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Coherent states of inverse oscillators and related problems

V G Bagrov[1], **D M Gitman**[2], **E S Macedo**[2] **and A S Pereira**[2]

[1] Department of Physics, Tomsk State University, Tomsk, Russia
[2] Institute of Physics, University of São Paulo, São Paulo, Brazil

E-mail: bagrov@phys.tsu.ru, gitman@if.usp.br, emacedo@if.usp.br and apereira@if.usp.br

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Abstract
We consider an example of a quantum system with a continuous and unbounded spectrum—a particle in the parabolic potential barrier. Conditionally, we call this system the inverse oscillator. Following the method used in our previous publication (Bagrov *et al* 2012 *J. Phys. A: Math. Theor.* **45** 125306), we construct different families of generalized coherent states (GCS). We discuss properties of the constructed GCS and their relation to what is already known in the literature about semiclassical states for the inverse oscillator. Then in the same manner, we construct families of GCS for the normal oscillator. Using special variables, we succeeded in constructing GCS for the normal oscillator that admit different limiting cases: free particle GCS, the usual Schrödinger, Glauber CS and GCS of the inverse oscillator.

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1. Introduction

Coherent states (CS) play an important role in modern quantum mechanics due to their fundamental theoretical importance and wide range of applications, e.g. in a semiclassical description of quantum systems, in quantization theory, in condensed matter physics, in radiation theory, in quantum computations and so on, see, e.g. [1]. It is well-known that a universal definition of CS and a constructive scheme of their construction for arbitrary physical systems is absent. Due to Glauber, Malkin, Man'ko and Dodonov [2–4] there exists a well-developed scheme of constructing CS for systems with quadratic Hamiltonians with discrete spectra and due to Perelomov [5] for systems with a given Lie group. Some nontrivial generalizations of the Glauber approach were developed by Klauder and Gazeau (see [6]). When constructing CS, one always tries to maintain the basic properties of already known CS for simple systems. In particular, CS have to form a complete system, they have to minimize uncertainty relations for some physical quantities (e.g. coordinates and momenta) at a fixed time instant, which means that some physical quantities, calculated with respect to time-dependent CS, have to move along the corresponding classical trajectories. It is also desirable for time-dependent CS to maintain their form under the time evolution, such that this evolution affects only their parameters.

In the literature [7–11], different approaches to constructing CS for systems with a continuous and unbounded spectrum were considered. In particular, some kinds of CS for the inverse oscillator were constructed in [12, 13]. The work of Barton [12] is mainly devoted to the quantum mechanics of a particle in the inverted oscillator potential. In this work, Barton defines CS of the inverse oscillator acting by a displacement operator on stationary states of the corresponding Hamiltonian. As a result, such CS are not normalizable and difficulties exist with proving their completeness. The Barton construction was essentially improved in the work by Klauder *et al* [13]. They constructed CS acting by the displacement operator on some square-integrable combinations of the stationary states. The choice of these combinations plays the role of a regularization. Such constructed CS are already normalized and, for some of the chosen regularizations, form complete systems. In the latter approach, one has infinite possibilities to choose the regularization; the procedure of lifting the regularization seems not to be simple in the general case.

In this paper we use a quite different method to construct families of generalized CS (GCS) for the inverse oscillator. Our construction of GCS is based on a simple fact that the Hilbert space is separable, so that a discrete oscillator-like basis exists there. This fact does not contradict the fact that we consider systems with continuous spectra. Then, even for systems with continuous spectra, one can always construct complete discrete sets of solutions of the Schrödinger equation. In turn, this allows one to introduce creation and annihilation operators such that the latter are integrals of motion (this part of the construction uses the Dodonov–Man'ko method for constructing integrals of motion, see [4]). With the help of these operators, one constructs a Fock space and already in such a space constructs GCS. Then we discuss properties of the constructed GCS and their relation with the CS of the normal oscillator.

2. Generalized coherent states of the inverse oscillator

Consider a one-dimensional particle moving in the parabolic potential barrier. Its classical Hamilton function is

$$H = \frac{P^2}{2m} - \frac{m\omega_0^2 x^2}{2}, \quad x \in \mathbb{R}, \tag{1}$$

where x, P are the coordinate and the momentum of the particle. Usually, such a system is called the inverse oscillator (by the analogy with the well-known oscillator for which the potential in the rhs of (1) is positive) in spite of the fact that it does not represent any oscillator motion. The parabolic potential implies an unbounded motion and a continuous energy spectrum in the quantum case.

The general solution of classical Hamilton equations has the form

$$\begin{aligned} x(t) &= x_0 \cosh\omega_0 t + \frac{P_0}{m\omega_0}\sinh\omega_0 t, \\ P(t) &= m x_0\omega_0 \sinh\omega_0 t + P_0\cosh\omega_0 t, \end{aligned} \tag{2}$$

where x_0 and P_0 are initial data for the phase-space variables at $t = 0$.

