

# *Stability of Standing Waves for the Logarithmic Schrödinger Equation with Attractive Delta Potential*

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ABSTRACT. We consider the one-dimensional logarithmic Schrödinger equation with a delta potential. Global well-posedness is verified for the Cauchy problem in  $H^1(\mathbb{R})$  and in an appropriate Orlicz space. In the attractive case, we prove orbital stability of the ground states via variational approach.

## 1. INTRODUCTION

The present paper is devoted to the analysis of existence and stability of the ground states for the following nonlinear Schrödinger equation with a delta potential:

$$(1.1) \quad i \partial_t u + (\partial_x^2 + \gamma \delta(x))u + u \log |u|^2 = 0,$$

where  $\gamma \in \mathbb{R}$  and  $u = u(x, t)$  is a complex-valued function of  $(x, t) \in \mathbb{R} \times \mathbb{R}$ . Here,  $\delta(x)$  is the delta measure at the origin. The parameter  $\gamma$  is real; when positive, the potential is called *attractive*, otherwise, *repulsive*.

In the absence of the delta potential, the equation (1.1) has been proposed in order to obtain a nonlinear equation which helps to quantify departures from the strictly linear regime, preserving some aspects of quantum mechanics, such as separability and additivity of total energy for non-interacting subsystems, the validity of the lower energy bound, and Planck's relation for all stationary states (see [9, 10]). This equation admits applications in quantum mechanics, quantum optics, nuclear physics, fluid dynamics, plasma physics, and Bose-Einstein condensation (see [23, 30] and references therein).

The formal expression  $-\partial_x^2 - \gamma \delta(x)$  which appears in (1.1) admits a precise interpretation as a self-adjoint operator  $H_\gamma$  on  $L^2(\mathbb{R})$ . Indeed, for  $u, v \in H^1(\mathbb{R})$

we have formally

$$\langle (-\partial_x^2 - \gamma \delta(x))u, v \rangle = t_\gamma(u, v),$$

where  $t_\gamma$  is the bilinear form defined on  $H^1(\mathbb{R})$  by

$$(1.2) \quad t_\gamma(u, v) = \operatorname{Re} \int_{\mathbb{R}} \partial_x u \partial_x \bar{v} \, dx - \gamma \operatorname{Re}[u(0) \overline{v(0)}].$$

It is clear that this form is bounded from below and closed on  $H^1(\mathbb{R})$ . Then, it is possible to show that the self-adjoint operator on  $L^2(\mathbb{R})$  associated with  $t_\gamma$  is given by (see [28, Theorem 10.7 and Example 10.7])

$$(1.3) \quad \begin{cases} H_\gamma v(x) = -\frac{d^2}{dx^2} v(x) & \text{for } x \neq 0, \\ v \in \operatorname{dom}(H_\gamma) = \left\{ v \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}) : \right. \\ \quad \left. v'(0+) - v'(0-) = -\gamma v(0) \right\}. \end{cases}$$

Notice that  $H_\gamma$  can also be defined via the theory of self-adjoint extensions of symmetric operator (see [4–6]). Now, from Albeverio *et. al.* (see [4, Chapter I.3] for details) we have the following spectral properties of  $H_\gamma$  which will be used in our local well-posedness theory for model (1.1): for  $\sigma_{\text{ess}}(H_\gamma)$  and  $\sigma_{\text{disc}}(H_\gamma)$  denoting the essential and discrete spectrum of  $H_\gamma$ , respectively, it is well known that  $\sigma_{\text{ess}}(H_\gamma) = [0, \infty)$ , for  $\gamma \neq 0$ ;  $\sigma_{\text{disc}}(H_\gamma) = \emptyset$  for  $\gamma < 0$ ;  $\sigma_{\text{disc}}(H_\gamma) = \{-\gamma^2/4\}$  for  $\gamma > 0$ .

Before presenting our results, let us first introduce some preliminaries. We consider the reflexive Banach space (see the [Appendix](#))

$$(1.4) \quad W(\mathbb{R}) = \{u \in H^1(\mathbb{R}) : |u|^2 \log |u|^2 \in L^1(\mathbb{R})\}.$$

By Lemma 5.3 in the [Appendix](#), we have that the operator

$$\begin{cases} W(\mathbb{R}) \rightarrow W'(\mathbb{R}), \\ u \mapsto \partial_x^2 u + \gamma \delta(x)u + u \log |u|^2, \end{cases}$$

is continuous and bounded. Here,  $W'(\mathbb{R})$  is the dual space of  $W(\mathbb{R})$ . Therefore, if  $u \in C(\mathbb{R}, W(\mathbb{R})) \cap C^1(\mathbb{R}, W'(\mathbb{R}))$ , then equation (1.1) makes sense in  $W'(\mathbb{R})$ . We define the energy functional

$$(1.5) \quad E(u) = \frac{1}{2} \|\partial_x u\|_{L^2}^2 - \frac{\gamma}{2} |u(0)|^2 - \frac{1}{2} \int_{\mathbb{R}} |u|^2 \log |u|^2 \, dx.$$

At least formally, we have that  $E$  is conserved by the flow of (1.1) (see Proposition 2.1). Moreover, by Proposition 5.4 in the [Appendix](#), we also have that  $E$  is well defined and of class  $C^1$  on  $W(\mathbb{R})$ .

We note that the use of the space  $W(\mathbb{R})$  is mainly due to the fact that the functional  $E$ , in general, fails to be finite and of class  $C^1$  on all  $H^1(\mathbb{R})$  (see Cazenave [13]).

The next proposition gives a result on the existence of weak solutions to (1.1) in the energy space  $W(\mathbb{R})$ . We recall that a global (weak) solution to (1.1) is a function  $u \in C(\mathbb{R}, W(\mathbb{R})) \cap C^1(\mathbb{R}, W'(\mathbb{R}))$  solving (1.1) in  $W'(\mathbb{R})$  for all  $t \in \mathbb{R}$ .

**Proposition 1.1.** *For any  $u_0 \in W(\mathbb{R})$ , there is a unique maximal solution  $u \in C(\mathbb{R}, W(\mathbb{R})) \cap C^1(\mathbb{R}, W'(\mathbb{R}))$  of (1.1) such that*

$$u(0) = u_0 \quad \text{and} \quad \sup_{t \in \mathbb{R}} \|u(t)\|_{W(\mathbb{R})} < \infty.$$

Furthermore, the conservation of energy and charge hold; that is,

$$E(u(t)) = E(u_0) \quad \text{and} \quad \|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2, \quad \text{for all } t \in \mathbb{R}.$$

The proof of Proposition 1.1 will be given in Section 2. In this paper, we are mainly interested in the study of the orbital stability of standing wave solutions  $u(x, t) = e^{i\omega t} \varphi(x)$  for (1.1), where  $\omega \in \mathbb{R}$ ,  $\gamma > 0$ , and  $\varphi \in W(\mathbb{R}) \cap \text{dom}(H_\gamma)$  is a real-valued function which has to solve the following stationary problem:

$$(1.6) \quad -\partial_x^2 \varphi - \gamma \delta(x) \varphi + \omega \varphi - \varphi \log |\varphi|^2 = 0 \quad \text{in } W'(\mathbb{R}).$$

As we will see later in Section 3, there exists a unique positive symmetric solution  $\varphi_{\omega, \gamma}$  (the soliton peak-Gausson profile) of (1.6) which is explicitly given for every  $\omega \in \mathbb{R}$  by

$$(1.7) \quad \varphi_{\omega, \gamma}(x) = e^{(\omega+1)/2} e^{-(1/2)(|x|+\gamma/2)^2}.$$

This solution is constructed from the known solution of (1.1) in the case where  $\gamma = 0$  (i.e.,  $\varphi_{\omega, 0}$ ) on each side of the defect pasted together at  $x = 0$  to satisfy continuity and the jump condition  $\varphi'_{\omega, \gamma}(0+) - \varphi'_{\omega, \gamma}(0-) = -\gamma \varphi_{\omega, \gamma}(0)$  at  $x = 0$ .

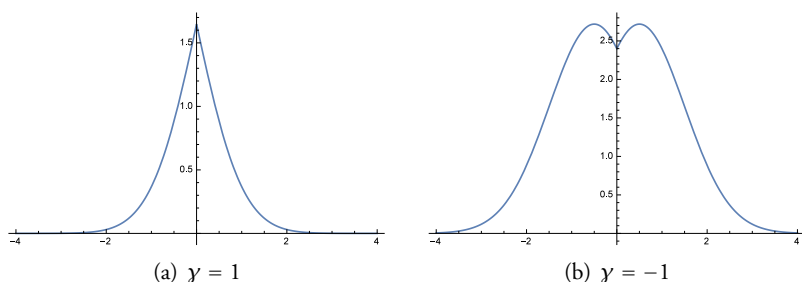
The dependence of  $\varphi_{\omega, \gamma}$  on  $\gamma$  can be seen in Figure 1.1 below. Notice that the sign of  $\gamma$  determines the profile of  $\varphi_{\omega, \gamma}$  near  $x = 0$ . Indeed, it has a “v” shape when  $\gamma < 0$ , and a “^” shape when  $\gamma > 0$ .

