x,t),t)$ for every

$(x,t) \in R^2$ is bijective with inverse $L^{-1}(\lambda,s) := (\lambda + y(s),s)$ for

all $(\lambda,s) \in R^2$. The Jacobian determinants are

$$\text{Det}J_L(x,t) = \text{Det}J_{L^{-1}}(\lambda,s) = 1.$$

With these notations we have the following result:

PROPOSITION 2.11 For every $G \in \mathcal{G}_s(R)$ the statemets below hold:

(I) $G\cdot y^* \approx 0$ if and only if $G \approx 0$.

(II) $G\cdot y^* = 0$ if and only if $G = 0$.

Proof. Let $\hat{G}$ be any representative of G , hence the map

$$\hat{G}\cdot y^* : (\varepsilon,x,t) \in I\times R^2 \longrightarrow \hat{G}(\varepsilon,x - y(t)) \in R$$

is a representative of $G\cdot y^*$. If $G\cdot y^* \approx 0$ then

$$(2.11.1) \quad \lim_{\varepsilon\to 0} \int_{R^2} \hat{G}(\varepsilon,x - y(t))\psi(x,t)dxdt = 0 \text{ for each } \psi \in \mathcal{D}(R^2)$$

Given $\psi \in \mathcal{D}(R)$, it is enough to show that

$$(2.11.2) \qquad \lim_{\varepsilon\to 0} \int_{-\infty}^{+\infty} \hat{G}(\varepsilon,\lambda)\psi(\lambda)d\lambda = 0$$

Fix $\psi_1 \in \mathcal{D}(R)$ with integral equal to 1 on R , then

$$\int_{-\infty}^{+\infty} \hat{G}(\varepsilon,\lambda)\psi(\lambda)d\lambda = \int_S G(\varepsilon,\lambda)\psi(\lambda)\psi_1(t)d\lambda dt \qquad (\varepsilon \in I)$$

where $S := \text{supp}(\psi)\times\text{supp}(\psi_1)$. Hence, by changing variables we

can write

$$\int_{-\infty}^{+\infty} \hat{G}(\epsilon,\lambda)\psi(\lambda)d\lambda = \int_{L^{-1}(S)} \hat{G}(\epsilon,x - y(t))\psi(x - y(t))\psi_s(t)dxdt \quad (\epsilon \in I).$$

Consider $\psi \in D(R^a)$ defined by $\psi(x,t) := \psi(x - y(t))\psi_s(t)$. From the above identity we get, as $\text{supp}(\psi) = L^{-1}(S)$

$$\int_{-\infty}^{+\infty} \hat{G}(\epsilon,\lambda)\psi(\lambda)d\lambda = \int_{\text{supp}(\psi)} \hat{G}(\epsilon,x - y(t))\psi(x,t)dxdt \quad (\epsilon \in I)$$

which implies (2.11.2) by (2.11.1), that is $G \approx 0$. Conversely, now we must show that (2.11.1) holds. Indeed, for a fixed $\psi \in D(R^a)$, the change of variable formula shows that

$$\int_{R^a} G(\epsilon,x - y(t))\psi(x,t)dxdt = \int_{L(\text{supp}(\psi))} G(\epsilon,\lambda)\psi(\lambda + y(t),t)d\lambda dt \quad (\epsilon \in I)$$

Let $\phi \in D(R^a)$ be defined by $\phi(\lambda,t) := \psi(\lambda + y(t),t)$. Since $\text{supp}(\phi) = L(\text{supp}(\psi))$, the above equality implies

(2.11.3) $$\int_{R^a} \hat{G}(\epsilon,x - y(t))\psi(x,t)dxdt = \int_{-\infty}^{+\infty} \hat{G}(\epsilon,\lambda)\psi(\lambda)d\lambda ,$$

where $\psi \in D(R^a)$ is defined by

$$\psi(\lambda) := \int_{-\infty}^{+\infty} \psi(\lambda,t)dt \quad \text{for every } \lambda \in R$$

The assumption $G \approx 0$ shows that (2.11.2) holds and this together with (2.11.3), implies (2.11.1). The proof of (I) is complete.

For the proof of (II), assume now that $G \cdot y^* = 0$, $K \subset\subset R$ and $p \in N$. We will show that we can find $\gamma \in \Gamma$ and $N \in N$ such that for each $q \geq N$ there are $c > 0$ and $\eta \in I$ satisfying

(2.11.4) $$|\hat{G}^{(p)}(\epsilon,\lambda)| \leq c\epsilon^{\gamma(q) - N} \quad \text{for every } (\epsilon,\lambda) \in]0,\eta[\times K .$$

Indeed, the assumption on $G \cdot y^*$ implies, since $K_s := \bar{L}^{-1}(K \times K)$ is a compact subset of R^a and $\alpha := (p,0) \in N^a$, that we can find

$\gamma \in \Gamma$ and $N \in \mathbb{N}$ such that for every $q \geq N$ there are $c > 0$ and $\eta \in I$ satisfying

$$(2.11.5) \quad |\partial^\alpha(\hat{G} \cdot y^*)(\varepsilon,x,t)| \leq c\varepsilon^{\gamma(q) - N} \text{ for all } (\varepsilon,x,t) \in]0,\eta[\times K$$

Given $\lambda \in K$, if $(x,t) := L^{-1}(\lambda,\lambda)$, we have $x = \lambda + y(t)$ and $\lambda = t$ and therefore

$$(2.11.6) \quad \hat{G}^{(p)}(\varepsilon,\lambda) = \hat{G}^{(p)}(\varepsilon,x - y(t)) = \partial^\alpha(G \cdot y^*)(\varepsilon,x,t)$$

which shows that (2.11.4) follows from (2.11.5).Conversely, assume that $G = 0$ and fix $K \subset\subset R^a$ and $(p,q) \in \mathbb{N}^2$. We must show that we can find $\gamma \in \Gamma$ and $N \in \mathbb{N}$ such that for every $q \geq N$ there are $c > 0$ and $\eta \in I$ satisfying (2.11.5). In fact, by using the second identity in (2.11.6) and the chain rule, for all $(\varepsilon,x,t) \in I\times R^a$, we get

$$(2.11.7) \quad \partial^\alpha(\hat{G} \cdot y^*)(\varepsilon,x,t) = \sum_{m=1}^{q} \sum_{\alpha} \bar{c}_p(\alpha,m)\hat{G}^{(p+m)}(\varepsilon,x-y(t))\prod_{i=1}^{m} y^{(\alpha_i)}(t)$$

where α runs over $I(q,m)$ and $\bar{c}_p(\alpha,m):=(-1)^{|\alpha|}c_p(\alpha,m)$ (see the proof of PROP.2.1(a) for the definitions of $I(q,m)$ and $c_p(\alpha,m)$). Let K_1 and K_2 be compact subsets of R such that $L(K) \subset K_1\times K_2$ and consider

$$(2.11.8) \quad c_1 := \sup_{\substack{s\in K \\ 1\leq m\leq q}} \sum_{\alpha} |\bar{c}_p(\alpha,m)| \prod_{i=1}^{m} |y^{(\alpha_i)}(s)| \ .$$

