

The dynamics of measurable pseudo-Anosov maps

by

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Abstract. We study the dynamics of measurable pseudo-Anosov homeomorphisms of surfaces, a generalization of Thurston's pseudo-Anosov homeomorphisms. A measurable pseudo-Anosov map has a transverse pair of full measure turbulations consisting of streamlines which are dense immersed lines: these turbulations are equipped with measures which are expanded and contracted uniformly by the homeomorphism. The turbulations need not have a good product structure anywhere, but have some local structure imposed by the existence of tartans: bundles of unstable and stable streamline segments which intersect regularly, and on whose intersections the product of the measures on the turbulations agrees with the ambient measure.

We prove that measurable pseudo-Anosov maps are transitive, have dense periodic points, sensitive dependence on initial conditions, and are ergodic with respect to the ambient measure.

Measurable pseudo-Anosov maps were introduced in our earlier paper [arXiv:2306.16059], where we constructed a parameterized family of non-conjugate examples on the sphere.

1. Introduction. One of the groundbreaking results proved by William Thurston in the late 1970s and early 1980s was his classification theorem for isotopy classes of surface homeomorphisms [24, 18]. Such a class contains a map of finite order, or it contains a pseudo-Anosov map, or the surface can be cut into pieces which carry maps of these two types.

Pseudo-Anosov maps are defined as having a transverse pair of invariant foliations, with finitely many pronged singularities, which carry a pair of holonomy invariant measures, one expanding and the other contracting under the map. In a precise sense, pseudo-Anosov maps are simplest in their isotopy class, which implies, among other things, that all maps in the class have

2020 *Mathematics Subject Classification*: Primary 37E30; Secondary 57K20.

Key words and phrases: surface homeomorphisms, chaos, pseudo-Anosov maps, measurable pseudo-Anosov maps, turbulations.

Received 1 August 2023; revised 16 February 2024 and 22 July 2024.

Published online 10 September 2024.

infinitely many periodic orbits, of infinitely many periods, and have positive topological entropy.

The classification theorem applies to surfaces of finite topological type, that is, to compact surfaces from which finitely many points have been removed. Dynamicists have often put this theorem to use in studying surface dynamics by finding periodic orbits, removing them from the surface, and applying Thurston’s theorem to the isotopy class on the punctured surface thus obtained [6, 7]. It then follows that the original map has at least as much dynamics as the “hidden pseudo-Anosov map”.

Powerful though it is, there is a fundamental shortcoming to this approach when studying the dynamics of parameterized families of maps: there are only countably many isotopy classes of homeomorphisms on a surface of finite topological type, even if we allow an arbitrary finite number of punctures. This means that it is not possible to capture all of the dynamical complexity of a whole parameterized family using only canonical representatives from the isotopy classes of maps in the family acting on the surface punctured at finite invariant sets. Thus we are faced with the task of finding a new class of maps that retains many of the properties of pseudo-Anosov maps but is rich enough to occur in every member of a parameterized family.

Our approach to this task was inspired by one of the standard approaches to constructing pseudo-Anosov maps. Beginning with a pseudo-Anosov isotopy class, a train track algorithm such as that of Bestvina–Handel [5, 19, 20] produces an expanding map on a 1-dimensional graph, its *invariant train track*. Embedding the inverse limit of the induced map acting on the track into the surface and then collapsing down appropriate complementary regions yields the associated pseudo-Anosov map. The holonomy invariant measures on the invariant foliations are derived from a pair of measures associated with the expanding map.

To work with a parameterized family, we can instead start with a family of piecewise expanding maps on a *fixed* graph and turn the same handle: take inverse limits, embed into a surface and collapse complementary regions. We studied the simplest such example, in which the graph is the interval and the family of maps is the tent family. This led to the appropriate generalization of pseudo-Anosov maps considered here, which we called *measurable pseudo-Anosov maps* (see [10, 11]).

Inverse limits of tent maps on the interval have been much studied by topologists since the early 20th century and yield very complicated continua. For example, when the parameter s is such that critical orbit of the tent map f_s of slope s is dense (a full measure, dense G_δ set of parameters), theorems of Bruin and of Raines [12, 22] imply that the inverse limit is nowhere locally the product of a Cantor set and an interval. In addition, the collection of

globally regular (bi-dense, continuous injective images of \mathbb{R}) path components is meager. In contrast, and of critical importance here, with respect to the natural invariant measure, the collection of globally regular path components has full measure. Moreover, there is a pair of measures, one expanding by factor s on path components of the inverse limit, and the other contracting in the “fiber” direction by $1/s$; these are built from Lebesgue measure on the interval and the unique absolutely continuous invariant measure of the tent map respectively. Further, there are positive measure sets, called *boxes*, with a Fubini-like splitting connecting these two measures with the natural, ambient, invariant measure [9].

Embedding the inverse limit as an attractor in the disk using the Brown–Barge–Martin procedure [3, 8] and then collapsing down complementary regions produces a homeomorphism, F_s , of the sphere [10]. The path components of the inverse limit become unstable sets while the fiber direction collapses down to stable sets. The ambient invariant measure on the inverse limit becomes an invariant Oxtoby–Ulam measure on the sphere. The pair of measures on the inverse limit become unstable and stable measures and the boxes become positive measure sets with a nice Fubini-like structure called *tartans*. It is shown in [11] that the resulting *unimodal measurable pseudo-Anosov* family $\{F_s\}_{s \in (\sqrt{2}, 2]}$ has the following properties:

- For each $s \in (\sqrt{2}, 2]$, the map F_s has topological entropy $\ln s$; in particular, no two members of the family are conjugate.
- Every member of the family is a measurable pseudo-Anosov map as defined here.
- For an uncountable dense set of values of s , F_s is a *generalized pseudo-Anosov map* as defined in [14, 15].
- For a countable discrete set of parameter values, F_s is a “classical” pseudo-Anosov map with finitely many one-pronged singularities.
- However, for a full Lebesgue measure subset of parameters $s \in (\sqrt{2}, 2]$, F_s is a “genuine” measurable pseudo-Anosov map in the sense that it is neither a classical pseudo-Anosov nor a generalized pseudo-Anosov map.

The definition of measurable pseudo-Anosov maps is general enough to include a large class of maps, among these, of course, the classical pseudo-Anosov maps introduced by Thurston, the generalized pseudo-Anosov maps from [14, 15], and the family just constructed. It has two fundamental components. The first is that the transverse pair of invariant foliations of a pseudo-Anosov map is generalized to a transverse pair of invariant *turbulations* – decompositions of almost all of the surface into continuous injective images of \mathbb{R} , called *streamlines*, themselves equipped with 1-dimensional measures – one of which is uniformly expanded, and the other uniformly contracted, under the dynamics. The second component is the ambient (2-dimensional)

Oxtoby–Ulam invariant measure and the tartans, which play the rôle of local charts. Each tartan has a product structure using arcs from the stable and unstable streamlines, and the measures on the streamlines are connected to the ambient invariant measure via a Fubini-like hypothesis. Tartans form a family of positive measure sets whose union has full measure.

The purpose of this article is to prove that this definition carries dynamical content by showing that measurable pseudo-Anosov maps have the following properties:

- They are transitive.
- The collection of their periodic orbits is dense.
- With a mild additional hypothesis, they are ergodic.

It is shown in [18] that Thurston’s pseudo-Anosov maps have these properties, and here we extend them to the much more general class of measurable pseudo-Anosov maps.

As we have explained, the definition of measurable pseudo-Anosov maps was predominantly guided by Thurston’s definition and the examples derived from the unimodal family, but we were also influenced by the higher-dimensional holomorphic analogs in [4, 21, 23]. In particular, the term “turbulation” is borrowed from comments in [21].

2. Turbulations, tartans, and measurable pseudo-Anosov maps.

In this section we define measurable pseudo-Anosov maps, together with their invariant measured turbulations and the tartans which give these turbulations local structure. Throughout the paper, Σ will denote a compact surface, and μ will denote an *Oxtoby–Ulam* (or *OU*) measure on Σ : one which is Borel, non-atomic, positive on open sets, and satisfies $\mu(\partial\Sigma) = 0$.

The definitions are lengthy, and we start by motivating and summarizing them. In place of the measured foliations which are left invariant by a pseudo-Anosov map, the invariant stable and unstable objects of a measurable pseudo-Anosov map are *measured turbulations*. Each turbulation is a union of disjoint immersed lines (i.e. continuous injective images of \mathbb{R}) called *streamlines*, which take the place of the leaves of pseudo-Anosov foliations. Each streamline is equipped with its own OU *stream measure*, which takes the place of the transverse measures on pseudo-Anosov foliations. The measurable pseudo-Anosov map will send streamlines to streamlines in each turbulation, expanding stream measures by a constant factor λ in the *unstable turbulation*, and contracting by a constant factor $1/\lambda$ in the *stable turbulation*.

