Indecomposable baric algebras, II*

Henrique Guzzo Junior
Instituto de Matemática e Estatística
Universidade de São Paulo
Caixa Postal 20570 – 01452-990, São Paulo, Brazil
E-mail: guzzo@ime.usp.br

1 Introduction

Baric algebras play a central role in the theory of genetic algebras. They were introduced by I.M.H. Etherington [3], aiming for an algebraic treatment of Population Genetics. But the whole class of baric algebras is too large, some conditions (usually with a background in Genetics) must be imposed in order to obtain a workable mathematical object. With this in mind, several classes of baric algebras have been defined: train, Bernstein, special triangular, etc. But there are relevant examples in Genetics which do not give rise to baric algebras, see [6]. As a sample of the recent work in the field of Genetic algebra, see [2], [7], [9], [10] and [11].

2 Join of baric algebras

Let F be a field of characteristic not 2, A an algebra over F, not necessarily associative, commutative or finite dimensional. If $\omega: A \longrightarrow F$ is a nonzero homomorphism, then the ordered pair (A, ω) will be called a baric algebra over F and ω its weight function. For $x \in A$, $\omega(x)$ is called the weight of x. The set $N = \{x \in A \mid w(x) = 0\}$ is a two-sided ideal of A of codimension 1.

A baric homomorphism from (A, ω) to (A', ω') is a homomorphism of F-algebras $\varphi: A \longrightarrow A'$ such that $\omega' \circ \varphi = \omega$. In particular, $\varphi(\ker\omega) \subseteq \ker\omega'$. If φ and φ' are baric homomorphisms, the same holds for $\varphi \circ \varphi'$ and for φ^{-1}, when φ is bijective.

Every baric algebra (A, ω) can be decomposed as $A = Fc \oplus N$, where c is any element of A with $\omega(c) = 1$: for $x \in A$, $x = \omega(x)c + (x - \omega(x)c)$ and $x - \omega(x)c \in \ker\omega$. From this, every left ideal of N, say J, such that $cJ \subseteq J$ is also a left ideal of A. Similarly for right ideals, with the condition $Jc \subseteq J$. The converse is also true. Many (but not all) baric algebras relevant in Genetics, have an idempotent e such that $\omega(e) = 1$. In this case, the subspace Fe is a commutative subalgebra of A.

In this paper, we will always assume the existence of an idempotent of weight 1. There is a natural method of obtaining such algebras. If N is any F-algebra, $\lambda: N \longrightarrow N$ and

*This is an announcement of the results of the paper "Indecomposable baric algebras" and "Indecomposable baric algebras, II", by myself and R. Costa. The first one appeared in Linear Algebra and its Applications, 183: 223-236 (1993). The second will appear soon in Linear Algebra and its Applications.

$\rho: N \longrightarrow N$ are F-linear mappings, define on the vector space $F \bigoplus N$ a multiplication and a weight function by

$$(\alpha, a)(\beta, b) = (\alpha\beta, ab + \alpha\lambda(b) + \beta\rho(a)); \qquad \omega(\alpha, a) = \alpha \tag{1}$$

where $\alpha, \beta \in F$, $a, b \in N$. Clearly ω is a non-zero homomorphism, $e = (1,0)$ is an idempotent of weight 1, $(1,0)(0,n) = (0, \lambda(n))$, $(0,n)(1,0) = (0, \rho(n))$, for all $n \in N$. We denote this algebra by $[N, \lambda, \rho]$. Every baric algebra (A, ω) with idempotent e of weight 1 is obtained by this method, taking $N = ker\omega$, $\lambda = L_e$ and $\rho = R_e$, where L and R are left and right multiplication operators. For easy reference, we denote this class by Ω.

