RT-MAT 2003-20

SYMMETRIC UNITS IN ALTERNATIVE LOOP RINGS

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Setembro 2003

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ABSTRACT. Let L be an RA loop, that is, a loop whose loop ring in any characteristic is an alternative, but not associative, ring. For $\alpha = \sum \alpha_\ell \ell$ in a loop ring RL, define $\alpha^\sharp = \sum \alpha_\ell \ell^{-1}$ and call α *symmetric* if $\alpha^\sharp = \alpha$. We find necessary and sufficient conditions under which the symmetric units are closed under multiplication (and hence form a subloop of the loop of units in RL) when R has characteristic two and when $R = \mathbb{Z}$ is the ring of rational integers.

1. INTRODUCTION

A *loop ring* is an algebraic object RL, constructed in the same way as a group ring, but in which the underlying loop L is not necessarily associative. This paper is concerned with loop rings which are alternative, but not associative. Loops which give rise to such loop rings (over rings of any characteristic) are called *RA (ring alternative) loops.* The best reference for information about RA loops and their loop rings is the monograph [4]. One property of RA loops used implicitly throughout is their *diassociativity*: any subloop generated by just two elements is associative (so parentheses to indicate order of multiplication in monomials are not required). It is also important to remember that an RA loop L possesses an element $s \neq 1$, which we always so label, which is both a unique nonidentity commutator and a unique nonidentity associator; that is, if $a, b \in L$ do not commute, then $ba = sab$ and, if $a, b, c \in L$ do not associate, then $(ab)c = [a(bc)]s$. (It is easy to see that s is necessarily central and of order 2.)

If $\alpha = \sum \alpha_\ell \ell$ is an element of a loop ring RL, we define $\alpha^\sharp = \sum \alpha_\ell \ell^{-1}$ and call α *symmetric* if $\alpha^\sharp = \alpha$.[1]

In this paper, with a given coefficient ring R fixed, we call an RA loop *admissible* if the product of symmetric units is symmetric; equivalently, if

1991 *Mathematics Subject Classification.* Primary 17D05; Secondary 20N05, 16S34.

The first author is again most grateful to FAPESP of Brasil and to the Instituto de Matemática e Estatística of the Universidade de São Paulo where he is always made to feel very welcome and where the environment for thinking about mathematics is marvellous.

This research was supported by the Natural Sciences and Engineering Research Council of Canada, Grant No. OGP0009087, and by FAPESP, Proc. 2000/07291-0 and CNPq., Proc. 300243/79-0(RN) of Brasil.

May 8, 2003.

[1] In group rings, the map $\alpha \mapsto \alpha^\sharp$ is denoted $\alpha \mapsto \alpha^*$. Since alternative loop rings have a canonical involution which is denoted $\alpha \to \alpha^*$, we use $\sharp$ instead of $*$ for the involution of central interest here.

the symmetric units form a subloop of the loop of units in RL. Let

$$S_1 = \{\ell \in L \mid \ell^2 = 1\}, \quad S_2 = \{\ell + \ell^{-1} \mid \ell^2 \neq 1\}, \text{ and } S = S_1 \cup S_2.$$

Then $\alpha \in RL$ is symmetric if and only if it is a linear combination of elements of S. Since the mapping $\alpha \mapsto \alpha^\sharp$ is an antiautomorphism of RL, the product of symmetric units α, β is symmetric if and only if α and β commute. In particular, this will be the case if the elements of S commute pairwise. The converse is true in certain situations.

For example, let $R = F$ be a field of characteristic two, let L be an RA 2-loop, and assume that symmetric units in FL commute. It is known that $\Delta(L)$, the augmentation of FL, is nilpotent [5], so $1 + \ell + \ell^{-1}$ is a unit for any $\ell \in L$ (because $\ell + \ell^{-1} \in \Delta(L)$). It follows that any two such elements must commute, hence any two elements of S commute.