Since $G = 0$ we can find $\gamma \in \Gamma$ and $N \in \mathbb{N}$ such that for every $q \geq N$ there are $c_m > 0$ and $\eta \in I$ satisfying

$$(2.11.9) \quad \sup_{\lambda\in K_1} \sum_{m=1}^{q} |\hat{G}^{(p+m)}(\varepsilon,\lambda)| \leq c_m\varepsilon^{\gamma(q)-N} \text{ whenever } \varepsilon \in]0,\eta[\ .$$

Given $(x,t) \in K$, if $(\lambda,s) = L(x,t)$ we have $s = t$ and $\lambda = x - y(t)$,

hence (2.11.7),(2.11.8) and (2.11.9) imply (2.11.5) for $c = c_1.c_2$. ∎

3 A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS FROM HYDRODYNAMICS WITH VISCOSITY

Assume that $\nu \in C^{\infty}(R_+^*, R_+^*)$ is a strictly increasing function which satisfies (2.5.1) (with $b = \infty$, since ν is defined in R_+^*) and consider the following system of non-linear PDE:

[S]
$$
\begin{cases}
(E_1) & \rho_t^- + (\rho u)_x \approx 0 \\
(E_2) & (\rho u)_t + (p + \rho u^2)_x \approx [(\nu \cdot \rho) u_x]_x \\
(E_3) & E_t + [(E + p)u]_x \approx [(\nu \cdot \rho) u u_x]_x \\
(E_4) & E \approx \lambda p + \frac{1}{2}\rho u^2
\end{cases}
$$

where ρ, u, p and E are in $G_s(R^2)$, $\lambda \in R^*$ and the symbol $\approx$ denotes the association relation (see DEF.1.1). We also consider another system [S̄] associated with [S], which consists of the equations (E_i) (i=2,3,4) of [S] and the equation (E_1) replaced by

$(\bar{E}_1)$ $\qquad\qquad \rho_t + (\rho u)_x = 0$.

A discontinuous (shock) solution (constant before and after the shock with constant velocity of propagation and macroscopic form of the shock invariable with the time) for the system [S] (or [S̄]) is any subset (ρ, u, p, E) of $G_s(R)$ given by

(S_1) $\quad \rho = \Delta\rho.H.y^* + \rho_1$, $\qquad\qquad (S_2)$ $\quad u = \Delta u.K.y^* + u_1$,

(S_3) $\quad p = \Delta p.L.y^* + p_1$, $\qquad\qquad (S_4)$ $\quad E = \Delta E.M.y^* + E_1$,

which are solutions of the system $[S]$ (or $[\bar{S}]$) and satisfy the following conditions:

(A_1) $y^* \in C^\infty(R)$ is the function defined in $[2.1]$, associated to the function $y : t \in R \longrightarrow ct \in R$, where $c \in R$ (that is $y^*(x,t) = x - ct$ for all $(x,t) \in R^2$).

(A_2) Let κ denote indistinctly ρ, u, p or E. There are real numbers κ_r and κ_1 verifying the condition $0 < \kappa_r < \kappa_1$ and we define $\Delta\kappa := \kappa_r - \kappa_1$ (hence $\Delta\kappa < 0$).

(A_3) H, K, L and M are proper Heaviside g.f.(see DEF.1.7).

(A_4) H has a representative $\hat{H}$ satisfying (2.10.1)(see PROP.2.10).

(A_5) K has the following property : $YK' \stackrel{\sim}{=} 0$ in $\varsigma_s(R^*)$ for every proper Heaviside g.f. Y.

Note that the Heaviside g.f. g_ρ in EX.1.5(c) is proper and satisfies the condition $(g_\rho)'|R^* = 0$, hence g_ρ has property (A_5). In what follows we will need the identities below, called jump conditions, which are relations among the real numbers $c, \rho_r, \rho_1, ..., E_r$ and E_1 introduced in (A_1) and (A_2):

(J_1) $c = \Delta u(1 + \dfrac{\rho_1}{\Delta\rho}) + u_1$.

(J_2) $\dfrac{\Delta p}{\Delta u^2} = \rho_1(1 + \dfrac{\rho_1}{\Delta\rho})$.

(J_3) $\Delta p(1 + \dfrac{u_1}{\Delta u}) = \rho_1 \dfrac{\Delta E}{\Delta\rho} - E_1 - p_1$.

(J_4) $\begin{cases} E_r = \lambda p_r + \frac{1}{2}\rho_r u_r^2 \\ E_1 = \lambda p_1 + \frac{1}{2}\rho_1 u_1^2 \end{cases}$.

Finally, by introducing the generalized functions ρ_*, u_*, P_* and E_*

in $\varsigma_s(R)$ defined by

[3.1]
$$p_* := \Delta_p H + p_1 , \qquad u_* := \Delta u K + u_1 ,$$
$$p_* := \Delta_p L + p_1 , \qquad E_* := \Delta E . M + E_1 ,$$

we can write the shock solutions $(S_1)-(S_4)$ in the following way

[3.2]
$$p = p_* \cdot y^* , \qquad u = u_* \cdot y^* ,$$
$$p = p_* \cdot y^* , \qquad E = E_* \cdot y^* .$$

Since H satisfies (A_0) and (A_4) and v verifies (2.5.1), from PROP. 2.10(II) it follows that the generalized function V defined in (2.10.2), namely

[3.3]
$$V = (\Delta v)^{-1}(v \cdot p_* - v_1) \in \varsigma_s(R)$$

is a proper Heaviside g.f.. With these assumptions and notations (which includes those of PROP.2.10), we have the following result:

LEMMA 3.1 (a) The following conditions are equivalent:

(i) $VK' \approx - \dfrac{v_1}{\Delta v} H$

(ii) $(v \cdot p_*)u'_* \approx 0$

(b) If the equivalent conditions of (a) hold, then the following conditions are equivalent:

(j) $(v \cdot p_*)u_* u'_* \approx 0$

(jj) $VKK' \approx - \dfrac{1}{2} \dfrac{v_1}{\Delta v} H$

Proof. (a) From [3.3] we get

(3.1.1)
$$v \cdot p_* = \Delta v V + v_1$$

which implies that (ii) is equivalent to

(3.1.2) $\qquad \Delta \nu \nabla u'_* + \nu_1 u'_* \widetilde{\sim} 0 .$

From the relations $\Delta u \neq 0$, $u'_* = \Delta u K'$ (see [3.1]) and $K' \widetilde{\sim} H'$ (see PROP.1.6) it follows that (3.1.2) is equivalent to (i).

