INFINITESIMAL BENDINGS FOR CLASSES OF TWO DIMENSIONAL SURFACES.

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Abstract. Infinitesimal bendings for classes of two-dimensional surfaces in $\mathbb{R}^3$ are investigated. The techniques used to construct the bending fields include reduction to solvability of Bers-Vekua type equations and systems of differential equations with periodic coefficients.

1. Introduction

This paper deals with infinitesimal bendings for classes of orientable surfaces. We consider a smooth surface $S \subset \mathbb{R}^3$ given by a position vector R over a region $\Omega \subset \mathbb{R}^2$. Thus

$$S = \{R(s,t) \in \mathbb{R}^3;\ (s,t) \in \Omega\},$$

where $R \in C^\infty(\Omega, \mathbb{R}^3)$. A one parameter deformation surface S_ϵ ($\epsilon \in \mathbb{R}$) given by the position vector

$$R_\epsilon(s,t) = R(s,t) + 2\sum_{j=1}^{m} \epsilon^j U_j(s,t)\,,$$

with $U_j \in C^k(\Omega, \mathbb{R}^3)$ ($k \in \mathbb{Z}^+$), is an infinitesimal bending of S of order $m \in \mathbb{Z}^+$ if the metrics of S and S_ϵ coincide to order m as $\epsilon \to 0$. That is,

$$dR_\epsilon^2(s,t) = dR^2(s,t) + o(\epsilon^m) \quad \text{as} \quad \epsilon \to 0.$$

The study of infinitesimal bendings of surfaces has a long and rich history and has many physical applications (see [N], [P], and [R]). For a complete overview we refer to the survey article by Sabitov [S] and the extensive references within.

The results of this paper are generalizations of those contained in [M1], [M3] that deal with infinitesimal bendings of surfaces with nonnegative curvature. For a surface with positive Gaussian curvature except at a finite number of planar points, we use the (complex) vector field of asymptotic directions and an associated Bers-Vekua type equation to construct non trivial infinitesimal bendings of any finite order (Theorem 3.2). For surfaces with nonnegative curvature given as a graph of a homogeneous function: $R(s,t) = (s,t,z(s,t))$ with z a homogeneous function, we construct infinitesimal bendings of higher orders through the solvability of associated systems of periodic differential equations, provided that two numbers attached to the surface satisfy a number theoretic condition (Theorems 4.2 and 4.3).

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In the final section, we consider a special class of surfaces defined as a graphs of function $s^{m+2} \pm t^{n+2}$ (a model for surfaces defined as graphs of functions $f(s)+g(t)$). We show (Theorem 5.1) that the space of real analytic infinitesimal bendings on the rectangle $|s| < \rho, |t| < \rho$ with $0 < \rho \leq \infty$, is isomorphic to the space $\mathcal{A}(\rho)^4$, where $\mathcal{A}(\rho)$ is the space of convergent power series of one variable with radius of convergence ρ.

2. Definitions and Equations for Bending Fields

Let S be a C^∞ surface in $\mathbb{R}^3$ over a domain $\Omega \subset \mathbb{R}^2$ given by

$$S = \{R(s,t) = (x(s,t), y(s,t), z(s,t))\,;\ (s,t) \in \Omega\}\,, \tag{2.1}$$

and S_ε a deformation of S, given by

$$S_\varepsilon = \left\{R_\varepsilon(s,t) = R(s,t) + 2\varepsilon U^1(s,t) + \ldots + 2\varepsilon^m U^m(s,t)\right\}, \tag{2.2}$$

where $U^j : \Omega \to \mathbb{R}^3$ is a C^k function for $j \in \{1, 2, \ldots, m\}$, for some $k \in \mathbb{N}$. The deformation S_ε is an *infinitesimal bending of S of order m* if its first fundamental form dR_ε^2 satisfies the following condition:

$$dR_\varepsilon^2 = dR^2 + o(\varepsilon^m),\ \text{ as } \varepsilon \to 0.$$

Since

$$dR_\varepsilon^2 = dR^2 + 4\varepsilon\left(dR \cdot dU^1\right) + \sum_{j=2}^{m} 4\varepsilon^j \left(dR \cdot dU^j + \sum_{i=1}^{m-1} dU^i \cdot dU^{m-i}\right) + o(\varepsilon^m),$$

then S_ε is an infinitesimal bending of order m if and only if

$$dR \cdot dU^1 = 0 \text{ and } dR \cdot dU^j = -\sum_{i=1}^{j-1} dU^i \cdot dU^{j-i}, \quad j = 2, 3, \ldots, m. \tag{2.3}$$

For each $j \in \{1, 2, \ldots, m\}$, set $U^j(s,t) = \left(u^j(s,t), v^j(s,t), w^j(s,t)\right)$. Equation (2.3) can be written as

$$\begin{cases} x_s u_s^j + y_s v_s^j + z_s w_s^j = F^j, \\ x_s u_t^j + y_s v_t^j + z_s w_t^j + x_t u_s^j + y_t v_s^j + z_t w_s^j = G^j, \\ x_t u_t^1 + y_t v_t^1 + z_t w_t^1 = H^j, \end{cases} \tag{2.4}$$

with $F^1 = G^1 = H^1 = 0$ and, when $j \geq 2$,

$$\begin{aligned} F^j &= -\sum_{i=1}^{j-1} \left(u_s^{j-i} u_s^i + v_s^{j-i} v_s^i + w_s^{j-i} w_s^i\right), \\ G^j &= -\sum_{i=1}^{j-1} \left(u_s^i u_t^{j-i} + v_s^i v_t^{j-i} + w_s^i w_t^{j-i} + u_t^i u_s^{j-i} + v_t^i v_s^{j-i} + w_t^i w_s^{j-i}\right), \\ H^j &= -\sum_{i=1}^{j-1} \left(u_t^{j-i} u_t^i + v_t^{j-i} v_t^i + w_t^{j-i} w_t^i\right). \end{aligned} \tag{2.5}$$

The trivial bendings of S are those generated through the rigid motions of the underlying space $\mathbb{R}^3$. In particular, the first order trivial infinitesimal bendings of S are given by $S_\varepsilon^{A,B} = \left\{R_\varepsilon^{A,B}(s,t) = R(s,t) + \varepsilon\left(A \times R(s,t) + B\right)\right\}$, where A, B are constants in $\mathbb{R}^3$ and $\times$ denotes the vector product in $\mathbb{R}^3$. A surface S is said to be *rigid* under infinitesimal bendings if it admits only trivial infinitesimal bendings.

Let N be the normal unit vector to S given by $\dfrac{R_s \times R_t}{\|R_s \times R_t\|}$ and e, f, g, the coefficients of the second fundamental form of S:

$$e = R_{ss} \cdot N, \qquad f = R_{st} \cdot N, \qquad g = R_{tt} \cdot N. \tag{2.6}$$

The Gaussian curvature of S is:

$$K(s,t) = \frac{eg - f^2}{\|R_s \times R_t\|^2}. \tag{2.7}$$

Throughout this work, except for the last section, we will assume that the Gaussian curvature of S is nonnegative: $K(s,t) \geq 0$, for all $(s,t) \in \Omega$.

3. Surfaces with Nonnegative Curvature and Flat Points

In this section, we assume that the parametrization domain $\Omega \subset \mathbb{R}^2$ of S is relatively compact and that $K > 0$ on $\overline{\Omega}$ except at finitely many points $p_1, \cdots p_n$ at which both principal curvatures vanish. We assume throughout that K vanishes uniformly only to a finite order at each flat point p_j (see (3.1) below)). We prove that such a surface S admits nontrivial infinitesimal bendings of any order. The idea is to use the complex vector field of asymptotic directions (see [M1], [M4]) to reduce the study of the bending equations into solving Bers-Vekua type equations (see also [AU] and [U] for the local deformation of surfaces near flat points).

Let S be given by (2.1) and e, f, g, the coefficients of its second fundamental form. We assume throughout this section the existence of $p_1, \cdots p_n \in \Omega$ such that

$$\begin{gathered} K(p) > 0 \quad \forall p \in \overline{\Omega} \setminus \{p_1, \cdots, p_n\} \quad \text{and} \\ \text{order}_{p_j}(K) = 2\,\text{order}_{p_j}(e) = 2\,\text{order}_{p_j}(g) \quad \forall j \in \{1, \cdots, n\}, \end{gathered} \tag{3.1}$$

where $\text{order}_p(F)$ denotes the order of vanishing of the function F at the point p.

The field of asymptotic directions is given by

$$L = g(s,t)\frac{\partial}{\partial s} + \lambda(s,t)\frac{\partial}{\partial t} \quad \text{where} \quad \lambda = -f + i\sqrt{eg - f^2}. \tag{3.2}$$

Proposition 3.1. *Let S be a surface with nonnegative curvature given by* (2.1) *and L be the vector field of asymptotic directions given by* (3.2). *For a solution $U^j = (u^j, v^j, w^j)$ of* (2.4) *with $j \in \mathbb{N}$, the $\mathbb{C}$-valued function $h^j = LR \cdot U^j$ satisfies the equation*

$$CLh^j = Ah^j - B\overline{h}^j + C\left[g^2F^j + g\lambda G^j + \lambda^2 H^j\right], \tag{3.3}$$

where

$$\begin{gathered} A = (LR \times \overline{L}R)(L^2R \times \overline{L}R), \quad B = (LR \times \overline{L}R)(L^2R \times LR), \\ C = (LR \times \overline{L}R)(LR \times \overline{L}R). \end{gathered} \tag{3.4}$$

Proof. Define functions φ^j and ψ^j by

$$\varphi^j = R_s \cdot U^j = x_s u^j + y_s v^j + z_s w^j, \tag{3.5}$$

$$\psi^j = R_t \cdot U^j = x_t u^j + y_t v^j + z_t w^j. \tag{3.6}$$

Then

$$\varphi^j_s = R_{ss} \cdot U^j + F^j = x_{ss}u^j + y_{ss}v^j + z_{ss}w^j + F^j, \tag{3.7}$$

$$\psi^j_t = R_{tt} \cdot U^j + H^j = x_{tt}u^j + y_{tt}v^j + z_{tt}w^j + H^j, \tag{3.8}$$

$$\varphi^j_t + \psi^j_s = 2R_{st} \cdot U^j + G^j = 2\left[x_{st}u^j + y_{st}v^j + z_{st}w^j\right] + G^j. \tag{3.9}$$