The quantum motion of the IO is described by the Schrödinger equation

$$\mathrm{i}\hbar\partial_t\Psi(x,t) = \mathcal{H}\Psi(x,t), \quad \mathcal{H} = -\frac{\hbar^2}{2m}\partial_x^2 - \frac{m\omega_0^2 x^2}{2}. \tag{3}$$

In the dimensionless variables q and τ,

$$q = \sqrt{\frac{m\omega_0}{\hbar}}x, \quad \tau = \frac{1}{2}\omega_0 t,$$

equation (3) takes the following form

$$\hat{S}\Psi = 0, \quad \hat{S} = \mathrm{i}\partial_\tau - \hat{H}, \quad \hat{H} = -\left(\partial_q^2 + q^2\right). \tag{4}$$

Let us introduce creation $\hat{a}^\dagger$ and annihilator $\hat{a}$ operators ($[\hat{a}, \hat{a}^\dagger] = 1$) as follows:

$$\hat{a} = \frac{q + \partial_q}{\sqrt{2}}, \quad \hat{a}^\dagger = \frac{q - \partial_q}{\sqrt{2}};$$
$$q = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}, \quad \partial_q = \frac{\hat{a} - \hat{a}^\dagger}{\sqrt{2}}. \tag{5}$$

Being written in terms of these operators, the Hamiltonian (4) takes the second canonical form for a quadratic combination of creation and annihilation operators,

$$\hat{H} = -(\hat{a}^2 + \hat{a}^{\dagger\,2}). \tag{6}$$

The rhs of equation (6) cannot be reduced to the first canonical form for a quadratic combination of creation and annihilation operators, which is the oscillator-like form, by any canonical transformation; this indicates that the spectrum of $\hat{H}$ is continuous and unbounded from below, see e.g. [14].

However, we can construct the corresponding Fock space and a discrete basis, which is useful for our consideration. To this end, we first construct the operator

$$\hat{A}(\tau) = f(\tau)\hat{a} + g(\tau)\hat{a}^\dagger + \varphi(\tau), \tag{7}$$

where $f(\tau)$, $g(\tau)$ and $\varphi(\tau)$ are some functions of τ, such that the operators $\hat{A}(\tau)$ and $\hat{A}^\dagger(\tau)$ are the integrals of motion of equation (4). This means that the operators $\hat{A}(\tau)$ and $\hat{A}^\dagger(\tau)$ have to obey the condition[3]

$$[\hat{S}, \hat{A}(\tau)] = [\hat{S}, \hat{A}^\dagger(\tau)] = 0. \tag{8}$$

One can easily see that equations (8) hold true if the functions $f(\tau)$, $g(\tau)$ and $\varphi(\tau)$ are solutions of the following set of equations

$$\mathrm{i}\dot{f} + 2g = 0, \quad \mathrm{i}\dot{g} - 2f = 0, \quad \mathrm{i}\dot{\varphi} = 0, \tag{9}$$

where $\dot{\xi} = \partial_\tau \xi$. The general solution of these equations reads

$$f(\tau) = c_1 \cosh 2\tau + \mathrm{i}c_2 \sinh 2\tau,$$
$$g(\tau) = c_2 \cosh 2\tau - \mathrm{i}c_1 \sinh 2\tau, \quad \varphi(\tau) = c_3, \tag{10}$$

where c_j, $j = 1, 2, 3$ are arbitrary complex constants. Without loss of generality, we can set $c_3 = 0$.

It follows from equations (5) and (10) that

$$[\hat{A}(\tau), \hat{A}^\dagger(\tau)] = \delta = |f(\tau)|^2 - |g(\tau)|^2 = |c_1|^2 - |c_2|^2. \tag{11}$$

For $\delta = 0$, the operator $\hat{A}(\tau)$ can be considered as a self-adjoint one.

[3] One can easily see that the well-known condition

$$\frac{1}{\mathrm{i}\hbar}[\hat{A}, \hat{H}] + \frac{\partial \hat{A}}{\partial t} = 0$$

for an operator $\hat{A}$ to be an integral of motion can be rewritten in the following form

$$\left[\mathrm{i}\hbar\frac{\partial}{\partial t} - \hat{H}, \hat{A}\right] = 0.$$

For $\delta > 0$, without loss of generality, we can set $\delta = 1$, which corresponds to the multiplication of $\hat{A}$ by a complex number. In this case, at any time instant τ, the operators $\hat{A}^\dagger(\tau)$ and $\hat{A}(\tau)$ are creation and annihilation operators, respectively, i.e.,

$$[\hat{A}(\tau), \hat{A}^\dagger(\tau)] = 1 \forall\ \tau. \tag{12}$$

For $\delta < 0$, one has to treat $\hat{B} = \hat{A}^\dagger$ as an annihilation operator and we return to the previous case $\delta > 0$.

For our purposes, we have to consider the case $\delta = 1$, which implies equation (12). Here a family of the operators $\hat{A}(\tau) = \hat{A}(\tau, c_1, c_2)$ is parametrized by complex numbers c_1 and c_2 that are restricted by the condition $|c_1|^2 - |c_2|^2 = 1$.

Using equation (7), we can express the operators $\hat{a}$ and $\hat{a}^\dagger$ via the operators $\hat{A}$ and $\hat{A}^\dagger$

$$\begin{aligned} \hat{a} &= f^*(\tau)\hat{A}(\tau) - g(\tau)\hat{A}^\dagger(\tau), \\ \hat{a}^\dagger &= f(\tau)\hat{A}^\dagger(\tau) - g^*(\tau)\hat{A}(\tau), \end{aligned} \tag{13}$$

and using equations (5), we can express the operators $\hat{q}$ and $\hat{p} = -\mathrm{i}\partial_q$ in term of the operators $\hat{A}$ and $\hat{A}^\dagger$

$$\hat{q} = \frac{(f-g)\hat{A}^\dagger + (f^* - g^*)\hat{A}}{\sqrt{2}}, \quad \hat{p} = \frac{(f^* + g^*)\hat{A} - (f+g)\hat{A}^\dagger}{\mathrm{i}\sqrt{2}}. \tag{14}$$

One can easily verify that expressions in the rhs of equations (13) and (14) do not depend on time due to equations (10).

GCS can be constructed as eigenfunctions of the annihilation operator $\hat{A}(\tau)$ that, at the same time, obey the Schrödinger equation (4).