For  $\gamma = 0$ , consider the equation (1.1) in higher dimensions:

$$(1.8) \quad i \partial_t u + \Delta u + u \log |u|^2 = 0.$$

This equation has been studied previously by several authors (see [8–10, 13, 16, 22] and the references therein). In particular, note the Gaussian shape standing-wave for (1.8) (introduced by Bialynicki-Birula and Mycielski in the '70 [9, 10])

$$\varphi_{\omega, N}(x) = e^{(\omega+N)/2} e^{-(1/2)|x|^2};$$

FIGURE 1.1.  $\varphi_{\omega, \gamma}$  as a function of  $x$  for  $\omega = 1$ 

in dimension  $N$ , it was shown by Cazenave in [10] that these Gaussian profiles are orbitally stable in  $W(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : |u|^2 \log |u|^2 \in L^1(\mathbb{R}^N)\}$  under radial perturbations for  $N \geq 2$ . In the case  $N \geq 1$ , we can use the Cazenave-Lions approach in [16] for showing stability on all  $W(\mathbb{R}^N)$ . For  $N \geq 3$ , d'Avenia, Montefusco, and Squassina in [8] showed the existence of infinitely many weak solutions for

$$-\Delta \psi_\omega + \omega \psi_\omega = \psi_\omega \log |\psi_\omega|^2,$$

and that the Gaussian profile  $\varphi_{\omega, N}$  is nondegenerated, that is,

$$\ker(\mathcal{L}) = \text{span}\{\partial_{x_i} \varphi_{\omega, N} : i = 1, 2, \dots, N\},$$

where  $\mathcal{L}u = -\Delta u + (|x|^2 + \omega - 2)u$  is the linearized operator for  $-\Delta u + \omega u = u \log(u^2)$ .

Concerning model (1.1), recently Angulo and Goloshchapova [6] proved that  $\varphi_{\omega, \gamma}$  given by (1.7) is orbitally stable for  $\gamma > 0$  in the “weighted space”

$$\tilde{W} = H^1(\mathbb{R}) \cap L^2(x^2 dx),$$

orbitally unstable in  $\tilde{W}$  for  $\gamma < 0$ , and orbitally stable in  $\tilde{W}_{\text{rad}}$  for any  $\gamma \neq 0$ . The stability analysis in [6] relies on the abstract theory by Grillakis, Shatah, and Strauss [21], and the analytic perturbation theory and extension theory of symmetric operators (see also [7] for applications of the extension theory in the case of star graphs). We mention that, since the key energetic functional  $E : W(\mathbb{R}) \rightarrow \mathbb{R}$  is not twice continuously differentiable at  $\varphi_{\omega, \gamma}$ , the approach elaborated in [21] cannot be done on  $W(\mathbb{R})$ , but can be done on the weighted space  $\tilde{W}$  which is continuously embedding in  $W(\mathbb{R})$ . This is the main reason why in [6] it was necessary to use the space  $\tilde{W}$  in the stability approach.

The purpose of this paper is to extend the results in [6] about the stability of  $\varphi_{\omega, \gamma}$  in (1.7) to the space  $W(\mathbb{R})$  for the case  $\gamma > 0$ . Our approach will be based

on a variational characterization of  $\varphi_{\omega\gamma}$ . This characterization can not be used to treat the case  $\gamma < 0$ , and it is left open (see Remark 4.5 below).

The basic symmetry associated with equation (1.1) is the phase-invariance (while the translation invariance does not hold due to the defect). Thus, the definition of stability takes into account only this type of symmetry, and is formulated as follows.

**Definition 1.2.** We say that a standing wave solution  $u(x, t) = e^{i\omega t}\varphi(x)$  of (1.1) is orbitally stable in  $W(\mathbb{R})$  if, for any  $\varepsilon > 0$ , there exists  $\eta > 0$  such that if  $u_0 \in W(\mathbb{R})$  and  $\|u_0 - \varphi\|_{W(\mathbb{R})} < \eta$ , then the solution  $u(t)$  of (1.1) with  $u(0) = u_0$  exists for all  $t \in \mathbb{R}$  and satisfies

$$\sup_{t \in \mathbb{R}} \inf_{\theta \in \mathbb{R}} \|u(t) - e^{i\theta}\varphi\|_{W(\mathbb{R})} < \varepsilon.$$

Otherwise, the standing wave  $e^{i\omega t}\varphi(x)$  is said to be unstable in  $W(\mathbb{R})$ .

Next, we state our main result in this paper.

**Theorem 1.3.** Let  $\omega \in \mathbb{R}$ . If  $\gamma > 0$ , then the standing wave  $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ , where  $\varphi_{\omega,\gamma}$  is given in (1.7), is orbitally stable in  $W(\mathbb{R})$ .

The proof of Theorem 1.3 is based on the variational characterization of the stationary solutions  $\varphi$  for (1.6) as minimizers of the action

$$S_{\omega,\gamma}(u) = E(u) + ((\omega + 1)/2)\|u\|_{L^2}^2$$

on the Nehari manifold

$$\{u \in W(\mathbb{R}) \setminus \{0\} : I_{\omega,\gamma}(u) = 0\}$$

with  $I_{\omega,\gamma}(u) = 2E(u) + \omega\|u\|_{L^2}^2$  (see Theorem 4.4), and the uniqueness of positive solutions (modulo rotations) for (1.6) given by the peak-Gausson profile (1.7) (see Proposition 3.1). We note that an analogous variational analysis has been used for NLS equations with point interactions on all lines by Fukuizumi and Jeanjean [18], Fukuizumi, Ohta, and Ozawa [19], Adami and Noja [1], Adami, Noja, and Visciglia [2], and on star graphs by Adami, Cacciapuoti, Finco, and Noja [3].

We also note that recently equation (1.8) has been considered with an external potential  $V$  satisfying specific conditions (see Ji and Szulkin [25] and Squassina and Szulkin [29]),

$$i\partial_t u + \Delta u - V(x)u + u \log |u|^2 = 0.$$

From the results of Ji and Szulkin in [25], it follows that there exist infinitely many profiles of the standing wave  $u(x, t) = e^{i\omega t}f_\omega$  (see also [29]) for when  $V$  is coercive. Specifically, the elliptic equation

$$-\Delta f_\omega + (V(x) + \omega)f_\omega = f_\omega \log |f_\omega|^2$$

has infinitely many solutions for  $V \in C(\mathbb{R}^N, \mathbb{R})$  such that  $\lim_{|x| \rightarrow \infty} V(x) = +\infty$ . Moreover, Ji and Szulkin also showed the existence of a ground state solution (a nontrivial positive solution with least possible energy) for bounded potential such that  $\omega + 1 + V_\infty > 0$ , in which

$$V_\infty := \lim_{|x| \rightarrow \infty} V(x) = \sup_{x \in \mathbb{R}^N} V(x)$$

and  $\sigma(-\Delta + V(x) + \omega + 1) \subset (0, +\infty)$  (here,  $\sigma(A)$  represents the spectrum of the linear operator  $A$ ). Thus, we see that in the case of a delta-potential  $V(x) = -\gamma\delta(x)$ , the later restriction on the “frequency”  $\omega$  is ineffective (see the proof of Theorem 4.4 below).

The rest of the paper is organized as follows. In Section 2, we give an idea of the proof of Proposition 1.1. In Section 3, we prove that the stationary problem (1.6) has a unique nonnegative nontrivial solution. Section 4 is devoted to giving a variational characterization of the stationary solutions of (1.6). In Section 5, we establish the proof of Theorem 1.3. In the Appendix, we include information about the space  $W(\mathbb{R})$  and the smooth property of the energy functional  $E$  in (1.5).

**Notation 1.4.** The space  $L^2(\mathbb{R}, \mathbb{C})$  will be denoted by  $L^2(\mathbb{R})$  and its norm by  $\|\cdot\|_{L^2}$ . This space will be endowed with the real scalar product

$$(u, v) = \operatorname{Re} \int_{\mathbb{R}} u \bar{v} \, dx, \quad \text{for } u, v \in L^2(\mathbb{R}).$$

The space  $H^1(\mathbb{R}, \mathbb{C})$  will be denoted by  $H^1(\mathbb{R})$  and its norm by  $\|\cdot\|_{H^1(\mathbb{R})}$ . The dual space of  $H^1(\mathbb{R})$  will be denoted by  $H^{-1}(\mathbb{R})$ . We denote by  $C_0^\infty(\mathbb{R} \setminus \{0\})$  the set of  $C^\infty$  functions from  $\mathbb{R} \setminus \{0\}$  to  $\mathbb{C}$  with compact support. Throughout this paper, the letter  $C$  will denote positive constants.

## 2. THE CAUCHY PROBLEM

In this section, we prove the well-posedness of the Cauchy Problem for (1.1) in the energy space  $W(\mathbb{R})$ . The idea of the proof is an adaptation of the proof of [14, Theorem 9.3.4]. Thus, we will approximate the logarithmic nonlinearity by a smooth nonlinearity, and as a consequence we construct a sequence of global solutions of the regularized Cauchy problem in  $C(\mathbb{R}, H^1(\mathbb{R}))$ , pass to the limit using standard compactness results, and extract a subsequence which converges to the solution of the limiting equation (1.1).

First, let us establish the following well-posedness result in  $H^1(\mathbb{R})$  associated with the NLS equation with a delta potential

$$(2.1) \quad \begin{cases} i \partial_t u + \partial_x^2 u + \gamma \delta(x) u + g(u) = 0, \\ u(0) = u_0 \in H^1(\mathbb{R}), \end{cases}$$

where  $g : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  is globally Lipschitz continuous on  $L^2(\mathbb{R})$  and where  $\operatorname{Im}(g(u), iu) = 0$ , and such that there exists  $G \in C(H^1(\mathbb{R}), \mathbb{R})$  with  $G' = g$ .