Let $R_s \times R_t = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha = \|R_s \times R_t\|$. It follows from (3.5) and (3.6) that

$$\begin{aligned}\alpha_3 u^j &= \alpha_1 w^j + \varphi^j y_t - \psi^j y_s,\\ \alpha_3 v^j &= \alpha_2 w^j - \varphi^j x_t + \psi^j x_s.\end{aligned}$$

By using these expressions, we can rewrite (3.7), (3.8) and (3.9) as

$$\begin{aligned}\alpha_3 \varphi_s^j &= \begin{vmatrix} x_{ss} & y_{ss} \\ x_t & y_t \end{vmatrix} \varphi^j - \begin{vmatrix} x_{ss} & y_{ss} \\ x_s & y_s \end{vmatrix} \psi^j + \alpha e w^j + \alpha_3 F^j\\ \alpha_3 \psi_t^j &= \begin{vmatrix} x_{tt} & y_{tt} \\ x_t & y_t \end{vmatrix} \varphi^j - \begin{vmatrix} x_{tt} & y_{tt} \\ x_s & y_s \end{vmatrix} \psi^j + \alpha g w^j + \alpha_3 H^j\\ \alpha_3(\varphi_t^j + \psi_s^j) &= \begin{vmatrix} x_{st} & y_{st} \\ x_t & y_t \end{vmatrix} 2\varphi^j - \begin{vmatrix} x_{st} & y_{st} \\ x_s & y_s \end{vmatrix} 2\psi^j + 2\alpha f w^j + \alpha_3 G^j.\end{aligned}$$

We can eliminate the function w^j in the system above (using Lemma 3.1 of [M2] and canceling α_3) and reduce it to the following system for φ^j and ψ^j:

$$g\varphi_s^j - e\psi_t^j = -\frac{[R_t \cdot (R_{ss} \times R_{tt})]}{\alpha}\varphi^j + \frac{[R_s \cdot (R_{ss} \times R_{tt})]}{\alpha}\psi^j + \left(gF^j - eH^j\right),$$

$$\begin{aligned}f(\varphi_s^j + \psi_t^j) - \frac{e+g}{2}(\varphi_t^j + \psi_s^j) = \frac{[R_t \cdot (R_{st} \times (R_{ss} + R_{tt})]}{\alpha}\varphi^j + f(F^j + H^j) -\\ - \frac{[R_s \cdot (R_{st} \times (R_{ss} + R_{tt})]}{\alpha}\psi^j - \frac{e+g}{2}G^j.\end{aligned}$$

We rewrite the system in a matrix form as

$$\begin{pmatrix} g & 0 \\ f & -\frac{e+g}{2} \end{pmatrix} \begin{pmatrix} \varphi^j \\ \psi^j \end{pmatrix}_s - \begin{pmatrix} 0 & e \\ \frac{e+g}{2} & -f \end{pmatrix} \begin{pmatrix} \varphi^j \\ \psi^j \end{pmatrix}_t = \underbrace{\begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}}_{\Xi} \begin{pmatrix} \varphi^j \\ \psi^j \end{pmatrix} + \\ + \begin{pmatrix} gF^j - eH^j \\ f(F^j + H^j) - \dfrac{e+g}{2}G^j \end{pmatrix}.$$

This system can be reduced further after multiplication by $\dfrac{2}{(e+g)}\begin{pmatrix} -\dfrac{e+g}{2} & 0 \\ -f & g \end{pmatrix}$ into

$$-g\begin{pmatrix} \varphi^j \\ \psi^j \end{pmatrix}_s - \begin{pmatrix} 0 & -e \\ g & -2f \end{pmatrix} \begin{pmatrix} \varphi^j \\ \psi^j \end{pmatrix}_t = \Lambda \begin{pmatrix} \varphi^j \\ \psi^j \end{pmatrix} + \begin{pmatrix} eH^j - gF^j \\ 2fH^j - gG^j \end{pmatrix}, \tag{3.10}$$

where $\Lambda = \dfrac{2}{(e+g)}\begin{pmatrix} -\dfrac{e+g}{2} & 0 \\ -f & g \end{pmatrix}\Xi$.

Note that λ is an eigenvalue for the transpose of $\begin{pmatrix} 0 & -e \\ g & -2f \end{pmatrix}$ with eigenvector $\eta = \begin{pmatrix} g \\ \lambda \end{pmatrix}$. After multiplying (3.10) by η^t and using $\lambda^2 + 2f\lambda + eg = 0$, we get

$$g(g\varphi_s^j + \lambda\psi_s^j) + \lambda(g\varphi_t^j + \lambda\psi_t^j) = -\eta^t \Lambda \begin{pmatrix} \varphi^j \\ \psi^j \end{pmatrix} + g^2 F^j + g\lambda G^j + \lambda^2 H^j. \tag{3.11}$$

Observe that $\eta^t \begin{pmatrix} \varphi^j \\ \psi^j \end{pmatrix} = g\varphi^j + \lambda\psi^j = h^j$, so that $h^j_s = \eta^t \begin{pmatrix} \varphi^j \\ \psi^j \end{pmatrix}_s + \eta^t_s \begin{pmatrix} \varphi^j \\ \psi^j \end{pmatrix}$, $h^j_t = \eta^t \begin{pmatrix} \varphi^j \\ \psi^j \end{pmatrix}_t + \eta^t_t \begin{pmatrix} \varphi^j \\ \psi^j \end{pmatrix}$ and (3.11) becomes

$$gh^j_s + \lambda h^j_t = g\eta^t_s \begin{pmatrix} \varphi^j \\ \psi^j \end{pmatrix} + \lambda\eta^t_t \begin{pmatrix} \varphi^j \\ \psi^j \end{pmatrix} - \eta^t \Lambda \begin{pmatrix} \varphi^j \\ \psi^j \end{pmatrix} + g^2F^j + g\lambda G^j + \lambda^2 H^j. \quad (3.12)$$

Since g is a real function,

$$(\lambda - \overline{\lambda})g\varphi^j = \lambda\overline{h}^j - \overline{\lambda}h^j, \qquad (\lambda - \overline{\lambda})\psi^j = h^j - \overline{h}^j. \quad (3.13)$$

Hence after multiplying (3.12) by $g(\lambda - \overline{\lambda})$, we get

$$g(\lambda - \overline{\lambda})Lh = Ph + Q\overline{h} + g(\lambda - \overline{\lambda})\left(g^2F^j + g\lambda G^j + \lambda^2 H^j\right), \quad (3.14)$$

where the coefficients P and Q are given by.

$$P = g(\lambda - \overline{\lambda})\frac{(L^2R \times \overline{L}R)\cdot(LR \times \overline{L}R)}{(LR \times \overline{L}R)\cdot(LR \times \overline{L}R)} = g(\lambda - \overline{\lambda})\frac{A}{C},$$

$$Q = -g(\lambda - \overline{\lambda})\frac{(L^2R \times LR)\cdot(LR \times \overline{L}R)}{(LR \times \overline{L}R)\cdot(LR \times \overline{L}R)} = g(\lambda - \overline{\lambda})\frac{B}{C}$$

(see [M2] for details). This completes the proof of the proposition. □

Remark 3.1. A direct calculation gives

$$\begin{aligned} L^2R \times LR &= (\lambda\cdot Lg - g\cdot L\lambda)(R_s \times R_t) + g^3(R_{ss} \times R_s) + \lambda^3(R_{tt} \times R_t) + \\ &\quad + g^2\lambda\left[(R_{ss} \times R_t) + 2(R_{st} \times R_s)\right] + g\lambda^2\left[2(R_{st} \times R_t) + (R_{tt} \times R_s)\right], \\ L^2R \times \overline{L}R &= (\overline{\lambda}\cdot Lg - g\cdot L\lambda)(R_s \times R_t) + g^3(R_{ss} \times R_s) + \lambda|\lambda|^2(R_{tt} \times R_t) + \\ &\quad + g^2\overline{\lambda}(R_{ss} \times R_t) + 2g^2\lambda(R_{st} \times R_s) + 2g|\lambda|^2(R_{st} \times R_t) + g\lambda^2(R_{tt} \times R_s), \\ LR \times \overline{L}R &= -g(\lambda - \overline{\lambda})(R_s \times R_t), \end{aligned} \quad (3.15)$$

which implies that $C = -4g^2(eg - f^2)\left\|R_s \times R_t\right\|^2$.

To continue, we need to understand the behavior of the coefficients A, B and C at the flat points $p_1, \ldots, p_n$. Since the order of contact of the surface S with the tangent plane at the flat points p_j is $m_j \geq 3$, by proceeding as in [M1] we can find local polar coordinates (r, θ) centered in p_j such that

$$\begin{gathered} e = r^{m_j-2}e^j_1(\theta) + r^{m_j-1}e^j_2(r,\theta), \quad f = r^{m_j-2}f^j_1(\theta) + r^{m_j-1}f^j_2(r,\theta), \\ g = r^{m_j-2}g^j_1(\theta) + r^{m_j-1}g^j_2(r,\theta). \end{gathered} \quad (3.16)$$

The hypothesis (3.1) implies that $e^j_1(\theta)g^j_1(\theta) - f^j_1(\theta)^2 > 0$ for all $\theta \in \mathbb{R}$. This fact, associated to (3.2), implies that

$$\lambda = r^{m_j-2}\lambda^j_1(\theta) + r^{m_j-1}\lambda^j_2(r,\theta). \quad (3.17)$$

Furthermore the vector field L can be normalized and written as

$$L = \Xi(r,\theta)r^{m_j-3}\left(\mu_j\frac{\partial}{\partial\theta} - ir\frac{\partial}{\partial r}\right), \quad (3.18)$$

where $\mu_j > 0$ is an invariant attached to L and $\Xi \neq 0$ everywhere.