Figure 1 is a schematic depiction of some streamlines of a typical transverse pair of turbulations in a subset of Σ . The reader should bear in mind that there may be backtracking (or “kinks”) at every scale and in every open

subset of Σ , so that the figure only shows the first level of complexity of the turbulations. Note that, unlike pseudo-Anosov foliations, which necessarily have leaves which end at pronged singularities, every streamline of a turbulation is a full immersed line; however, the streamlines are not required to cover the whole of Σ but only a full μ -measure subset of it (so, for example, the invariant foliations of a pseudo-Anosov map become invariant turbulations when we throw away the singular leaves).

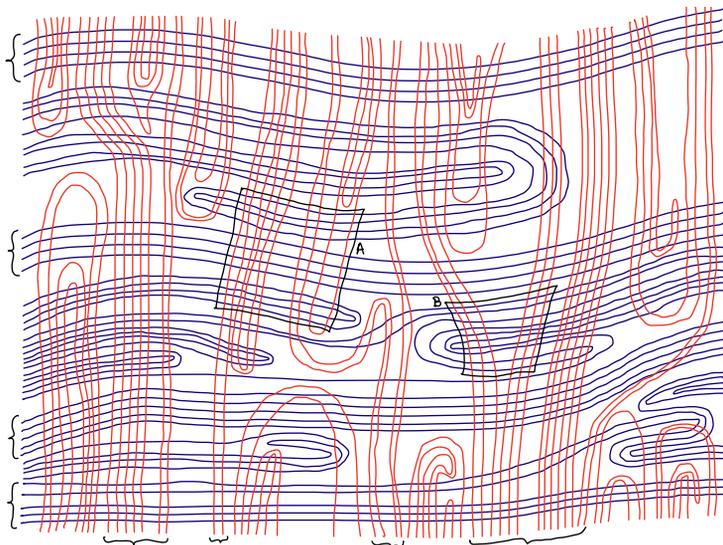


Fig. 1. Invariant turbulations of a measurable pseudo-Anosov map (schematic)

The turbulations may not define a product structure in any open subset. Nevertheless, suitable collections of *stream arcs* of the two turbulations do define a product structure on the set of their intersections. On the largest scale in Figure 1, the stream arcs indicated by braces on the left and bottom of the figure intersect regularly: each stream arc of one turbulation intersects each stream arc of the other turbulation exactly once. We call such a collection of stream arcs a *tartan*, and the stream arcs which constitute it are called *fibers*; a tartan R endows the set R^{th} of its intersection points with a product structure, and we require the measure on R^{th} induced by the product of the stream measures on the fibers to agree with the ambient measure μ ; this is called *compatibility* of R . Provided that $\mu(R^{\text{th}}) > 0$, the tartan has given structure to a significant subset of Σ . Tartans play the same structural rôle as charts and changes of charts in the theory of foliations, and are reminiscent of product structures in the differentiable dynamics of Axiom A maps.

There are many other tartans in Figure 1: for example, the stream arcs in the regions labeled A and B – or at least those stream arcs which have been drawn – are fibers of tartans (some of whose intersections are also intersections of a larger tartan). We say that a pair of turbulations is *full* if there is a countable collection of tartans which gives structure to a full μ -measure subset of Σ (the union of their intersection sets has full measure), and in addition every non-empty open subset of Σ contains a tartan R with $\mu(R^\text{th}) > 0$. The latter condition need not follow from the former; additional regularity conditions which ensure that it does are discussed in Section 5.

These are the essential ingredients for the definition of a measurable pseudo-Anosov map: it should preserve a transverse full pair of measured turbulations, expanding the streamlines of one and contracting the streamlines of the other by the same uniform factor. There are some additional technical conditions, the most important of them being:

- by analogy with pseudo-Anosov foliations, every streamline of an invariant turbulations is dense in Σ ;
- stream arcs of small measure have small diameter (with respect to some fixed metric on Σ) – this condition is called *tameness*.

It turns out that holonomy invariance of the measures within tartans is a consequence of compatibility (Lemma 2.6): if R is a compatible tartan and B is a subset of R^th contained in a fiber of one of the turbulations, then the stream measure of B is preserved as it is pushed into other fibers along fibers of the second turbulation.

We now proceed with the formal definitions.

DEFINITIONS 2.1 (Measured turbulations, streamlines, dense streamlines, stream measure, transversality, stream arcs). A *measured turbulation* (\mathcal{T}, ν) on Σ is a partition of a full μ -measure Borel subset of Σ into immersed lines called *streamlines*, together with an OU measure ν_ℓ on each streamline ℓ , which assigns finite measure to closed arcs in ℓ .

- We refer to the measures ν_ℓ as *stream measures* to distinguish them from the ambient measure μ on Σ .
- We say that the measured turbulation *has dense streamlines* if every streamline is dense in Σ .
- A pair of measured turbulations is *transverse* if they are topologically transverse on a full μ -measure subset of Σ (to be precise, there is a full measure subset of Σ whose every point x is contained in streamlines ℓ and ℓ' of the two turbulations which intersect transversely at x).
- Given distinct points x and y of a streamline ℓ of \mathcal{T} , we write $[x, y]_\ell$, or just $[x, y]$ if the streamline is irrelevant or clear from the context, for the (unoriented) closed arc in ℓ with endpoints x and y , and $[x, y)_\ell$ and $(x, y)_\ell$

for the arcs obtained by omitting one or both endpoints of $[x, y]_\ell$. We refer to these as *stream arcs* of the turbulation.

- The *measure* of a stream arc is its stream measure.

We impose a regularity condition on turbulations which requires that stream arcs of small measure are small. Note that the following definition is independent of the choice of metric on Σ : since Σ is compact, it is equivalent to the topological condition that for every neighborhood N of the diagonal $\{(x, x) : x \in \Sigma\}$ in $\Sigma \times \Sigma$ there is some $\delta > 0$ such that if $[x, y]$ is a stream arc with $\nu([x, y]) < \delta$, then $(x, y) \in N$. However, the metric formulation is usually easier to apply directly.

DEFINITION 2.2 (Tame turbulation). Let d be any metric on Σ compatible with its topology. A measured turbulation (\mathcal{T}, ν) on Σ is *tame* if for every $\epsilon > 0$ there is some $\delta > 0$ such that if $[x, y]$ is a stream arc with $\nu([x, y]) < \delta$, then $d(x, y) < \epsilon$.

Let (\mathcal{T}^s, ν^s) and (\mathcal{T}^u, ν^u) be a transverse pair of measured turbulations on Σ . With a view to what is to come, we refer to them as *stable* and *unstable* turbulations, and similarly apply these adjectives to their streamlines, stream measures, stream arcs, etc., although dynamics will not enter the picture until the very end of this section. If x and y lie in the same streamline of one or both turbulations, we write, for example, $[x, y]_s$ and $[x, y]_u$ to denote the stream arcs of the stable or unstable turbulation with x and y as endpoints.

DEFINITIONS 2.3 (Tartan R , fibers, R^\natural , positive measure tartan, oriented tartan). A *tartan* $R = (R^s, R^u)$ consists of Borel subsets R^s and R^u of Σ , which are disjoint unions of stable and unstable stream arcs respectively, having the following properties:

- Every arc of R^s intersects every arc of R^u exactly once, either transversely or at an endpoint of one or both of these arcs.
- There is a consistent orientation of the arcs of R^s and R^u : the arcs can be oriented so that every arc of R^s (respectively R^u) crosses the arcs of R^u (respectively R^s) in the same order.
- The measures of the arcs of R^s and R^u are bounded above.
- There is an open topological disk $U \subseteq \Sigma$ which contains $R^s \cup R^u$.

We refer to the arcs of R^s and R^u as the *stable* and *unstable fibers* of R , to distinguish them from other stream arcs. We write R^\natural for the set of *tartan intersection points*:

$$R^\natural = R^s \cap R^u.$$

We say that R is a *positive measure tartan* if $\mu(R^\natural) > 0$. An *oriented tartan* is a tartan together with a consistent orientation of its stable and unstable fibers.

Note that R^s and R^u are subsets of Σ , but they decompose uniquely as disjoint unions of stable/unstable stream arcs, so where necessary or helpful they can be viewed as the collections of their fibers.