We summarize.

Remark 1.1. An RA 2-loop over a field of characteristic two is admissible if and only if the elements of S commute pairwise.

2. The Modular Case

In this section, the coefficient ring is always a field F of characteristic two.

Proposition 2.1. *Let L be an admissible RA loop with unique commutator/associator s. Then every element of order 2 in L is central and, if $\ell \in L$ is not central, then ℓ has order 4.*

Proof. We follow some arguments in [1]. Assume $\ell \in L$ has order 2. Then ℓ commutes with any $k \in L$ satisfying $k^2 = 1$. If $k^2 \neq 1$, then ℓ commutes with $k + k^{-1}$. The equation $\ell(k + k^{-1}) = (k + k^{-1})\ell$ implies

$$\ell k + \ell k^{-1} = k\ell + k^{-1}\ell.$$

If $\ell k \neq k\ell$, then $\ell k = k^{-1}\ell$, so $\ell k \ell = k^{-1}\ell^2 = k^{-1}$, which implies that $k\ell$ has order 2. Thus ℓ and $k\ell$ commute, so ℓ and k commute, a contradiction. All this shows that elements $\ell \in L$ with $\ell^2 = 1$ are central.

Next, suppose the noncentral element ℓ does not have order 4. Then $\ell^4 \neq 1$ (since $\ell^2 = 1$ implies centrality). Choose k such that $k\ell \neq \ell k$. The loop $G = \langle k, \ell \rangle$ is a nonabelian admissible group not of exponent 4. The proof of [1, Lemma 2] establishes that the set

$$A = \{t \in L \mid t^4 \neq 1\}$$

is commutative and, if $b \notin \langle A \rangle$, then $b^{-1}ab = a^{-1}$ for all $a \in A$. Now $\ell \in A$, so $k \notin \langle A \rangle$ (since $k\ell \neq \ell k$). Thus $k^{-1}\ell k = \ell^{-1}$ while, on the other hand, $k^{-1}\ell k = sk$. Thus $\ell^2 = s$ and $\ell^4 = s^2 = 1$, a contradiction which completes the proof. □

Corollary 2.2. *An admissible RA loop has exponent 4.*

Proof. Because of the Proposition, it suffices to show that if $a \in L$ is central, then $a^4 = 1$. So take $a \in L$ be central and $b \in L$ not central (so $b^4 = 1$). Then ab is not central, so $(ab)^4 = 1$. Since $(ab)^4 = a^4b^4 = a^4$, the result follows. □

Now let $k, \ell \in L$ with $k\ell \neq \ell k$. (Thus $k\ell = s\ell k$.) In particular $k^2 \neq 1$ and $\ell^2 \neq 1$. Since $(k+k^{-1})(\ell+\ell^{-1}) = (\ell+\ell^{-1})(k+k^{-1})$, we obtain

$$k\ell + k\ell^{-1} + k^{-1}\ell + k^{-1}\ell^{-1} = \ell k + \ell k^{-1} + \ell^{-1}k + \ell^{-1}k^{-1}.$$

Since $k\ell \notin \{k\ell^{-1}, k^{-1}\ell, \ell k\}$, we must have

$$k\ell \in \{k^{-1}\ell^{-1}, \ell k^{-1}, \ell^{-1}k, \ell^{-1}k^{-1}\}.$$

If $k\ell = k^{-1}\ell^{-1}$, then $k^2 = \ell^{-2}$. Let $x = k^{-1}$ and $y = k\ell$. Then $y^2 = k\ell k\ell = sk^2\ell^2 = s$ and $x^{-1}yx = k^2\ell k^{-1} = \ell^{-1}k^{-1} = y^{-1}$. Note that $\langle k, \ell\rangle = \langle x, y\rangle$.

If $k\ell = \ell k^{-1}$, then $\ell^{-1}k\ell = k^{-1}$.

If $k\ell = \ell^{-1}k$, then $k\ell k^{-1} = \ell^{-1}$.