In these coordinates we have

$$\begin{aligned} Lg &= \Xi(r,\theta) r^{2m_j-5}\zeta_1^j(\theta) + r^{2m_j-4}\zeta_2^j(r,\theta);\\ L\lambda &= \Xi(r,\theta) r^{2m_j-5}\vartheta_1^j(\theta) + r^{2m_j-4}\vartheta_2^j(r,\theta). \end{aligned} \tag{3.19}$$

Using (3.15), (3.16), (3.17) and (3.19), we deduce that

$$\begin{aligned} L^2R\times LR &= \Xi(r,\theta) r^{3m_j-7}\varsigma_1^j(\theta)(R_s\times R_t) + r^{3m_j-6}\varsigma_2^j(r,\theta);\\ L^2R\times \overline{L}R &= \Xi(r,\theta) r^{3m_j-7}\varkappa_1^j(\theta)(R_s\times R_t) + r^{3m_j-6}\varkappa_2^j(r,\theta);\\ LR\times \overline{L}R &= r^{2m_j-4}\kappa_1^j(\theta)(R_s\times R_t) + r^{2m_j-3}\kappa_2^j(r,\theta). \end{aligned}$$

Hence A, B, and C can be written as:

$$\begin{aligned} A &= (L^2R\times \overline{L}R)(LR\times \overline{L}R) = \Xi(r,\theta) r^{5m_j-11}\varrho_1^j(\theta)\,\|R_s\times R_t\|^2 + r^{5m_j-10}\varrho_2^j(r,\theta);\\ B &= (L^2R\times LR)(LR\times \overline{L}R) = \Xi(r,\theta) r^{5m_j-11}\nu_1^j(\theta)\,\|R_s\times R_t\|^2 + r^{5m_j-10}\nu_2^j(r,\theta);\\ C &= (LR\times \overline{L}R)(LR\times \overline{L}R) = r^{4m_j-8}\mu_1^j(\theta)\,\|R_s\times R_t\|^2 + r^{4m_j-9}\mu_2^j(r,\theta). \end{aligned} \tag{3.20}$$

Since $\|R_s\times R_t\|^2$ is always strictly positive, we infer from (3.20) that

$$\begin{aligned} \frac{A}{C}(r,\theta) &= \Xi(r,\theta) r^{m_j-3}a_1^j(\theta) + r^{m_j-2}a_2^j(r,\theta);\\ \frac{B}{C}(r,\theta) &= \Xi(r,\theta) r^{m_j-3}b_1^j(\theta) + r^{m_j-2}b_2^j(r,\theta). \end{aligned} \tag{3.21}$$

The following result about the first integral of L (proved in [M1]) will be used.

Lemma 3.1. [M1] *There exists an injective function $Z:\overline{\Omega}\to\mathbb{C}$ satisfying the following conditions:*

(1) Z is C^∞ on $\overline{\Omega}\setminus\{p_1,p_2,\dots,p_n\}$.
(2) $LZ=0$ on $\overline{\Omega}$.
(3) For every $j=1,2,,\dots,n$, there exists $\mu_j>0$ and polar coordinates (r,θ) centered at p_j such that

$$Z(r,\theta) = Z(0,0) + r^{\mu_j}\cdot e^{i\theta} + O(r^{2\mu_j}) \tag{3.22}$$

in a neighborhood of p_j.

We use the first integral Z of L given by (3.22) to transform the equation (3.3) to a Bers-Vekua type equation. The following notation will be used: for each $\ell\in\{1,2,\dots,n\}$, let $\zeta_\ell = Z(p_\ell)$ and $D(\zeta)=\prod_{\ell=1}^n(\zeta-\zeta_\ell)$. The pushforward via Z of a function f defined in Ω will be denoted $\widetilde{f}$: $\widetilde{f} = f\circ Z^{-1}$.

Proposition 3.2. *Let Z as in Lemma 3.1. A function h^j is a solution of (3.3) in Ω if and only if its Z-pushforward $\widetilde{h}^j$ satisfies the following equation in $Z(\Omega)$:*

$$\frac{\partial \widetilde{h}^j}{\partial\bar\zeta} = \frac{P(\zeta)}{D(\zeta)}\widetilde{h}^j + \frac{Q(\zeta)}{D(\zeta)}\overline{\widetilde{h}}^j + \frac{1}{D(\zeta)}\left(S_1(\zeta)\widetilde{F}^j + S_2(\zeta)\widetilde{G}^j + S_3(\zeta)\widetilde{H}^j\right), \tag{3.23}$$

with the following conditions being satisfied:

1) $P,Q\in L^\infty(Z(\Omega))\cap C^\infty(Z(\Omega)\setminus\{\zeta_1,\zeta_2\,\dots,\zeta_n\})$; morevoer, we are able to write in a small neighborhood of each ζ_j the coefficients as

$$P(\zeta) = \chi_{1,j}(\sigma) + \rho^{\epsilon_j}\kappa_j^1(\rho,\sigma),\ \ Q(\zeta) = \chi_{2,j}(\sigma) + \rho^{\epsilon_j}\kappa_j^2(\rho,\sigma),$$

where $\zeta=\rho e^{i\sigma}$, both $\chi_{1,j}$ and $\chi_{2,j}$ are 2π-periodic and $\epsilon_j>0$, for every j.

2) S_1, S_2, S_3 are bounded in $Z(\Omega)$ and vanish to order $\dfrac{m_\ell - 1}{\mu_\ell}$ at ζ_ℓ, for every ℓ, where m_ℓ and μ_ℓ are the positive numbers associated with the point p_ℓ.

Proof. The Z-pushforward of (3.3) gives

$$\widetilde{(L\overline{Z})}\,\frac{\partial \widetilde{h}^j}{\partial \overline{\zeta}} = \frac{\widetilde{A}}{\widetilde{C}}\,\widetilde{h}^j - \frac{\widetilde{B}}{\widetilde{C}}\,\overline{\widetilde{h}^j} + \widetilde{M}, \tag{3.24}$$

where

$$M = g^2 F^j + g\lambda G^j + \lambda^2 H^j. \tag{3.25}$$

Since the vector field L is elliptic outside the points ζ_j and C does not vanish outside these points, (3.24) has the form (3.23) in $Z(\Omega)\setminus\{\zeta_1, \cdots, \zeta_n\}$. Let us verify the proposition near each point ζ_j. It follows from (3.18) and (3.22) that

$$\overline{L}Z = -2i\Xi(r,\theta)\left[r^{m_j-3+\mu_j}e^{-i\theta} + r^{m_j-3}O(r^{2\mu_j})\right] \tag{3.26}$$

The next step is to understand how composition with Z^{-1} acts over each term. By assuming that $Z(0,0) = 0$ and setting $\rho = |Z|$, it follows from (3.22) that we are able to write $\rho(r,\theta) = r^{\mu_j}\left(1 + r^{\mu_j}J_1^j(r,\theta)\right)^{1/2}$, where $J_1^j(r,\theta)$ is a continuous and bounded function. By the binomial theorem, if r is sufficiently small we have

$$\rho(r,\theta) = r^{\mu_j}\left(1 + r^{\mu_j}J_2^j(r,\theta)\right) = r^{\mu_j} + r^{2\mu_j}J_2^j(r,\theta) = y + y^2 J_3^j(y,\theta),$$

if we denote $y = r^{\mu_j}$. As a consequence of the proof of Lemma 3.1, Z is a C^1 function in terms of y and θ. Thus, by possibly taking r even smaller we are able to solve ρ in terms of y, which allows us to deduce that $r^{\mu_j} = \rho + \rho^2 K_1(\rho,\theta)$, with K_1 continuous and bounded. This implies that

$$r = \rho^{1/\mu_j}\left(1 + \rho K_1^j(\rho,\theta)\right)^{1/\mu_j} = \rho^{1/\mu_j} + \rho^{1+1/\mu_j}K_2^j(\rho,\theta). \tag{3.27}$$

We deduce from (3.16), 3.17, (3.21), (3.25) and (3.27) that, for some $\varepsilon_j > 0$,

$$\begin{aligned}
\widetilde{M}(\rho,\sigma) &= \widetilde{\Xi}(\rho,\sigma)\left[\rho^{\frac{2m_j-4}{\mu_j}}\gamma_1(\rho,\sigma)\widetilde{F^j} + \rho^{\frac{2m_j-4}{\mu_j}}\gamma_2(\rho,\sigma)\widetilde{G^j} + \rho^{\frac{2m_j-4}{\mu_j}}\gamma_3(\rho,\sigma)\widetilde{H^j}\right],\\
\frac{\widetilde{A}}{\widetilde{C}}(\rho,\sigma) &= \widetilde{\Xi}(\rho,\sigma)\left[\rho^{\frac{m_j-3}{\mu_j}}\alpha_1^j(\sigma) + \rho^{\frac{m_j-3}{\mu_j}+\varepsilon_j}\alpha_2^j(\rho,\sigma)\right],\\
\frac{\widetilde{B}}{\widetilde{C}}(\rho,\sigma) &= \widetilde{\Xi}(\rho,\sigma)\left[\rho^{\frac{m_j-3}{\mu_j}}\beta_1^j(\sigma) + \rho^{\frac{m_j-3}{\mu_j}+\varepsilon_j}\beta_2^j(\rho,\sigma)\right].
\end{aligned} \tag{3.28}$$

By using expression (3.27) in (3.26), we obtain

$$\widetilde{L\overline{Z}}(\rho,\sigma) = \widetilde{\Xi}(\rho,\sigma)\left[\rho^{\frac{m_j-3}{\mu_j}+1}\psi_1^j(\sigma) + \rho^{\frac{m_j-3}{\mu_j}+2}\psi_2^j(\rho,\theta)\right]. \tag{3.29}$$

It follows from (3.28) and (3.29) that (3.24) can be rewritten as

$$\begin{aligned}
\frac{\partial \widetilde{h}^j}{\partial \overline{\zeta}} = &\left[\left(\frac{\chi_1^j(\sigma) + \rho^{\epsilon_j}\chi_2^j(\rho,\theta)}{\rho e^{i\sigma}}\right)\widetilde{h}^j + \left(\frac{\kappa_1^j(\sigma) + \rho^{\epsilon_j}\kappa_2^j(\rho,\theta)}{\rho e^{i\sigma}}\right)\overline{\widetilde{h}^j}\right] + \\
&+ \left[\frac{\rho^{\frac{m_j-1}{\mu_j}}\iota_1^j(\rho,\sigma)\widetilde{F^j}}{\rho e^{i\sigma}} + \frac{\rho^{\frac{m_j-1}{\mu_j}}\iota_2^j(\rho,\sigma)\widetilde{G^j}}{\rho e^{i\sigma}} + \frac{\rho^{\frac{m_j-1}{\mu_j}}\iota_3^j(\rho,\sigma)\widetilde{H^j}}{\rho e^{i\sigma}}\right],
\end{aligned} \tag{3.30}$$

where $\epsilon_j = \min\{\varepsilon_j, 1\}$. Observe that $\rho e^{i\sigma} = \zeta - \zeta_j$. Hence, in order to obtain an expression as in (3.23), it is sufficient to multiply both the numerator and the denominator of each term in the right hand-side of (3.30) by $D_j(\zeta) = \prod_{\substack{\ell=1,\dots,n \\ \ell\neq j}} (\zeta - \zeta_\ell)$, which finalizes the proof. □

We can now proceed to the main result of the section.