First let us consider eigenvectors $|z, \tau\rangle$ of the annihilation operator $\hat{A}(\tau)$ corresponding to the eigenvalue z

$$\hat{A}(\tau)|z, \tau\rangle = z|z, \tau\rangle. \tag{15}$$

Taking the operators $\hat{a}^\dagger$ and $\hat{a}$ that enter into expression (7) in the coordinate representation (4), we obtain the first order differential equation for the eigenvectors $|z, \tau\rangle$ in the q-representation

$$[(f+g)q + (f-g)\partial_q]\langle q|z, \tau\rangle = \sqrt{2}z\langle q|z, \tau\rangle.$$

It has the following solution

$$\langle q|z, \tau\rangle = \exp\left[-\frac{1}{2}\frac{f+g}{f-g}(q - \langle q\rangle)^2 + \mathrm{i}\langle p\rangle q + \mathrm{i}\phi(\tau)\right], \tag{16}$$

where

$$\begin{aligned} \langle q\rangle &\equiv \langle z, \tau|\hat{q}|z, \tau\rangle = \frac{(f-g)z^* + (f^* - g^*)z}{\sqrt{2}} = q_0 \cosh 2\tau + p_0 \sinh 2\tau, \\ \langle p\rangle &\equiv \langle z, \tau|\hat{p}|z, \tau\rangle = \frac{(f^* + g^*)z - (f+g)z^*}{\mathrm{i}\sqrt{2}} = q_0 \sinh 2\tau + p_0 \cosh 2\tau, \\ q_0 &= \langle q\rangle|_{\tau=0} = \frac{1}{\sqrt{2}}[(c_1 - c_2)z^* + (c_1^* - c_2^*)z], \\ p_0 &= \langle p\rangle|_{\tau=0} = \frac{\mathrm{i}}{\sqrt{2}}[(c_1 + c_2)z^* - (c_1^* + c_2^*)z]. \end{aligned}$$

In fact, we have a family of states $|z, \tau\rangle$ parametrized by the complex numbers c_1 and c_2 for which $|c_1|^2 - |c_2|^2 = 1$. For a given set, c_1 and c_2, there is a one-to-one correspondence between the complex number z and the initial data q_0 and p_0

$$z = \frac{1}{\sqrt{2}}[(c_1 + c_2)q_0 + \mathrm{i}(c_1 - c_2)p_0].$$

The factor function $\phi(\tau)$ has to be determined such that the states (16) satisfy the Schrödinger equation (4) and are then normalized. The first step is to find $\phi(\tau)$ from the Schrödinger equation (4)

$$\phi(\tau) = \frac{\mathrm{i}}{2}\ln(f-g) - \frac{1}{2}\langle p\rangle\langle q\rangle + C. \tag{17}$$

Then we find the constant C from the normalization condition

$$\langle z,\tau|z,\tau\rangle = 1 \Rightarrow C = \frac{\mathrm{i}}{4}\ln\pi. \tag{18}$$

Thus, we obtain the coordinate representation for the states $|z,\tau\rangle$

$$\begin{aligned}\langle q|z,\tau\rangle &= \frac{1}{\sqrt{\sqrt{\pi}(f-g)}}\exp\left[-\frac{1}{2}\frac{f+g}{f-g}(q-\langle q\rangle)^2 + \frac{\mathrm{i}}{2}\langle p\rangle(2q-\langle q\rangle)\right]\\ &= \frac{1}{\sqrt{\sqrt{\pi}(f-g)}}\exp\left[-\frac{1}{2}\frac{f+g}{f-g}\left(q-\frac{\sqrt{2}z}{f+g}\right)^2 + \frac{f^*+g^*}{f+g}\frac{z^2}{2} - \frac{|z|^2}{2}\right].\end{aligned} \tag{19}$$

We call such defined states the GCS of the inverse oscillator. As has already been mentioned, in fact, we deal with a family of states parametrized by two complex numbers c_1 and c_2 that obey the restriction $|c_1|^2 - |c_2|^2 = 1$.

Let us consider a technically different way of constructing the GCS. Here, we construct the vacuum state first, which is $|0,\tau\rangle$. The expression for this state follows from equation (19) at $z = 0$

$$\langle q|0,\tau\rangle = \frac{1}{\sqrt{\sqrt{\pi}(f-g)}}\exp\left(-\frac{f+g}{f-g}\frac{q^2}{2}\right). \tag{20}$$

By its method of construction, this state satisfies the Schrödinger equation, $\hat{S}|0,\tau\rangle = 0$, and is annihilated by the annihilation operator $\hat{A}(\tau)\,|0,\tau\rangle = 0$.

Then, using the creation operator $\hat{A}^\dagger(\tau)$, we construct a discrete system $|n,\tau\rangle$ as follows

$$|n,\tau\rangle = \frac{[\hat{A}^\dagger(\tau)]^n}{\sqrt{n!}}|0,\tau\rangle. \tag{21}$$

This system is complete at any fixed time instant τ,

$$\sum_{n=0}^{\infty}|n,\tau\rangle\langle n,\tau| = 1, \tag{22}$$

and each member of the system obeys the Schrödinger equation. Indeed, the vacuum state obeys this equation and since the operator $\hat{A}^\dagger(\tau)$ commutes with the operator $\hat{S}$, the state $|n,\tau\rangle$ satisfies the Schrödinger equation as well

$$\hat{S}|n,\tau\rangle = \frac{[\hat{A}^\dagger(\tau)]^n}{\sqrt{n!}}\hat{S}|0,\tau\rangle = 0.$$