If $k\ell = \ell^{-1}k^{-1} = (k\ell)^{-1}$, then $(k\ell)^2 = 1$, so $k\ell$ is central, implying $k\ell = \ell k$, which is not true.

To summarize, if $k\ell \neq \ell k$, then the group generated by k and ℓ is

$$H = \langle x, y \mid x^4 = y^4 = 1, y^{-1}xy = x^{-1}\rangle. \tag{2.1}$$

Note that this is the group (of order 16) labelled $16\Gamma_2c_2$ in the Hall and Senior catalogue of groups of 2-power order [7].

It is known that any torsion RA loop is the direct product of an indecomposable loop (which is necessarily a 2-loop) and a possibly trivial abelian group [4, Proposition V.1.1], [3, Theorem 6]. Thus, if we can classify the (finite or torsion) indecomposable admissible RA loops, then we have in fact classified all admissible (finite or torsion) RA loops because of the proposition which follows. Part of the proof we present requires an elementary fact about the loop ring of a direct product.

Remark 2.3. Suppose $\alpha \in RL$ and $\beta \in RK$ are elements of loop rings RL and RK and suppose that $\alpha\beta = 0 \in R[L \times K]$. Then $\alpha = 0$ or $\beta = 0$. To see this, note that writing $\alpha = \sum \alpha_\ell \ell$ and $\beta = \sum \beta_k k$ gives $\alpha\beta = \sum \alpha_\ell\beta_k \ell k$, which is a linear combination of distinct elements of the loop $L \times K$.

Proposition 2.4. *The direct product $L \times A$ of an RA 2-loop L and an abelian group A is admissible if and only if L is admissible and A has exponent 2.*

Proof. If L is admissible and A has exponent 2, elements of order 2 in $L \times A$ are certainly central, so, to show that $L \times A$ is admissible, it suffices to show that elements of the form $\ell a + (\ell a)^{-1}$, $\ell \in L, a \in A$ commute. This is immediate because $\ell a + (\ell a)^{-1} = \ell a + \ell^{-1}a = (\ell + \ell^{-1})a$, with a central and elements of the form $\ell + \ell^{-1}$ commuting pairwise.

Now assume that $L \times A$ is admissible. Since a subloop of an admissible loop is admissible, L is admissible, so it remains only to show that A has exponent 2. Suppose this is not the case. Take $a \in A$ with $a^2 \neq 1$ and let $k \in L$ be a noncentral element with $k^2 \neq s$. (The existence of such k is a consequence of the fact that any noncentral element lives in a group H defined in (2.1).) There exists $\ell \in L$ such that $k\ell \neq \ell k$. We claim that $\ell a + (\ell a)^{-1}$ does not commute with $k + k^{-1}$. This can be established by showing that $\ell^{-1}(a + a^{-1})$ and $k + k^{-1}$ do not commute, or, equivalently, that

$$[\ell^{-1}(a+a^{-1})](k+k^{-1}) + (k+k^{-1})[\ell^{-1}(a+a^{-1})] \neq 0, \tag{2.2}$$

because

$$\ell a + (\ell a)^{-1} = \ell a + \ell^{-1}a^{-1} = (\ell + \ell^{-1})a + \ell^{-1}(a + a^{-1})$$

and the first term on the right here commutes with $k + k^{-1}$ because A is central and L is admissible. The sum in (2.2) is

$$(\ell^{-1}k + \ell^{-1}k^{-1} + k\ell^{-1} + k^{-1}\ell^{-1})(a + a^{-1}),$$

which is the product of elements in FL and FA. Using remark 2.3, this is 0 if and only if one of the two factors is 0. Since $a + a^{-1} \neq 0$, we have only to prove that

$$\ell^{-1}k + \ell^{-1}k^{-1} + k\ell^{-1} + k^{-1}\ell^{-1} \neq 0.$$

Since k and ℓ do not commute, this element is

$$\ell^{-1}k + \ell^{-1}k^{-1} + s\ell^{-1}k + s\ell^{-1}k^{-1} = (1+s)\ell^{-1}(k + k^{-1}).$$