Theorem 3.2. *Let $S \subset \mathbb{R}^3$ be a smooth surface given by* (2.1) *with curvature K satisfying* (3.1). *For every $k, m \in \mathbb{N}$, there exist functions*

$$U^1, U^2, \dots, U^m : \Omega \to \mathbb{R}^3 \quad \text{with} \quad U^j \in C^k(\Omega) \cap C^\infty(\Omega \setminus \{p_1, p_2, \dots, p_n\}),$$

such that each U^j vanishes to order $\geq k$ at each point $p_1, \cdots p_n$ and such that the deformation surface S_ϵ given by (2.2) *is a nontrivial infinitesimal bending of S of order m.*

Proof. We prove this result by induction on the order m; in general the idea is to solve (3.23) and show it has a solution $\widetilde{h}^j$ that vanishes, to any prescribed order, at the points $\zeta_1, \cdots, \zeta_n$. Then use $h^j = \widetilde{h}^j \circ Z$ to recover the field U^j through $h^j = LR \cdot U^j$ (Proposition 3.1).

Let $j = 1$; then $F^1 = G^1 = H^1 = 0$, which turns the expression (3.23) into

$$\frac{\partial \widetilde{h}^1}{\partial \overline{\zeta}} = \frac{P(\zeta)}{D(\zeta)} \widetilde{h}^1 + \frac{Q(\zeta)}{D(\zeta)} \overline{\widetilde{h}}^1 . \tag{3.31}$$

Applying *Theorem* 2.3 of [LVM], for any $l \in \mathbb{N}$ we are able to find a non-trivial solution for (3.31) in $C^l(Z(\Omega)) \cap C^\infty(Z(\Omega) \backslash \{\zeta_1, \cdots, \zeta_n\})$, such that it vanishes to order $\geq l$ at each point $\zeta_1, \cdots \zeta_n$. Therefore, as a consequence of Proposition 3.2, $h^1 = \widetilde{h}^1 \circ Z$ solves (3.3) and vanishes to order $l\mu_i$ at the points $p_1, \dots, p_n$.

Next we construct the field U^1 from h^1. As a consequence of (3.13), we have $\varphi^1 = \dfrac{\lambda \overline{h}^1 - \overline{\lambda} h^1}{(\lambda - \overline{\lambda}) g}$ and $\psi^1 = \dfrac{h^1 - \overline{h}^1}{(\lambda - \overline{\lambda})}$. These functions would be well defined at the flat points $p_1, \cdots, p_n$ provided that h^1 vanishes to higher orders than those the vanishing of the denominators. This is indeed the case when $(l \cdot \mu_i) - (m_i - 2) \geq 0$, since both g and $\lambda - \overline{\lambda}$ have order of vanishing $m_i - 2$ at p_i (see 3.16 and 3.17).

The field $U^1 = (u^1, v^1, w^1)$ is related to φ^1 and ψ^1 via relations (3.5), (3.6) and (3.7), which form the system

$$\begin{pmatrix} \varphi^1 \\ \psi^1 \\ \varphi^1_s \end{pmatrix} = \begin{pmatrix} x_s & y_s & z_s \\ x_t & y_t & z_t \\ x_{ss} & y_{ss} & z_{ss} \end{pmatrix} \begin{pmatrix} u^1 \\ v^1 \\ w^1 \end{pmatrix} = M(s,t) \begin{pmatrix} u^1 \\ v^1 \\ w^1 \end{pmatrix} .$$

The determinant of the matrix M is given by $R_{ss} \cdot (R_s \times R_t) = \|R_s \times R_t\| \, e$. Thus

$$\begin{aligned} u^1 &= \frac{1}{\|R_s \times R_t\| \, e} \left(\psi^1 \begin{vmatrix} y_s & z_s \\ y_{ss} & z_{ss} \end{vmatrix} - \varphi^1 \begin{vmatrix} y_t & z_t \\ y_{ss} & z_{ss} \end{vmatrix} + \varphi^1_s \begin{vmatrix} y_s & z_s \\ y_t & z_t \end{vmatrix} \right), \\ v^1 &= \frac{1}{\|R_s \times R_t\| \, e} \left(-\varphi^1 \begin{vmatrix} x_t & z_t \\ x_{ss} & z_{ss} \end{vmatrix} + \psi^1 \begin{vmatrix} x_s & z_s \\ x_{ss} & z_{ss} \end{vmatrix} - \varphi^1_s \begin{vmatrix} x_s & z_s \\ x_t & z_t \end{vmatrix} \right), \\ w^1 &= \frac{1}{\|R_s \times R_t\| \, e} \left(\varphi^1 \begin{vmatrix} x_t & y_t \\ x_{ss} & y_{ss} \end{vmatrix} - \psi^1 \begin{vmatrix} x_s & y_s \\ x_{ss} & y_{ss} \end{vmatrix} + \varphi^1_s \begin{vmatrix} x_s & y_s \\ x_t & y_t \end{vmatrix} \right). \end{aligned} \tag{3.32}$$

These functions are well defined if $\varphi^1, \psi^1, \varphi^1_s$ vanish to an order greater than $m_i - 2$ at each p_i. In this case u^1, v^1, and w^1 vanish to order

$$[l\mu_i - (m_i - 1)] - (m_i - 2) = l\mu_i - 2m_i + 3. \tag{3.33}$$

Now consider $m^\star = \max\{m_i,\ i = 1, \dots, n\}$ and $\mu_\star = \min\{\mu_i,\ i = 1, \dots, n\}$. Given $k \in \mathbb{N}$, by choosing $l \geq \dfrac{2m^\star - 3 + k}{\mu_\star}$, we have $l\mu_i - 2m_i + 3 \geq k$ for each $i \in \{1, 2, \dots, n\}$. This implies (from (3.33)) that $U^1 = (u^1, v^1, w^1)$ vanishes to order greater than k at each p_i. Hence $U^1 \in C^k(\Omega) \cap C^\infty(\Omega \setminus \{p_1, p_2, \dots, p_n\})$.

It remains to show that U^1 is not trivial. Suppose by contradiction that

$$U^1 = A \times R(s,t) + B, \quad A, B \in \mathbb{R}^3.$$

Then $U^1_s(p_1) = A \times R_s(p_1) = 0$ and $U^1_t(p_1) = A \times R_t(p_1) = 0$, which implies that $A = 0$. Since U^1 vanishes at p_0, then $B = 0$ and $U^1 \equiv 0$, which ends the first case.

Suppose next that the statement holds to order up to $j-1$; then for every $\ell \in \mathbb{N}$ there exist functions $U^1, U^2, \dots, U^{j-1} \in C^\infty(\Omega \setminus \{p_1, \cdots, p_n\}, \mathbb{R}^3) \cap C^\ell(\Omega, \mathbb{R}^3)$, such that each field U^r vanishes to order ℓ at each point p_i and

$$R^{j-1}_\epsilon(s,t) = R(s,t) + 2\sum_{r=1}^{j-1} \epsilon^r U^r(s,t)$$

is an infinitesimal bending of order $j-1$ of S.

Since $U^1, \cdots, U^{j-1}$ vanish to order ℓ at the points p_i, the functions F^r, G^r, and H^r given by (2.5) vanish to order $2\ell - 2$. Hence their Z-Pushforwards $\widetilde{F}^r$, $\widetilde{G}^r$, and $\widetilde{H}^r$ vanish to order $\dfrac{2\ell-2}{\mu_i}$ at the points $\zeta_i = Z(p_i)$. Thus (by Proposition 3.2) the nonhomogeneous term of equation (3.23) vanishes to order $\rho_i = \dfrac{2\ell + m_i - 3}{\mu_i} - 1$ at ζ_i.

By taking ℓ sufficiently large and applying *Theorem* 2.1 of [LVM], we obtain a solution $\widetilde{h}^j$ of (3.23) that vanishes to order q at each point $\zeta_1, \cdots, \zeta_n$, for some q that will be chosen later. Therefore $h^j = \widetilde{h}^j \circ Z$ solves (3.3) and vanishes to order $q\mu_i$ at the point p_i, for every $i = 1, \cdots, n$. Now we construct the field U^j from h^j. Once again $\varphi^j = \dfrac{\lambda\overline{h}^j - \overline{\lambda}h^j}{(\lambda - \overline{\lambda})g}$, $\psi^j = \dfrac{h^j - \overline{h}^j}{(\lambda - \overline{\lambda})}$ and these functions would be well defined if $q\mu_i - (m_i - 2) \geq 0$. Once again the field $U^j = (u^j, v^j, w^j)$ is related to φ^j and ψ^j via relations (3.5), (3.6) ,(3.7) and in this case

$$u^j = \frac{1}{\|R_s \times R_t\|\, e}\left(\psi^j \begin{vmatrix} y_s & z_s \\ y_{ss} & z_{ss} \end{vmatrix} - \varphi^j \begin{vmatrix} y_t & z_t \\ y_{ss} & z_{ss} \end{vmatrix} + (\varphi^j_s - F^j)\begin{vmatrix} y_s & z_s \\ y_t & z_t \end{vmatrix}\right),$$

$$v^j = \frac{1}{\|R_s \times R_t\|\, e}\left(-\varphi^j \begin{vmatrix} x_t & z_t \\ x_{ss} & z_{ss} \end{vmatrix} + \psi^j \begin{vmatrix} x_s & z_s \\ x_{ss} & z_{ss} \end{vmatrix} - (\varphi^j_s - F^j)\begin{vmatrix} x_s & z_s \\ x_t & z_t \end{vmatrix}\right),$$

$$w^j = \frac{1}{\|R_s \times R_t\|\, e}\left(\varphi^j \begin{vmatrix} x_t & y_t \\ x_{ss} & y_{ss} \end{vmatrix} - \psi^j \begin{vmatrix} x_s & y_s \\ x_{ss} & y_{ss} \end{vmatrix} + (\varphi^j_s - F^j)\begin{vmatrix} x_s & y_s \\ x_t & y_t \end{vmatrix}\right).$$

These functions will be well defined if $\varphi^j, \psi^j, \varphi^j_s$ and F^j vanish to an order greater than $m_i - 2$ at each p_i. In this situation u^j, v^j, and w^j vanish to order

$$r_i = \min\{[q\mu_i - (m_i - 1)], 2\ell - 2\} - (m_i - 2). \tag{3.34}$$

Let $m^\star$ and $\mu_\star$ as in the previous case. Given $k \in \mathbb{N}$, take ℓ large enough so

$$2\ell \geq k + m^\star \quad \text{and} \quad q \geq \frac{k + 2m^\star - 3}{\mu_\star}.$$