**Proposition 2.1.** *For any  $u_0 \in H^1(\mathbb{R})$ , there is a unique maximal solution  $u \in C(\mathbb{R}, H^1(\mathbb{R})) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}))$  of (2.1) such that  $u(0) = u_0$ . Furthermore, the conservation of charge and energy hold; that is, for all  $t \in \mathbb{R}$ ,  $\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2$  and*

$$\mathcal{E}(u(t)) = \mathcal{E}(u_0), \quad \text{where } \mathcal{E}(u) = \frac{1}{2} \|\partial_x u\|_{L^2}^2 - \frac{\gamma}{2} |u(0)|^2 - G(u).$$

*Proof.* The idea will be to check the assumptions of Theorem 3.3.1 and Theorem 3.7.1 in [14] in order to obtain the local result. Indeed, first we note that  $H_\gamma$  defined in (1.3) satisfies  $H_\gamma \geq -m$ , where  $m = \gamma^2/4$  if  $\gamma > 0$ , and  $m = 0$  if  $\gamma < 0$ . Thus, we have that  $A \equiv -H_\gamma - m$  is a self-adjoint operator,  $A \leq 0$  on  $X = L^2(\mathbb{R})$  with domain  $\operatorname{dom}(A) = \operatorname{dom}(H_\gamma)$ . Moreover, in our case the norm on  $H^1(\mathbb{R})$

$$\|u\|_{X_A}^2 = \|\partial_x u\|_{L^2}^2 + (m+1)\|u\|_{L^2}^2 - \gamma|u(0)|^2$$

is equivalent to the usual  $H^1(\mathbb{R})$ -norm. Next, it is easy to see that the conditions (3.7.1), (3.7.3)–(3.7.6) in [14, Section 3.7] hold, choosing  $r = \rho = 2$ , because we are in the one-dimensional case. Also, the condition (3.7.2) follows easily from the self-adjoint property of  $A$ . Finally, we need to show there is uniqueness for the problem (2.1). Therefore, let  $I$  be an interval containing 0, and let  $u_1, u_2 \in L^\infty(I, H^1(\mathbb{R})) \cap W^{1,\infty}(I, H^1(\mathbb{R}))$  be two solutions of (2.1). It follows that (see [14, Remark 3.7.2])

$$u_2(t) - u_1(t) = i \int_0^t e^{-iA(t-s)} (g(u_2(s)) - g(u_1(s))) \, ds \quad \text{for all } t \in I.$$

Since  $g$  is globally Lipschitz continuous on  $L^2(\mathbb{R})$ , there exists a constant  $C$  such that

$$\|u_2(t) - u_1(t)\|_{L^2}^2 \leq C \int_0^t \|u_2(s) - u_1(s)\|_{L^2}^2 \, ds,$$

and therefore the uniqueness follows by Gronwall's lemma. Therefore, we obtain that the initial value problem (2.1) is locally well posed in  $(H^1(\mathbb{R}), \|\cdot\|_{X_A})$ . Moreover, we have the conservation of charge and energy.

Finally, from Theorem 3.3.1 and Theorem 3.7.1 in [14] we see easily that the solution of (2.1) is global in  $H^1(\mathbb{R})$ . This finishes the proof.  $\square$

**Remark 2.2.** For the completeness of the exposition we recall that, for  $\gamma < 0$ , the unitary group  $G_\gamma(t) = e^{-itH_\gamma}$  associated with equation (2.1) is given explicitly by the formula (see [24])

$$G_\gamma(t)\varphi(x) = e^{it\partial_x^2}\varphi(x) + e^{it\partial_x^2}(\varphi * \rho_\gamma)(|x|),$$

where

$$\rho_Y(x) = -\frac{Y}{2}e^{(Y/2)x}\chi_+^0.$$

Here,  $\chi_+^0$  denotes the characteristic function of  $[0, +\infty)$ , and  $e^{it\partial_x^2}$  represents the free group of Schrödinger ( $Y = 0$ ). For the case  $Y > 0$ , we refer to [17]. Thus, an explicit formula for the group  $e^{itA}$  can be obtained.

*Proof of Proposition 1.1.* The proof is an adaptation of the proof of Theorem 9.3.4 in [14]. We only discuss the modifications that are not sufficiently clear in our case. We first regularize the logarithmic nonlinearity near the origin. Indeed, for  $z \in \mathbb{C}$ , we define the functions  $a_m$  and  $b_m$  by

$$a_m(z) = \begin{cases} a(z), & \text{if } |z| \geq \frac{1}{m}; \\ mza\left(\frac{1}{m}\right), & \text{if } |z| \leq \frac{1}{m}; \end{cases}$$

$$b_m(z) = \begin{cases} b(z), & \text{if } |z| \leq m; \\ \frac{z}{m}b(m), & \text{if } |z| \geq m, \end{cases}$$

where the functions  $a$  and  $b$  are defined in the Appendix (5.10). Moreover, set  $f_m(u) = b_m(u) - a_m(u)$  for  $u \in H^1(\mathbb{R})$ . We note that the function  $f_m$  is globally Lipschitz  $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ . For a given initial data  $u_0 \in H^1(\mathbb{R})$ , we consider the regularized Cauchy problem

$$(2.2) \quad \begin{cases} i\partial_t u^m + (\partial_x^2 + Y\delta(x))u^m + f_m(u^m) = 0, \\ u^m(0) = u_0. \end{cases}$$

Applying Proposition 2.1, we see that for every  $m \in \mathbb{N}$ , there exists a unique global (weak) solution  $u^m \in C(\mathbb{R}, H^1(\mathbb{R})) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}))$  of (2.2) which satisfies

$$E_m(u^m(t)) = E_m(u_0) \text{ and } \|u^m(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2, \quad \text{for all } t \in \mathbb{R},$$

where

$$E_m(u) = \frac{1}{2}\|\partial_x u(t)\|_{L^2}^2 - \frac{Y}{2}|u(0, t)|^2 + \frac{1}{2} \int_{\mathbb{R}} \Phi_m(|u|) dx - \frac{1}{2} \int_{\mathbb{R}} \Psi_m(|u|) dx,$$

and the functions  $\Phi_m$  and  $\Psi_m$  are defined by

$$\Phi_m(z) = \frac{1}{2} \int_0^{|z|} a_m(s) ds \quad \text{and} \quad \Psi_m(z) = \frac{1}{2} \int_0^{|z|} b_m(s) ds.$$

Arguing in the same way as in the proof of [14, Theorem 9.3.4], Step 2, we deduce that the sequence of approximating solutions  $u^m$  is bounded in the space



$L^\infty(\mathbb{R}, H^1(\mathbb{R}))$ . It also follows from the NLS equation (2.2) that the sequence  $\{u^m|_{\Omega_k}\}_{m \in \mathbb{N}}$  is bounded in the space  $W^{1,\infty}(\mathbb{R}, H^{-1}(\Omega_k))$ , where  $\Omega_k = (0, k)$ . Therefore, we have that  $\{u^m\}_{m \in \mathbb{N}}$  satisfies the assumptions of Lemma 9.3.6 in [14]. Let  $u$  be the limit of  $u^m$ .

We show that the limiting function  $u \in L^\infty(\mathbb{R}, H^1(\mathbb{R}))$  is a weak solution of (1.1). We first write a weak formulation of the NLS equation (2.2). Indeed, for any test functions  $\psi \in C_0^\infty(\mathbb{R}_x)$  and  $\varphi \in C_0^\infty(\mathbb{R}_t)$ , we have

$$(2.3) \quad - \int_{\mathbb{R}} [\langle iu^m, \psi \rangle \varphi'(t) + t_y(u^m, \psi) \varphi(t)] dt \\ + \int_{\mathbb{R}} \int_{\mathbb{R}} f_m(u^m) \psi(x) \varphi(t) dx dt = 0.$$

We pass to the limit as  $n \rightarrow \infty$  in the integral formulation (2.3), and obtain the following integral equation (see proof of [14, Theorem 9.3.4], Step 3):

$$(2.4) \quad - \int_{\mathbb{R}} [\langle iu, \psi \rangle \varphi'(t) + t_y(u, \psi) \varphi(t)] dt \\ + \int_{\mathbb{R}} \int_{\mathbb{R}} u \log |u|^2 \psi(x) \varphi(t) dx dt = 0.$$

Moreover, it is easy to see that  $u \in L^\infty(\mathbb{R}, L^A(\mathbb{R}))$  and  $u(0) = u_0$ . Therefore, by integral equation (2.4),  $u \in L^\infty(\mathbb{R}, W(\mathbb{R}))$  is a weak solution of the equation (1.1). In particular, from Lemma 5.3 in the Appendix, we deduce that  $u \in W^{1,\infty}(\mathbb{R}, W'(\mathbb{R}))$ .

Now, we show uniqueness of the solution in the class

$$L^\infty(\mathbb{R}, W(\mathbb{R})) \cap W^{1,\infty}(\mathbb{R}, W'(\mathbb{R})).$$

Indeed, let  $u$  and  $v$  be two solutions of (1.1) in that class. Taking the difference of the two equations and taking the  $W(\mathbb{R}) - W'(\mathbb{R})$  duality product with  $i(u - v)$ , we see that

$$\langle u_t - v_t, u - v \rangle_{W(\mathbb{R}), W'(\mathbb{R})} = - \operatorname{Im} \int_{\mathbb{R}} (u \log |u|^2 - v \log |v|^2) (\bar{u} - \bar{v}) dx.$$

Thus, from [14, Lemma 9.3.5] we obtain

$$\|u(t) - v(t)\|_{L^2} \leq 8 \int_0^t \|u(s) - v(s)\|_{L^2}^2 ds.$$

Therefore, the uniqueness of the solution follows by Gronwall's lemma.