(b) The definition of u_* ([3.1]) shows that (j) is equivalent to

(3.1.3) $\qquad \Delta u (\nu \cdot P_*) K u'_* + u_1 (\nu \cdot P_*) u'_* \widetilde{\sim} 0 .$

Since $\Delta u \neq 0$ and $u'_* = \Delta u K'$, the condition (i) of (a) shows that (3.1.3) is equivalent to

(3.1.4) $\qquad (\nu \cdot P_*) K K' \widetilde{\sim} 0 .$

By PROP.1.8(c) we have $K^= \widetilde{\sim} K$ and hence, by the chain rule, we get $K K' \widetilde{\sim} \frac{1}{2} K'$ which together with (3.1.1) and $\Delta \nu \neq 0$, implies that (3.1.4) is equivalent to (jj). ∎

In the sequel we will prove that the system [S] has shock solutions if and only if the jump conditions $(J_1)-(J_4)$ of the classical case without viscosity (that is $\nu = 0$) hold. We will show also that the situation is rather different for the associated system [S̄] which may not have shock solutions. More precisely, we will prove that, in the particular case $\nu(x) = x^2$ for all $x \in R^*_+$, the corresponding system [S̄] has not shock solutions. We recall that the generalized functions ρ, u, P and E defined by $(S_1), (S_2), (S_3)$ and (S_4) respectively, satisfy the conditions $(A_i) (1 \leq i \leq 5)$.

PROPOSITION 3.2 If ρ and u are given by (S_1) and (S_2) respectively then the following conditions are equivalent:

(I) ρ and u are solutions of the equation (E_1).

(II) The jump condition (J_1) holds.

Proof. The relations [3.2], the chain rule and the PROP.2.11 (I) imply that $\rho_t + (\rho u)_x \approx 0$ if and only if $(\rho_* u_* - c\rho_*)' \approx 0$. On the other hand, by [3.1] we get

$$(\rho_* u_* - c\rho_*)' = (u_1 - c)\Delta\rho H' + \rho_1 \Delta u K' + \Delta\rho\Delta u(HK)',$$

and since K' and $(HK)'$ are associated with H' (see PROP.1.8(c) and PROP.1.2(c)) it follows that

$$(\rho_* u_* - c\rho_*)' \approx \Delta\rho[-(c - u_1) + (1 + \frac{\rho_1}{\Delta\rho})\Delta u]H'.$$

Hence, the result follows from PROP.1.6(c) and (e). ∎

PROPOSITION 3.3 If ρ,u and p are given by (S_1),(S_2) and (S_3) respectively then the following conditions are equivalent:

(i) ρ,u and p are solutions of the equations (E_1) and (E_2).

(ii) We have the following statements: (a) The jump conditions (J_1) and (J_2) hold , (b) The generalized functions H, K and V (see [3.3]) satisfy the relation

(3.3.1) $$VK' \approx - \frac{v_1}{\Delta v} H' \qquad \text{in } \mathcal{G}_s(R).$$

Proof. Firstly we will give some preliminary computations. Since $[(v \cdot \rho)u_x]_x = [(v \cdot \rho_*)u_*']\cdot y^*$ and $(\rho u)_t + (\rho u^2 + p)_x =$

$(-c\rho_* u_* + \rho_* u_*^2 + p_*)' \cdot y^*$, by PROP.2.11(I) it follows that

$$(3.3.2) \quad \begin{vmatrix} (\rho u)_t + (\rho u^2 + p)_x \cong [(\nu \cdot \rho)u_x]_x & \text{if and only if} \\ \\ (-c\rho_* + \rho_* u_*^2 + p_*)' \cong [(\nu \cdot \rho_*)u_*'] . \end{vmatrix}$$

On the other hand , by [3.1] we have

$$-c\rho_* + \rho_* u_*^2 + p_* = a_0 + a_1 H + a_2 K + a_3 L + a_4 HK + a_5 K^2 + a_6 HK^2$$

where the real numbers a_i $(0 \le i \le 6)$ are defined by

$$a_0 = p_1 + \rho_1 u_1(u_1 - c) , \quad a_1 = u_1 \Delta\rho(u_1 - c) , \quad a_2 = \rho_1 \Delta u(2u_1 - c) ,$$

$$a_3 = \Delta p , \quad a_4 = \Delta\rho\Delta u(2u_1 - c) , \quad a_5 = \rho_1 \Delta u^2 , \quad a_6 = \Delta\rho\Delta u^2 .$$

Since K', L', $(HK)'$, $(K^2)'$ and $(HK^2)'$ are associated with H' (see PROP.1.8(c) and PROP.1.2(c)), the preceding identity implies

$$(3.3.3) \qquad (-c\rho_* + \rho_* u_*^2 + p_*)' \cong aH'$$

where $a = \sum a_i$. From (3.3.2) and (3.3.3) it follows that

$(\rho u)_t + (\rho u^2 + p)_x \cong [(\nu \cdot \rho)u_x]_x$ if and only if $[(\nu \cdot \rho_*)u_*'] \cong aH'$ hence, by integration (see PROP. 1.2(d) or [A-B],4.3.2) we get

$$(3.3.4) \quad \begin{vmatrix} (\rho u)_t + (\rho u^2 + p)_x \cong [(\nu \cdot \rho)u_x]_x & \text{if and only if} \\ \\ (\nu \cdot \rho_*)u_*' \cong aH + z , \text{ for some generalized constant } z. \end{vmatrix}$$

From the definition of a_j $(0 \le j \le 6)$ we obtain

$$a = \Delta p + \Delta\rho u_1(u_1 - c) + \Delta u u_1(\Delta\rho + \rho_1) + \Delta u(u_1 - c)(\Delta\rho + \rho_1) + \Delta u^2(\Delta\rho + \rho_1).$$

Now, if the condition (J_4) holds we can write

$$a = \Delta p + \Delta u^2(\Delta\rho + \rho_1)(-\frac{\rho_1}{\Delta\rho}) = \Delta p - \rho_1 \Delta u^2 (1 + \frac{\rho_1}{\Delta\rho}).$$

We can summarize the above considerations as follows:

(3.3.5)   If (J_*) holds then $(J_\square)$ holds if and only if $a = 0$.