This is 0 if and only if $(1+s)(k+k^{-1}) = 0$. This is not the case, however, because $(1+s)(k+k^{-1}) = k+k^{-1}+sk+sk^{-1}$ and $k^2 \neq 1$, $s \neq 1$, $k^2 \neq s$. □

In light of Proposition 2.4, we turn our attention to the classification of finite indecomposable RA loops as found in Chapter V of [4]. (See also [8].) In examining these references, it is helpful to note a few facts about RA loops.

An RA loop has the form $L = G \cup Gu$, where G is (certain kind of) group, u is an element not in G, $g \to g^*$ is an involution on G, g_0 is an element in $\mathcal{Z}(G)$ (the centre of G) which is fixed by the involution, and multiplication in L is defined by

$$g(hu) = (hg)u$$
$$(gu)h = (gh^*)u$$
$$(gu)(hu) = g_0h^*g$$

(See [4, §II.5].) In consequence, such L is denoted $M(G, *, g_0)$. It will prove useful also to know that for any $x \in L$,

$$x^* = \begin{cases} x & \text{if } x \text{ is central} \\ sx & \text{if } x \text{ is not central} \end{cases}$$

[4, Theorem IV.3.1]. If L is finite and indecomposable (that is, not a nontrivial direct product), the group $G = D \times C$ is the direct product of an indecomposable group D and a cyclic group C, which might be trivial. The group D is generated by two noncommuting elements x, y.

Assume now that L is admissible. From what we have shown, D must be the group H described in (2.1). Since $\mathcal{Z}(H) = C_2 \times C_2$ and elements in L of order 2 are central, Table 3 on p. 142 of [4] shows clearly that L must be of type $\mathcal{L}_4$, with $D = D_5 = H$ and $s = g_0$. Since the unique nonidentity commutator/associator of L is in D, Theorem V.1.7 of [4] shows that C is in fact trivial. Thus $G = H$ has order 16 and $L = M(16\Gamma_2C_2, *, s)$ (a Moufang loop of order 32).[2]

Theorem 2.5. *Let L be a finite RA 2-loop and let F be a field of characteristic two. Then L is admissible if and only if $L = L_0 \times A$, where $L_0 = M(16\Gamma_2c_2, *, s)$ and A is an abelian group of exponent two.*

Proof. The RA 2-loop L can be written as the product $L_0 \times A$ of an indecomposable RA loop L_0 and an abelian group A. If this is admissible, then $L_0 = M(16\Gamma_2c_2, *, s)$ as shown above and A has exponent two, by Proposition 2.4. Conversely, if $L_0 = M(16\Gamma_2c_2, *, s)$ and A is an abelian group of exponent 2, again appealing to Proposition 2.4, to prove that $L_0 \times A$ is admissible, it is sufficient to prove that L_0 is admissible. The reader may check that in $H = D_5$, as presented in (2.1), the set $C = \{1, x, x^2, x^3, y^2, xy^2, x^2y^2, x^3y^2\}$ is an abelian subgroup of index 2 and that $t^{-1}ct = c^{-1}$ for every $c \in C$ and every $t \notin C$. By [1], $G = D_5 = 16\Gamma_2c_2$ is admissible. In particular, elements of order 2 in G are central. By the definition of multiplication in $L = G \cup Gu$, if $g \in G$, then $(gu)^2 = (gu)(gu) = g_0g^*g = sg^*g$. If $g^2 = 1$, then g is central so $g^* = g$ and $sg^*g = s \neq 1$. If $g^2 \neq 1$, then g is not central, so $g^* = sg$ and $sg^*g = s^2g^2 = g^2 \neq 1$. Thus the only elements of order 2 in $L = G \cup Gu$ are in G and hence central. It remains only to verify that two kinds of pairs of elements commute in FL.