NOTATION 2.4 ($\mathfrak{s}(x)$, $\mathfrak{u}(x)$, $\mathfrak{s} \pitchfork \mathfrak{u}$, E^s , E^u , $\psi^{s,u}$, $\nu_{\mathfrak{s}}$ and $\nu_{\mathfrak{u}}$, product measure $\nu_{\mathfrak{s} \times \mathfrak{u}}$, holonomy maps $h_{\mathfrak{s},\mathfrak{s}'}$, $h_{\mathfrak{u},\mathfrak{u}'}$). Let R be a tartan; $x, y \in R^{\pitchfork}$; \mathfrak{s} and \mathfrak{s}' be stable fibers of R ; and \mathfrak{u} and \mathfrak{u}' be unstable fibers of R . We write (see Figure 2):

- ν_x^s and ν_x^u for the stream measures on the stable and unstable streamlines through x ;
- $\mathfrak{s}(x)$ and $\mathfrak{u}(x)$ for the stable and unstable fibers of R containing x ;
- $\mathfrak{s} \pitchfork \mathfrak{u} = \mathfrak{u} \pitchfork \mathfrak{s} \in R^{\pitchfork}$ for the unique intersection point of \mathfrak{s} and \mathfrak{u} ;
- $E^s = \mathfrak{s} \cap R^u = \mathfrak{s} \cap R^{\pitchfork}$ and $E^u = \mathfrak{u} \cap R^s = \mathfrak{u} \cap R^{\pitchfork}$;
- $\psi^{s,u}: E^s \times E^u \rightarrow R^{\pitchfork}$ for the bijection $(x, y) \mapsto \mathfrak{u}(x) \pitchfork \mathfrak{s}(y)$;
- $\nu_{\mathfrak{s}}$ and $\nu_{\mathfrak{u}}$ for the (restrictions of the) stream measures on E^s and E^u ;
- $\nu_{\mathfrak{s} \times \mathfrak{u}}$ for the product measure $\nu_{\mathfrak{s}} \times \nu_{\mathfrak{u}}$ on $E^s \times E^u$;
- $h_{\mathfrak{s},\mathfrak{s}'}: E^s \rightarrow E^{s'}$ (respectively $h_{\mathfrak{u},\mathfrak{u}'}: E^u \rightarrow E^{u'}$) for the *holonomy map* given by $x \mapsto \mathfrak{u}(x) \pitchfork \mathfrak{s}'$ (respectively $y \mapsto \mathfrak{s}(y) \pitchfork \mathfrak{u}'$).

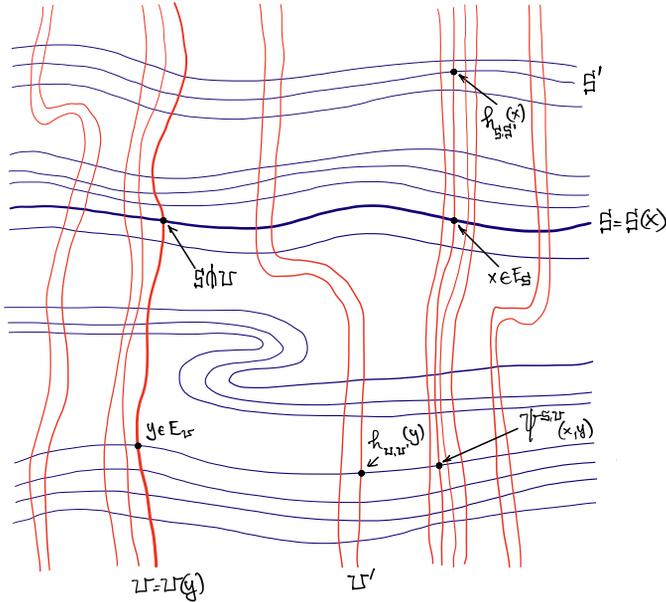


Fig. 2. Illustration of Notation 2.4

Note that ν_x^s and ν_y^u are measures on entire streamlines, whereas $\nu_{\mathfrak{s}}$ and $\nu_{\mathfrak{u}}$ are restrictions of such measures to subsets of R^{\pitchfork} .

DEFINITION 2.5 (Compatible tartan). We say that a tartan $R = (R^s, R^u)$ is *compatible* (with the ambient measure μ) if, for all stable and unstable fibers \mathfrak{s} and \mathfrak{u} , the bijection $\psi^{s,u}: E^s \times E^u \rightarrow R^\natural$ is bi-measurable, and $\psi_*^{s,u} \nu_{\mathfrak{s} \times \mathfrak{u}} = \mu|_{R^\natural}$.

If R is compatible and $A \subset R^\natural$ is Borel, then using Fubini's theorem in the product $E^s \times E^u$ and pushing forward to R^\natural we have, for any stable and unstable fibers \mathfrak{s} and \mathfrak{u} ,

$$(1) \quad \mu(A) = \int_{E^s} \nu_{\mathfrak{u}}(A^{\mathfrak{u}}(x)) d\nu_{\mathfrak{s}}(x) = \int_{E^u} \nu_{\mathfrak{s}}(A^{\mathfrak{s}}(y)) d\nu_{\mathfrak{u}}(y),$$

where

$$\begin{aligned} A^{\mathfrak{u}}(x) &= \{y \in E^u : \mathfrak{u}(x) \pitchfork \mathfrak{s}(y) \in A\}, \\ A^{\mathfrak{s}}(y) &= \{x \in E^s : \mathfrak{u}(x) \pitchfork \mathfrak{s}(y) \in A\}. \end{aligned}$$

LEMMA 2.6 (Holonomy invariance). *Let R be a compatible positive measure tartan, \mathfrak{s} and \mathfrak{s}' be stable fibers of R , and $B \subset E^s$ be $\nu_{\mathfrak{s}}$ -measurable. Then $B' = h_{\mathfrak{s},\mathfrak{s}'}(B)$ is $\nu_{\mathfrak{s}'}$ -measurable, and $\nu_{\mathfrak{s}'}(B') = \nu_{\mathfrak{s}}(B)$. The analogous statement holds for holonomies $h_{\mathfrak{u},\mathfrak{u}'}$.*

Proof. Let \mathfrak{u} be any unstable fiber. Then $\nu_{\mathfrak{u}}(E^u) > 0$ by (1) with $A = R^\natural$.

Since x and $h_{\mathfrak{s},\mathfrak{s}'}(x)$ lie on the same unstable fiber $\mathfrak{u}(x)$ for all $x \in B$, we have $\psi^{s,u}(B \times E^u) = \psi^{s',u}(B' \times E^u)$. Then $\nu_{\mathfrak{s}}$ -measurability of B implies μ -measurability of $\psi^{s,u}(B \times E^u) = \psi^{s',u}(B' \times E^u)$, and hence $\nu_{\mathfrak{s}'}$ -measurability of B' .

As $\mu(\psi^{s,u}(B \times E^u)) = \nu_{\mathfrak{s}}(B)\nu_{\mathfrak{u}}(E^u)$ and $\mu(\psi^{s',u}(B' \times E^u)) = \nu_{\mathfrak{s}'}(B')\nu_{\mathfrak{u}}(E^u)$, the result follows. ■

It is useful to have a version of (1) reflecting the fact that, in compatible tartans, the stream measures are the disintegration of the ambient measure onto fibers:

$$(2) \quad \mu(A) = \int_{E^s} \nu_{\mathfrak{u}(x)}(A \cap \mathfrak{u}(x)) d\nu_{\mathfrak{s}}(x) = \int_{E^u} \nu_{\mathfrak{s}(y)}(A \cap \mathfrak{s}(y)) d\nu_{\mathfrak{u}}(y).$$

This follows because

$$\begin{aligned} \nu_{\mathfrak{s}(y)}(A \cap \mathfrak{s}(y)) &= \nu_{\mathfrak{s}}(h_{\mathfrak{s}(y),\mathfrak{s}}(A \cap \mathfrak{s}(y))) \\ &= \nu_{\mathfrak{s}}(\{\mathfrak{u}(x) \pitchfork \mathfrak{s} : x \in A \cap \mathfrak{s}(y)\}) \\ &= \nu_{\mathfrak{s}}(\{x' \in E^s : \mathfrak{u}(x') \pitchfork \mathfrak{s}(y) \in A\}) = \nu_{\mathfrak{s}}(A^{\mathfrak{s}}(y)), \end{aligned}$$

and likewise $\nu_{\mathfrak{u}(x)}(A \cap \mathfrak{u}(x)) = \nu_{\mathfrak{u}}(A^{\mathfrak{u}}(x))$ for any $x \in E^s$.

DEFINITION 2.7 (Stable and unstable widths $w^s(R)$, $w^u(R)$). Taking $B = \mathfrak{s} \cap R^\natural = E^s$ in Lemma 2.6, we see that if R is compatible and of positive measure then $\nu_{\mathfrak{s}}(E^s) > 0$ is independent of the stable fiber \mathfrak{s} ; and similarly $\nu_{\mathfrak{u}}(E^u) > 0$ is independent of the unstable fiber \mathfrak{u} . We write $w^s(R)$

and $w^u(R)$ for these quantities (the stable and unstable *widths* of R), and note that $\mu(R^\natural) = w^s(R)w^u(R)$ by (1).