On the other hand, by [3.3] we have

$$(\nu \cdot P_*) u'_* = \Delta \nu \nabla u'_* + \nu_1 u'_*$$

and since $u'_* = \Delta u K'$, $K' \overset{\sim}{=} 0$ in $\mathscr{S}_S(R^*)$(see PROP.1.6(d')) and $\nabla K' \overset{\sim}{=} 0$ in $\mathscr{S}_S(R^*)$ by the assumption $(A_\square)$, from the above identity we get

(3.3.6) $(\nu \cdot P_*) u'_* \overset{\sim}{=} 0$ in $\mathscr{S}_S(R^*)$.

(i) $\Longrightarrow$ (ii): Since ρ and u are solutions of (E_*), by PROP.3.2 the condition (J_*) holds and since ρ, u and p are solutions of $(E_\square)$, by (3.3.4) we have

(3.3.7) $(\nu \cdot P_*) u'_* \overset{\sim}{=} aH + z$ in $\mathscr{S}_S(R)$.

By restriction to R^*_- and using (3.3.6) and PROP.1.6(d), from (3.3.7) we get $z \overset{\sim}{=} 0$, hence we can write (3.3.7) as follows

(3.3.8) $(\nu \cdot P_*) u'_* \overset{\sim}{=} aH$ in $\mathscr{S}_S(R)$.

Now, by restriction to R^*_+, from (3.3.8) we get

$$(\nu \cdot P_*) u'_* \overset{\sim}{=} aH \text{ in } \mathscr{S}_S(R^*_+),$$

hence, by (3.3.6) and a minor modification in the proof of the PROP.1.6(e) we have $a = 0$, which shows by (3.3.8) that

(3.3.9) $(\nu \cdot P_*) u'_* \overset{\sim}{=} 0$ in $\mathscr{S}_S(R)$.

From (3.3.9) and LEMMA 3.1 it follows (3.3.1). Since $a = 0$ and the condition (J_*) holds, by (3.3.5) it follows that $(J_\square)$ holds.

(ii) $\Longrightarrow$ (i): Since the conditions (J_*) and $(J_\square)$ hold, by PROP.3.2

and (3.3.5) it follows that ρ and u are solutions of (E_1) and $a=0$.

From (3.3.1) and LEMMA 3.1 it follows that (3.3.9) holds, which

implies $(v \cdot \rho_*)u'_* \overset{\approx}{} 0 = aH$. Hence, (3.3.4) (with $z = 0$) shows

that ρ,u and p are solutions of (E_2). ▮

PROPOSITION 3.4 If ρ, u, p and E are given by (S_1),(S_2),(S_3) and

(S_4) respectively, then the following conditions are equivalent:

(i) ρ,u,p and E are solutions of the equations (E_1),(E_2) and (E_3).

(ii) We have the following statements: (a) The jump conditions

(J_1),(J_2) and (J_3) hold, (b) The generalized functions H, K and V

(see [3.3]) satisfy the relation (3.3.1) and

$$(3.4.1) \qquad VKK' \overset{\approx}{} - \frac{1}{2} \frac{v_1}{\Delta v} H' \qquad \text{in } \varsigma_s(R) .$$

Proof. The argument is similar to that of the proof of the PROP.3.

3. The relations [3.2] and the chain rule imply the two identities

$$E_t + [(E + p)u]_x = [(E_* + p_*)u_* - cE_*]' \cdot y^*$$

$$[(v \cdot \rho)uu_x]_x = [(v \cdot \rho_*)u_* u'_*]' \cdot y^*$$

Hence, by PROP.2.11(I) we get

$$(3.4.2) \quad \begin{vmatrix} E_t + [(E + p)u]_x \overset{\approx}{} [(v \cdot \rho)uu_x]_x & \text{if and only if} \\ [(E_* + p_*)u_* - cE_*]' \overset{\approx}{} [(v \cdot \rho_*)u_* u'_*]' . \end{vmatrix}$$

Now, from [3.1] we get

$$(E_* + p_*)u_* - cE_* = b_0 + b_1 K + b_2 L + b_3 M + b_4 KL + b_5 KM$$

where the real numbers b_j ($0 \leq j \leq 5$) are defined by

$$b_0 = (E_1 + p_1)u_1 - cE_1 \ , \ b_1 = \Delta u(E_1 + p_1) \ , \ b_2 = \Delta p u_1$$

$$b_3 = \Delta E(u_1 - c) \ , \ b_4 = \Delta u \Delta p \ \text{and} \ b_5 = \Delta u \Delta E \ .$$

Since K', L', M', $(KL)'$ and $(KM)'$ are associated to H', by derivation

of the preceding identity we obtain

(3.4.3) $\qquad\qquad [(E_* + p_*)u_* - cE_*]' \ \tilde{\approx} \ bH'$

where $b = \sum b_j$. From (3.4.2) and (3.4.3) it follows that

$E_t + [(E + p)u]_x \ \tilde{\approx} \ [(\nu \cdot \rho)uu_x]_x$ if and only if $[(\nu \cdot \rho_*)u_* u'_*]' \cdot \tilde{\approx}$

bH' hence, by integration we have

(3.4.4) $\left| \begin{array}{l} E_t + [(E + p)u]_x \ \tilde{\approx} \ [(\nu \cdot \rho)uu_x]_x \quad \text{if and only if} \\[2mm] (\nu \cdot \rho_*)u_* u'_* \ \tilde{\approx} \ bH + z, \ \text{for some generalized constant } z \ . \end{array} \right.$

From the definition of b_j $(0 \le j \le 5)$ it follows

$$b = \Delta p(\Delta u + u_1) + \Delta E(\Delta u - c + u_1) + \Delta u(E_1 + p_1)$$

and since the condition (J_4) implies $\Delta u - c + u_1 = -p_1 \dfrac{\Delta u}{\Delta p}$ we can

write

$$b = \Delta p(\Delta u + u_1) - \Delta u(p_1 \frac{\Delta E}{\Delta p} - E_1 - p_1) \ .$$

Therefore, we have the following statement:

(3.4.5) If (J_4) holds then (J_3) holds if and only if $b = 0$.