Then we can construct the GCS as follows

$$\mathcal{D}(z)|0,\tau\rangle = \exp\left(-\frac{|z|^2}{2}\right)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n,\tau\rangle, \tag{23}$$

where the displacement operator $\mathcal{D}(z)$ has the form

$$\mathcal{D}(z) = \exp[z\hat{A}^\dagger(\tau) - z^*\hat{A}(\tau)]. \tag{24}$$

The displacement operator is a unitary operator and an integral of motion due to equations (8), $[\hat{S},\mathcal{D}(z)] = 0$. This is why the states (23) obey the Schrödinger equation, $\hat{S}|z,\tau\rangle = 0$, and are normalized. In addition, using the properties of the displacement operator, one can easily see that the states (23) are eigenvectors of the annihilation operator, $\hat{A}(\tau)|z,\tau\rangle = z|z,\tau\rangle$. This is why the set $|z,\tau\rangle$ is an analogue of the GCS (16). Indeed, calculating the sum in equation (23) we obtain

$$\mathcal{D}(z)\langle q|0,\tau\rangle = \mathrm{e}^{-\frac{|z|^2}{2}}\sum_{n=0}^{\infty}\frac{[z\hat{A}(\tau)]^n}{n!}\langle q|0,\tau\rangle = \frac{\mathrm{e}^{-|z|^2/2}}{\sqrt{\sqrt{\pi}(f-g)}} \times \exp\left(-\frac{f+g}{f-g}\frac{q^2}{2}\right)\sum_{n=0}^{\infty}\left(\sqrt{\frac{f^*-g^*}{f-g}}\frac{z}{\sqrt{2}}\right)^n\frac{1}{n!}H_n\left(\frac{q}{|f-g|}\right),$$

where $H_n(x)$ are Hermite polynomials. Then

$$\mathcal{D}(z)\langle q|0,\tau\rangle = \frac{1}{\sqrt{\sqrt{\pi}(f-g)}}\exp\left[-\frac{1}{2}\frac{f+g}{f-g}\left(q-\frac{\sqrt{2}z}{f+g}\right)^2 + \frac{f^*+g^*}{f+g}\frac{z^2}{2} - \frac{|z|^2}{2}\right]. \quad (25)$$

The rhs of equation (25) coincides with the rhs of (19) such that

$$|z,\tau\rangle = \mathcal{D}(z)|0,\tau\rangle. \quad (26)$$

The constructed GCS $|z,\tau\rangle$ at any fixed τ obey the usual completeness relations in the Hilbert space of functions that depend on q

$$\int\int\langle q|z,\tau\rangle\langle z,\tau|q'\rangle\,\mathrm{d}^2z = \pi\delta(q-q'), \quad \mathrm{d}^2z = \mathrm{d}\,\mathrm{Re}\,z\,\mathrm{d}\,\mathrm{Im}\,z,\ \forall\tau. \quad (27)$$

The overlap of two GCS with different z reads

$$\langle z',\tau|z,\tau\rangle = \exp\left(z'^*z - \frac{|z'|^2+|z|^2}{2}\right). \quad (28)$$

Let us calculate variances $\sigma_i,\ i=1,2,3$ in the state $|z,\tau\rangle$. To this end, we use relations between the operators $\hat{q}$ and $\hat{p} = -\mathrm{i}\partial_q$, and the creation and annihilation operators $\hat{A}^\dagger(t)$ and $\hat{A}(t)$, which follow from (14). We have

$$\sigma_1 \equiv \langle(\hat{\Delta}q)^2\rangle = \frac{|f-g|^2}{2}, \quad \hat{\Delta}q = \hat{q} - \langle q\rangle,$$
$$\sigma_2 \equiv \langle(\hat{\Delta}p)^2\rangle = \frac{|f+g|^2}{2}, \quad \hat{\Delta}p = \hat{p} - \langle p\rangle,$$
$$\sigma_3 \equiv \frac{1}{2}\langle\hat{\Delta}q\hat{\Delta}p + \hat{\Delta}p\hat{\Delta}q\rangle = \frac{\mathrm{i}}{2}(gf^* - g^*f). \quad (29)$$

It is well-known that condition $\delta = 1$ can be satisfied by choosing

$$c_1 = \mathrm{e}^{\mathrm{i}\beta}\cosh\alpha, \quad c_2 = \mathrm{e}^{-\mathrm{i}\beta}\sinh\alpha, \quad (30)$$

where α and β are arbitrary real constants. In these terms we obtain

$$2\sigma_1 = \sin^2\left(\frac{\pi}{4}+\beta\right)\cosh(4\tau+2\alpha) + \cos^2\left(\frac{\pi}{4}+\beta\right)\cosh(4\tau-2\alpha) - \sinh 2\alpha\cos 2\beta,$$
$$2\sigma_2 = \sin^2\left(\frac{\pi}{4}+\beta\right)\cosh(4\tau+2\alpha) + \cos^2\left(\frac{\pi}{4}+\beta\right)\cosh(4\tau-2\alpha) + \sinh 2\alpha\cos 2\beta,$$
$$2\sigma_3 = \sin^2\left(\frac{\pi}{4}+\beta\right)\sinh(4\tau+2\alpha) + \cos^2\left(\frac{\pi}{4}+\beta\right)\sinh(4\tau-2\alpha). \quad (31)$$

It is seen from (31) that for any $c_{1,2}$ (α and β) the variances grow exponentially with time. However, in all the constructed GCS, the Schrödinger–Robertson uncertainty relation [15] is minimized,

$$\sigma_1\sigma_2 - \sigma_3^2 = 1/4, \quad (32)$$

such that we deal with some kind of squeezed states. But the Heisenberg uncertainty relation is not minimized and the product $\sigma_1\sigma_2$ depends on time.