We claim that the weak solution  $u$  of (1.1) satisfies both conservation of charge and energy. Indeed, by weak lower semicontinuity of the  $H^1(\mathbb{R})$ -norm,

by Fatou's lemma, and by arguing in the same way as in Step 3 of the proof of Theorem 9.3.4 in [14], we deduce that

$$(2.5) \quad E(u(t)) \leq E(u_0) \text{ and } \|u(t)\|_{L^2(\mathbb{R})}^2 = \|u_0\|_{L^2}^2 \quad \text{for all } t \in \mathbb{R}.$$

Now, fix  $t_0 \in \mathbb{R}$ . Let  $\varphi_0 = u(t_0)$ , and let  $w$  be the solution of (1.1) with  $w(0) = \varphi_0$ . By uniqueness, we see that  $w(\cdot - t_0) = u(\cdot)$  on  $\mathbb{R}$ . From (2.5), we deduce in particular that

$$E(u_0) \leq E(\varphi_0) = E(u(t_0)).$$

Therefore, we have that both  $\|u(t)\|_{L^2}^2$  and  $E(u(t))$  are constant on  $\mathbb{R}$ . Finally, the inclusion  $u \in C(\mathbb{R}, W(\mathbb{R})) \cap C^1(\mathbb{R}, W'(\mathbb{R}))$  follows from conservation laws. This completes the proof.  $\square$

### 3. STATIONARY PROBLEM

This section is devoted to showing that the set

$$\mathcal{A}_{\omega, \gamma} = \left\{ \varphi \in W(\mathbb{R}) \setminus \{0\} : \right. \\ \left. \varphi \text{ is a solution of the stationary problem (1.6) in } W'(\mathbb{R}) \right\}$$

is given (modulo rotations) by  $\varphi_{\omega, \gamma}$  in (1.7). More exactly, we have the following result.

**Proposition 3.1.** *Let  $\gamma \in \mathbb{R} \setminus \{0\}$  and  $\omega \in \mathbb{R}$ . Then, (1.6) has a unique nonnegative nontrivial solution. Thus,  $\varphi_{\omega, \gamma}$  is this solution, and therefore we have  $\mathcal{A}_{\omega, \gamma} = \{e^{i\theta} \varphi_{\omega, \gamma} : \theta \in \mathbb{R}\}$ .*

For  $\gamma = 0$ , the set of solutions of the stationary problem (1.6) is well known. In particular, modulo translation and phase, there exists a unique positive solution, which is explicitly known. Indeed (see, e.g., [10, Appendix D]),

$$\mathcal{A}_{\omega, 0} = \{e^{i\theta} \varphi_{\omega, 0}(\cdot - \gamma) : \theta \in \mathbb{R}, \gamma \in \mathbb{R}\}.$$

Before giving the proof of Proposition 3.1, we have the following basic properties of the solutions of (1.6).

**Lemma 3.2.** *Let  $\gamma \in \mathbb{R} \setminus \{0\}$ ,  $\omega \in \mathbb{R}$ , and  $v \in \mathcal{A}_{\omega, \gamma}$ . Then,  $v$  verifies the following:*

$$(3.1) \quad v \in C^j(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R}), \quad j = 1, 2,$$

$$(3.2) \quad -v'' + \omega v - v \log |v|^2 = 0, \quad \text{on } \mathbb{R} \setminus \{0\},$$

$$(3.3) \quad v'(0+) - v'(0-) = -\gamma v(0),$$

$$(3.4) \quad v'(x), v(x) \rightarrow 0, \quad \text{as } |x| \rightarrow \infty.$$

*Proof.* The proof of item (3.1) follows by a standard bootstrap argument by using test functions  $\xi \in C_0^\infty(\mathbb{R} \setminus \{0\})$  (see, e.g., [14, Chapter 8]). Indeed, from (1.6) applied with  $\xi v$ , we deduce that

$$-(\xi v)'' + \omega \xi v = -\xi'' v - 2\xi' v' + \xi v \log(|v|^2),$$

in the sense of distributions on  $\mathbb{R} \setminus \{0\}$ . The righthand side is in  $L^2(\mathbb{R})$ , and so  $\xi v \in H^2(\mathbb{R})$ . This implies that  $v$  is in  $C^2(\mathbb{R} \setminus \{0\})$  and is a classical solution of this equation on  $\mathbb{R} \setminus \{0\}$ , from which (3.1) and (3.2) follow. To prove (3.3), we consider  $\xi \in W(\mathbb{R})$  such that  $\xi \in C_0^\infty(\mathbb{R})$  and for  $\varepsilon > 0$  we have  $\text{supp } \xi \subset [-\varepsilon, \varepsilon]$  and  $\xi(0) = 1$ . Therefore, by considering  $v$  and  $\xi$  real-valued functions without loss of generality, we have for  $0 < \delta < \varepsilon$  that

$$\begin{aligned} 0 &= \langle -v'' - \gamma \delta(x)v + \omega v - v \log |v|^2, \xi \rangle \\ &= \lim_{\delta \rightarrow 0} \int_{-\varepsilon}^{-\delta} v'(x) \xi'(x) dx + \lim_{\delta \rightarrow 0} \int_{\delta}^{\varepsilon} v'(x) \xi'(x) dx - \gamma v(0) \xi(0) \\ &\quad + \omega \int_{-\varepsilon}^{\varepsilon} v(x) \xi(x) dx - \int_{-\varepsilon}^{\varepsilon} v(x) \log |v(x)|^2 \xi(x) dx \\ &= v'(0-) - v'(0+) - \int_{-\varepsilon}^{0-} v'' \xi dx - \int_{0+}^{\varepsilon} v'' \xi dx - \gamma v(0) \\ &\quad + \omega \int_{-\varepsilon}^{\varepsilon} v(x) \xi(x) dx - \int_{-\varepsilon}^{\varepsilon} v \log |v|^2 \xi dx \rightarrow v'(0-) - v'(0+) - \gamma v(0), \end{aligned}$$

as  $\varepsilon \rightarrow 0$ . This proves the jump condition. Finally, since  $v \in H^1(\mathbb{R})$ , it follows that  $v(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Thus, by (3.2),  $v''(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ , and so  $v'(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . This completes the proof of Lemma 3.2.  $\square$

**Remark 3.3.** As an example of test function  $\xi \in W(\mathbb{R})$  used in the proof of Lemma 3.2, with the properties that for  $\varepsilon > 0$  we have  $\text{supp } \xi \subset [-\varepsilon, \varepsilon]$  and  $\xi(0) = 1$ , is the following:

$$\xi(x) = \begin{cases} e e^{\varepsilon^2/(x^2 - \varepsilon^2)}, & \text{if } |x| < \varepsilon \\ 0, & \text{if } |x| \geq \varepsilon. \end{cases}$$

We now give the proof of Proposition 3.1. Our proof is inspired by the techniques of [18, Lemma 26].

*Proof of Proposition 3.1.* By construction, we have that  $\varphi_{\omega, \gamma} \in \mathcal{A}_{\omega, \gamma}$ , and so

$$\{e^{i\theta} \varphi_{\omega, \gamma} : \theta \in \mathbb{R}\} \subseteq \mathcal{A}_{\omega, \gamma}.$$

Now, let  $\varphi \in \mathcal{A}_{\omega, \gamma}$ . Arguing as in [26, Lemma 3], we can show that there exist  $\theta_0 \in \mathbb{R}$  and a positive function  $v$  such that  $\varphi(x) = e^{i\theta_0} v(x)$  for all  $x \in \mathbb{R}$ .

We will prove that  $v(x) = \varphi_{\omega, y}(x)$ . Indeed, it is clear, by Lemma 3.2, that the properties (3.1)–(3.4) also hold for  $v(x)$ . Let  $f(s) = -\omega s + s \log(s^2)$  and  $F(s) = \int_0^s f(t) dt$ . Multiplying the equation (3.2) by  $v'(x)$  and integrating from  $x = 0$  and  $R > 0$  yields

$$-\frac{1}{2}\{v'(R)\}^2 + \frac{1}{2}\{v'(0+)\}^2 - F(v(R)) + F(v(0+)) = 0.$$

Now, letting  $R \rightarrow \infty$ , we have

$$(3.5) \quad \frac{1}{2}\{v'(0+)\}^2 + F(v(0+)) = 0.$$

Arguing in the same way on  $(-\infty, 0]$ , we conclude that

$$\frac{1}{2}\{v'(0-)\}^2 + F(v(0-)) = 0.$$

Since  $v \in W(\mathbb{R})$ ,  $v$  is continuous at 0, and so we must have  $|v'(0-)| = |v'(0+)|$ . Now, if we suppose that  $v'(0-) = v'(0+)$ , then  $v(0) = 0$  by (3.3). Then, for the case  $v'(0-) = v'(0+) = 0$ , we obtain immediately  $v \equiv 0$  on  $\mathbb{R}$ . On the other hand, for  $v'(0-) = v'(0+) \neq 0$ , we have that  $v$  becomes negative close to 0. Therefore, since  $v$  is a positive solution, this is a contradiction. Hence, we need to have  $v'(0-) = -v'(0+)$ , from which we infer immediately that