Case 1: Let $\alpha = g + g^{-1}$ and $\beta = hu + (hu)^{-1}$, $g, h \in G$. The rules for multiplication in L give $(hu)^{-1} = u^{-1}h^{-1} = suh^{-1} = s(h^{-1})^*u$. So,

$$\begin{aligned}\alpha\beta &= (g + g^{-1})(hu + s(h^{-1})^*u)\\ &= g(hu) + sg[(h^{-1})^*u] + g^{-1}(hu) + sg^{-1}[(h^{-1})^*u]\\ &= \left[hg + s(h^{-1})^*g + hg^{-1} + s(h^{-1})^*g^{-1}\right]u\\ &= \left[h(g + g^{-1}) + s(h^{-1})^*(g + g^{-1})\right]u\\ &= \left[(h + s(h^{-1})^*)(g + g^{-1})\right]u = \left[t(g + g^{-1})\right]u,\end{aligned}$$

with $t = h + s(h^{-1})^*$. Also

$$\begin{aligned}\beta\alpha &= (hu + s(h^{-1})^*u)(g + g^{-1})\\ &= (hu)g + (hu)g^{-1} + [s(h^{-1})^*u]g + [s(h^{-1})^*u]g^{-1}\\ &= \left[hg^* + h(g^{-1})^* + s(h^{-1})^*g^* + s(h^{-1})^*(g^{-1})^*\right]u\\ &= \left[(h + s(h^{-1})^*)g^* + (h + s(h^{-1})^*)(g^{-1})^*\right]u\end{aligned}$$

[2] In the classification and labelling of the Moufang loops of order less than 64, this loop has also been denoted 32/65 [6].

$$= \left[(h+s(h^{-1})^*)(g+g^{-1})^*\right]u = \left[t(g+g^{-1})^*\right]u.$$

If h is not central, then $s(h^{-1})^* = ssh^{-1} = h^{-1}$ and the elements $t(g+g^{-1})$, $t(g+g^{-1})^*$ clearly commute; while such is clearly also the case if h is central. Thus $\alpha\beta = \beta\alpha$.

Case 2: Let $\alpha = gu + (gu)^{-1}$ and $\beta = hu + (hu)^{-1}$, $g, h \in G$. We have $(gu)^{-1} = s(g^{-1})^*u$ and $(hu)^{-1} = s(h^{-1})^*u$, so

$$\begin{aligned}
\alpha\beta &= [gu + s(g^{-1})^*u][hu + s(h^{-1})^*u] \\
&= (gu)(hu) + s(gu)[(h^{-1})^*u] + s[(g^{-1})^*u]hu + [(g^{-1})^*u][(h^{-1})^*u] \\
&= g_0h^*g + sg_0h^{-1}g + sg_0h^*(g^{-1})^* + g_0h^{-1}(g^{-1})^* \\
&= sh^*g + h^{-1}g + h^*(g^{-1})^* + sh^{-1}(g^{-1})^* \qquad \text{(because } g_0 = s\text{)} \\
&= (sh^* + h^{-1})g + (h^* + sh^{-1})(g^{-1})^*.
\end{aligned}$$

Since $h^* + sh^{-1} = s(sh^* + h^{-1})$, we have

$$\alpha\beta = (sh^* + h^{-1})(g + s(g^{-1})^*). \tag{2.3}$$

By symmetry,

$$\beta\alpha = (sg^* + g^{-1})(h + s(h^{-1})^*). \tag{2.4}$$

If g is not central, then $g^* = sg$ and $(g^{-1})^* = sg^{-1}$, so $g+s(g^{-1})^* = g+g^{-1} = sg^* + g^{-1}$ and, if h is not central, then $h + s(h^{-1})^* = h + h^{-1} = h + s(h^{-1})^*$. In all cases, each factor on the right of (2.3) commutes with each factor on the right of (2.4), implying that α and β commute. This completes the proof. □

Interestingly, Theorem 2.5 can be extended verbatim to torsion loops, as we proceed to show. First, we observe that the concepts of "torsion" and "local finiteness" are the same for RA loops.