We have also the relation

(3.4.6) $\qquad\qquad (\nu \cdot \rho_*)u_* u'_* \ \tilde{\approx} \ 0 \qquad \text{in } \varsigma_s(R^*)$

which follows from a minor modification in the proof of (3.3.6).

(i) $\Longrightarrow$ (ii): Since ρ,u and p are solutions of (E_1) and (E_2), by

PROP.3.3 , the conditions $(J_4),(J_2)$ and (3.3.1) hold. Since ρ,u,p

and E are solutions of (E_1), (E_2) and (E_3), by (3.4.4) it follows

$$(\nu \cdot p_*) u_* u_*' \;\tilde{}\; bH + z \qquad \text{in } \varsigma_s(R) .$$

Hence, by a minor modification in the proof of the implications

$(3.3.7) \Longrightarrow (3.3.8) \Longrightarrow (3.3.9)$, (by applying (3.4.6) and PROP.1.6

(d),(e)), we get $b = 0$ and $z \;\tilde{}\; 0$ which implies

(3.4.7) $\qquad (\nu \cdot p_*) u_* u_*' \;\tilde{}\; 0 \qquad$ in $\varsigma_s(R) .$

Since (3.3.1) holds, the relation (3.4.7) together with LEMMA 3.1

implies (3.4.1). By (3.4.5), since $b = 0$, we have that (J_3) holds.

(ii) $\Longrightarrow$ (i): Since (J_1), (J_2) and (3.3.1) hold, by PROP.3.3 it

follows that ρ, u and p are solutions of (E_1) and (E_2). On the

other hand, since (J_1) and (J_3) hold, by (3.4.5) we have $b = 0$

hence, from (3.3.1), (3.4.1) and LEMMA 3.1(b) it follows that

(3.4.7) holds and therefore $(\nu \cdot p_*) u_* u_*' \;\tilde{}\; 0 = bH$ in $\varsigma_s(R)$. Hence,

by (3.4.4) (with $z = 0$), it follows that ρ, u, p and E are solutions

of (E_3). ∎

PROPOSITION 3.5 If ρ, u, p and E are given by (S_1), (S_2), (S_3) and

(S_4) respectively , then the following conditions are equivalent:

(i) ρ, u, p and E are solutions of (E_4),

(ii) The jump condition (J_4) holds.

Proof. The relations [3.2] imply

(3.5.1) $\qquad \lambda p + \dfrac{1}{2} \rho u^2 = (\lambda p_* + \dfrac{1}{2} \rho_* u_*^2) \cdot y^* .$

From [3.1] it follows

$$\lambda P_* + \frac{1}{2}\rho_* u_*^2 = d_0 + d_1 H + d_2 K + d_3 L + d_4 HK + d_5 K^2 + d_6 HK^2$$

where the real numbers d_j ($0 \le j \le 6$) are defined by

(3.5.2)
$$
\begin{vmatrix}
d_0 = \lambda P_1 + \frac{1}{2}\rho_1 u_1^2 \ , \ d_1 = \frac{1}{2} u_1^2 \Delta\rho \ , \ d_2 = \rho_1 u_1 \Delta u \ , \\
d_3 = \lambda\Delta\rho \ , \ d_4 = u_1 \Delta\rho\Delta u \ , \ d_5 = \frac{1}{2}\rho_1 \Delta u^2 , \ d_6 = \frac{1}{2}\Delta\rho\Delta u^2 .
\end{vmatrix}
$$

From the identity above and (3.5.1) , by the same argument used in PROP.3.3 and PROP.3.4 , it follows that

$$\lambda P_* + \frac{1}{2}\rho_* u_*^2 \; \widetilde{\approx} \; d_0 + dH \qquad \text{in } \varsigma_s(R)$$

where $d = \sum d_j$. Now , from (3.5.1) and PROP.2.11(I) we get

(3.5.3) $E \; \widetilde{\approx} \; \lambda P + \frac{1}{2}\rho u^2$ if and only if $E_* \; \widetilde{\approx} \; d_0 + dH$ in $\varsigma_s(R)$.

On the other hand, the definition of d and (3.5.2) show that

$$d = (\lambda P_r + \frac{1}{2}\rho_r u_r^2) - (\lambda P_1 + \frac{1}{2}\rho_1 u_1^2) \ ,$$

which, by the definition of ΔE , implies

(3.5.4) $\Delta E = d$ and $E_1 = \lambda P_1 + \frac{1}{2}\rho_1 u_1^2$ if and only if (J_4) holds.

(i) $\Longrightarrow$ (ii): From (i), (3.5.3) and the relation $H \; \widetilde{\approx} \; M$ it follows that $\Delta E . H + E_1 \; \widetilde{\approx} \; \Delta E . M + E_1 = E_* \; \widetilde{\approx} \; d_0 + dH$, hence

(3.5.5) $$(d - \Delta E)H \; \widetilde{\approx} \; E_1 - d_0 \qquad \text{in } \varsigma_s(R) .$$

Therefore, by restriction to R_-^* , the PROP.1.6(d) shows that

(3.5.6) $$E_1 = d_0 = \lambda P_1 + \frac{1}{2}\rho_1 u_1^2$$

and by (3.5.5) we get $(d - \Delta E)H \; \widetilde{\approx} \; 0$ in $\varsigma_s(R)$. Now, by restriction to R_+^* , the PROP.1.6(d) and (e)(with a minor modification) we have

(3.5.7) $\Delta E = d$.

From (3.5.6),(3.5.7) and (3.5.4), it follows that (J_4) holds .

(ii) $\Longrightarrow$ (i): From (ii) and (3.5.4) it follows that (3.5.6) 'and

(3.5.7) holds. Since $H \approx M$ we have $E_{\mu} \approx \Delta E.H + E_1$ in $\varsigma_s(R)$, hence

$E_{\mu} \approx dH + \Delta P_1 + \frac{1}{2} P_1 u_1^2 = dH + d_{\bullet}$, which shows (i) by (3.5.3).∎

We can summarize the PROP.3.4 and 3.5 in the following result:

PROPOSITION 3.6 For the generalized functions ρ, u, p and E in

$\varsigma_s(R^{\bullet})$ defined by (S_4), (S_2), (S_3) and (S_4) respectively, the

following conditions are equivalent:

(i) ρ, u, p and E are solutions of the system [S],

(ii) The jump conditions $(J_4),(J_2),(J_3)$ and (J_4) holds and the

generalized functions H, K and V (see [3.3]) satisfy the following

relations in $\varsigma_s(R)$:

(3.6.1) $VK' \approx -\frac{v_1}{\Delta v} H'$ and $VKK' \approx -\frac{1}{2}\frac{v_1}{\Delta v} H'$. ∎

Next, we will show that 'there are systems [S] without shock

solutions. We need the following result:

LEMMA 3.7 Let ρ and u be the generalized functions in $\varsigma_s(R^{\bullet})$ given

by (S_4) and (S_2) respectively and suppose that ρ and u are

solutions of the equation $(\bar{E}_4)$. Then there exists a generalized

constant $z \in \varsigma_s(R)$ such that

(3.7.1) $\rho_{\mu}^2 u_{\mu}' = z\rho_{\mu}'$ and $z \approx \rho_1(u_1 - c)$.