3. GCS of the normal oscillator and their relations to GCS of the free particle and inverse oscillator

3.1. Preliminary construction

Here we construct GCS of the normal oscillator and study their relation to the GCS of the inverse oscillator and to free particle GCS. Here, in doing this, we mainly use the coordinate representation. For generality, we add a linear term that describes an external force into the Hamiltonian. The corresponding Schrödinger equation has the form

$$\mathrm{i}\hbar\partial_t\Psi(x,t)=\hat{H}\Psi(x,t),\quad \hat{H}=-\frac{\hbar^2}{2m}\partial_x^2+\alpha x+\frac{m\omega_0^2x^2}{2}. \tag{33}$$

Here $x,\ t$ are the position and time of the particle with the mass m. It is convenient to introduce dimensionless coordinate q, time τ, and constants b and ω as follows

$$q=xl^{-1},\quad \tau=\frac{\hbar}{2ml^2}t,\quad b=\frac{\sqrt{2}\alpha ml^3}{\hbar^2},\quad \omega^2=\frac{m^2\omega_0^2l^4}{\hbar^2}, \tag{34}$$

so that $\omega_0 t=2\omega\tau$. In the new variables equation (33) takes the form

$$\hat{S}\psi(\tau;q)=0,\quad \hat{S}=\mathrm{i}\partial_\tau-\hat{H}, \tag{35}$$

where

$$\hat{H}=-\partial_q^2+\omega^2q^2+\sqrt{2}bq,\qquad \Psi(x,t)=\psi(q,\tau). \tag{36}$$

Then we introduce creation and annihilation operators by equations (5). In terms of these operators, the Hamiltonian $\hat{H}$ takes the form

$$\hat{H}=\tfrac{1}{2}[(1+\omega^2)(\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a})-(1-\omega^2)(\hat{a}^2+\hat{a}^{\dagger 2})]+b(\hat{a}+\hat{a}^\dagger). \tag{37}$$

Then as before, we construct the integral of motion $\hat{A}(\tau)$ according to equation (7). In the case under consideration, we have the following equations for the functions $f(\tau), g(\tau)$, and $\varphi(\tau)$,

$$\mathrm{i}\dot{f}+(1+\omega^2)f+(1-\omega^2)g=0,\quad \mathrm{i}\dot{g}-(1-\omega^2)f-(1+\omega^2)g=0,\quad \mathrm{i}\dot{\varphi}+b(f-g)=0.$$

They can be represented as follows:

$$\dot{f}+\dot{g}=2\mathrm{i}\omega^2(f-g),\quad \dot{f}-\dot{g}=2\mathrm{i}(f+g), \tag{38}$$

such that $J=(f+g)^2-\omega^2(f-g)^2=\text{const}$ is the first integral of the set (38). The general solution of the latter set has the form

$$\begin{aligned}
f(\tau)&=c_1\cos 2\omega\tau+\mathrm{i}\big[c_1+c_2+\omega^2(c_1-c_2)\big]\frac{\sin 2\omega\tau}{2\omega},\\
g(\tau)&=c_2\cos 2\omega\tau-\mathrm{i}\big[c_1+c_2-\omega^2(c_1-c_2)\big]\frac{\sin 2\omega\tau}{2\omega},\\
\varphi(\tau)&=b\frac{\sin\omega\tau}{\omega}\left[\mathrm{i}(c_1-c_2)\cos\omega\tau-(c_1+c_2)\frac{\sin\omega\tau}{\omega}\right]+c_3;\\
J&=(c_1+c_2)^2-\omega^2(c_1-c_2)^2.
\end{aligned} \tag{39}$$

Here $c_j,\ j=1,2,3$ are arbitrary complex constants. Without loss of generality, we can set $c_3=0$.

Note that nontrivial solutions of the set (38) that satisfy the condition $f(\tau)=g(\tau)$ do not exist. Therefore, if $\hat{A}\neq 0$ then $f-g\neq 0$.

Another important observation is: the introduced variables allow one to consider the limit $\omega\to 0$, which corresponds to the limit $\omega_0\to 0$. Therefore, we can obtain a free particle case in the solutions (39), which is nontrivial for the oscillator problem.

One can see that the functions (39) obey the following useful relations

$$
\begin{aligned}
&f(\tau)+g(\tau)=(c_1+c_2)\cos 2\omega\tau+\mathrm{i}(c_1-c_2)\omega\sin 2\omega\tau,\\
&f(\tau)-g(\tau)=(c_1-c_2)\cos 2\omega\tau+\mathrm{i}(c_1+c_2)\frac{\sin 2\omega\tau}{\omega},\\
&\varphi(\tau)=\frac{b}{2\omega^2}[f(\tau)+g(\tau)-c_1-c_2],\quad |f(\tau)|^2-|g(\tau)|^2=|c_1|^2-|c_2|^2=\delta,\\
&\frac{f+g}{2(f-g)}+\frac{f^*+g^*}{2(f^*-g^*)}=\frac{1}{|f-g|^2},\quad \frac{\varphi}{(f-g)}+\frac{\varphi^*}{(f^*-g^*)}=\frac{2b}{|f-g|^2}\left(\frac{\sin\omega t}{\omega}\right)^2. \qquad (40)
\end{aligned}
$$

If $\delta > 0$, then without loss of generality (multiplying $\hat{A}$ by a complex number), we can set $\delta = 1$. In this case $\hat{A}^\dagger$ and $\hat{A}$ are creation and annihilation operators.

For $\delta = 0$, the operator $\hat{A}(\tau)$ can be considered as a self-adjoint one. For $\delta > 0$, without loss of generality, we can set $\delta = 1$, which corresponds to the multiplication of $\hat{A}$ by a complex number. In this case, at any time instant τ, the operators $\hat{A}^\dagger(\tau)$ and $\hat{A}(\tau)$ are creation and annihilation operators, respectively, i.e., equation (12) holds true. For $\delta < 0$, one has to treat $\hat{B} = \hat{A}^\dagger$ as an annihilation operator and we return to the previous case $\delta > 0$. As in the previous section, we will consider the case $\delta = 1$, which implies equation (12). Here a family of the operators $\hat{A}(\tau) = \hat{A}(\tau, c_1, c_2)$ is parametrized by complex numbers c_1 and c_2 that are restricted by the condition $|c_1|^2 - |c_2|^2 = 1$.