Lemma 2.6. *An RA loop is torsion if and only if it is locally finite.*

Proof. A locally finite loop is always torsion. On the other hand, remember that if x and y are any two elements in an RA loop, then $yx = xy$, or $yx = sxy$ (s the unique nonidentity commutator/associator), and if x, y, z are any three elements, then $(xy)z = x(yz)$ or $(xy)z = sx(yz)$. Since s is central of order 2, any element in a subloop K generated by elements $x_1, x_2, \ldots, x_n$ can be written in the form $s^\epsilon(\cdots((x_1^{i_1}x_2^{i_2})x_3^{i_3})\cdots x_n^{i_n})$, $i_j \in \mathbb{Z}$, $\epsilon = 0, 1$, so, if L is torsion, then K is finite. □

Suppose that $L = M(G, *, g_0)$ is an admissible torsion RA loop. Clearly G is an admissible group, so by [1], $G = H \times E$ is the direct product of an elementary abelian group and a group H which is one of four types.

i. H has an abelian subgroup A of index 2 and an element b of order 4 such that $b^{-1}ab = a^{-1}$ for all $a \in A$;

ii. $H = Q_8 \times C$ is the direct product of the quaternion group Q_8 and a cyclic group C of order 4, or the direct product of two quaternion groups;

iii. H is the central product of the group $\langle x, y \mid x^4 = y^4 = 1, x^2 = (y, x)\rangle$ with a quaternion group;

iv. H is isomorphic to either

$$32\Gamma_4c_3 = \langle x, y, u \mid x^4 = y^4 = 1,\\ x^2 = (y,x), y^2 = (u^2 = (u,x), x^2y^2 = (u,y)\rangle$$

$$64\Gamma_{13}a_5 = \langle x, y, u, v \mid x^4 = y^4 = (v,u) = 1,\\ x^2 = v^2 = (y,x) = (v,y), y^2 = u^2 = (u,x),\\ x^2y^2 = (u,y) = (v,x)\rangle.$$

(The latter two groups are denoted H_{32} and H_{245}, respectively, in [1].)

Since an RA loop has a unique commutator, groups of types iii and iv are quickly eliminated. Suppose $G = Q_8 \times C \times E$ is a group with $H = Q_8 \times C$ of type ii. Since the unique nonidentity commutator/associator of G lies in Q_8, $M(G, *, g_0) = M(Q_8, *, g_0) \times (C \times E)$ by [4, Proposition V.1.6]. This loop is not admissible by Proposition 2.4. It follows that $G = H \times E$, with H of type i. Since the unique nonidentity commutator of G is necessarily in H, we must have E of exponent 2, quoting Proposition 2.4 again. Thus G has an abelian subgroup (which we also call) A of index 2 such that $x^{-1}ax = a^{-1}$ for all $a \in A$ and $x \notin A$. Since $[G\colon \mathcal{Z}(G)] = 4$ (the group G defining an RA loop is a $C_2 \times C_2$ extension of its centre), $A \subseteq \mathcal{Z}$ is not possible, so choose $a \in A \setminus \mathcal{Z}$ and x such that $ax \neq xa$. Since A is abelian, $x \notin A$, so $x^{-1}ax = a^{-1}$ on the one hand and $x^{-1}ax = sa$ on the other. It follows that $a^2 = s$ and a has order 4. Since A is a 2-group and a has maximal order in A, $A = \langle a\rangle \times A_0$ for some subgroup A_0. Suppose some element $t \in A_0$ has order 4. Then t and x cannot commute; otherwise, $x^{-1}tx = t = t^{-1}$ would imply that $t^2 = 1$, which is not true. Thus $x^{-1}tx = st = t^{-1}$, so $t^2 = s = a^2 \in \langle a\rangle \cap A_0 = \{1\}$. This contradiction shows that A_0 has exponent two (and hence is central).