Proof. By the assumption (A_4) there is a representative $\hat{H}$ of H verifying $(2.10.1)$ (see PROP.2.10), let $\hat{P}_* = \Delta\rho\hat{H} + \rho_1$. From PROP. 2.10 (I) and PROP.2.8 (b) it follows that P_* is an invertible element of $\mathcal{G}_s(\mathbb{R})$ and, by PROP.2.8(d), we know that $\log P_*$ is well defined, $\log P_* \in \mathcal{G}_s^{\log}(\mathbb{R})$ (see REM.2.9 (a)) and therefore, $e^{\log P_*}$ is also well defined. Since ρ and u are solutions of $(\bar{E}_1)$, the identity $\rho_t + (\rho u)_x = (\rho_* u_* - c\rho_*)' \cdot y^*$ and PROP.2.11(II) imply $\rho_* u_*' + \rho_*' u_* = c\rho_*$. Therefore, by the definition of u_* (see [3.1]) we can write

$(3.7.2)$ $\qquad K' + (P_*^{-1}\rho_*')K = (\Delta u)^{-1}(c - u_1)P_*^{-1}\rho_*'$.

The moderate function $\hat{F} := \log \hat{P}_*$ is a representative of $F := \log P_*$. Since $F \in \mathcal{G}_s^{\log}(\mathbb{R})$, $F' = \rho_*^{-1}\rho_*'$ and $e^{\hat{F}} = \hat{P}_*$, by the THEOREM 2.3, any solution of the differential equation $(3.7.2)$ has a representative given by

$(3.7.3)$ $\qquad \hat{K}(\varepsilon,x) = (c-u_1)[\hat{P}_*(\varepsilon,x)\Delta u]^{-1}[\hat{z}_0(\varepsilon)+\hat{P}_*(\varepsilon,x)-\hat{P}_*(\varepsilon,a)]$

for each $(\varepsilon,x) \in I\times\mathbb{R}$, where $\hat{z}_0 \in \mathcal{G}_{M,s}(\mathbb{R})$ and we can assume that $a < 0$. Since ρ and u are clearly solutions of (E_1), by PROP.3.2, it follows that the condition (J_1) holds, which implies that

$$(\Delta u)^{-1}(c - u_1) = 1 + \frac{\rho_1}{\Delta\rho} < 0.$$

Taking $x = a$ in $(3.7.3)$ we get

$$\lim_{\varepsilon\to 0}\hat{z}_0(\varepsilon) = (c - u_1)^{-1}(\Delta u)[\Delta\rho.\lim_{\varepsilon\to 0}\hat{H}(\varepsilon,a) + \rho_1].\lim_{\varepsilon\to 0}\hat{K}(\varepsilon,a)$$

and, since by the assumption (A_0), $\hat{R}$ and $\check{R}$ satisfy the condition (PH2) (see DEF.1.7) and $a < 0$, it follows that $\lim_{\epsilon \longrightarrow 0} \hat{z}_0(\epsilon) = 0$. We consider the moderate function

$$\hat{z}_1 : \epsilon \in I \longmapsto \hat{P}_*(\epsilon, a) - \hat{z}_0(\epsilon) \in R,$$

then the above computations show that $\lim_{\epsilon \longrightarrow 0} \hat{z}_1(\epsilon) = P_1$, hence $z_1 \stackrel{\sim}{\sim} P_1$, where z_1 denotes the class of $\hat{z}_1$ in $\bar{R}_\epsilon$. Hence, we can write the class K of $\check{R}$ (see (3.7.3)) in the following way

$$K = (\Delta u)^{-1}(c - u_1)\rho_*^{-1}(P_* - z_1),$$

wich implies $u_* = c + (c - u_1)z_1\rho_*^{-1}$ and therefore we have $u'_* = (u_1 - c)z_1\rho_*^{-2}\rho'_*$. Hence, it is enough to define $z := (u_1 - c)z_1$. ∎

In the result below we consider the particular case $\nu(x) = x^2$.

PROPOSITION 3.8 The system

$(\bar{E}_1)$ $\rho_t + (\rho u)_x = 0$, (E_2^*) $(\rho u)_t + (p + \rho u^2)_x \stackrel{\sim}{\sim} (\rho^2 u_x)_x$

has no shock solutions.

Proof Assume that there are generalized functions ρ, u and p in $\varsigma_\epsilon(R^2)$ given by $(S_1),(S_2)$ and (S_3), verifying the conditions (A_i) $(1 \leq i \leq 5)$, which are solutions of $(\bar{E}_1)$ and (E_2^*). By LEMMA 3.7 there is a generalized constant $z \in \varsigma_\epsilon(R)$ satisfying (3.7.1). Since ρ, u and p are clearly solutions of (E_1) and (E_2^*), by PROP.3.3 the generalized functions H,K and V verify the condition (3.3.1) and by LEMMA 3.1(since $\nu(x) = x^2$ for all $x \in R$) we have $\rho_*^2 u'_* \stackrel{\sim}{\sim} 0$

In $\varsigma_s(R)$. Therefore, by (3.7.1) it follows that $zp'_* \overset{\sim}{\cdot} 0$ in $\varsigma_s(R)$, hence by integration (see PROP.1.2(d)), there exists a generalized constant $z_0 \in \varsigma_s(R)$ such that