In order that the ambient measure μ be globally compatible with the stream measures, it is clearly necessary for a full measure subset of Σ to be covered by intersections of positive measure compatible tartans; this is condition (a) in the following definition. However, a corresponding topological condition, given by part (b) of the definition, is also useful: it follows from (a) if an additional regularity condition is imposed on tartans or on the turbulations. We return to this point in Section 5.

DEFINITION 2.8 (Full turbulations). We say that a transverse pair (\mathcal{T}^s, ν^s) and (\mathcal{T}^u, ν^u) is *full* if

- (a) there is a countable collection R_i of (positive measure) compatible tartans with $\mu(\bigcup_i R_i^\natural) = 1$;
- (b) for every non-empty open subset U of Σ , there is a positive measure compatible tartan $R = (R^s, R^u)$ with $R^s \cup R^u \subset U$.

We are finally in a position to define measurable pseudo-Anosov maps.

DEFINITIONS 2.9 (Image turbulation, measurable pseudo-Anosov turbulations, measurable pseudo-Anosov map, dilatation). If $F: \Sigma \rightarrow \Sigma$ is a μ -preserving homeomorphism, we write $F(\mathcal{T}, \nu)$ for the measured turbulation whose streamlines are $\{F(\ell) : \ell \in \mathcal{T}\}$, with measures $\nu_{F(\ell)} = F_*(\nu_\ell)$.

A pair $(\mathcal{T}^s, \nu^s), (\mathcal{T}^u, \nu^u)$ of measured turbulations on Σ are *measurable pseudo-Anosov turbulations* if they are transverse, tame, full, and have dense streamlines.

A μ -preserving homeomorphism $F: \Sigma \rightarrow \Sigma$ is *measurable pseudo-Anosov* if there is a pair $(\mathcal{T}^s, \nu^s), (\mathcal{T}^u, \nu^u)$ of measurable pseudo-Anosov turbulations and a number $\lambda > 1$, called the *dilatation* of F , such that $F(\mathcal{T}^s, \nu^s) = (\mathcal{T}^s, \lambda\nu^s)$ and $F(\mathcal{T}^u, \nu^u) = (\mathcal{T}^u, \lambda^{-1}\nu^u)$.

Note that the positions of λ and λ^{-1} here are not errors: they differ from those familiar in the definition of pseudo-Anosov maps since we are using measures along streamlines rather than transverse to them.

REMARK 2.10. If $F: \Sigma \rightarrow \Sigma$ is a measurable pseudo-Anosov map, then so is any homeomorphism topologically conjugate to it.

3. Subtartans. This short section describes some useful constructions of subtartans.

DEFINITION 3.1 (Subtartan). Let $R = (R^s, R^u)$ and $S = (S^s, S^u)$ be tartans. We say that S is a *subtartan* of R if every stable (respectively unstable) fiber of S is contained in a stable (respectively unstable) fiber of R .

The simplest way to obtain a subtartan of R is just to discard some of its fibers: if S^s and S^u are Borel subsets of R^s and R^u which are unions of fibers of R , then S is a subtartan of R , which is clearly compatible if R is.

Another construction is to take a rectangle bounded by segments of two stable and two unstable fibers of R , and to form a subtartan whose fibers are the intersections of fibers of R with this rectangle.

DEFINITION 3.2 ($R(x, y)$). Let R be a compatible tartan, and let $x, y \in R^\natural$ lie on different stable and unstable fibers of R . Let D be the closed disk bounded by the arcs $[x, \mathfrak{s}(x) \pitchfork \mathfrak{u}(y)]_s$, $[\mathfrak{s}(x) \pitchfork \mathfrak{u}(y), y]_u$, $[y, \mathfrak{s}(y) \pitchfork \mathfrak{u}(x)]_s$, and $[\mathfrak{s}(y) \pitchfork \mathfrak{u}(x), x]_u$. We define $R(x, y)$ to be the subtartan of R whose stable and unstable fibers are the intersections of the stable and unstable fibers of R with D .

It is easily seen that $R(x, y)$ is compatible, and that $w^s(R(x, y)) = \nu_{\mathfrak{s}(x)}([x, \mathfrak{s}(x) \pitchfork \mathfrak{u}(y)]_s)$ and $w^u(R(x, y)) = \nu_{\mathfrak{u}(x)}([x, \mathfrak{s}(y) \pitchfork \mathfrak{u}(x)]_u)$.

We can also construct subtartans which consist of all the fibers of R which intersect given stable and unstable fibers in prescribed points. We start by defining how to *trim* a tartan to remove “loose” segments of fibers.

DEFINITION 3.3 ($\text{trim}(R)$). A tartan R can be *trimmed*, to yield a subtartan $\text{trim}(R)$, by throwing away the ends of its fibers which contain no intersections. Each stable fiber \mathfrak{s} of R yields a stable fiber \mathfrak{s}' of $\text{trim}(R)$, which is the minimal (open, closed, or half-open) subarc of \mathfrak{s} which contains E^s , and analogously for unstable fibers.

DEFINITION 3.4 ($R(I^s, I^u)$). Let $R = (R^s, R^u)$ be a positive measure oriented compatible tartan, and let \mathfrak{s} and \mathfrak{u} be stable and unstable fibers of R .

For each $x \in E^s = \mathfrak{s} \cap R^\natural$, we denote by ι_x the initial (with respect to the given orientation of \mathfrak{s}) stream arc of \mathfrak{s} with terminal endpoint x (which may be open or closed at its initial point, according as \mathfrak{s} is). Given a subinterval $I^s \subset [0, 1]$, define $A^s(I^s) \subset E^s$ by

$$A^s(I^s) = \{x \in E^s : \nu_{\mathfrak{s}}(\iota_x \cap R^\natural) / w^s(R) \in I^s\}.$$

Likewise, if $I^u \subset [0, 1]$ is a subinterval, we define a corresponding subset $A^u(I^u)$ of E^u . We then define $R(I^s, I^u)$ by trimming the subtartan of R whose stable fibers are $\mathfrak{s}(y)$ for $y \in A^u(I^u)$, and whose unstable fibers are $\mathfrak{u}(x)$ for $x \in A^s(I^s)$.

The definition is independent of the choice of fibers \mathfrak{s} and \mathfrak{u} by Lemma 2.6.

For example, $S = R([1/3, 2/3], [1/3, 2/3])$ is the “middle thirds” subtartan of R , with $\mu(S^\natural) = \mu(R^\natural)/9$ (in contrast to the middle thirds Cantor set, here we *retain* the middle thirds of the stable and unstable fibers).

4. Density. Let α be an oriented stable or unstable stream arc with stream measure ν . Then there is a *stream metric* d_α on α , defined by $d_\alpha(x, y) = \nu([x, y]_\alpha)$, with respect to which α is measure-preserving and orientation-preserving isometric to a finite interval I in \mathbb{R} with Lebesgue measure – in one dimension, metrics and OU measures are equivalent. In particular, we can apply the Lebesgue density theorem to stream arcs. In this section we present some consequences which will be used later. In Lemma 4.3 we show that, for a compatible tartan R , the set $\text{den}_1(R)$ of points of R^\natural which are (two-sided) density points of R^\natural along both stable and unstable fibers has full measure; moreover, the same is true for $\text{den}_2(R)$, the set of points which are density points of $\text{den}_1(R)$ along both fibers. Lemma 4.5 concerns points x for which all small stream arcs with endpoint x contain a high density of points of R^\natural .

NOTATION 4.1. Let $\varphi: I \rightarrow \alpha$ be such a measure-preserving and orientation-preserving isometry from an interval in \mathbb{R} to an oriented stream arc. Pick some $x = \varphi(\hat{x}) \in \alpha$. If $a \in \mathbb{R}$ is small enough that $\hat{x} + a \in I$, then we denote by $x + a$ the point $\varphi(\hat{x} + a)$ of α .

When we write, for example, $[x, x + a]_s$ (respectively $[x, x + a]_u$), we mean the stream arc with endpoints x and $x + a$ on the stable (respectively unstable) streamline through x ; by construction, these arcs have stream measure $|a|$. In practice the orientation of α will be irrelevant, since the definitions which follow are invariant under $a \mapsto -a$.

DEFINITION 4.2 ($\text{den}_k(R)$). Let R be a compatible tartan. We define

$$\text{den}_k(R) \subset R^\natural,$$

the *set of level k density points of R* , inductively for $k \geq 0$ by $\text{den}_0(R) = R^\natural$ and $\text{den}_k(R) = \text{den}_k^s(R) \cap \text{den}_k^u(R)$, where

$$\text{den}_k^{s/u}(R) = \left\{ x \in \text{den}_{k-1}(R) : \lim_{a \rightarrow 0} \frac{\nu_x^{s/u}([x, x + a]_{s/u} \cap \text{den}_{k-1}(R))}{|a|} = 1 \right\}.$$

That is, an element x of $\text{den}_k(R)$ is an element of R^\natural which is a (two-sided) density point of $\text{den}_{k-1}(R)$ along both $\mathfrak{s}(x)$ and $\mathfrak{u}(x)$.