$$(3.6) \quad v'(0+) = -\frac{y}{2}v(0).$$

For  $c > 0$ , we set

$$P(c) = \frac{y^2 - 4\omega - 4}{8}c^2 + c^2 \log(c).$$

It is clear that this function has a unique zero  $c_0 > 0$ ,  $c_0 = e^{(\omega+1)/2}e^{-y^2/8}$ . Direct computations show that, by (3.5) and (3.6), the function  $v$  in  $x = 0$  satisfies  $P(v(0)) = 0$ . Therefore,

$$(3.7) \quad v(0) = c_0 = e^{(\omega+1)/2}e^{-y^2/8}.$$

We note that the initial value problem on  $(0, \infty)$  for the equation (3.2) with (3.7) and (3.6) as initial conditions has a unique solution. Indeed, the solution is unique for  $x > 0$  close to 0 since  $f \in C([0, \infty)) \cap C^1((0, \infty))$ . A similar argument can be applied on  $(-\infty, 0)$ , and so the solution of (3.2) is unique in  $(-\infty, 0)$ . Moreover, we remark that  $v(0) = \varphi_{\omega, y}(0)$ . By the uniqueness, we see that  $v(x) = \varphi_{\omega, y}(x)$  on  $\mathbb{R}$ . This proves

$$\mathcal{A}_{\omega, y} \subseteq \{e^{i\theta}\varphi_{\omega, y} : \theta \in \mathbb{R}\}.$$

□

## 4. EXISTENCE OF A GROUND STATE

The idea of this section is to give a variational characterization of the stationary solutions for (1.6). This characterization will be used in the stability theory in  $W(\mathbb{R})$  of the orbit generated by  $\varphi_{\omega,y}$ ,  $\mathcal{A}_{\omega,y}$ . To establish our main result (Theorem 1.3), we need to establish some preliminaries.

**Definition 4.1.** For  $y \in \mathbb{R}$  and  $\omega \in \mathbb{R}$ , we define the following functionals of class  $C^1$  on  $W(\mathbb{R})$ :

$$S_{\omega,y}(u) = \frac{1}{2} \|\partial_x u\|_{L^2}^2 + \frac{\omega+1}{2} \|u\|_{L^2}^2 - \frac{y}{2} |u(0)|^2 - \frac{1}{2} \int_{\mathbb{R}} |u|^2 \log |u|^2 dx,$$

$$I_{\omega,y}(u) = \|\partial_x u\|_{L^2}^2 + \omega \|u\|_{L^2}^2 - y |u(0)|^2 - \int_{\mathbb{R}} |u|^2 \log |u|^2 dx.$$

We note that for  $u \in W(\mathbb{R})$  the derivative of  $S_{\omega,y}$  in  $u$  is given by

$$S'_{\omega,y}(u) = -\partial_x^2 u - y \delta(x)u + \omega u - u \log |u|^2,$$

in the sense that for  $h \in W(\mathbb{R})$ ,

$$\begin{aligned} S'_{\omega,y}(u)(h) &= \langle S'_{\omega,y}(u), h \rangle \\ &= \operatorname{Re} \left[ \int_{\mathbb{R}} u' \overline{h'} dx - \int_{\mathbb{R}} u \bar{h} \log |u|^2 dx + \omega \int_{\mathbb{R}} u \bar{h} dx \right] \\ &\quad - y \operatorname{Re}(u(0) \overline{h(0)}). \end{aligned}$$

Therefore, from Lemma 3.2 we have immediately that  $\varphi \in \mathcal{A}_{\omega,y}$  if and only if  $\varphi \in W(\mathbb{R}) \setminus \{0\}$  and  $S'_{\omega,y}(\varphi) = 0$ . Indeed, since for  $\varphi \in \mathcal{A}_{\omega,y}$  we have for every  $h \in W(\mathbb{R})$

$$\operatorname{Re} \int_{\mathbb{R}} \varphi' \overline{h'} dx = \operatorname{Re}[(\varphi'(0-) - \varphi'(0+)) \overline{h(0)}] - \int_{-\infty}^{0-} \varphi'' \bar{h} dx - \int_{0+}^{+\infty} \varphi'' \bar{h} dx,$$

we obtain immediately from (3.2)–(3.3) that  $S'_{\omega,y}(\varphi)(h) = 0$ . The other implication is trivial.

Next, we consider the minimization problem

$$\begin{aligned} (4.1) \quad d_y(\omega) &= \inf \{ S_{\omega,y}(u) : u \in W(\mathbb{R}) \setminus \{0\}, I_{\omega,y}(u) = 0 \} \\ &= \frac{1}{2} \inf \{ \|u\|_{L^2}^2 : u \in W(\mathbb{R}) \setminus \{0\}, I_{\omega,y}(u) = 0 \}, \end{aligned}$$

and define the set of ground states by

$$\mathcal{N}_{\omega,y} = \{ \varphi \in W(\mathbb{R}) : S_{\omega,y}(\varphi) = d_y(\omega), I_{\omega,y}(\varphi) = 0 \}.$$

**Remark 4.2.** The set  $\{u \in W(\mathbb{R}) \setminus \{0\} : I_{\omega,y}(u) = 0\}$  is called the Nehari manifold. By definition, we have  $I_{\omega,y}(u) = \langle S'_{\omega,y}(u), u \rangle$ . Thus, the above set is a one-codimension manifold that contains all stationary points of  $S_{\omega,y}$ .

**Remark 4.3.** We have the relation  $\mathcal{N}_{\omega,y} \subseteq \mathcal{A}_{\omega,y}$ . Indeed, let  $u \in \mathcal{N}_{\omega,y}$ . Then, there is a Lagrange multiplier  $\Lambda \in \mathbb{R}$  such that  $S'_{\omega,y}(u) = \Lambda I'_{\omega,y}(u)$ , so we have  $\langle S'_{\omega,y}(u), u \rangle = \Lambda \langle I'_{\omega,y}(u), u \rangle$ . The fact that  $\langle S'_{\omega,y}(u), u \rangle = I_{\omega,y}(u) = 0$  and  $\langle I'_{\omega,y}(u), u \rangle = -2\|u\|_{L^2(\mathbb{R})}^2 < 0$  implies  $\Lambda = 0$ ; that is,  $S'_{\omega,y}(u) = 0$ , and so  $u \in \mathcal{A}_{\omega,y}$ .

For  $y > 0$ , the existence of minimizers for (4.1) is obtained through variational techniques (see [2], [18], [19]). More precisely, we will show the following theorem.

**Theorem 4.4.** *Let  $y > 0$ . There exists a minimizer of  $d_y(\omega)$  for any  $\omega \in \mathbb{R}$ . In addition, the infimum is achieved at solutions to (1.6). More precisely, the set of minimizers for the problem (4.1) is given by  $\mathcal{N}_{\omega,y} = \{e^{i\theta} \varphi_{\omega,y} : \theta \in \mathbb{R}\}$ .*

**Remark 4.5.** We note that  $d_y(\omega)$  has no minimizer when  $y < 0$ ; we can see this by contradiction. Suppose  $u_{\omega,y}$  is a minimizer of  $d_y(\omega)$ . From Remark 4.3, it is clear there exists  $\theta_0 \in \mathbb{R}$  such that  $u_{\omega,y}(x) = e^{i\theta_0} \varphi_{\omega,y}(x)$ . In particular, we have that  $\lim_{|x| \rightarrow \infty} |u_{\omega,y}(x)| = 0$  and  $|u_{\omega,y}(x)| > 0$  on all  $x \in \mathbb{R}$ . Now, let  $\tau_y u_{\omega,y}(x) \equiv u_{\omega,y}(x - y)$  for any  $y \in \mathbb{R}$ . By direct computations, we see that

$$I_{\omega,y}(\tau_y u_{\omega,y}) - I_{\omega,y}(u_{\omega,y}) = y[|\varphi_{\omega,y}(0)|^2 - |\varphi_{\omega,y}(y)|^2],$$

and therefore we have that  $I_{\omega,y}(\tau_y u_{\omega,y}) < 0$  for  $|y|$  sufficiently large. Thus, there is  $\lambda \in (0, 1)$  such that  $I_{\omega,y}(\lambda \tau_y u_{\omega,y}) = 0$ . Then, by (4.1) we have

$$d_y(\omega) \leq \frac{1}{2} \|\lambda \tau_y u_{\omega,y}\|_{L^2}^2 < \frac{1}{2} \|\tau_y u_{\omega,y}\|_{L^2}^2 = \frac{1}{2} \|u_{\omega,y}\|_{L^2}^2 = d_y(\omega),$$

which is a contradiction.

To prove Theorem 4.4 we need several preliminary lemmas. In the first lemma, we recall the logarithmic Sobolev inequality. For a proof we refer to [27, Theorem 8.14].