(3.8.1) $$zp_* \overset{\sim}{\ } z_0 \quad \text{in } \varsigma_s(R).$$

Since $\hat{H}$ satisfies (PH2) and (2.10.1) (see $(A_*),(A_4)$) the dominated convergence theorem shows that $p_* \overset{\sim}{\ } p_1$ in $\varsigma_s(R^*_-)$ (resp. $p_* \overset{\sim}{\ } p_r$ in $\varsigma_s(R^*_+)$) which together with $z \overset{\sim}{\ } p_1(u_1 - c)$ (see (3.7.1)) implies $zp_* \overset{\sim}{\ } p_1^2(u_1 - c)$ in $\varsigma_s(R^*_-)$ (resp. $zp_* \overset{\sim}{\ } p_1 p_r(u_1 - c)$ in $\varsigma_s(R^*_+)$) (here we use the fact : If $f \in \varsigma_s(\Omega)$, $z \in \bar{R}_s$, $f \overset{\sim}{\ } 0$ and $z \overset{\sim}{\ } \alpha \in R$, then $zf \overset{\sim}{\ } 0$). Hence, by restriction to R^*_- (resp. R^*_+) in (3.8.1), we get $z_0 \overset{\sim}{\ } p_1^2(u_1 - c)$ (resp. $z_0 \overset{\sim}{\ } p_1 p_r(u_1 - c)$) in $\bar{R}_s$. Therefore, we get $p_1^2(u_1 - c) \overset{\sim}{\ } p_1 p_r(u_1 - c)$ in $\bar{R}_s$ which implies

(3.8.2) $$p_1^2(u_1 - c) = p_1 p_r(u_1 - c) .$$

By PROP.3.3 we know that the condition (J_4) holds, hence $u_1 - c > 0$ therefore, from (3.8.2) it follows that $p_1 = p_r$ .This is a contradiction in view of the assumption (A_2). ∎

From the PROP.3.8 we get:

THEOREM 3.9 The system

$$\left\{ \begin{array}{l} (\bar{E}_1) \quad \rho_t + (\rho u)_x = 0 \\[2mm] (E_2^*) \quad (\rho u)_t + (p + \rho u^2)_x \approx (\rho^2 u_x)_x \\[2mm] (E_3^*) \quad E_t + [(E + p)u]_x \approx (\rho^2 u u_x)_x \\[2mm] (E_4) \quad E \approx \lambda p + \frac{1}{2} \rho u^2 \end{array} \right.$$

has no shock solutions. ∎

When we have the equality in (E_4) (instead of the association) we get the following result:

PROPOSITION 3.10 For the generalized functions ρ, u, p and E in $G_s(R^2)$ defined by $(S_1),(S_2),(S_3)$ and (S_4) respectively, the following conditions are equivalent:

(i) ρ, u, p and E are solutions of the equation

$$(\bar{E}_4) \qquad E = \lambda p + \frac{1}{2} \rho u^2 \quad,$$

(ii) The condition (J_4) holds and we have the identity

$$\Delta E . M = d_1 H + d_2 K + d_3 L + d_4 H K + d_5 K^2 + d_6 H K^2$$

where the real numbers d_j $(1 \le j \le 6)$ are given by (3.5.2)(see the proof of PROP.3.5). In this case we have $\Delta E = \sum_{j=1}^{6} d_j$.

Proof. The argument is a minor modification of the proof of PROP.3.5 by applying PROP.2.11(II). ∎

Finally, we will complete the information contained in PROP.3.6 by showing that there are proper Heaviside g.f. H, K and V verifying the conditions (A_4) and (A_5) and the relations (3.6.1). Let us

recall that $\nu \in C^\infty(R_+^*, R_+^*)$ is a strictly increasing function which satisfies the condition (2.5.1) with $b = +\infty$ (see PROP.2.5).

LEMMA 3.11 Let $\theta \in R$ such that $\theta > 1$ and let $\mu : I \longrightarrow [1,\theta[$ be a function such that $\lim_{\varepsilon \longrightarrow \theta} \mu(\varepsilon) = \theta$. Then, there exists $\hat{V} \in \mathcal{E}_{M,s}[R]$ satisfying the condition (PH2) (see DEF.1.7) and

(3.11.1) $\theta \leq \hat{V}(\varepsilon,x) \leq \mu(\varepsilon)$ for every $(\varepsilon,x) \in IxR$.

Moreover, if V is the class of $\hat{V}$ in $\mathcal{G}_s(R)$ and $K := \mathfrak{g}_\rho \in \mathcal{G}_s(R)$ is the Heaviside g.f. of EX. 1.5(c), then we have:

(3.11.2) $VK' \approx \theta K'$ and $VKK' \approx \frac{1}{2} VK'$ in $\mathcal{G}_s(R)$.

Proof. Fix $\rho \in \mathcal{D}(R)$ as in EX.1.5(c) and let denote by $z(\varepsilon,.)$ the characteristic function of the interval $[-\frac{3}{2}\varepsilon, \frac{3}{2}\varepsilon] \subset R$. Then, we have $u(\varepsilon,.) := z(\varepsilon,.)* \rho_\alpha \in C^\infty(R)$, where $\alpha := \frac{1}{4}\varepsilon$ and $*$ denotes as usual the convolution product. It is well known that

 supp $u(\varepsilon,..) \subset [-2\varepsilon, 2\varepsilon]$, $u(\varepsilon,..) \equiv 1$ in $[-\varepsilon,\varepsilon]$ and $\theta \leq u(\varepsilon,..) \leq 1$ for every $\varepsilon \in I$. Next, we define the two following functions:

$v(\varepsilon,x) := \mu(\varepsilon)u(\varepsilon,x)$ for every $(\varepsilon,x) \in IxR$

and

$w(\varepsilon,x) := \begin{cases} u(\varepsilon,x) & \text{for each } (\varepsilon,x) \in IxR_+ \\ 1 & \text{for each } (\varepsilon,x) \in IxR_-. \end{cases}$

Then we set $\hat{V}(\varepsilon,x) := 1 + w(\varepsilon,x)[v(\varepsilon,x) - 1]$ for every $(\varepsilon,x) \in IxR$ and it is easy to see that $\hat{V}$ satisfies (3.11.1) and