Suppose (A_1, ω_1) and (A_2, ω_2) belong to Ω , with idempotents e_1 and e_2 resp, so $A_1 = Fe_1 \bigoplus N_1$ and $A_2 = Fe_2 \bigoplus N_2$. Consider $N = N_1 \bigoplus N_2$, endowed with the componentwise multiplication. Let $\lambda, \rho: N \longrightarrow N$ be the linear operators

$$\lambda(n_1, n_2) = (e_1 n_1, e_2 n_2), \qquad \rho(n_1, n_2) = (n_1 e_1, n_2 e_2)$$

So in $[N, \lambda, \rho]$, according to (1), we have the multiplication:

$$(\alpha, n_1, n_2)(\alpha', n_1', n_2') = (\alpha\alpha', n_1 n_1' + \alpha e_1 n_1' + \alpha' n_1 e_1, n_2 n_2' + \alpha e_2 n_2' + \alpha' n_2 e_2)$$

This algebra is called the join of (A_1, ω_1) and (A_2, ω_2) and is denoted $(A_1 \vee A_2, \omega_1 \vee \omega_2)$ or simply $A_1 \vee A_2$. The idempotent $(1,0,0)$ is the join of e_1 and e_2, denoted $e_1 \vee e_2$. It is not difficult to prove that this construction is independent of the idempotents e_1 and e_2.

Proposition 1 *Let F be a field, (A_i, ω_i) F-algebras in Ω, $i = 1,2,3$. We have the following baric isomorphisms:*

(i) $(F \vee A_1, id_F \vee \omega_1) \cong (A_1, \omega_1)$

(ii) $(A_1 \vee A_2, \omega_1 \vee \omega_2) \cong (A_2 \vee A_1, \omega_2 \vee \omega_1)$

(iii) $((A_1 \vee A_2) \vee A_3, (\omega_1 \vee \omega_2) \vee \omega_3) \cong (A_1 \vee (A_2 \vee A_3), \omega_1 \vee (\omega_2 \vee \omega_3))$

Condition (iii) allows us to define recursively the join $(A_1 \vee \ldots \vee A_n, \omega_1 \vee \ldots \vee \omega_n)$ of n members of Ω.

3 The Krull-Schmidt Theorem

Suppose (A, ω) is a baric algebra with idempotent e of weight 1, so $A = Fe \bigoplus N$ where $N = ker\omega$. The additive group $(N, +)$ can be endowed with a structure of abelian M-group, see [8, Chap V, Def.1]. The set M is formed by all right and left multiplications R_a and L_a, where a belongs to $A \cup F$ and $\psi(a, T) = T(a)$, $a \in ker\omega$, $T \in M$. In this case, the M-subgroups of $(N, +)$ are the two-sided ideals of the algebra A, contained in N. These are exactly the two-sided ideals of N which are invariant under L_e and R_e (in short, invariant).

According to [8, Chap. V, § 12] an abelian M-group N is decomposable if there are two non trivial M-subgroups N_1 and N_2 of N such that $N = N_1 \bigoplus N_2$. In our context, this concept is translated to the following definition:

Definition 1 *A baric algebra (A,ω) with an idempotent of weight 1 is decomposable if there are non trivial two-sided ideals N_1 and N_2 of A, both contained in $N = ker\omega$, such that $N = N_1 \bigoplus N_2$. Otherwise, it is indecomposable.*

Clearly all two dimensional algebras are indecomposable.

Theorem 1 *For any member (A,ω) of Ω the following conditions are equivalent:*

a) *(A,ω) is a decomposable baric algebra;*

b) *There are (A_1,ω_1) and (A_2,ω_2) in Ω such that $\dim A_i \geq 2$ and $(A,\omega) \cong (A_1 \vee A_2, \omega_1 \vee \omega_2)$;*

c) *The aditive group $N = \ker\omega$ is a decomposable M-group.*

Definition 2 *A baric algebra (A,ω), with $N = ker\omega$, satisfies d.c.c. (resp. a.c.c.) if the M-group $(N,+)$ satisfies d.c.c. (resp. a.c.c.), as stated in [12, p.153].*

Proposition 2 *If (A,ω) is a baric algebra in Ω satisfying d.c.c, there exist m indecomposable baric subalgebras (A_i,ω_i) of (A,ω) such that $(A,\omega) \cong (A_1 \vee \ldots \vee A_m, \omega_1 \vee \ldots \vee \omega_m)$.*

Theorem 2 (Krull-Schmidt) *Suppose $(A,\omega) \in \Omega$ satisfies both d.c.c. and a.c.c. and let $(A_1,\omega_1), \ldots, (A_n,\omega_n), (B_1,\gamma_1) \ldots, (B_m,\gamma_m)$ be indecomposable members of Ω such that*

$$(A,\omega) \cong (A_1 \vee \ldots \vee A_n, \omega_1 \vee \ldots \vee \omega_n)$$

$$(A,\omega) \cong (B_1 \vee \ldots \vee B_m, \gamma_1 \vee \ldots \vee \gamma_m)$$

Then $n = m$ and for some permutation $i \mapsto i'$ of indices, we have $(A_i,\omega_i) \cong (B_{i'},\gamma_{i'})$ for all $i = 1, \ldots, n$.