**Lemma 4.6.** *Let  $f$  be any function in  $H^1(\mathbb{R}) \setminus \{0\}$  and  $\alpha$  be any positive number. Then,*

$$\int_{\mathbb{R}} |f(x)|^2 \log |f(x)|^2 dx \leq \frac{\alpha^2}{\pi} \|f'\|_{L^2}^2 + (\log \|f\|_{L^2}^2 - (1 + \log \alpha)) \|f\|_{L^2}^2.$$

**Lemma 4.7.** *Let  $y > 0$  and  $\omega \in \mathbb{R}$ . Then, the quantity  $d_y(\omega)$  is positive and satisfies*

$$(4.2) \quad d_y(\omega) \geq \sqrt{\frac{\pi}{8}} e^{\omega+1} e^{-y^2/2}.$$

*Proof.* Let  $u \in W(\mathbb{R}) \setminus \{0\}$  be such that  $I_{\omega, \gamma}(u) = 0$ . By Hölder's inequality, we have

$$(4.3) \quad 2\gamma|u(0)|^2 \leq \gamma^2 \|u\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2.$$

Moreover, using (4.3), the logarithmic Sobolev inequality with  $\alpha = \sqrt{\pi/2}$ , and  $I_{\omega, \gamma}(u) = 0$ , we obtain

$$\left( \omega - \frac{\gamma^2}{2} + 1 + \log \left( \sqrt{\frac{\pi}{2}} \right) \right) \|u\|_{L^2}^2 \leq (\log \|u\|_{L^2}^2) \|u\|_{L^2}^2,$$

which implies that

$$\|u\|_{L^2}^2 \geq \sqrt{\frac{\pi}{2}} e^{\omega+1} e^{-\gamma^2/2}.$$

Finally, by the definition of  $d_\gamma(\omega)$  given in (4.1), we get (4.2).  $\square$

**Lemma 4.8.** *Let  $\gamma > 0$ . The following inequality holds for any  $\omega \in \mathbb{R}$ :*

$$(4.4) \quad d_\gamma(\omega) < d_0(\omega).$$

*Proof.* We first note that by [16, Remark II.3], the profile standing-wave  $\varphi_{\omega,0}$  is a minimizer of

$$d_0(\omega) = \inf\{S_{\omega,0}(u) : u \in W(\mathbb{R}) \setminus \{0\}, I_{\omega,0}(u) = 0\},$$

that is,  $S_{\omega,0}(\varphi_{\omega,0}) = d_0(\omega)$  and  $I_{\omega,0}(\varphi_{\omega,0}) = 0$ . On the other hand, easy computations permit us to obtain

$$I_{\omega,\gamma}(\varphi_{\omega,0}) = I_{\omega,0}(\varphi_{\omega,0}) - \gamma|\varphi_{\omega,0}(0)|^2 = -\gamma e^{(\omega+1)} < 0.$$

Thus, there exists  $0 < \lambda < 1$  such that  $I_{\omega,\gamma}(\lambda\varphi_{\omega,0}) = 0$ . Therefore, by the definition of  $d_\gamma(\omega)$ , we see that

$$d_\gamma(\omega) \leq S_{\omega,\gamma}(\lambda\varphi_{\omega,0}) = \lambda^2 S_{\omega,0}(\varphi_{\omega,0}) < S_{\omega,0}(\varphi_{\omega,0}) = d_0(\omega),$$

and the proof of the lemma is finished.  $\square$

**Remark 4.9.** When  $\gamma < 0$  we have  $d_\gamma(\omega) = d_0(\omega)$  for any  $\omega \in \mathbb{R}$ . Indeed, let  $u \in W(\mathbb{R}) \setminus \{0\}$  be such that  $I_{\omega,0}(u) = 0$ . By direct computations,  $I_{\omega,\gamma}(u) = -\gamma|u(0)|^2 > 0$ . Then, there is  $s > 1$  such that  $I_{\omega,\gamma}(su) = 0$ . Then, since  $S_{\omega,\gamma}(su) = s^2 S_{\omega,0}(u) \geq S_{\omega,0}(u) \geq d_0(\omega)$ , we obtain from the definition of  $d_\gamma(\omega)$  given in (4.1), that  $d_0(\omega) \leq d_\gamma(\omega)$ . On the other hand, we define  $w_n(x) = \varphi_{\omega,0}(x - n)$  for  $n \in \mathbb{N}$ . It is clear that  $d_0(\omega) = S_{\omega,0}(w_n)$  and  $I_{\omega,\gamma}(w_n) = -\gamma|\varphi_{\omega,0}(n)|^2 > 0$ . Thus, there exists  $\lambda_n > 1$  such that

$I_{\omega,Y}(\lambda_n w_n) = 0$  for any  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} \lambda_n = 1$ . Then, by the definition of  $d_Y(\omega)$  and from  $I_{\omega,0}(w_n) = 0$ , we obtain

$$d_Y(\omega) \leq S_{\omega,Y}(\lambda_n w_n) = \lambda_n^2 S_{\omega,0}(w_n) = \lambda_n^2 d_0(\omega),$$

which implies that  $d_Y(\omega) \leq d_0(\omega)$  and thus  $d_Y(\omega) = d_0(\omega)$ .

The following lemma is a variant of the Brézis-Lieb lemma from [11].

**Lemma 4.10.** *Let  $\{u_n\}$  be a bounded sequence in  $W(\mathbb{R})$  such that  $u_n \rightarrow u$  almost everywhere in  $\mathbb{R}$ . Then,  $u \in W(\mathbb{R})$  and*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \{|u_n|^2 \log |u_n|^2 - |u_n - u|^2 \log |u_n - u|^2\} dx = \int_{\mathbb{R}} |u|^2 \log |u|^2 dx.$$

*Proof.* First, by (5.9) in the Appendix, recall  $|z|^2 \log |z|^2 = A(|z|) - B(|z|)$  for every  $z \in \mathbb{C}$ . By the weak-lower semicontinuity of the  $L^2(\mathbb{R})$ -norm, and by the Fatou lemma, we have  $u \in W(\mathbb{R})$ . It is clear that the sequence  $\{u_n\}$  is bounded in  $L^A(\mathbb{R})$ . Since  $A$  in (5.9) is a convex and increasing function with  $A(0) = 0$ , it follows from the Brézis-Lieb lemma [11, Theorem 2 and Examples (b)] that

$$(4.5) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}} |A(|u_n|) - A(|u_n - u|) - A(|u|)| dx = 0.$$

On the other hand, by the continuous embedding  $W(\mathbb{R}) \hookrightarrow H^1(\mathbb{R})$ , we have that  $\{u_n\}$  is also bounded in  $H^1(\mathbb{R})$ . An easy calculation shows that the function  $B$  defined in (5.9) is convex, increasing, and nonnegative with  $B(0) = 0$ . Furthermore, by the Hölder and Sobolev inequalities, for any  $u, v \in H^1(\mathbb{R})$  we have the following key inequality:

$$(4.6) \quad \int_{\mathbb{R}} |B(|u(x)|) - B(|v(x)|)| dx \leq C(1 + \|u\|_{H^1(\mathbb{R})}^2 + \|v\|_{H^1(\mathbb{R})}^2) \|u - v\|_{L^2}.$$

Then, the function  $B$  satisfies the hypotheses of Theorem 2 and Examples (b) in [11], and therefore,

$$(4.7) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}} |B(|u_n|) - B(|u_n - u|) - B(|u|)| dx = 0.$$

Thus, the result follows from (4.5) and (4.7).  $\square$

*Proof of Theorem 4.4.* We use the argument in [19, Proposition 3] (see [1]). Let  $\{u_n\} \subseteq W(\mathbb{R})$  be a minimizing sequence for  $d_Y(\omega)$ ; then, the sequence  $\{u_n\}$  is bounded in  $W(\mathbb{R})$ . Indeed, it is clear that the sequence  $\|u_n\|_{L^2}^2$  is bounded. Moreover, using (4.3) and the logarithmic Sobolev inequality, and recalling that  $I_{\omega,Y}(u_n) = 0$ , we obtain

$$\left(\frac{1}{2} - \frac{\alpha^2}{\pi}\right) \|u'_n\|_{L^2}^2 \leq \log\left(\frac{e^{Y^2/2} e^{-(\omega+1)}}{\alpha}\right) \|u_n\|_{L^2}^2 + (\log \|u_n\|_{L^2}^2) \|u_n\|_{L^2}^2.$$



Taking  $\alpha > 0$  sufficiently small, we see that  $\|u'_n\|_{L^2}^2$  is bounded, so the sequence  $\{u_n\}$  is bounded in  $H^1(\mathbb{R})$ . Using  $I_{\omega,y}(u_n) = 0$  again along with (4.6), we obtain

$$\|u'_n\|_{L^2}^2 + \int_{\mathbb{R}} A(|u_n(x)|) dx \leq C,$$

which implies, by (5.11) in the [Appendix](#), that the sequence  $\{u_n\}$  is bounded in  $W(\mathbb{R})$ . Furthermore, since  $W(\mathbb{R})$  is a reflexive Banach space, there is  $\varphi \in W(\mathbb{R})$  such that, up to a subsequence,  $u_n \rightharpoonup \varphi$  weakly in  $W(\mathbb{R})$  and  $u_n(x) \rightarrow \varphi(x)$  for almost every  $x \in \mathbb{R}$ .