Following the scheme used in the previous section, first we find eigenvectors $|z,\tau\rangle$ of the annihilation operator $\hat{A}(\tau)$ corresponding to the eigenvalue z, i.e., we solve the equation (15). The corresponding solution has the form

$$
\begin{aligned}
&\langle q|z,\tau\rangle=\psi_z^{c_1,c_2}(q,\tau)=F(\tau)\exp R,\\
&R=\frac{\sqrt{2}(z-\varphi)}{f-g}q-\frac{f+g}{2(f-g)}q^2,
\end{aligned} \qquad (41)
$$

where, at this stage, $F(\tau)$ is an arbitrary function of τ which depends on complex parameters c_1, c_2, z. Further, we are going to determine this function demanding that the eigenvectors $\psi_z^{c_1,c_2}(q,\tau)$ satisfy the Schrödinger equation (35). Such states will be GCS in the case under consideration.

It should be noted that many properties of the GCS can be obtained without knowledge of the specific form of the function $F(\tau)$.

Using equations (40), we find the modulus square of the state (41),

$$
\begin{aligned}
&|\psi_z^{c_1,c_2}(q,\tau)|^2=|F(\tau)|^2\exp(R+R^*),\\
&R+R^*=-\frac{q^2-2qq(\tau)}{|f-g|^2}=-\tilde{q}^2+\frac{q^2(\tau)}{|f-g|^2},\quad \tilde{q}=\frac{q-q(\tau)}{|f-g|},
\end{aligned} \qquad (42)
$$

where the real function $q(\tau)$ reads

$$
\begin{aligned}
q(\tau)=\frac{1}{\sqrt{2}}[(z-\varphi)(f^*-g^*)+(z^*-\varphi^*)(f-g)]=-\sqrt{2}b\left(\frac{\sin\omega\tau}{\omega}\right)^2+\big[(c_1-c_2)z^*\\
+(c_1^*-c_2^*)z\big]\frac{\cos 2\omega\tau}{\sqrt{2}}+\mathrm{i}\big[(c_1+c_2)z^*-(c_1^*+c_2^*)z\big]\frac{\sin 2\omega\tau}{\sqrt{2}\omega}. \qquad (43)
\end{aligned}
$$

As a consequence of equations (40) and (43) we have

$$
\begin{aligned}
&\dot{q}(\tau)=\mathrm{i}\sqrt{2}[(f+g)(z^*-\varphi^*)-(f^*+g^*)(z-\varphi)]=\mathrm{i}\sqrt{2}[(f+g)z^*\\
&\qquad-(f^*+g^*)z]-\sqrt{2}b\frac{\sin 2\omega\tau}{\omega},\qquad \ddot{q}(\tau)=-4\omega^2q(\tau)-2\sqrt{2}b,\\
&\dot{q}^2(t)+4\omega^2q^2(\tau)+4\sqrt{2}bq(\tau)=2[(c_1-c_2)z^*+(c_1^*-c_2^*)z]^2\omega^2\\
&\qquad-2[(c_1+c_2)z^*-(c_1^*+c_2^*)z]^2+4b[(c_1-c_2)z^*+(c_1^*-c_2^*)z]=\text{const}. \qquad (44)
\end{aligned}
$$

It follows from (42)

$$I=\int_{-\infty}^{\infty}|\psi_z^{c_1,c_2}(q,\tau)|^2\mathrm{d}q=\sqrt{\pi}|f-g||F(\tau)|^2\exp\left[\frac{q^2(t)}{|f-g|^2}\right]. \tag{45}$$

It is convenient to introduce the function $\widetilde{\gamma}(\tau)$ instead of the function $F(\tau)$ as follows

$$F(\tau)=\frac{\exp\widetilde{\gamma}(\tau)}{\sqrt{(f-g)\sqrt{\pi}}}. \tag{46}$$

Then

$$\psi_z^{c_1,c_2}(q,\tau)=\frac{\exp(R+\widetilde{\gamma})}{\sqrt{(f-g)\sqrt{\pi}}}, \tag{47}$$

and

$$I=\exp Q,\quad Q=\frac{q^2(\tau)}{|f-g|^2}+\widetilde{\gamma}+\widetilde{\gamma}^*.$$

Because the Hamiltonian (37) is self-adjoint, we expect that $I=\text{const}$. This is why we can set $I=1$ $(Q=0)$.

Using equations (42) and (47), we obtain

$$R+\widetilde{\gamma}=-\widetilde{q}^2/2+\mathrm{i}\Gamma,\quad \Gamma=\mathrm{Im}(R+\widetilde{\gamma}), \tag{48}$$

such that

$$\psi_z^{c_1,c_2}(q,\tau)=\frac{\exp(R+\widetilde{\gamma})}{\sqrt{(f-g)\sqrt{\pi}}}=\frac{\exp(\mathrm{i}\Gamma-\widetilde{q}^2/2)}{\sqrt{(f-g)\sqrt{\pi}}}. \tag{49}$$

Substituting representation (49) into the Schrödinger equation, we fix the quantities and (see the appendix)

$$\widetilde{\gamma}=-\frac{\Omega_1+\mathrm{i}b^2\Omega_2}{J}+\widetilde{\gamma}_0,\quad Q=\text{const}.$$

One can verify that the GCS (49) form an overcomplete set for any fixed time instant τ,

$$\int\psi_z^{c_1,c_2}(q,\tau)\psi_z^{*c_1,c_2}(q',\tau)\,\mathrm{d}^2z=\pi\delta(q-q'). \tag{50}$$