LEMMA 4.3. *Let R be a compatible tartan. Then $\mu(\text{den}_k(R)) = \mu(R^\natural)$ for all $k \geq 0$.*

Proof. The proof is by induction on k , with the case $k = 0$ vacuous. We can assume that $\mu(R^\natural) > 0$.

Let \mathfrak{u} be any unstable fiber of R , and define

$$T = \{y \in E^{\mathfrak{u}} : \nu_{\mathfrak{s}(y)}(\text{den}_{k-1}(R) \cap \mathfrak{s}(y)) = w^s(R)\}.$$

Since $\mu(R^\natural) - \mu(\text{den}_{k-1}(R)) = 0$ by the inductive hypothesis, we have, by (2),

$$\int_{E^u} (w^s(R) - \nu_{\mathfrak{s}(y)}(\text{den}_{k-1}(R) \cap \mathfrak{s}(y))) d\nu_u(y) = 0,$$

and so $\nu_u(T) = \nu_u(E^u) = w^u(R)$.

Let $y \in T$. Since $\text{den}_{k-1}^s(R) \cap \mathfrak{s}(y) = \text{den}_{k-1}^s(R) \cap E^{\mathfrak{s}(y)}$, we have

$$\text{den}_k^s(R) \cap \mathfrak{s}(y) = \left\{ x \in \text{den}_{k-1}(R) \cap E^{\mathfrak{s}(y)} : \lim_{a \rightarrow 0} \frac{\nu_x^s([x, x+a]_s \cap (\text{den}_{k-1}(R) \cap E^{\mathfrak{s}(y)}))}{|a|} = 1 \right\}.$$

Therefore, by the Lebesgue density theorem we have, writing ℓ for the streamline containing $\mathfrak{s}(y)$,

$$\begin{aligned} \nu_{\mathfrak{s}(y)}(\text{den}_k^s(R) \cap \mathfrak{s}(y)) &= \nu_\ell(\text{den}_k^s(R) \cap \mathfrak{s}(y)) = \nu_\ell(\text{den}_{k-1}(R) \cap E^{\mathfrak{s}(y)}) \\ &= \nu_{\mathfrak{s}(y)}(\text{den}_{k-1}(R) \cap \mathfrak{s}(y)) = w^s(R). \end{aligned}$$

It follows from (2) that

$$\mu(\text{den}_k^s(R)) = \int_T \nu_{\mathfrak{s}(y)}(\text{den}_k^s(R) \cap \mathfrak{s}(y)) d\nu_u(y) = w^u(R)w^s(R) = \mu(R^\natural).$$

The same argument shows that $\mu(\text{den}_k^u(R)) = \mu(R^\natural)$, and the result follows. ■

We now restrict to level 1 density points, and consider the set of those points $x \in R^\natural$ for which the density of intersections in all small stable and unstable stream arcs with endpoint x has at least some prescribed value.

DEFINITION 4.4 ($X_{\delta,\eta}(R)$). Let R be a compatible tartan. For $\delta \in (0, 1)$ and $\eta > 0$, we define

$$X_{\delta,\eta}^{s/u}(R) = \{x \in R^\natural : \nu_x^{s/u}([x, x+a]_{s/u} \cap R^\natural) \geq (1-\delta)|a| \text{ for all } a \in (-\eta, \eta)\},$$

and

$$X_{\delta,\eta}(R) = X_{\delta,\eta}^s(R) \cap X_{\delta,\eta}^u(R).$$

LEMMA 4.5. Let R be a compatible tartan and $\delta \in (0, 1)$. Then

$$\lim_{\eta \rightarrow 0} \mu(X_{\delta,\eta}(R)) = \mu(R^\natural).$$

Proof. Note that if $\eta' < \eta$ then $X_{\delta,\eta}(R) \subset X_{\delta,\eta'}(R)$. Let

$$X_\delta(R) = \bigcup_{\eta > 0} X_{\delta,\eta}(R) = \bigcup_{k \geq 1} X_{\delta,1/k}(R).$$

By continuity of measure, it suffices to show that $\mu(X_\delta(R)) = \mu(R^\natural)$. This follows from Lemma 4.3 since $\text{den}_1(R) \subset X_\delta(R)$, as can be easily shown. ■

5. Positive measure tartans in open sets. Recall (Definition 2.8) that for a transverse pair of turbulations to be *full* there are two requirements: first, that there is a countable collection of compatible tartans whose intersections cover a full measure subset of Σ ; and second, that every non-empty open set contains a positive measure compatible tartan. We now discuss conditions which, combined with the first of these requirements, imply the second.

Let U be a non-empty open subset of Σ . Since μ is OU, we have $\mu(U) > 0$, and hence, by the first requirement, there is a tartan R with $\mu(R^\natural \cap U) > 0$. By Lemma 4.3 it follows that $\mu(\text{den}_2(R) \cap U) > 0$. Let $x \in \text{den}_2(R) \cap U$.

For each $i \geq 1$, pick $a_i > 0$ such that $[x, x + a_i]_s$ has diameter less than $1/i$. By definition of $\text{den}_2(R)$, we can pick $y_i \in (x, x + a_i]_s \cap \text{den}_1(R)$; moreover, we have $\nu_{\mathfrak{s}(x)}([x, y_i]_s \cap R^\natural) \geq \nu_{\mathfrak{s}(x)}([x, y_i]_s \cap \text{den}_1(R)) > 0$.

Similarly, given i , for each $j \geq 1$ we pick $b_{i,j} > 0$ such that $[y_i, y_i + b_{i,j}]_u$ has diameter less than $1/j$. Since $y_i \in \text{den}_1(R)$, we can further pick $z_{i,j} \in (y_i, y_i + b_{i,j}]_u \cap R^\natural$ such that $\nu_{\mathfrak{u}(y_i)}([y_i, z_{i,j}]_u \cap R^\natural) > 0$.

For each i , the diameter of $[x, \mathfrak{s}(z_{i,j}) \pitchfork \mathfrak{u}(x)]_u$ tends to zero as $j \rightarrow \infty$. For, given $\epsilon > 0$, let $\delta > 0$ be such that $[x, x + \delta]_u$ has diameter less than ϵ , and let $w \in (x, x + \delta]_u \cap R^\natural$ (such a point exists since $x \in \text{den}_2^u(R)$). Let j be large enough that $1/j$ is less than the diameter of $[y_i, \mathfrak{u}(y_i) \pitchfork \mathfrak{s}(w)]_u$; then $z_{i,j} \in [y_i, \mathfrak{u}(y_i) \pitchfork \mathfrak{s}(w)]_u$, so that $\mathfrak{s}(z_{i,j}) \pitchfork \mathfrak{u}(x) \in [x, x + \delta]_u$ by Definition 2.3(b).

Each pair $i, j \geq 1$ yields a compatible subtartan $R(x, z_{i,j})$ of R , with measure $\nu_{\mathfrak{s}(x)}([x, y_i]_s \cap R^\natural) \nu_{\mathfrak{u}(y_i)}([y_i, z_{i,j}]_u \cap R^\natural) > 0$. If i and j are sufficiently large then the arcs $[x, y_i]_s$, $[y_i, z_{i,j}]_u$, and $[x, \mathfrak{s}(z_{i,j}) \pitchfork \mathfrak{u}(x)]_u$ are contained in U . Thus it suffices to show that we can choose i and j so that the remaining bounding arc $[z_{i,j}, \mathfrak{s}(z_{i,j}) \pitchfork \mathfrak{u}(x)]_s$ of $R(x, z_{i,j})$ is also contained in U . This is the step that requires an additional regularity condition on tartans or on the measured turbulations.

The simplest such condition just says that the result we need is true:

DEFINITION 5.1 (Regular tartan). We say that a tartan R is *regular* if for all $x \in R^\natural$ and all neighborhoods U of x there is some $\delta > 0$ such that if $y \in \mathfrak{s}(x) \cap R^\natural$ and $z \in \mathfrak{u}(x) \cap R^\natural$ with $\nu_{\mathfrak{s}(x)}([x, y]_s) < \delta$ and $\nu_{\mathfrak{u}(x)}([x, z]_u) < \delta$, then $R(y, z) \subseteq U$ (that is, all of the fibers of $R(y, z)$ are contained in U).