Next, we show  $\varphi$  is nontrivial. Suppose, by contradiction, that  $\varphi \equiv 0$ . Since the embedding  $H^1(-1, 1) \hookrightarrow C[-1, 1]$  is compact, we see  $u_n(0) \rightarrow \varphi(0) = 0$ . Thus, since  $I_{\omega,y}(u_n) = 0$ , we obtain

$$(4.8) \quad \lim_{n \rightarrow \infty} I_{\omega,0}(u_n) = y \lim_{n \rightarrow \infty} |u_n(0)|^2 = 0.$$

Define the sequence  $v_n(x) = \lambda_n u_n(x)$  with

$$\lambda_n = \exp\left(\frac{I_{\omega,0}(u_n)}{2\|u_n\|_{L^2}^2}\right),$$

where  $\exp(x)$  represents the exponential function. Then, it follows from (4.8) that  $\lim_{n \rightarrow \infty} \lambda_n = 1$ . Moreover, an easy calculation shows that  $I_{\omega,0}(v_n) = 0$  for any  $n \in \mathbb{N}$ . Thus, by the definition of  $d_y(\omega)$ , it follows that

$$d_0(\omega) \leq \frac{1}{2} \lim_{n \rightarrow \infty} \|v_n\|_{L^2}^2 = \frac{1}{2} \lim_{n \rightarrow \infty} \{\lambda_n^2 \|u_n\|_{L^2}^2\} = d_y(\omega).$$

But this is contrary to (4.4), and therefore we conclude that  $\varphi$  is nontrivial.

Now, we prove that  $I_{\omega,y}(\varphi) = 0$ . First, assume that  $I_{\omega,y}(\varphi) < 0$ , by contradiction. By elementary computations, we can see there is  $0 < \lambda < 1$  such that  $I_{\omega,y}(\lambda\varphi) = 0$ . Then, from the definition of  $d_y(\omega)$  and the weak lower semicontinuity of the  $L^2(\mathbb{R})$ -norm, we have

$$d_y(\omega) \leq \frac{1}{2} \|\lambda\varphi\|_{L^2}^2 < \frac{1}{2} \|\varphi\|_{L^2}^2 \leq \frac{1}{2} \liminf_{n \rightarrow \infty} \|u_n\|_{L^2}^2 = d_y(\omega),$$

which is impossible. On the other hand, assume that  $I_{\omega,y}(\varphi) > 0$ . Since the embedding  $W(\mathbb{R}) \hookrightarrow H^1(\mathbb{R})$  is continuous, we see that  $u_n \rightharpoonup \varphi$  weakly in  $H^1(\mathbb{R})$ . Thus, we have

$$(4.9) \quad \|u_n\|_{L^2}^2 - \|u_n - \varphi\|_{L^2}^2 - \|\varphi\|_{L^2}^2 \rightarrow 0,$$

$$(4.10) \quad \|u'_n\|_{L^2}^2 - \|u'_n - \varphi'\|_{L^2}^2 - \|\varphi'\|_{L^2}^2 \rightarrow 0,$$

as  $n \rightarrow \infty$ . Combining (4.9), (4.10), and Lemma 4.10 leads to

$$\lim_{n \rightarrow \infty} I_{\omega, \gamma}(u_n - \varphi) = \lim_{n \rightarrow \infty} I_{\omega, \gamma}(u_n) - I_{\omega, \gamma}(\varphi) = -I_{\omega, \gamma}(\varphi),$$

which combined with  $I_{\omega, \gamma}(\varphi) > 0$  gives us that  $I_{\omega, \gamma}(u_n - \varphi) < 0$  for sufficiently large  $n$ . Thus, by (4.9), and applying the same argument as above, we see that

$$d_Y(\omega) \leq \frac{1}{2} \lim_{n \rightarrow \infty} \|u_n - \varphi\|_{L^2}^2 = d_Y(\omega) - \frac{1}{2} \|\varphi\|_{L^2}^2,$$

which is a contradiction because  $\|\varphi\|_{L^2}^2 > 0$ . Then, we deduce that  $I_{\omega, \gamma}(\varphi) = 0$ . Finally, by the weak lower semicontinuity of the  $L^2(\mathbb{R})$ -norm, we have

$$(4.11) \quad d_Y(\omega) \leq \frac{1}{2} \|\varphi\|_{L^2}^2 \leq \frac{1}{2} \liminf_{n \rightarrow \infty} \|u_n\|_{L^2}^2 = d_Y(\omega),$$

which implies, by the definition of  $d_Y(\omega)$ , that  $\varphi \in \mathcal{N}_{\omega, \gamma}$ . Moreover, by Remark 4.3 and Proposition 3.1, there exists  $\theta \in \mathbb{R}$  such that  $\varphi(x) = e^{i\theta} \varphi_{\omega, \gamma}(x)$ . This concludes the proof of Theorem 4.4.  $\square$

## 5. STABILITY OF THE GROUND STATES

This section is devoted to the proof of Theorem 1.3. We first prove compactness of the minimizing sequences.

**Lemma 5.1.** *Let  $\{u_n\} \subseteq W(\mathbb{R})$  be a minimizing sequence for  $d_Y(\omega)$ . Then, up to a subsequence, there is  $\theta \in \mathbb{R}$  such that  $u_n \rightarrow e^{i\theta} \varphi_{\omega, \gamma}$  in  $W(\mathbb{R})$ .*

*Proof.* By Theorem 4.4, we see there is  $\varphi \in \mathcal{N}_{\omega, \gamma}$  such that, up to a subsequence,  $u_n \rightharpoonup \varphi$  weakly in  $W(\mathbb{R})$  and  $u_n(x) \rightarrow \varphi(x)$  for almost every  $x \in \mathbb{R}$ . Furthermore, by (4.1) and (4.11) we have  $u_n \rightarrow \varphi$  in  $L^2(\mathbb{R})$ . Then, since the sequence  $\{u_n\}$  is bounded in  $H^1(\mathbb{R})$ , from (4.6) we obtain

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} B(|u_n(x)|) dx = \int_{\mathbb{R}} B(|\varphi(x)|) dx.$$

Thus, since  $I_{\omega, \gamma}(u_n) = I_{\omega, \gamma}(\varphi) = 0$  for any  $n \in \mathbb{N}$ , we obtain

$$(5.1) \quad \lim_{n \rightarrow \infty} \left\{ \|u'_n\|_{L^2}^2 + \int_{\mathbb{R}} A(|u_n(x)|) dx \right\} = \|\varphi'\|_{L^2}^2 + \int_{\mathbb{R}} A(|\varphi(x)|) dx.$$

Moreover, by (5.1), the weak lower semicontinuity of the  $L^2(\mathbb{R})$ -norm, and the Fatou lemma, we deduce (see, e.g., [22, Lemma 12 in chapter V])

$$(5.2) \quad \lim_{n \rightarrow \infty} \|u'_n\|_{L^2}^2 = \|\varphi'\|_{L^2}^2,$$

$$(5.3) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}} A(|u_n(x)|) \, dx = \int_{\mathbb{R}} A(|\varphi(x)|) \, dx.$$

Since  $u_n \rightharpoonup \varphi$  weakly in  $H^1(\mathbb{R})$ , it follows from (5.2) that  $u_n \rightarrow \varphi$  in  $H^1(\mathbb{R})$ . Finally, by Proposition 5.2 (ii) (Appendix) and (5.3) we have  $u_n \rightarrow \varphi$  in  $L^4(\mathbb{R})$ . Thus, by definition of the  $W(\mathbb{R})$ -norm, we infer that  $u_n \rightarrow \varphi$  in  $W(\mathbb{R})$ . Thus, by Remark 4.3 and Proposition 3.1, there exists  $\theta \in \mathbb{R}$  such that  $\varphi(x) = e^{i\theta} \varphi_{\omega, y}(x)$ . This finishes the proof.  $\square$

*Proof of Theorem 1.3.* We argue by contradiction. Suppose that  $\varphi_{\omega, y}$  is not stable in  $W(\mathbb{R})$ . Then, there exist  $\varepsilon > 0$  and two sequences  $\{u_{n,0}\} \subset W(\mathbb{R})$ ,  $\{t_n\} \subset (0, \infty)$  such that

$$(5.4) \quad \|u_{n,0} - \varphi_{\omega, y}\|_{W(\mathbb{R})} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

$$(5.5) \quad \inf_{\theta \in \mathbb{R}} \|u_n(t_n) - e^{i\theta} \varphi_{\omega, y}\|_{W(\mathbb{R})} \geq \varepsilon, \quad \text{for any } n \in \mathbb{R},$$

where  $u_n$  is the solution of (1.1) with initial data  $u_{n,0}$  (see Proposition 1.1). Set  $v_n(x) = u_n(t_n, x)$ . By (5.4) and conservation laws, we obtain

$$(5.6) \quad \|v_n\|_{L^2}^2 = \|u_n(t_n)\|_{L^2}^2 = \|u_{n,0}\|_{L^2}^2 \rightarrow \|\varphi_{\omega, y}\|_{L^2}^2,$$

$$(5.7) \quad E(v_n) = E(u_n(t_n)) = E(u_{n,0}) \rightarrow E(\varphi_{\omega, y}),$$

as  $n \rightarrow \infty$ . In particular, it follows from (5.6) and (5.7) that, as  $n \rightarrow \infty$ ,

$$(5.8) \quad S_{\omega, y}(v_n) \rightarrow S_{\omega, y}(\varphi_{\omega, y}) = d_y(\omega).$$