It follows from (3.34) that for such choices of ℓ and q, we have $r_i \geq k$ for each i and the field $U^j = (u^j, v^j, w^j)$ vanishes to order k at each p_i. This completes the proof. □

4. Infinitesimal Bendings of Graphs of Homogeneous Functions

In this section we study infinitesimal bendings of surfaces given as graphs of homogeneous functions. We prove that any such generic surface with nonnegative curvature has nontrivial infinitesimal bendings of high orders. The surfaces considered here are given by

$$S = \left\{ R(s,t) = (s, t, z(s,t))\,;\ (s,t) \in \mathbb{R}^2 \right\}, \tag{4.1}$$

where z is a nonnegative homogeneous function of order $m \geq 2$ (with m not necessarily an integer), such that $z \in C^\infty(\mathbb{R}^2 \setminus \{0\})$. We assume that S has nonnegative curvature and no asymptotic curves. The Gaussian curvature of S is

$$K(s,t) = \frac{z_{ss} z_{tt} - z_{st}^2}{(1 + z_s^2 + z_t^2)^2}\,.$$

It is more convenient here to use polar coordinates $s = r\cos\theta$, $t = r\sin\theta$, so that $z = r^m P(\theta)$, with $P(\theta) \in C^\infty(S^1)$. The assumption that S has no asymptotic curves implies that $P(\theta) > 0$ for all θ (if $P(\theta_0) = 0$, then the ray $\theta = \theta_0$ would be an asymptotic curve). The second derivatives of z in polar coordinates are:

$$\begin{aligned}
z_{ss} &= r^{m-2}\left[(m-1)m\cos^2\theta P - 2(m-1)\sin\theta\cos\theta P' + m\sin^2\theta P + P''\sin^2\theta\right],\\
z_{st} &= r^{m-2}\left[(m-2)m\sin\theta\cos\theta P + (m-1)\cos(2\theta)P' - P''\sin\theta\cos\theta\right],\\
z_{tt} &= r^{m-2}\left[(m-1)m\sin^2\theta P + 2(m-1)\sin\theta\cos\theta P' + m\cos^2\theta P + P''\cos^2\theta\right].
\end{aligned}$$

It follows that

$$z_{ss} z_{tt} - z_{st}^2 = r^{2m-4}(m-1)\left[m^2 P^2(\theta) + mP(\theta)P''(\theta) - (m-1)P'(\theta)^2\right].$$

From now on we will assume that $K \geq 0$ and K vanishes on at most a finite number of rays $\theta = \theta_1, \cdots, \theta_\ell$. This is equivalent to

$$\left[m^2 P^2(\theta) + mP(\theta)P''(\theta) - (m-1)P'(\theta)^2\right] > 0, \quad \forall \theta \notin \{\theta_1, \cdots, \theta_\ell\}. \tag{4.2}$$

We write in a more convenient form the equation related to the bending fields $U^j = (u^j, v^j, w^j)$. The functions φ^j and ψ^j related to U^j by (3.5) and (3.6) take the form

$$\varphi^j = u^j + z_s w^j, \quad \psi^j = v^j + z_t w^j\,. \tag{4.3}$$

In this situation equation (3.10) leads to the following system in polar coordinates

$$\frac{1}{r}\begin{pmatrix}\varphi^j\\ \psi^j\end{pmatrix}_\theta = A(\theta)\begin{pmatrix}\varphi^j\\ \psi^j\end{pmatrix}_r + T(\theta)\begin{pmatrix}F^j\\ G^j\\ H^j\end{pmatrix}, \tag{4.4}$$

where $A(\theta) = \dfrac{1}{(m^2-m)r^{m-2}P(\theta)} \begin{pmatrix} \alpha_{11}(r,\theta) & \alpha_{12}(r,\theta) \\ \alpha_{21}(r,\theta) & \alpha_{22}(r,\theta) \end{pmatrix}$, with

$$\alpha_{11}(r,\theta) = (z_{tt} - z_{ss}) \sin\theta\cos\theta + 2z_{st}\cos^2\theta, \qquad \alpha_{12}(r,\theta) = -z_{ss},$$
$$\alpha_{21}(r,\theta) = z_{tt}, \qquad \alpha_{22}(r,\theta) = (z_{tt} - z_{ss}) \sin\theta\cos\theta - 2z_{st}\sin^2\theta,$$

and $T(\theta)$ is the matrix $T(\theta) = \dfrac{1}{(m^2-m)r^{m-2}P(\theta)} \begin{pmatrix} \beta_1(r,\theta) & \beta_2(r,\theta) & \beta_3(r,\theta) \\ \gamma_1(r,\theta) & \gamma_2(r,\theta) & \gamma_3(r,\theta) \end{pmatrix}$, with

$$\begin{array}{ll} \beta_1(r,\theta) = (-2z_{st}\cos\theta - z_{tt}\sin\theta), & \gamma_1(r,\theta) = -z_{tt}\cos\theta, \\ \beta_2(r,\theta) = z_{ss}\cos\theta, & \gamma_2(r,\theta) = -z_{tt}\sin\theta, \\ \beta_3(r\theta) = z_{ss}\sin\theta, & \gamma_3(r,\theta) = z_{ss}\cos\theta + 2z_{st}\sin\theta. \end{array} \tag{4.5}$$

Note that it follows from the homogeneity of z that the matrices A and T are independent of the variable r.

Now we use the change of variable $\rho = rP(\theta)^{1/m}$ to transform (4.4) into

$$\frac{1}{\rho}\begin{pmatrix} \varphi^j \\ \psi^j \end{pmatrix}_\theta = \Lambda(\theta)\begin{pmatrix} \varphi^j \\ \psi^j \end{pmatrix}_\rho + \widetilde{T}(\theta)\begin{pmatrix} \widetilde{F}^j(\rho,\theta) \\ \widetilde{G}^j(\rho,\theta) \\ \widetilde{H}^j(\rho,\theta) \end{pmatrix}, \tag{4.6}$$

where $\Lambda(\theta) = \dfrac{1}{(m^2-m)r^{m-2}P(\theta)} \begin{pmatrix} z_{st} & -z_{ss} \\ z_{tt} & -z_{st} \end{pmatrix}$ has trace 0,

$$\widetilde{T}(\theta) = \frac{T(\theta)}{P(\theta)^{1/m}} \quad \text{and} \quad \begin{pmatrix} \widetilde{F}^j(\rho,\theta) \\ \widetilde{G}^j(\rho,\theta) \\ \widetilde{H}^j(\rho,\theta) \end{pmatrix} = \begin{pmatrix} F^j\left(\frac{\rho}{P(\theta)^{1/m}},\theta\right) \\ G^j\left(\frac{\rho}{P(\theta)^{1/m}},\theta\right) \\ H^j\left(\frac{\rho}{P(\theta)^{1/m}},\theta\right) \end{pmatrix}. \tag{4.7}$$

Remark 4.1. For $j = 1$, $F^1 = G^1 = H^1 = 0$, and the homogenous system (4.6) is studied in [M3]. It is proved that for every $n \in \mathbb{N}$, the surface S has nontrivial first order infinitesimal bendings fields $U^1 \in C^n(\mathbb{R}^2)$ of the form

$$U^1(r,\theta) = \left(r^{\lambda_p}a^1(\theta), r^{\lambda_p}b^1(\theta), r^{\lambda_p+1-m}c^1(\theta)\right), \tag{4.8}$$

with $a^1, b^1, c^1 \in C^\infty(S^1)$ and λ_p an eigenvalue of the system

$$X'\theta) = \lambda\Lambda(\theta)X(\theta). \tag{4.9}$$

With the first bending U^1 as given by (4.8), it follows from (2.5) that

$$\begin{aligned} F^2(r,\theta) &= r^{2\lambda_p-2}f_1(\theta) + r^{2\lambda_p-2m}f_2(\theta), \\ G^2(r,\theta) &= r^{2\lambda_p-2}g_1(\theta) + r^{2\lambda_p-2m}g_2(\theta), \\ H^2(r,\theta) &= r^{2\lambda_p-2}h_1(\theta) + r^{2\lambda_p-2m}h_2(\theta), \end{aligned} \tag{4.10}$$

where f_i, g_i, $h_i \in C^\infty(S^1)$ ($i = 1,2$). For such expressions of F^2, G^2 and H^2, equation (4.6) becomes

$$\frac{1}{\rho}\begin{pmatrix} \varphi^2 \\ \psi^2 \end{pmatrix}_\theta = \Lambda(\theta)\begin{pmatrix} \varphi^2 \\ \psi^2 \end{pmatrix}_\rho + \rho^{2\lambda_p-2}V_1(\theta) + \rho^{2\lambda_p-2m}V_2(\theta), \tag{4.11}$$

with V_1, $V_2 \in C^\infty(S^1,\mathbb{R}^2)$. We seek solutions of (4.11) in the form

$$\begin{pmatrix} \varphi^2 \\ \psi^2 \end{pmatrix} = \rho^{2\lambda_p-1}X_1(\theta) + \rho^{2\lambda_p-2m+1}X_2(\theta), \tag{4.12}$$

with $X_1,\, X_2 \in C^\infty(S^1, \mathbb{R}^2)$. This leads to the following equations for $X_1,\, X_2$:

$$\frac{dX_1}{d\theta} = (2\lambda_p - 1)\Lambda(\theta)X_1 + V_1(\theta) \qquad \frac{dX_2}{d\theta} = (2\lambda_p - 2m + 1)\Lambda(\theta)X_2 + V_2(\theta)\,. \tag{4.13}$$

It follows from the classical theory of differential equations with periodic coefficients (see Section 2.9 of [YS] for instance) that if V_1 and V_2 are not zero and if both $(2\lambda_p - 1)$ and $(2\lambda_p - 2m+1)$ are not eigenvalues of the periodic system (4.9), then (4.13) has periodic solutions X_1 and X_2 (note that if $V_i = 0$, the corresponding equation has a trivial periodic solution). The next step is to understand the asymptotic behavior of the spectrum of (4.9) and show that there exist arbitrarily large $p \in \mathbb{N}$ such that $(2\lambda_p - 1)$ and $(2\lambda_p - 2m + 1)$ are indeed not eigenvalues of (4.9).

Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and write system (4.9) in the standard Hamiltonian form

$$JX'(\theta) = \lambda H(\theta)X(\theta), \tag{4.14}$$

with

$$H(\theta) = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = J\Lambda = \frac{1}{(m^2 - m)r^{m-2}P(\theta)} \begin{pmatrix} z_{tt} & -z_{st} \\ -z_{st} & z_{ss} \end{pmatrix}. \tag{4.15}$$

The assumption on the curvature K given by (4.2) implies that the matrix H is positive except possibly at points $\theta_1, \cdots, \theta_\ell \in S^1$. It follows then from the Oscillation Theorem (see Theorem V in [YS] pg. 766) that the spectrum (4.14) (or equivalently (4.9)) consists of a sequence

$$\lambda_1^- \le \lambda_1^+ < \lambda_2^- \le \lambda_2^+ < \ldots \lambda_j^- \le \lambda_j^+ < \ldots, \tag{4.16}$$

with $\lim\limits_{j\to\infty} \lambda_j^\pm = \infty$. To get the asymptotic behavior of $\lambda_j^\pm$ we introduce the following functions:

$$b(\theta) = \sqrt{\det(H(\theta))} = \frac{1}{(m^2 - m)r^{m-2}P(\theta)}\sqrt{z_{tt}z_{ss} - z_{st}^2}\,, \tag{4.17}$$

and

$$\begin{aligned} c_1(\theta) &= \frac{-h_{21}}{b}\,\frac{h_{11}'}{h_{11}} = \frac{z_{st}}{\sqrt{z_{tt}z_{ss} - z_{st}^2}}\,\frac{d\log(z_{tt})}{d\theta} \\ c_2(\theta) &= \frac{-h_{21}}{b}\,\frac{h_{22}'}{h_{22}} = \frac{z_{st}}{\sqrt{z_{tt}z_{ss} - z_{st}^2}}\,\frac{d\log(z_{ss})}{d\theta}. \end{aligned} \tag{4.18}$$

Proposition 4.1. *Let b, c_1 and c_2 be as in* (4.17), (4.18) *and suppose that c_1, c_2 are elements of $L^1([0, 2\pi])$. Let*

$$b_1 = \int_0^{2\pi} b(\theta)d\theta \quad \text{and} \quad b_2 = -\frac{1}{4}\int_0^{2\pi} (c_1(\theta) - c_2(\theta))\, d\theta. \tag{4.19}$$

The eigenvalues (4.16) *of* (4.9) *have the following asymptotic behavior as $j \to \infty$:*

$$\lambda_j^\pm = \frac{j\pi}{b_1} + \frac{b_2}{b_1} + O\left(\frac{1}{j}\right). \tag{4.20}$$

Proof. Note that $b(\theta) > 0$ for $\theta \neq \theta_i$, $i = 1, \cdots, \ell$; this implies $b_1 > 0$. For $\varepsilon > 0$, consider the Hamiltonian $H_\varepsilon(\theta) = H(\theta) + \varepsilon I = \begin{pmatrix} h_{11}+\varepsilon & h_{12} \\ h_{21} & h_{22}+\varepsilon \end{pmatrix}$. Since z_{tt} and z_{ss} are nonnegative, then $h_{11} + \varepsilon$ and $h_{22} + \varepsilon$ are strictly positive and

$$\mathrm{Det}(H_\varepsilon) = \mathrm{Det}(H) + \varepsilon \mathrm{Trace}(H) + \varepsilon^2 > 0\,, \quad \forall \theta \in \mathbb{R}\,.$$

Thus matrix H_ε is positive for every $\varepsilon > 0$ and the asymptotic behavior of the spectrum of the equation

$$JX'_\varepsilon = \lambda H_\varepsilon(\theta) X_\varepsilon \tag{4.21}$$

is given by (see p.776 [YS])

$$\lambda^\pm_{\varepsilon j} = \frac{j\pi}{b_1^\varepsilon} + \frac{b_2^\varepsilon}{b_1^\varepsilon} + O\left(\frac{1}{j}\right),$$

where

$$b_1^\varepsilon = \int_0^{2\pi} \sqrt{\mathrm{Det}(H_\varepsilon)}\, d\theta \quad \text{and} \quad b_2^\varepsilon = \frac{-1}{4} \int_0^{2\pi} \frac{h_{12}}{\mathrm{Det}(H_\varepsilon)} \left(\frac{h'_{11}}{h_{11}+\varepsilon} - \frac{h'_{22}}{h_{22}+\varepsilon} \right) d\theta\,.$$

Moreover, the hypothesis $c_1, c_2 \in L^1$ and the Dominated Convergence Theorem imply that $b_1 = \lim_{\varepsilon\to 0^+} b_1^\varepsilon$ and $b_2 = \lim_{\varepsilon\to 0^+} b_2^\varepsilon$. Finally, since the monodromy matrix of (4.21) is analytic with respect to ε, it follows that for j large, $\lambda^\pm_j = \lim_{\varepsilon\to 0^+} \lambda^\pm_{\varepsilon j}$ and the asymptotic expansion (4.20) follows. □

Now we can prove the main results of this section.

Theorem 4.2. *Let $S \subset \mathbb{R}^3$ be the graph of a homogeneous function $z(s,t)$ of order $m \geq 2$ and satisfying* (4.2). *Suppose that conditions in Proposition 4.1 hold and b_1, b_2 given by* (4.19) *satisfy*

$$b_1 - b_2 \notin \pi\mathbb{Z} \quad \textit{and} \quad (2m-1)b_1 - b_2 \notin \pi\mathbb{Z}\,. \tag{4.22}$$

Then for every $k \in \mathbb{N}$, there exist U^1, $U^2 \in C^k(\mathbb{R}^2, \mathbb{R}^3)$ such that the deformation of S given by the position vector

$$R(s,t) + 2\epsilon U^1(s,t) + \epsilon^2 U^2(s,t)$$

is a nontrivial infinitesimal bending of order 2.

Proof. Let $k \in \mathbb{N}$; we already have the first field $U^1 \in C^k$ given by (4.8) (with $p \in \mathbb{N}$ large enough). Now we construct $U^2 = (u^2, v^2, w^2)$ through the functions φ^2, ψ^2 given by (4.12) and satisfying (4.11). Let

$$\delta_1 = \min\left\{ \left| \frac{a\pi}{b_1} + \frac{b_2}{b_1} - 1 \right|,\ a \in \mathbb{Z} \right\}, \quad \delta_2 = \min\left\{ \left| \frac{a\pi}{b_1} + \frac{b_2}{b_1} + 1 - 2m \right|,\ a \in \mathbb{Z} \right\}$$

and $\delta = \min\{\delta_1, \delta_2\}$. It follows from (4.22) that $\delta > 0$. Thanks to the asymptotic expansion (4.20), we can choose p large enough so that

$$\lambda^\pm_p = \frac{p\pi}{b_1} + \frac{b_2}{b_1} + r_p, \tag{4.23}$$

with $|r_p| < \dfrac{\delta}{6}$. Now we show that there is no $q \in \mathbb{N}$ such that

$$2\lambda^\pm_p - 1 = \lambda^\pm_q\,.$$

It follows from (4.20) that

$$2\lambda_p^{\pm} - 1 - \lambda_q^{\pm} = (2p-q)\frac{\pi}{b_1} + \frac{b_2}{b_1} - 1 + 2r_p - r_q.$$

We can assume with no loss of generality that $q > p$ and thus $|r_q| < \dfrac{\delta}{6}$. This implies that

$$\left|2\lambda_p^{\pm} - 1 - \lambda_q^{\pm}\right| \geq \left|(2p-q)\frac{\pi}{b_1} + \frac{b_2}{b_1} - 1\right| - |2r_p - r_q| \geq \frac{\delta}{2} > 0\,.$$

A similar argument shows that there is no $q \in \mathbb{N}$ such that

$$2\lambda_p^{\pm} - 2m + 1 = \lambda_q^{\pm}.$$

As a consequence, the periodic system (4.13) has solutions $X_1(\theta)$, $X_2(\theta)$. We have therefore $\begin{pmatrix}\varphi^2\\ \psi^2\end{pmatrix}$ a solution of (4.11) in the form (4.12). Using (3.7), (3.8), and (3.9), we get

$$w^2 = \frac{\varphi_s^2 + \psi_t^2 - F^2 - H^2}{z_{ss} + z_{tt}} = r^{2\lambda_p - m}\gamma_1(\theta) + r^{2\lambda_p - 3m+2}\gamma_2(\theta), \tag{4.24}$$

and then from (4.3) and (4.24) we obtain

$$\begin{aligned} u^2 &= r^{2\lambda_p - 1}\alpha_1(\theta) + r^{2\lambda_p - 2m+1}\alpha_2(\theta),\\ v^2 &= r^{2\lambda_p - 1}\beta_1(\theta) + r^{2\lambda_p - 2m+1}\beta_2(\theta), \end{aligned} \tag{4.25}$$

with $\alpha_i,\, \beta_i,\, \gamma_i\ \in C^\infty(S^1)$ for $i = 1, 2$. By taking $p \in \mathbb{N}$ large enough, we get $U^1,\, U^2\ \in C^k(\mathbb{R}^2, \mathbb{R}^3)$. □

Theorem 4.3. *Let $S \subset \mathbb{R}^3$ be the graph of a homogeneous function $z(s,t)$ of order $m \geq 2$ and satisfying* (4.2)*. Suppose that conditions in Proposition 4.1 hold and $b_1,\ b_2$ given by* (4.19) *satisfy*

$$(b_2\mathbb{N} + b_1\mathbb{Z}) \cap (mb_1\mathbb{N} + \pi\mathbb{Z}) = \emptyset. \tag{4.26}$$

Then for every $k, l \in \mathbb{N}$, there exist $U^1, \cdots, U^l\ \in C^k(\mathbb{R}^2, \mathbb{R}^3)$ such that the deformation of S given by the position vector

$$R(s,t) + 2\epsilon U^1(s,t) + \cdots 2\epsilon^l U^l(s,t)$$

is a nontrivial infinitesimal bending of order l.