A second condition, which is perhaps more natural, in that it relates metric to stream measure within tartans, is:

DEFINITION 5.2 (Bi-Lipschitz regular tartan). We say that a tartan R is *stable* (resp. *unstable*) *bi-Lipschitz regular* if there is a metric d on Σ compatible with its topology and a constant $K \geq 1$ such that whenever x and y lie on the same stable (resp. unstable) fiber of R , we have

$$\frac{1}{K} d(x, y) \leq \nu_s([x, y]_s) \text{ (resp. } \nu_u([x, y]_u)) \leq K d(x, y).$$

A final condition is of interest because it can be expressed solely in terms of a single turbulation.

DEFINITION 5.3 (Partial flowboxes). A turbulation \mathcal{T} has *partial flowboxes* if whenever

- x and y lie on the same streamline ℓ of \mathcal{T} ;
- there are arcs α_x and α_y in Σ , transverse to ℓ at x and y , which contain transverse intersections with streamlines of \mathcal{T} arbitrarily close to x and y on the same side of ℓ ,

then every neighborhood U of the stream arc $[x, y]$ contains other stream arcs with endpoints on α_x and α_y .

LEMMA 5.4 (Conditions for positive measure tartans in open sets). *Let (\mathcal{T}^s, ν^s) and (\mathcal{T}^u, ν^u) be a transverse pair of measured turbulations which satisfy Definition 2.8(a), namely there is a countable collection R_i of compatible tartans with $\mu(\bigcup_i R_i^{\text{th}}) = 1$. Then the pair also satisfies Definition 2.8(b), and so it is full, if any of the following conditions holds:*

- (a) *the tartans in Definition 2.8(a) can be chosen to be regular; or*
- (b) *the tartans in Definition 2.8(a) can be chosen to be either stable or unstable bi-Lipschitz regular; or*
- (c) *one of the turbulations has partial flowboxes.*

Proof. Let U be a non-empty open subset of Σ . As in the introduction to this section, we can find a tartan R and a point $x \in R^{\text{th}} \cap U$, and points $y \in \mathfrak{s}(x)$, $z \in \mathfrak{u}(y)$, and $w = \mathfrak{s}(y) \text{th} \mathfrak{u}(x)$ arbitrarily close to x such that the subtartan $R(x, z)$ is compatible and has positive measure, and moreover $[x, y]_s$, $[x, w]_u$, and $[y, z]_u$ are contained in U . Thus it is only required to show that y , z , and w can be chosen close enough to x that $[z, w]_s$ is also contained in U .

This is clear if R is regular: if i and j are sufficiently large, then the points y and w satisfy $\nu_{\mathfrak{s}(x)}([x, y]_s) < \delta$ and $\nu_{\mathfrak{u}(x)}([x, w]_u) < \delta$, where $\delta > 0$ is provided by Definition 5.1.

If R is stable bi-Lipschitz regular, then let d be the metric and K be the Lipschitz constant provided by Definition 5.2. Let $\epsilon > 0$ be small enough that $B_d(x, 2\epsilon) \subset U$, and let $\delta = \epsilon/K^2$. Pick y , z , and w close enough to x that $d(z, w) < \delta$ and $d(x, z) < \epsilon$. Then $\nu_s([z, w]_s) < K\delta$, so that $\nu_s([z, v]_s) < K\delta$ for all $v \in [z, w]_s$, and thus $d(z, v) < K^2\delta = \epsilon$ for all $v \in [z, w]_s$. Hence every $v \in [z, w]_s$ has $d(x, v) \leq d(x, z) + d(z, v) < 2\epsilon$, and $[z, w]_s$ is contained in U as required. The unstable bi-Lipschitz regular case is analogous.

Finally, if the stable turbulation has partial flowboxes, then taking α_x and α_y in Definition 5.3 to be $\mathfrak{u}(x)$ and $\mathfrak{u}(y)$ immediately yields the result. If the unstable turbulation has partial flowboxes, exchange the rôles of the stable and unstable turbulations in the first paragraph of the proof. ■

We finish this section with a first consequence of fullness.

LEMMA 5.5. *Let $(\mathcal{T}^s, \nu^s), (\mathcal{T}^u, \nu^u)$ be measurable pseudo-Anosov turbulations on Σ , and let ℓ_s and ℓ_u be stable and unstable streamlines. Then $\ell_s \cap \ell_u$ is dense in Σ .*

Proof. Suppose for a contradiction that there is a non-empty open subset U of Σ with $\ell_s \cap \ell_u \cap U = \emptyset$.

By fullness there is a positive measure compatible tartan R contained in U . Any two stable and any two unstable fibers of R contain stable arcs s_1, s_2 and unstable arcs u_1, u_2 which bound an open disk $V \subset U$. There are infinitely many such disks; choose one which does not contain an entire end of either ℓ_s or ℓ_u (recall that the streamlines are dense, but not necessarily bi-dense).

Since ℓ_s is dense in Σ , it contains a point x of V . By the choice of V , the closure of the path component of $\ell_s \cap V$ containing x is an arc α with endpoints in $u_1 \cup u_2$. The endpoints of α cannot both lie on the same u_i , since then α and u_i would bound a disk which ℓ_u cannot enter, contradicting the denseness of ℓ_u . We conclude that there is an arc $\alpha \subset \bar{V}$ of ℓ_s with endpoints on u_1 and u_2 .

Analogously, there is an arc $\beta \subset \bar{V}$ of ℓ_u with endpoints on s_1 and s_2 . Then α and β intersect in $V \subset U$, which is the required contradiction. ■

6. Topological dynamics of measurable pseudo-Anosov maps.

In this section we prove our main results about the topological dynamics of measurable pseudo-Anosov maps: that they are topologically transitive (Theorem 6.1) and that they have dense periodic points (Theorem 6.2). It follows (see [2]) that they also exhibit sensitive dependence on initial conditions, and are therefore chaotic in the sense of Devaney [17].

THEOREM 6.1. *Measurable pseudo-Anosov maps are topologically transitive.*

Proof. (See Figure 3.) Let $F: \Sigma \rightarrow \Sigma$ be measurable pseudo-Anosov with dilatation $\lambda > 1$, and let U and V be non-empty open subsets of Σ . We need to show that there is some $n > 0$ with

$$F^n(U) \cap V \neq \emptyset.$$

By fullness, there is a positive measure compatible tartan R contained in U . We write $S = R([0, 1], [1/3, 2/3])$, so that $w^s(S) = w^s(R)$ and $w^u(S) = w^u(R)/3$, and hence $\mu(S^{\text{th}}) = \mu(R^{\text{th}})/3 > 0$.

Applying the Poincaré recurrence theorem to F^{-1} yields a point $x_0 \in S^{\text{th}}$ and a sequence $n_i \rightarrow \infty$ such that $x_i = F^{-n_i}(x_0) \in S^{\text{th}}$ for all i .

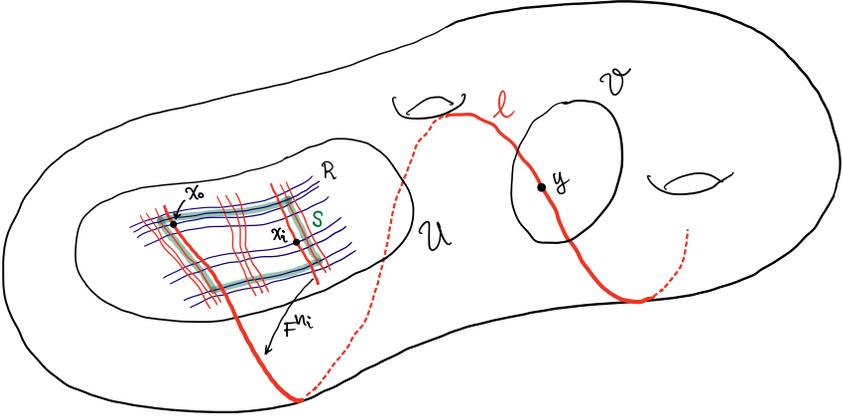


Fig. 3. Proof of topological transitivity

Since the unstable streamline ℓ through x_0 is dense in Σ , it contains some point $y \in V$. Write $K = \nu_\ell^u([x_0, y]_\ell)$, and pick i large enough that $\lambda^{n_i} w^u(R)/3 > K$.

Let $u \subset U$ be the unstable fiber of R which contains x_i . Then $x_0 \in F^{n_i}(u)$, and each of the two components of $F^{n_i}(u) \setminus \{x_0\}$ has stream measure at least $\lambda^{n_i} w^u(R)/3 > K$. Hence $y \in F^{n_i}(u) \subset F^{n_i}(U)$, so that $F^{n_i}(U) \cap V \neq \emptyset$ as required. ■

The proof of density of periodic points is more challenging, and makes use of the density result of Lemma 4.5.

THEOREM 6.2. *Measurable pseudo-Anosovs have dense periodic points.*

Proof. Let $F: \Sigma \rightarrow \Sigma$ be measurable pseudo-Anosov with dilatation λ , and let U be a non-empty open subset of Σ . We need to show that there is a periodic point of F in U .