One can easily find that

$$\langle q\rangle\equiv\langle z,\tau|\hat{q}|z,\tau\rangle=q(\tau),\quad \langle p\rangle\equiv\langle z,\tau|\hat{p}|z,\tau\rangle=\frac{1}{2}\frac{\mathrm{d}q(\tau)}{\mathrm{d}\tau},\quad \hat{p}=-\mathrm{i}\partial_q. \tag{51}$$

Calculating the variances σ_i, $i=1,2,3$, we obtain representations (29) and

$$J=\sigma_1\sigma_2-\sigma_3^2=1/4. \tag{52}$$

3.2. Limiting cases

We stress that due to the convenient choice of the variables (34), we obtain the possibility to consider different important limiting cases, in particular the limit $\omega_0\to 0$ that is equivalent to the limit $\omega\to 0$. Indeed, the solution (39) admits such a limit. The limiting expressions are GCS of the free particle (see [10]):

$$\psi_z^{c_1,c_2}(q,\tau)=\frac{\exp(R_0+\widetilde{\gamma}_0)}{\sqrt{(f_0-g_0)\sqrt{\pi}}},$$

where

$$R_0 = -\frac{f_0 + g_0}{2(f_0 - g_0)}q^2 + \frac{\sqrt{2}z}{f_0 - g_0}q, \quad f_0 = \lim_{\omega\to 0} f = c_1 + \mathrm{i}[c_1 + c_2]\tau,$$

$$g_0 = \lim_{\omega\to 0} g = c_2 - \mathrm{i}[c_1 + c_2]\tau, \qquad \widetilde{\gamma}_0 = \lim_{\omega\to 0} \widetilde{\gamma} = -\frac{z^2}{(f_0 - g_0)(f_0 + g_0)}. \tag{53}$$

Then

$$\psi_z^{c_1,c_2}(q,\tau) = \frac{1}{\sqrt{(a_2 + 2\mathrm{i}a_1\tau)\sqrt{\pi}}} \exp\left[\frac{\left(-\frac{a_1q^2}{2} + \sqrt{2}zq - \frac{z^2}{a_1}\right)}{a_2 + 2\mathrm{i}a_1\tau}\right],$$
$$a_1 = c_1 + c_2, \quad a_2 = c_1 - c_2.$$

Calculating the variances σ_i we obtain

$$\begin{aligned} \sigma_1 &= \frac{1}{2d} + 2d(\tau - \tau_0)^2, \quad \sigma_2 = d/2, \quad \sigma_3 = d(\tau - \tau_0); \\ d &= |c_1 + c_2|^2 = \mathrm{e}^{2\alpha}\cos^2\beta + \mathrm{e}^{-2\alpha}\sin^2\beta, \\ \tau_0 &= \frac{\mathrm{i}(c_1c_2^* - c_2c_1^*)}{2\,|c_1 + c_2|^2} = -\frac{\sinh 2\alpha \sin 2\beta}{2d}. \end{aligned} \tag{54}$$

It is seen from (54) that for any $c_{1,2}$ (α and β) the variances $\sigma_{1,3}$ grow with time. However, in all the constructed free particle GCS, the Schrödinger–Robertson uncertainty relation [15] is minimized, $\sigma_1\sigma_2 - \sigma_3^2 = 1/4$, such that we deal again with some kind of squeezed states. But the Heisenberg uncertainty relation is minimized only at $\tau = \tau_0$,

$$\sigma_1\sigma_2 = 1/4 + d^2(\tau - \tau_0)^2.$$

Choosing α and β, one can fix any values for the quantities $d > 0$ and τ_0.

By setting $\omega^2 = 1$, which is equivalent to choosing $l^2 = \hbar/m\omega_0$ in equations (34) and by setting $b = 0$, $c_1 = 1$, and $c_2 = 0$, we obtain from equations (39) that

$$f(\tau) = \exp 2\mathrm{i}\tau, \quad g(\tau) = \varphi(\tau) = 0. \tag{55}$$

One can easily see that in this case the GCS (49) are reduced to well-known CS for the harmonic oscillator obtained first by Schrödinger [16] and then by Glauber [2]. In this case, we fix the variable transformation (34). Such fixed new variables do not exist in the possibility of considering the limit $\omega_0 \to 0$. This explains the fact of why the CS for free particles were not obtained before from the Schrödinger CS.

Another important property of the variables (34) and solutions (39) is that they are well defined for all complex values ω, in particular for pure imaginary ones. This allows us to set $\omega = \mathrm{i}$, $b = 0$, then $\cos 2\omega\tau \to \cosh 2\tau$, $\frac{\sin 2\omega\tau}{\omega} \to \sinh 2\tau$. In this case, instead of (39) we have expressions (40) that correspond to the inverse oscillator case.

4. Conclusion

First of all we would like to stress that our construction of GCS for the inverse oscillator that has a continuous energy spectrum is based on a simple fact that the Hilbert space is separable, so that there exists a discrete oscillator-like basis there. As has already been mentioned, this fact does not contradict the fact that we consider a system with a continuous spectrum. Thus, we construct a complete discrete set of solutions of the Schrödinger equation and introduce creation and annihilation operators that are integrals of motion. With the help of these operators we construct a Fock space and in such a space we construct GCS as eigenfunctions of the introduced annihilation operators.

The constructed GCS form a normalized and overcomplete set of states in the Hilbert space. We demonstrate that mean values of the phase-space variables calculated with respect to time-dependent GCS move along the classical trajectories of the inverse oscillator. The GCS minimize the Schrödinger–Robertson uncertainty relation [15] at any time instant and are, by definition, squeezed states. The Heisenberg uncertainty relation is minimized only at an initial time instant.

It should be noted that the GCS constructed by us are quite different from the CS of the inverse oscillator proposed in the works [12, 13], are normalizable and obey the majority of properties that must be typical of CS.