Proof. We use an induction argument on the order l. Suppose that there exist $U^1, \cdots, U^{l-1}\ \in C^k(\mathbb{R}^2, \mathbb{R}^3)$ such that

$$R(s,t) + 2\epsilon U^1(s,t) + \cdots 2\epsilon^{l-1}U^{l-1}(s,t)$$

is a nontrivial infinitesimal bending of order $l-1$ and for $q \in \{1, \cdots, l-1\}$, $U^q(r,\theta) = (u^q(r,\theta), v^q(r,\theta), w^q(r,\theta))$, with

$$\begin{aligned} u^q(r,\theta) &= r^{q\lambda_p - m\nu_1 + \mu_1}\alpha_1(\theta) + \cdots + r^{q\lambda_p - m\nu_\sigma + \mu_\sigma}\alpha_\sigma(\theta),\\ v^q(r,\theta) &= r^{q\lambda_p - m\nu_1 + \mu_1}\beta_1(\theta) + \cdots + r^{q\lambda_p - m\nu_\sigma + \mu_\sigma}\beta_\sigma(\theta),\\ w^q(r,\theta) &= r^{q\lambda_p - m(\nu_1+1) + \mu_1 + 1}\gamma_1(\theta) + \cdots + r^{q\lambda_p - m(\nu_\sigma+1) + \mu_\sigma + 1}\gamma_\sigma(\theta)\,, \end{aligned} \tag{4.27}$$

where $\sigma,\, \nu_1,\, \mu_1, \cdots, \nu_\sigma,\, \mu_\sigma$ are integers that depend only on the index q,
$\alpha_i,\, \beta_i,\, \gamma_i\ \in C^\infty(S^1)$ for $i = 1, \cdots, \sigma$, and where λ_p is a spectral value of the equation (4.10) that can be chosen arbitrarily large. Note that the components of

the bending fields U^1 and U^2 constructed in the proof of Theorem 4.2 have the form given in (4.27) (see (4.8), (4.24), and (4.25)).

It follows from (4.27) that the functions F^l, G^l, and H^l defined by (2.5) have the form

$$\begin{aligned} F^l(r,\theta) &= r^{l\lambda_p - ms_1 + t_1} f_1(\theta) + \cdots + r^{l\lambda_p - ms_\tau + t_\tau} f_\tau(\theta), \\ G^l(r,\theta) &= r^{l\lambda_p - ms_1 + t_1} g_1(\theta) + \cdots + r^{l\lambda_p - ms_\tau + t_\tau} g_\tau(\theta), \\ H^l(r,\theta) &= r^{l\lambda_p - ms_1 + t_1} h_1(\theta) + \cdots + r^{l\lambda_p - ms_\tau + t_\tau} h_\tau(\theta), \end{aligned} \tag{4.28}$$

where τ, s_1, t_1, $\cdots$, s_τ, t_τ are integers that depend only on the index l, and where f_i, g_i, $h_i \in C^\infty(S^1)$ for $i = 1, \cdots, \tau$. With F^l, G^l, and H^l as in (4.28), we construct the next bending field $U^l = (u^l, v^l, w^l)$ through the functions φ^l and ψ^l given by (4.3). In this situation equation, (4.6) becomes

$$\frac{1}{\rho}\begin{pmatrix}\varphi^l \\ \psi^l\end{pmatrix}_\theta = \Lambda(\theta)\begin{pmatrix}\varphi^l \\ \psi^l\end{pmatrix}_\rho + \rho^{l\lambda_p - ms_1 + t_1} V_1(\theta) + \cdots + \rho^{l\lambda_p - ms_\tau + t_\tau} V_\tau(\theta)\,, \tag{4.29}$$

with $V_1, \cdots, V_\tau \in C^\infty(S^1, \mathbb{R}^2)$. We seek solutions of (4.29) in the form

$$\begin{pmatrix}\varphi^l \\ \psi^l\end{pmatrix} = \rho^{l\lambda_p - ms_1 + t_1 + 1} X_1(\theta) + \cdots + \rho^{l\lambda_p - ms_\tau + t_\tau + 1} X_\tau(\theta), \tag{4.30}$$

with $X_1, \cdots, X_\tau \in C^\infty(S^1, \mathbb{R}^2)$. This leads to the following periodic differential equations for the X_i's:

$$X_i'(\theta) = (l\lambda_p - ms_i + t_i + 1)\ \Lambda(\theta)\, X_i(\theta) + V_i(\theta)\,. \tag{4.31}$$

By using the asymptotic expansion (4.23) of the spectral value λ_p (with r_p arbitrarily small for p large enough), an argument similar to that used in the proof of Theorem 4.2 shows that for p large enough, $l\lambda_p - ms_\sigma + t_\sigma + 1$ cannot be a spectral value. Indeed, if there were arbitrarily large $p \in \mathbb{N}$ such that

$$l\lambda_p - ms_\sigma + t_\sigma + 1 = \lambda_q\,,$$

then necessarily

$$(l-1)b_2 + (t_\sigma + 1)b_1 = mb_1 s_\sigma + (q - lp)\pi \ + r_{p,q},$$

with $r_{p,q} \to 0$ when $p, q \to \infty$, which contradicts hypothesis (4.26). This implies that the system of equations (4.31) has a periodic solution and so there exist φ^l, ψ^l as in (4.30) satisfying (4.29). Since $w^l = \dfrac{\varphi^l_s + \psi^l_t - F^l - H^l}{z_{ss} + z_{tt}}$, then it follows from (4.28) and (4.30) that

$$\begin{cases} w^l &= \displaystyle\sum_{j=1}^{l} r^{l\lambda_p - m(s_j+1) + t_j + 2}\gamma_j(\theta)\,, \\ u^l &= \varphi^l - z_s w^l = \displaystyle\sum_{j=1}^{l} r^{l\lambda_p - ms_j + t_j + 1}\alpha_j(\theta)\,, \\ v^l &= \psi^l - z_t w^l = \displaystyle\sum_{j=1}^{l} r^{l\lambda_p - ms_j + t_j + 1}\beta_j(\theta)\,. \end{cases}$$

Thus for a given $k \in \mathbb{N}$, if p is large enough, $U^l = (u^l, v^l, w^l)$ as in (4.27) is in $C^k(\mathbb{R}^2, \mathbb{R}^3)$ and $R(s,t) + 2\displaystyle\sum_{j=1}^{l} \epsilon^j U^j(s,t)$ is an infinitesimal bending of order l. $\square$

Remark 4.4. When $m \in \mathbb{N}$, condition (4.26) reduces to $(b_2\mathbb{N} + b_1\mathbb{Z}) \cap \pi\mathbb{Z} = \emptyset$.

5. Analytic Infinitesimal Bendings of a Class of Surfaces

We describe here the structure of all analytic infinitesimal bendings of a particular class of surfaces given as the graph of functions of the form $s^{m+2} \pm t^{n+2}$, with m, n positive integers. We show that the space of such infinitesimal bendings is generated by four arbitrary analytic functions of one real variable. The basic ingredient needed is the equation in $\mathbb{R}^2$ given by

$$x^m w_{yy} + \varepsilon y^n w_{xx} = 0, \quad \varepsilon = 1 \text{ or } -1. \tag{5.1}$$

To describe the analytic solutions of (5.1) we will need some technical lemmas.

Lemma 5.1. *The double series* $\displaystyle\sum_{m,k\geq 0}\left(\frac{(m+k)!}{m!\,k!}\right)^2 X^k Y^m$ *is uniformly convergent in the bidisc* $|X| < 1/4$ *and* $|Y| < 1/4$.

Proof. Note that $\dfrac{(m+k)!}{m!\,k!} \leq 2^{m+k}$. Therefore

$$\sum_{m,k\geq 0}\left(\frac{(m+k)!}{m!\,k!}\right)^2 |X|^k|Y|^m \leq \sum_{m,k\geq 0}(4|X|)^m(4|Y|)^k$$

and the conclusion follows. ∎

We will use the following notations: D stands for the operator $D = \dfrac{1}{z^m}\dfrac{d^2}{dz^2}$ where z denotes a complex or real variable; and for α, $\beta \in \mathbb{Z}^+$,

$$A^\alpha_\beta = \prod_{k=1}^{\beta}[k(\alpha+2)-1][k(\alpha+2)] \quad \text{and} \quad B^\alpha_\beta = \prod_{k=1}^{\beta}[k(\alpha+2)][k(\alpha+2)+1].$$

We will also denote $A^\alpha_0 = 1$ and $B^\alpha_0 = 1$.

Lemma 5.2. *Let $h(z)$ be a holomorphic function in the disc $D(0,R)$ and let $M(X,Y)$ be the function defined by*

$$M(X,Y) = \sum_{j\geq 0} H_j(X^{m+2})Y^{j(n+2)},$$

where

$$H_j(X^{m+2}) = \frac{1}{A^n_j}D^j\left[h(X^{m+2})\right].$$

There exists a positive constant $C = C(m,n)$ that depends only on m, n such that the function M is holomorphic for $|X| < CR^{1/(m+2)}$ and $|Y| < CR^{1/(n+2)}$. In particular, if h is an entire function in $\mathbb{C}$, then M is an entire function in $\mathbb{C}^2$.

Proof. First note that for $q \geq 0$, $D(X^{q(m+2)}) = [q(m+2)]\,[q(m+2)-1]X^{(q-1)(m+2)}$ and it is easily verified that for $j \leq q$

$$D^j\left(X^{q(m+2)}\right) = \frac{A^m_q}{A^m_{q-j}}X^{(q-j)(m+2)}.$$

Let $0<\rho<R$ be arbitrary. For $j\geq 0$ and $|X|^{m+2}<\rho$ we have

$$\begin{aligned} D^j\left[h(X^{m+2})\right] &= \frac{1}{2\pi i}\int_{|\zeta|=\rho} h(\zeta) D^j\left(\frac{1}{\zeta - X^{m+2}}\right) d\zeta \\ &= \frac{1}{2\pi i}\sum_{q\geq j}\int_{|\zeta|=\rho}\frac{h(\zeta)}{\zeta^{1+q}} D^j(X^{q(m+2)})\, d\zeta \\ &= \frac{1}{2\pi i}\sum_{p\geq 0}\frac{A^m_{p+j}}{A^m_p}\int_{|\zeta|=\rho}\frac{h(\zeta)}{\zeta^{1+j}}\left(\frac{X^{m+2}}{\zeta}\right)^p d\zeta\,. \end{aligned}$$

Thus

$$\left|D^j\left[h(X^{m+2})\right]\right| \leq ||h|| \sum_{p\geq 0}\frac{A^m_{p+j}}{A^m_p}\frac{1}{\rho^j}\left(\frac{|X|^{m+2}}{\rho}\right)^p,$$

where $||h||$ denotes the maximum of h in the disc. It follows that

$$\begin{aligned} |M(X,Y)| &\leq \sum_{j\geq 0}|H_j(X^{m+2})|\,|Y^{n+2}|^j \\ &\leq ||h|| \sum_{j\geq 0}\sum_{p\geq 0}\frac{A^m_{p+j}}{A^m_p A^n_j}\left(\frac{|X|^{m+2}}{\rho}\right)^p\left(\frac{|Y|^{n+2}}{\rho}\right)^j. \end{aligned} \tag{5.2}$$