By fullness, there is a positive measure compatible oriented tartan R contained in U . We will show that there is a positive integer n and an open rectangle $V \subset U$ such that (see Figure 4)

- (V1) ∂V is the union of two arcs α and α' contained in stable fibers of R , and two arcs β and β' contained in unstable fibers of R ;
- (V2) the F^n -images of these boundary arcs are also contained in stable and unstable fibers of R ; and
- (V3) $F^n(\beta)$ and $F^n(\beta')$ both intersect both α and α' .

It follows that the fixed point index $\text{index}(F^n, V)$ of V under F^n is either $+1$ or -1 (depending on whether or not the orientations of the stable boundary arcs, and the unstable boundary arcs, are preserved by F^n), so that there is a fixed point of F^n in $V \subset U$ as required.

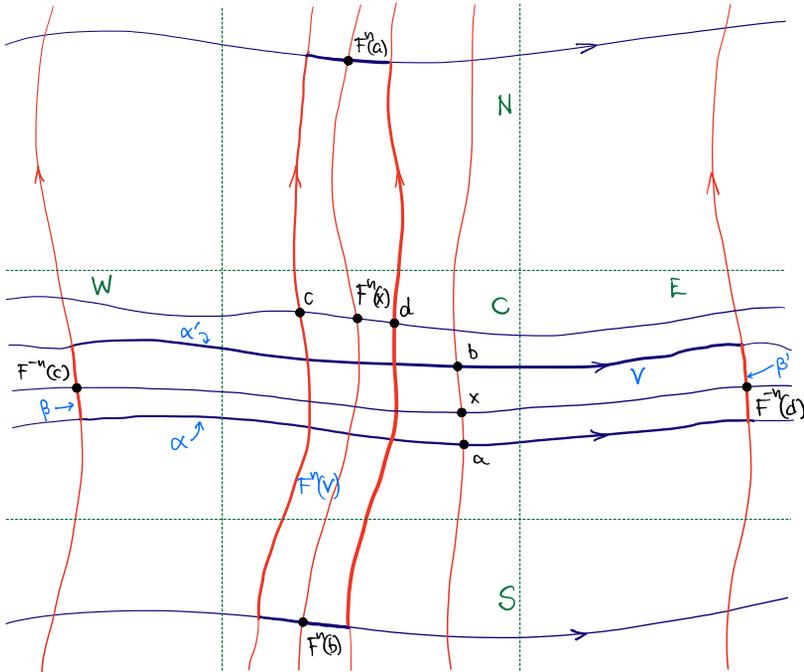


Fig. 4. Proof of Theorem 6.2: the rectangle V and its image under F^n ; the subdivision of R ; the points x and $F^n(x)$ and the construction of $a, b, c,$ and d

Note that if (V1) holds, then we immediately see from the definition of measurable pseudo-Anosov maps that the F^n -images of $\alpha, \alpha', \beta,$ and β' are stream arcs, but that this is not enough: we need these images to be contained in fibers of R so that we can control how they lie with respect to the original arcs.

Define subtartans $C, W, E, S,$ and N of R (see Figure 4) by

$$\begin{aligned} C &= R([1/3, 2/3], [1/3, 2/3]), \\ W &= R([0, 1/3], [1/3, 2/3]), \\ E &= R([2/3, 1], [1/3, 2/3]), \\ S &= R([1/3, 2/3], [0, 1/3]), \\ N &= R([1/3, 2/3], [2/3, 1]), \end{aligned}$$

the Central, West, East, South, and North subtartans of $1/3$ the stable and unstable widths of R , where the stable streamlines of R run from West to East and the unstable streamlines from South to North. We will construct the rectangle V so that the unstable boundary arcs β and β' lie in W and E , while the image stable boundary arcs $F^n(\alpha)$ and $F^n(\alpha')$ lie in S

and N (not necessarily respectively in each case): this will ensure that (V3) holds.

Let

- L be an upper bound on the stream measures of the fibers of R (which exists by Definitions 2.3(c));
- $\delta \in (0, 1)$ be such that $\delta < \min(w^s(R), w^u(R))/(4\lambda L)$;
- $\eta_0 \in (0, L)$ be such that $\mu(X_{\delta, \eta_0}(C)) > 0$ (see Definition 4.4 and Lemma 4.5);
- $n_0 > 0$ be such that $\eta_0 \in [\lambda^{-n_0}L, \lambda^{-n_0+1}L)$.

By the Poincaré recurrence theorem applied to $X_{\delta, \eta_0}(C)$, there is an integer $n > n_0$ and a point $x \in X_{\delta, \eta_0}(C)$ such that $F^n(x) \in X_{\delta, \eta_0}(C)$.

Pick $\eta \in (\lambda^{-n}L, \lambda^{-n+1}L)$. Since $\eta_0 \geq \lambda^{-n_0}L$ and $n > n_0$, we have $\eta < \eta_0$, so that $X_{\delta, \eta}(C) \supset X_{\delta, \eta_0}(C)$, and in particular $\{x, F^n(x)\} \subset X_{\delta, \eta}(C)$.

Note that $X_{\delta, \eta}(C) \subset C^{\text{th}}$, so that x and $F^n(x)$ are contained in both stable and unstable fibers of C , and hence in stable and unstable fibers of R .

We now use the point x to build the rectangle V , starting with its stable boundary arcs α and α' : we will find points a and b of the unstable streamline of x , on either side of x and within stream measure η of x , which will lie on these arcs. The key properties are

- $a, b \in C^{\text{th}} \subset R^{\text{th}}$, so that they lie on stable fibers of R ;
- $F^n(a)$ and $F^n(b)$ lie in $S^{\text{th}} \subset R^{\text{th}}$ and $N^{\text{th}} \subset R^{\text{th}}$ (not necessarily respectively), so that (a) they also lie on stable fibers of R , and (b) $F^n(s)$ and $F^n(t)$ lie in S and N , so that the image rectangle $F^n(V)$ stretches across C in the unstable direction.

Let \mathbf{u} be the unstable fiber of R through $F^n(x)$. We have $\nu_{F^n(x)}^u(\mathbf{u}) \leq L$ by definition of L ; since also $\lambda^{-n}L < \eta$, we have $F^{-n}(\mathbf{u}) \subset (x - \eta, x + \eta)_u$.

We first find the point $a \in (x - \eta, x)_u$. By definition of $X_{\eta, \delta}(C)$, we have

$$(3) \quad \nu_x^u((x - \eta, x)_u \cap C^{\text{th}}) \geq (1 - \delta)\eta.$$

We will assume that F^{-n} sends \mathbf{u} into the unstable fiber through x preserving the fiber orientation, so that $F^{-n}(S^s \cap \mathbf{u}) \subset (x - \eta, x)_u$: if it reverses fiber orientation, then we need only replace S with N (Figure 4 depicts the orientation-reversing case). Now

$$(4) \quad \nu_x^u(F^{-n}(S^s \cap \mathbf{u})) = \lambda^{-n}\nu_{F^n(x)}^u(S^s \cap \mathbf{u}) = \lambda^{-n}w^u(S) > \lambda^{-n}w^u(R)/4 > \delta\eta,$$

where the final inequality comes from $\delta < w^u(R)/(4\lambda L)$ and $\eta < \lambda^{-n+1}L$.

It follows from (3) and (4) that there is some $a \in (x - \eta, x)_u$ which lies both in C^{th} and in $F^{-n}(S^s \cap \mathbf{u})$. That is, $a \in C^{\text{th}}$ and $F^n(a) \in S^{\text{th}} \supset S^s \cap \mathbf{u}$ as required.

Applying the same argument to $(x, x + \eta)_u$ yields a point $b \in C^{\text{th}}$ with $F^n(b) \in N^{\text{th}}$ (or S^{th} if F^{-n} reverses fiber orientation).

An exactly analogous argument applied to $(F^n(x) - \eta, F^n(x) + \eta)_s$ yields points $c, d \in C^{\text{th}} \subset R^{\text{th}}$ such that $F^{-n}(c)$ and $F^{-n}(d)$ lie in $W^{\text{th}} \subset R^{\text{th}}$ and $E^{\text{th}} \subset R^{\text{th}}$ (not necessarily respectively).

The rectangle V whose boundary is contained in the stable fibers of R through a and b , and the unstable fibers of R through $F^{-n}(c)$ and $F^{-n}(d)$, then has the required properties. ■

7. Ergodicity. In this section we use a variant of the Hopf argument (see, for example, [1, 13, 16]) to prove ergodicity of measurable pseudo-Anosov maps. In classical uses of this argument, such as to prove the ergodicity of volume-preserving Anosov diffeomorphisms, the main effort is in proving that the invariant foliations are absolutely continuous. This makes it possible to use a Fubini theorem to show that Birkhoff averages of L^1 functions are constant a.e. in charts. Since the manifold is connected and each point is contained in an open chart, this implies that the global Birkhoff average is constant a.e., yielding ergodicity.