Now $x^2 \in A$, so we can write $x^2 = a^{i_0}a_0$, $a_0 \in A_0$, $i_0 \in \mathbb{Z}$. If $a_0 \neq 1$, this element has maximal order in A_0, so $A_0 = \langle a_0\rangle \times A_1$ for some group A_1, a factorization of A_0 which evidently also holds if $a_0 = 1$. Thus $A = \langle a\rangle \times \langle a_0\rangle \times A_1$. Let $B = \langle a, x, a_0\rangle$ be the group generated by a, x and a_0. We claim that $G = B \times A_1$. Since $A \subsetneq BA_1 \subseteq G$ and $[G\colon A] = 2$, we have $G = BA_1$. The subgroup A_1 is normal since it is central and so is B, because a_0 is central and $s = a^2 \in B$ (implying, for example, that, for any t, $t^{-1}xt = x$ or a^2x is in B).

To show that $B \cap A_1 = \{1\}$, let $b = a_0^{\epsilon}a^ix^j \in B \cap A_1$. If $j = 2j_1$ is even, then $b = a_0^{\epsilon}a^i(x^2)^{j_1} = a_0^{\epsilon}a^i(a^{i_0}a_0)^j \in (\langle a_0\rangle \times \langle a\rangle) \cap A_1 = \{1\}$. If $j = 2j_1 + 1$ is odd, then $b = a_0^{\epsilon}a^ia^{i_0j_1}a_0^{j_1}x \in A_1$ implies $x \in \langle a_0\rangle \times \langle a\rangle \times A_1 = A$, a contradiction.

We have shown that when $L = M(G, *, g_0)$ is torsion, the group $G = B \times A_1$ is the direct product of a finite group B and an abelian group A_1 of exponent two. Suppose $g_0 = (b, a_1)$ has a component $a_1 \in A_1$ which is different from 1. Writing $A_1 = \langle a_1 \rangle \times A_2$ and replacing B by $B \times \langle a_1 \rangle$, we can assume that $g_0 \in B$. Thus $M(B \times A_1, *, g_0) = M(B, *, g_0) \times A_1$ [4, Proposition V.1.6] with $M(B, *, g_0)$ finite (and admissible). So we obtain the following theorem.

Theorem 2.7. *When the ring of coefficients is a field of characteristic two, a torsion RA loop L is admissible if and only if $L = M(16\Gamma_2 c_2, *, s) \times A$ is the direct product of the loop $M(16\Gamma_2 c_2, *, s)$ and an abelian group of exponent two.*

3. Admissibility over Z

In this brief section, we observe that when the ring of coefficients is the ring of rational integers, an RA loop is admissible essentially when a group is admissible. In any RA loop L, the set $T(L)$ of torsion units forms a subloop [4, Lemma VIII.4.1] and, if L is admissible, it can be shown exactly as in [2] that every subloop of $T(L)$ is normal in $T(L)$; thus $T(L)$ is an abelian group or a Moufang Hamiltonian loop, without elements of odd order (see again [2]).

In particular, we have the following analogue of Theorem 2.7 in this situation.

Theorem 3.1. *If L is a torsion (equivalently, locally finite) RA loop, then L is admissible if and only if L is an abelian group or a Moufang Hamiltonian 2-loop.*

Proof. We have already established necessity. On the other hand, it is easy to see that abelian groups and Moufang Hamiltonian 2-loops are admissible, in the latter case because, in a Hamiltonian loop, if $\ell^2 = ne1$, then $\ell + \ell^{-1} = \ell + s\ell = \ell + \ell^*$, and such elements are central in any RA loop [4, Corollary III.4.3]. □

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