Another important result, from our point of view, is the following. It is known that there exist Hamiltonians of some physical systems which depend on some parameters, such that depending on the choice of these parameters, the corresponding energy spectrum may be discrete or continuous. As examples one can mention a charge particle in the magnetic field and the harmonic oscillator. In the first case, by switching off the magnetic field we transform the system with a discrete spectrum to a free particle that has a continuous spectrum. In the second case, by switching off the frequency ω we again transform the oscillator with a discrete spectrum to a free particle that has a continuous spectrum. Moreover, changing the real frequency ω to an imaginary quantity $\mathrm{i}\omega$, we transform the normal oscillator to the inverse one with a continuous spectrum. However, it is known that one meets difficulties to fulfil such limits in the corresponding wavefunctions, let us say in wavefunctions of stationary states. In the case of the charged particle in the constant magnetic field only special states admit the zero-field limit, see [17]. It turns out that our approach to constructing GCS and an appropriate choice of variables allowed us to construct special GCS of the normal oscillator (see section 3) in which one can fulfil both above mentioned limits, to the free particle case and to the inverse oscillator case.

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Appendix

We start with deriving the following relations

$$\partial_q \mathrm{e}^R = \frac{\sqrt{2}(Z-\varphi)-(f+g)q}{f-g}\mathrm{e}^R, \quad \partial_q^2 \mathrm{e}^R = (\sqrt{2}bq+\omega^2 q^2 - \Phi)\,\mathrm{e}^R,$$

where

$$\Phi = \frac{f+g}{f-g} + \left[\omega^2 - \left(\frac{f+g}{f-g}\right)^2\right]q^2 + \sqrt{2}\left[b + 2\frac{(f+g)(z-\varphi)}{(f-g)^2}\right]q - 2\left(\frac{z-\varphi}{f-g}\right)^2. \quad \text{(A.1)}$$

Then one can verify that

$$\hat{H}\psi_z^{c_1,c_2}(q,\tau) = \Phi\psi_z^{c_1,c_2}(q,\tau). \quad \text{(A.2)}$$

With the help of equations (38), we find the relations

$$\mathrm{i}\frac{\partial}{\partial\tau}\frac{1}{\sqrt{f-g}} = \frac{1}{\sqrt{f-g}}\frac{f+g}{f-g}, \quad \mathrm{i}\frac{\partial R}{\partial\tau} = \Phi + 2\left(\frac{z-\varphi}{f-g}\right)^2 - \frac{f+g}{f-g}, \quad \text{(A.3)}$$

and as a consequence we obtain

$$\mathrm{i}\frac{\partial}{\partial \tau}\psi_z^{c_1,c_2}(q,\tau) = \left[\Phi + 2\left(\frac{z-\varphi}{f-g}\right)^2 + \mathrm{i}\frac{\mathrm{d}\widetilde{\gamma}}{\mathrm{d}\tau}\right]\psi_z^{c_1,c_2}(q,\tau). \tag{A.4}$$

Taking into account equation (A.2), we see that the functions $\psi_z^{c_1,c_2}(q,\tau)$ satisfy the Schrödinger equation (35) if $\widetilde{\gamma}(t)$ obeys the relation

$$\frac{\mathrm{d}\widetilde{\gamma}}{\mathrm{d}\tau} = 2\mathrm{i}\left(\frac{z-\varphi}{f-g}\right)^2. \tag{A.5}$$

To solve this equation, we will use the relations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{f+g}{f-g} = -\frac{2\mathrm{i}J}{(f-g)^2}, \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{1}{f-g} = -\frac{2\mathrm{i}(f+g)}{(f-g)^2}, \tag{A.6}$$

derived from equations (38), and their consequences

$$\frac{\mathrm{d}\Omega_1}{\mathrm{d}\tau} = -2\mathrm{i}J\left(\frac{z-\varphi}{f-g}\right)^2 - \mathrm{i}b^2(f-g)^2, \qquad \frac{\mathrm{d}\Omega_2}{\mathrm{d}\tau} = (f-g)^2,$$
$$\Omega_1 = \frac{f+g}{f-g}(z-\varphi)^2 + b(f-g)(z-\varphi), \quad \Omega_2 = \frac{J}{2\omega^2}\left(\frac{\sin 4\omega\tau}{4\omega} - \tau\right)$$
$$+(c_1-c_2)^2\frac{\sin 4\omega\tau}{4\omega} + 2\mathrm{i}(c_1^2-c_2^2)\left(\frac{\sin 2\omega\tau}{2\omega}\right)^2. \tag{A.7}$$

Relations (A.7) allow one to transform equation (A.5) and thus to obtain its solution. Namely,

$$J\frac{\mathrm{d}\widetilde{\gamma}}{\mathrm{d}\tau} = -\frac{\mathrm{d}}{\mathrm{d}t}(\Omega_1 + \mathrm{i}b^2\Omega_2) \quad\Longrightarrow\quad \widetilde{\gamma} = -\frac{\Omega_1 + \mathrm{i}b^2\Omega_2}{J} + \widetilde{\gamma}_0,$$

where $\widetilde{\gamma}_0$ is a complex constant that may depend on c_1, c_2 and z.

Finally, we can verify independently that the quantity Q introduced in equation (47) is a constant, i.e., the relation

$$\frac{\mathrm{d}Q}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{q^2(\tau)}{|f-g|^2} + \widetilde{\gamma} + \widetilde{\gamma}^*\right] = 0 \tag{A.8}$$

holds true. This can be done easily with the help of equations (43), (44), and

$$\frac{\mathrm{d}|f-g|^2}{\mathrm{d}\tau} = 4\mathrm{i}(gf^* - fg^*), \tag{A.9}$$

which follows from (38).

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