In the case of measurable pseudo-Anosov maps, the compatibility conditions on a tartan R allow a Fubini argument to show that Birkhoff averages are constant a.e. in R^{th} . However, since tartans are not open sets, we need an additional hypothesis connecting tartans in order to deal with the global averages.

Let $F: \Sigma \rightarrow \Sigma$ be measurable pseudo-Anosov with dilatation $\lambda > 1$, and let $\phi: \Sigma \rightarrow \mathbb{R}$ be continuous. By Birkhoff's ergodic theorem, the limits

$$\phi^+(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(F^i(x)) \quad \text{and} \quad \phi^-(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(F^{-i}(x))$$

exist and are equal μ -almost everywhere.

Since continuous functions are dense in L^1 , by the converse of Birkhoff's ergodic theorem, it suffices for ergodicity to show that for every continuous ϕ the functions ϕ^+ and ϕ^- are constant almost everywhere.

Throughout this section we consider a fixed such function ϕ .

LEMMA 7.1. *Suppose that x and y lie in the same stable (respectively unstable) streamline ℓ , and that $\phi^+(x)$ (respectively $\phi^-(x)$) exists. Then $\phi^+(y)$ (respectively $\phi^-(y)$) exists and is equal to $\phi^+(x)$ (respectively $\phi^-(x)$).*

Proof. If x and y lie in the same stable streamline ℓ with $\nu_\ell([x, y]_s) = K$, then we have $\nu_{F^i(\ell)}([F^i(x), F^i(y)]_s) = K/\lambda^i$.

Given $\epsilon > 0$, let $N_\epsilon = \{(a, b) \in \Sigma \times \Sigma : |\phi(a) - \phi(b)| < \epsilon\}$, a neighborhood of the diagonal. By tameness there is a $\delta > 0$ with $(a, b) \in N_\epsilon$ whenever the stream measure of $[a, b]_s$ is less than δ ; therefore $|\phi(F^i(x)) - \phi(F^i(y))| < \epsilon$

for all sufficiently large i . It follows that $\phi^+(y) = \phi^+(x)$ as required. The argument when x and y are on the same unstable streamline is analogous. ■

DEFINITIONS 7.2 (Z, \sim). Let

$$Z = \{x \in \Sigma : \phi^+(x) \text{ and } \phi^-(x) \text{ exist and are equal}\},$$

so that $\mu(Z) = 1$ by Birkhoff's ergodic theorem, as noted above.

Let $x, y \in Z$. We write $x \sim y$ if there is a finite collection $\{\alpha_1, \dots, \alpha_k\}$ of stream arcs with $x \in \alpha_1$ and $y \in \alpha_k$ such that $\alpha_i \cap \alpha_{i+1}$ contains a point $z_i \in Z$ for $1 \leq i \leq k - 1$.

Observe that \sim is an equivalence relation on Z , and ϕ^+ and ϕ^- are constant on equivalence classes. Indeed, writing $z_0 = x$ and $z_k = y$, we have for each $1 \leq i \leq k$ either $\phi^+(z_{i-1}) = \phi^+(z_i)$ or $\phi^-(z_{i-1}) = \phi^-(z_i)$ by Lemma 7.1; and hence $\phi^\pm(z_{i-1}) = \phi^\pm(z_i)$ since $z_{i-1}, z_i \in Z$.

It follows that F is ergodic if \sim has a full measure equivalence class.

LEMMA 7.3. *Suppose that R is a positive measure compatible tartan. Then $\phi^+ = \phi^-$ is constant almost everywhere in R^\natural .*

Proof. Let \mathfrak{s} and \mathfrak{u} be any stable and unstable fibers of R , and let $V = R^\natural \cap Z$, a full measure subset of R^\natural .

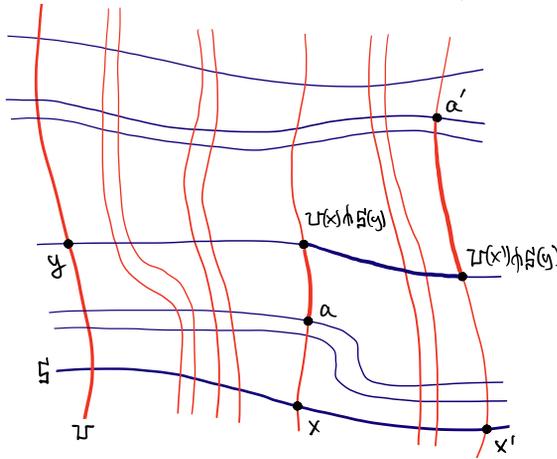


Fig. 5. Stream arcs showing that $a \sim \alpha'$

By (1),

$$\int_{E^s} \nu_u(E^u) d\nu_s(x) = \mu(R^\natural) = \mu(V) = \int_{E^s} \nu_u(V^u(x)) d\nu_s(x),$$

so that $W = \{x \in E^s : \nu_u(V^u(x)) = \nu_u(E^u)\}$ has full ν_s -measure in E^s . Let S be the subtartan of R obtained by discarding the unstable fibers which do

not intersect W . Then $\mu(S^{\text{th}}) = \mu(R^{\text{th}})$, and $\phi^+ = \phi^-$ is constant on $S^{\text{th}} \cap Z$, a full measure subset of R^{th} . For if $a, a' \in S^{\text{th}} \cap Z$, then let $x, x' \in W$ be such that $a \in \mathbf{u}(x)$ and $a' \in \mathbf{u}(x')$, and let $y \in V^u(x) \cap V^u(x')$. Then the stream arcs $[a, \mathbf{u}(x) \pitchfork \mathbf{s}(y)]_u$, $[\mathbf{u}(x) \pitchfork \mathbf{s}(y), \mathbf{u}(x') \pitchfork \mathbf{s}(y)]_s$, and $[\mathbf{u}(x') \pitchfork \mathbf{s}(y), a']_u$ connect a and a' and intersect at points of $V \subset Z$, so that $a \sim a'$ as required (see Figure 5). ■

In order to establish ergodicity, we need some means of relating the Birkhoff averages of ϕ on different positive measure tartans. The following is a natural, though strong, condition:

DEFINITIONS 7.4 (Tartan connection hypothesis). We say that measurable pseudo-Anosov turbulations (\mathcal{T}^s, ν^s) , (\mathcal{T}^u, ν^u) satisfy the *tartan connection hypothesis* if for any compatible positive measure tartans R and S the stable streamlines containing fibers of R intersect the unstable streamlines containing fibers of S in positive measure.

THEOREM 7.5. *If the turbulations of a measurable pseudo-Anosov map F satisfy the tartan connection hypothesis, then F is ergodic.*

Proof. Let R_i be a countable collection of positive measure compatible tartans with $\mu(\bigcup_i R_i^{\text{th}}) = 1$.

By Lemma 7.3 there are constants ϕ_i such that $\phi^+ = \phi^- = \phi_i$ almost everywhere in R_i^{th} , and it suffices to show that $\phi_i = \phi_j$ for all i and j .

Given i and j , let S_i and S_j be the positive measure compatible subtartans of R_i and R_j obtained by discarding the stable fibers of R_i on which $\phi^+ \neq \phi_i$ (or ϕ^+ does not exist), and the unstable fibers of R_j on which $\phi^- \neq \phi_j$ (or ϕ^- does not exist).

By the tartan connection hypothesis, the intersection A of the union of the stable streamlines containing fibers of S_i with the union of the unstable streamlines containing fibers of S_j has positive measure, so that $\mu(A \cap Z) > 0$. However, if $a \in A \cap Z$ then $\phi^+(a) = \phi_i$ and $\phi^-(a) = \phi_j$ by Lemma 7.1. Since $a \in Z$, we have $\phi_i = \phi_j$ as required. ■

Clearly Theorem 7.5 is also true under the weaker condition that for any compatible positive measure tartans R and S there exists a chain $R = T_0, T_1, \dots, T_k = S$ of compatible tartans such that, for each i , the stable streamlines containing fibers of T_i intersect the unstable streamlines containing fibers of T_{i+1} in positive measure. This is worth mentioning because this weaker condition is in turn implied by the *tartan chain hypothesis*: that for any compatible positive measure tartans R and S there is a chain as above such that $\mu(T_i^{\text{th}} \cap T_{i+1}^{\text{th}}) > 0$ for each i .

Acknowledgements. The authors acknowledge the support of São Paulo Research Foundation (FAPESP) grant number 2016/25053-8. This

work is also partially supported by the Simons Foundation Award No. 663281 granted to the Institute of Mathematics of the Polish Academy of Sciences for the years 2021–2023.

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