Next, combining (5.6) and (5.8) leads us to  $I_{\omega, y}(v_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Define the sequence  $f_n(x) = \rho_n v_n(x)$  with

$$\rho_n = \exp \left( \frac{I_{\omega, y}(v_n)}{2\|v_n\|_{L^2}^2} \right).$$

It is clear that  $\lim_{n \rightarrow \infty} \rho_n = 1$  and  $I_{\omega, y}(f_n) = 0$  for any  $n \in \mathbb{R}$ . Furthermore, since the sequence  $\{v_n\}$  is bounded in  $W(\mathbb{R})$ , we get that  $\|v_n - f_n\|_{W(\mathbb{R})} \rightarrow 0$  as  $n \rightarrow \infty$ . Then, by (5.8), we have that  $\{f_n\}$  is a minimizing sequence for  $d_y(\omega)$ . Thus, by Lemma 5.1, up to a subsequence, there is  $\theta_0 \in \mathbb{R}$  such that  $f_n \rightarrow e^{i\theta_0} \varphi_{\omega, y}$  in  $W(\mathbb{R})$ . Therefore, by using the triangular inequality, we have

$$\|u_n(t_n) - e^{i\theta_0} \varphi_{\omega, y}\|_{W(\mathbb{R})} \leq \|v_n - f_n\|_{W(\mathbb{R})} + \|f_n - e^{i\theta_0} \varphi_{\omega, y}\|_{W(\mathbb{R})} \rightarrow 0,$$

as  $n \rightarrow \infty$ , which is in contradiction to (5.5). This finishes the proof.  $\square$

## APPENDIX

The functional of energy in (1.5), in general, fails to be finite and of class  $C^1$  on  $H^1(\mathbb{R})$ . Because of this loss of smoothness, to study existence of solutions to (1.1) and (1.6), it is convenient to work in a suitable Banach space endowed with a Luxemburg-type norm in order to make functional  $E$  well defined and  $C^1$  smooth. Thus, define

$$F(z) = |z|^2 \log |z|^2 \quad \text{for every } z \in \mathbb{C},$$

and as in [13], define the functions  $A, B$  on  $[0, \infty)$  by

$$(5.9) \quad \begin{aligned} A(s) &= \begin{cases} -s^2 \log(s^2), & \text{if } 0 \leq s \leq e^{-3}; \\ 3s^2 + 4e^{-3}s - e^{-6}, & \text{if } s \geq e^{-3}; \end{cases} \\ B(s) &= F(s) + A(s). \end{aligned}$$

Furthermore, let functions  $a, b$  be defined by

$$(5.10) \quad \begin{aligned} a(z) &= \frac{z}{|z|^2} A(|z|), \\ b(z) &= \frac{z}{|z|^2} B(|z|), \end{aligned}$$

for  $z \in \mathbb{C}, z \neq 0$ . Notice we have  $b(z) - a(z) = z \log |z|^2$ . It follows that  $A$  is a nonnegative convex and increasing function, and

$$A \in C^1([0, +\infty)) \cap C^2((0, +\infty)).$$

The Orlicz space  $L^A(\mathbb{R})$  corresponding to  $A$  is defined by

$$L^A(\mathbb{R}) = \{u \in L^1_{\text{loc}}(\mathbb{R}) : A(|u|) \in L^1(\mathbb{R})\},$$

equipped with the Luxemburg norm

$$\|u\|_{L^A(\mathbb{R})} = \inf \left\{ k > 0 : \int_{\mathbb{R}} A(k^{-1}|u(x)|) dx \leq 1 \right\}.$$

Here, as usual  $L^1_{\text{loc}}(\mathbb{R})$  is the space of all locally Lebesgue integrable functions. It is proved in [13, Lemma 2.1] that  $A$  is a Young function which is  $\Delta_2$ -regular, and  $(L^A(\mathbb{R}), \|\cdot\|_{L^A})$  is a separable reflexive Banach space.

Next, consider the reflexive Banach space  $W(\mathbb{R}) = H^1(\mathbb{R}) \cap L^A(\mathbb{R})$  equipped with the usual norm  $\|u\|_{W(\mathbb{R})} = \|u\|_{H^1(\mathbb{R})} + \|u\|_{L^A(\mathbb{R})}$ . We can see here that  $W(\mathbb{R}) = \{u \in H^1(\mathbb{R}) : |u|^2 \log |u|^2 \in L^1(\mathbb{R})\}$  (see (1.4)). This follows from the definition of the spaces  $L^A(\mathbb{R})$  and  $W(\mathbb{R})$  (see [13, Proposition 2.2] for more details). Furthermore, one has the following chain of continuous embedding:

$$W(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \hookrightarrow W'(\mathbb{R}),$$

where  $W'(\mathbb{R}) = H^{-1}(\mathbb{R}) + (L^A(\mathbb{R}))'$  is the dual space of  $W(\mathbb{R})$  equipped with the usual norm.

Next, we list some properties of the Orlicz space  $L^A(\mathbb{R})$  that we have used through our manuscript. (For a proof of such statements, see Lemma 2.1 of [13].)

**Proposition 5.2.** *Let  $\{u_m\}$  be a sequence in  $L^A(\mathbb{R})$ . The following facts hold:*

- (i) *If  $u_m \rightarrow u$  in  $L^A(\mathbb{R})$ , then  $A(|u_m|) \rightarrow A(|u|)$  in  $L^1(\mathbb{R})$  as  $m \rightarrow \infty$ .*
- (ii) *Let  $u \in L^A(\mathbb{R})$ . If  $u_m(x) \rightarrow u(x)$  for almost every  $x \in \mathbb{R}$  and if*

$$\lim_{m \rightarrow \infty} \int_{\mathbb{R}} A(|u_m(x)|) dx = \int_{\mathbb{R}} A(|u(x)|) dx.$$

*Then,  $u_m \rightarrow u$  in  $L^A(\mathbb{R})$  as  $m \rightarrow \infty$ .*

- (iii) *For any  $u \in L^A(\mathbb{R})$ , we have*

$$(5.11) \quad \min \{ \|u\|_{L^A(\mathbb{R})}, \|u\|_{L^A(\mathbb{R})}^2 \} \leq \int_{\mathbb{R}} A(|u(x)|) dx \\ \leq \max \{ \|u\|_{L^A(\mathbb{R})}, \|u\|_{L^A(\mathbb{R})}^2 \}.$$

The following lemma is the foundation for showing the  $C^1$ -property of the energy functional  $E$  in (1.5) on  $W(\mathbb{R})$ .

**Lemma 5.3.** *The operator  $L : u \rightarrow \partial_x^2 u + \gamma \delta(x)u + u \log |u|^2$  is continuous from  $W(\mathbb{R})$  to  $W'(\mathbb{R})$ . The image under  $L$  of a bounded subset of  $W(\mathbb{R})$  is a bounded subset of  $W'(\mathbb{R})$ .*

*Proof.* Note that, as usual, the operator  $(-\partial_x^2 - \gamma \delta(x))u$  is naturally extended to  $H^1(\mathbb{R}) \rightarrow H^{-1}(\mathbb{R})$  via the relation (see (1.2))

$$\langle (-\partial_x^2 - \gamma \delta(x))u, v \rangle = t_\gamma(u, v), \quad \text{for } u, v \in H^1(\mathbb{R}).$$

Now, using  $W(\mathbb{R}) \hookrightarrow H^1(\mathbb{R})$ , we get that the linear operator  $u \rightarrow -\partial_x^2 u - \gamma \delta(x)u$  is continuous from  $W(\mathbb{R})$  to  $W'(\mathbb{R})$ . Thus, since  $u \rightarrow u \log |u|^2$  is continuous and bounded from  $W(\mathbb{R})$  to  $W'(\mathbb{R})$  (see [13, Lemma 2.6]), it follows that the operator  $L : W(\mathbb{R}) \rightarrow W'(\mathbb{R})$  is continuous and bounded. Lemma 5.3 is thus proved.  $\square$

From Lemma 5.3, we have the following consequence.

**Proposition 5.4.** *The operator  $E : W(\mathbb{R}) \rightarrow \mathbb{R}$  is of class  $C^1$ , and for  $u \in W(\mathbb{R})$ , the Fréchet derivative of  $E$  in  $u$  exists and is given by*

$$E'(u) = -\partial_x^2 u - \gamma \delta(x)u - u \log |u|^2 - u \in W'(\mathbb{R}).$$

*Proof.* We first show that  $E$  is continuous. Notice that

$$(5.12) \quad E(u) = \frac{1}{2} t_\gamma(u) + \frac{1}{2} \int_{\mathbb{R}} A(|u|) dx - \frac{1}{2} \int_{\mathbb{R}} B(|u|) dx.$$

The first term in the righthand side of (5.12) is continuous from  $H^1(\mathbb{R})$  to  $\mathbb{R}$ , and it follows from Proposition 5.2 (i) that the second term is continuous from  $L^4(\mathbb{R})$  to  $\mathbb{R}$ . Moreover, by (4.6) we get that the third term in the righthand side of (5.12) is continuous from  $H^1(\mathbb{R})$  to  $\mathbb{R}$ . Therefore,  $E \in C(W(\mathbb{R}), \mathbb{R})$ . Now, direct calculations show that, for  $u, v \in W(\mathbb{R})$ ,  $t \in (-1, 1)$  (see Proposition 2.7 in [13]),

$$\lim_{t \rightarrow 0} \frac{E(u + tv) - E(u)}{t} = \langle -\partial_x^2 u - \gamma \delta(x)u - u \log |u|^2 - u, v \rangle_{W(\mathbb{R}) - W'(\mathbb{R})}.$$

Thus,  $E$  is Gâteaux differentiable. Then, by Lemma 5.3 we see that  $E$  is Fréchet differentiable and  $E'(u) = -\partial_x^2 u - \gamma \delta(x)u - u \log |u|^2 - u$ .  $\square$

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