Now we estimate the coefficient $\dfrac{A^m_{p+j}}{A^m_p A^n_j}$. For α, $\beta\in\mathbb{Z}^+$, we have

$$\begin{aligned} A^\alpha_\beta &= (\alpha+2)^{2\beta}\prod_{k=1}^{\beta}k^2\left(1-\frac{1}{k(\alpha+2)}\right) \leq (\alpha+2)^{2\beta}(\beta!)^2, \\ A^\alpha_\beta &\geq (\alpha+2)^{2\beta}\prod_{k=1}^{\beta}k^2\left(1-\frac{1}{(\alpha+2)}\right) = (\alpha+2)^{\beta}(\alpha+1)^{\beta}(\beta!)^2\,. \end{aligned}$$

These inequalities imply that

$$\frac{A^m_{p+j}}{A^m_p A^n_j} \leq \left(\frac{m+2}{m+1}\right)^p\left(\frac{(m+2)^2}{(n+2)(n+1)}\right)^j\left(\frac{(p+j)!}{p!\,j!}\right)^2. \tag{5.3}$$

It follows from estimates (5.2) and (5.3) that

$$|M(X,Y)| \leq ||h|| \sum_{j,p}\frac{((p+j)!)^2}{(p!)^2\,(j!)^2}\left[\frac{m+2}{m+1}\frac{|X|^{m+2}}{\rho}\right]^p\left[\frac{(m+2)^2}{(n+2)(n+1)}\frac{|Y|^{n+2}}{\rho}\right]^j. \tag{5.4}$$

Finally the conclusion follows from (5.4) and Lemma 5.1 where the constant C can be taken as

$$C = \min\left[\left(\frac{m+1}{4(m+2)}\right)^{1/(m+2)}, \left(\frac{(n+2)(n+1)}{4(m+2)^2}\right)^{1/(n+2)}\right]. \tag{5.5}$$

□

Then we have the following proposition

Proposition 5.1. *Let $R > 0$ and C given in (5.5). A function w is an analytic solution of (5.1) for $|x|, |y| < R$ if and only if there exist h^1, h^2, h^3 and h^4 analytic functions of one real variable t in the interval $|t| < \min\left\{\left(\frac{R}{C}\right)^{m+2}, \left(\frac{R}{C}\right)^{n+2}\right\}$ such that*

$$w(x,y) = \sum_{p=0}^{\infty}\left[H_p^1(x^{m+2}) + xH_p^2(x^{m+2}) + yH_p^3(x^{m+2}) + xyH_p^4(x^{m+2})\right] y^{p(n+2)}\,, \tag{5.6}$$

where

$$\begin{aligned} H_p^i(x^{m+2}) &= \frac{(-\varepsilon)^p}{A_p^n}\, D^p\left(h^i(x^{m+2})\right) \qquad \text{for } i = 1,\ 2\,; \\ H_p^i(x^{m+2}) &= \frac{(-\varepsilon)^p}{B_p^n}\, D^p\left(h^i(x^{m+2})\right) \qquad \text{for } i = 3,\ 4\,, \end{aligned} \tag{5.7}$$

In particular, w is analytic on $\mathbb{R}^2$ if and only if h_1, h_2, h_3 and h_4 are analytic on $\mathbb{R}$.

Proof. Suppose that $w(x,y)$ is an analytic solution of (5.1). We expand w with respect to y as:

$$w(x,y) = \sum_{j=0}^{\infty} \alpha_j(x) y^j\,, \tag{5.8}$$

where the α_j are real analytic functions of x. Equation (5.1) leads to

$$\sum_{j=0}^{n-1}(j+2)(j+1)x^m\alpha_{j+2}(x)y^j + \sum_{j=n}^{\infty}\left[(j+2)(j+1)x^m\alpha_{j+2}(x) + \varepsilon\alpha''_{j-n}(x)\right] y^j = 0.$$

Therefore,

$$\begin{cases} \alpha_k(x) &= 0 \quad \text{for } k = 2, \cdots, n+1\,, \\ \alpha_k(x) &= \dfrac{-\varepsilon}{k(k-1)} D\alpha_{k-(n+2)} \quad \text{for } k \geq n+2\,, \end{cases} \tag{5.9}$$

where D is the operator defined above. It follows at once by induction from (5.9) that $\alpha_k = 0$ whenever k and $k-1 \notin (n+2)\mathbb{Z}^+$.

For $k = n+2$, we have $\alpha_{n+2} = \dfrac{-\varepsilon}{A_1} D\alpha_0$ and then for $k = 2(n+2)$ we get

$$\alpha_{2(n+2)} = \frac{-\varepsilon}{[2(n+2)][2(n+2)-1]} D\alpha_{n+2} = \frac{(-\varepsilon)^2}{A_2} D^2\alpha_0\,.$$

An induction shows that

$$\alpha_{p(n+2)} = \frac{(-\varepsilon)^p}{A_p^n} D^p\alpha_0\,, \quad \alpha_{p(n+2)+1} = \frac{(-\varepsilon)^p}{B_p^n} D^p\alpha_1\,. \tag{5.10}$$

Since each $\alpha_{p(n+2)}$ and $\alpha_{p(n+2)+1}$ is an analytic function of x, then (5.10) imposes restrictions on the power series representations $\alpha_0(x) = \sum a_j x^j$ and $\alpha_1(x) = \sum b_j x^j$. Indeed, we have

$$\alpha_{n+2}(x) = \frac{-\varepsilon}{A_1^n} D\alpha_0(x) = \frac{-\varepsilon}{A_1^n}\sum_{j=2}^{\infty} j(j-1)a_j x^{j-(m+2)},$$

and so $a_2 = \cdots = a_{m+1} = 0$. This gives

$$a_{n+2}(x) = \frac{-\varepsilon}{A_1^n} \sum_{k=m+2}^{\infty} [k + (m+2)]\,[k + (m+1)]\, a_{k+(m+2)} x^k \,.$$

By repeating the above argument and an induction on the relation $\alpha_{(p+1)(n+2)} = MD\alpha_{p(n+2)}$ (with M nonzero constant), we can prove that

$$a_j = 0 \text{ if } j \neq q(m+2) \text{ and } j \neq q(m+2)+1 \text{ with } q \in \mathbb{Z}\,.$$

We have then

$$\alpha_0(x) = \sum_{r=0}^{\infty} a_{r(m+2)} x^{r(m+2)} + \sum_{r=0}^{\infty} a_{r(m+2)+1} x^{r(m+2)+1}\,.$$

Let $h^1(t) = \sum_{r=0}^{\infty} a_{r(m+2)} t^r$ and $h^2(t) = \sum_{r=0}^{\infty} a_{r(m+2)+1} t^r$. Then

$$\alpha_0(x) = h^1(x^{m+2}) + x h^2(x^{m+2})\,. \tag{5.11}$$

Similar arguments show that there exist analytic functions $h^3(t)$ and $h^4(t)$ such that

$$\alpha_1(x) = h^3(x^{m+2}) + x h^4(x^{m+2})\,. \tag{5.12}$$

Expression (5.6) of the proposition follows from (5.8), (5.9), (5.10), (5.11), and (5.12) and the convergence follows from Lemma 5.2. Conversely, given analytic functions h^1, h^2, h^3 and h^4 as in the statement of the proposition, it is clear that the function $w(x, y)$ defined by (5.6) is an analytic solution of the (5.1) □

For $m, n \in \mathbb{Z}^+$, let $S_{m,n} \subset \mathbb{R}^3$ be the surface given by

$$S_{m,n} = \left\{(s, t, s^{m+2} + \varepsilon t^{n+2}), \quad (s, t) \in \mathbb{R}^2\right\}\,.$$

Denote by $\mathrm{IB}_{m,n}(\rho)$ the space of real analytic infinitesimal bendings of $S_{m,n}$ in the square $|s| < \rho$, $|t| < \rho$, with $0 < \rho \leq \infty$, and by $\mathcal{A}(\rho)$ the space of $\mathbb{R}$-valued real analytic functions in the interval $(-\rho\,,\ \rho)$. Then we have the following theorem.

Theorem 5.1. $\mathrm{IB}_{m,n}(\rho)$ *is isomorphic to* $\mathcal{A}(\rho)^4$.

Proof. Let $U(s, t) = (u(s, t), v(s, t), w(s, t)) \in \mathrm{IB}_{m,n}(\rho)$. In this particular case the system of equations (2.4) becomes

$$\begin{aligned} u_s + (m+2)s^{m+1} w_s &= 0 \\ u_t + v_s + (m+2)s^{m+1} w_t + \varepsilon(n+2)t^{n+1} w_s &= 0 \\ v_t + \varepsilon(n+2)t^{n+1} w_t &= 0 \end{aligned}$$

As in [V] we can reduce this system into a single equation for w to obtain

$$(m+2)(m+1)s^m w_{tt} + \varepsilon(n+2)(n+1)t^n w_{ss} = 0\,, \tag{5.13}$$

and any solution of (5.13) gives rise to an infinitesimal bending U. Equation (5.13) is equivalent to (5.1) through the linear change of variables $x = Ps$ and $y = Qt$ where

$$P = [(m+2)(m+1)]^{\frac{n}{mn-4}}\, [(n+2)(n+1)]^{\frac{2}{mn-4}}\,,$$

$$Q = [(n+2)(n+1)]^{\frac{m}{mn-4}}\, [(m+2)(m+1)]^{\frac{2}{mn-4}}\,,$$

when $mn \neq 4$. When $mn = 4$, another simple linear change of variables transforms equation (5.13) to (5.1). Proposition 5.1 establish an isomorphism between real analytic solutions of (5.13) and $\mathcal{A}(C\rho^\mu)^4$ (for some C and μ positive) and thus between $\mathrm{IB}_{m,n}(\rho)$ and $\mathcal{A}(C\rho^\mu)^4$. Finally, since $\mathcal{A}(C\rho^\mu)$ is clearly isomorphic to $\mathcal{A}(\rho)$, this completes the proof. □

Remark 5.2. It should be noted that surfaces given as graphs of functions of the form $f(s) + g(t)$ are bendable and some bendings are given in [D]. However, the above theorem characterizes all real analytic infinitesimal bendings.

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