(3.11.3) $\hat{V}(\varepsilon,x) = \mu(\varepsilon)$ for each $(\varepsilon,x) \in Ix[-\varepsilon,\varepsilon]$

and $\hat{U}(\epsilon,x) \equiv 0$ (resp. $\hat{U}(\epsilon,x) \equiv 1$) whenever $x \leqslant -2\epsilon$ (resp. $x \geqslant 2\epsilon$), which implies that $\hat{U}$ satisfies (PH2). It remains to show that $\hat{U}$ is moderate and for this it is enough to prove that u is moderate which follows from the definition of u. Indeed, for every $k \in \mathbb{N}$ we have $|u^{(k)}(\epsilon,x)| \leqslant 4^k \epsilon^{-k} \int |\rho^{(k)}(s)| ds$ $(x \in \mathbb{R})$. Finally, we will use (3.11.3) for to prove (3.11.2). Fix $\rho \in \mathcal{D}(\mathbb{R})$, $\hat{g}_\rho$ and $\hat{g}'_\rho = \hat{f}_\rho$ as in EX. 1.5(c). Since $\hat{R}' = \hat{f}_\rho$ it follows that $\hat{R}'$ is a representative of $K = f_\rho$ hence, letting $\ast \in \mathcal{D}(\mathbb{R})$ arbitrary, the first relation of (3.11.2) follows from $\lim_{\epsilon \to 0} I(\epsilon) = 0$, where

$$I(\epsilon) := \int_{-\infty}^{+\infty} [\hat{U}(\epsilon,x) - \theta]\hat{R}'(\epsilon,x)\ast(x)dx.$$

Since $\mathrm{supp}\,\hat{R}'(\epsilon,.) = [-\epsilon,\epsilon]$, by (3.11.3) it follows:

$$I(\epsilon) = [\mu(\epsilon) - \theta]\int_{-\infty}^{+\infty} \rho_\epsilon(x)\ast(x)dx \longrightarrow \theta \quad \text{if } \epsilon \longrightarrow \theta$$

because $\lim_{\epsilon \to 0} \mu(\epsilon) = \theta$ by hipothesis. For the second relation of (3.11.2), letting $\ast \in \mathcal{D}(\mathbb{R})$ arbitrary, it suffices to show that $\lim_{\epsilon \to 0} J(\epsilon) = 0$, where

$$J(\epsilon) := \int_{-\infty}^{+\infty} \hat{U}(\epsilon,x)\hat{R}'(\epsilon,x)[\hat{R}(\epsilon,x) - \tfrac{1}{2}]\ast(x)dx.$$

From (3.11.3) we get $J(\epsilon) = \mu(\epsilon)\int_{-\infty}^{+\infty} \hat{R}'(\epsilon,x)[\hat{R}(\epsilon,x) - \tfrac{1}{2}]\ast(x)dx$ and since $KK' \simeq \tfrac{1}{2} K'$ (see PROP.1.8(c) and PROP.1.2(c)) we conclude that the above integral tends to θ when $\epsilon \longrightarrow \theta$. ∎

LEMMA 3.12 Let $(\rho_r,\rho_1) \in \mathbb{R}^\ast$ with $0 < \rho_r < \rho_1$, $\delta \in]0,+\infty]$ and $\nu \in C^\infty(\mathbb{R}^\ast_+,\mathbb{R}^\ast_+)$ an strictly increasing function such that $\mathrm{Im}(\nu) =]0,\delta[$.

Moreover, assume that ν^{-1} satisfies the condition (2.5.1) and that (see the notations of PROP.2.10) ν satisfies one of the conditions (2.10.3),(b) or (2.10.4),(b). Then, there is $V \in \mathcal{G}_s(\mathbb{R})$ satisfying the following conditions:

(I) The generalized function H defined by (2.10.5) is a proper Heaviside g.f. and H has a representative verifying the condition (2.10.1).

(II) If $K := \vartheta_\rho \in \mathcal{G}_s(\mathbb{R})$ is the Heaviside g.f. of EX.1.5(c), then V, H and K satisfy the condition (3.6.1) (see PROP.3.6).

Proof. With the notations of PROP.2.10 we define $\theta := \dfrac{\nu_1}{\Delta\nu}$, then $\theta > 1$. Since $\nu_r \epsilon \leq \nu_r$ (resp. $\nu(\rho_r \epsilon) \leq \nu_r$ because ν is strictly incresing) we have $(\Delta\nu)^{-1} \cdot \nu_r \epsilon + \theta \geq 1$ (resp. $(\Delta\nu)^{-1} \cdot \nu(\rho_r \epsilon) + \theta \geq 1$) for each $\epsilon \in I$. Assume now that ν satisfies (2.10.3),(b) and consider the function $\mu : \epsilon \in I \longrightarrow (\Delta\nu)^{-1} \cdot \nu_r \epsilon + \theta \in [1, \theta[$. Then, by the LEMMA 3.11 there exists $V \in \mathcal{G}_s(\mathbb{R})$ satisfying (3.11.2) and a representative $\hat{V}$ verifying the conditions (PH2),(3.11.1) and hence, the definition of μ shows that in addition $\hat{V}$ satisfies (2.10.3),(a). Analogously, if ν satisfies (2.10.4),(b), by definig $\mu : \epsilon \in I \longrightarrow (\Delta\nu)^{-1} \cdot \nu(\rho_r \epsilon) + \theta \in [1, \theta[$, from the LEMMA 3.11 it follows that there exists $V \in \mathcal{G}_s(\mathbb{R})$ verifying (3.11.2) and a representative $\hat{V}$ of V which satisfies (PH2) and (2.10.4),(b). By

concequence, there exists $V \in \mathcal{G}_{3}(R)$ satisfying (3.11.2) and $\dot{U}$, a representative of V, verifying (PH2) in such a way that $(v, \dot{U})$ satisfies (2.10.3) or (2.10.4), hence V is a proper Heaviside g.f. because (3.11.1) implies (PH1). Now, the statement (I) follows from PROP.2.10(III) and the statement (II) follows from (3.11.2). ∎

From PROP.3.6 and LEMMA3.12 we get the following existence result:

THEOREM 3.13 Let $\beta \in]0, +\infty]$, $v \in C^{\infty}(R_+^*, R_+^*)$ an strictly incresing function such that v and v^{-1} satisfy the condition (2.5.1), $\text{Im}(v)$ = $]0, \beta[$ and v satisfies one of the conditions (2.10.3),(b) or (2.10.4),(b). Then, for the generalized functions ρ, u, p and E in $\mathcal{G}_{\mathcal{E}}(R)$ defined by (S_1), (S_2), (S_3) and (S_4) respectively, where H,K and U are as in LEMMA 3.12, the following conditions are equivalent:

(i) ρ, u, p and E are shock solutions of the system [S].

(ii) The jump conditions (J_i) $(1 \le i \le 4)$ hold. ∎

REMARK It is easy to see that the functions $v(x) := x^{\alpha}$ ($\alpha \in R_+^*$), $v(x) := a^x - 1$ ($a > 1$), $v(x) := \text{sh } x$, $v(x) := \text{ch } x$, for $x > 0$, are simple examples of functions $v \in C^{\infty}(R_+^*, R_+^*)$ which satisfy the hypothesis of THEOREM 3.13.

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