It is a consequence of this theorem that the classification of baric algebras, belonging to a subclass of Ω closed under the join operation, is reduced to the determination of the indecomposable algebras, provided a.c.c. and d.c.c. hold. The main classes studied in genetic algebra fall in this case. Of course, a.c.c. and d.c.c. hold for finite dimensional algebras but they are independent of each other in the infinite dimensional case.

4 Some closed classes

The proof of the Krull-Schmidt theorem applies equally well to every subclass Ω' of Ω having the following closure property: $(A_1 \vee A_2, \omega_1 \vee \omega_2) \in \Omega'$ if and only if both (A_1,ω_1) and (A_2,ω_2) belong to Ω'. We call such subclasses "closed". We will show in the sequel that the most important classes of genetic algebras are closed.

Theorem 3 *Given a train polynomial $p(x) = x^n + \gamma_1\omega(x)x^{n-1} + \ldots + \gamma_{n-1}\omega(x)^{n-1}x$, where $\gamma_1, \ldots, \gamma_{n-1} \in F$ satisfy $1 + \gamma_1 + \ldots + \gamma_{n-1} = 0$, the class of all right train algebras satisfying $p(x) = 0$ and having a nonzero idempotent, is closed.*

Corollary *If A_1 (resp. A_2) satisfies the train polynomial p_1 (resp. p_2) then $A_1 \vee A_2$ satisfies the train polynomial p, the least common multiple of p_1 and p_2.*

A baric algebra (A, ω) is Bernstein of order n if it is commutative and satisfies the identity $x^{[n+2]} = \omega(x)^{2^n} x^{[n+1]}$ for all x. Here $x^{[n]}$ denotes plenary powers, defined by $x^{[1]} = x$ and $x^{[k+1]} = x^{[k]} x^{[k]}$ for $k \geq 1$. See, for instance, [7] for recent developments in this subject. In an entirely similar way, we can prove the following theorem.

Theorem 4 *For every $n \geq 1$, the class of all Bernstein algebras of order n over a field F is closed.*

Suppose $f = f(x_1, \ldots, x_n)$ is a nonassociative polynomial over the field F and consider the variety defined by f, that is, the class of all F-algebras A satisfying $f(a_1, \ldots, a_n) = 0$, all $a_i \in A$. See [5] for a detailed study of varieties.

Theorem 5 *The subclass of the variety defined by f, consisting of those algebras which are baric and have an idempotent of weight 1, is closed.*

It is obvious that the intersection of closed classes is also closed and this implies, in the light of Theorem 5, that, for every variety of F-algebras, the subclass consisting of those algebras which are baric and have an idempotent of weight 1, is closed.

A finite dimensional commutative baric algebra (A, ω) is genetic over F, in Gonshor's sense, if there exists a (canonical) basis $\{a_0, a_1, \ldots, a_n\}$ of A such that, if $a_i a_j = \sum_k \gamma_{ijk} a_k$ for $i, j = 0, 1, \ldots, n$, then $\gamma_{000} = 1$, $\gamma_{0jk} = 0$ for $k < j$ and $\gamma_{ijk} = 0$ for $0 < i, j$ and $k \leq \max\{i, j\}$. See [12] for more information. Recall that nonzero idempotents must have weight 1. It is also well known that we can take a_0 to be an idempotent of A, when they exist.

Theorem 6 *The class of all genetic algebras over F, in Gonshor's sense, which have a nonzero idempotent (necessarily of weight 1) is closed.*

We remark that this theorem can be stated and proved for left or right Gonshor algebras, see [12, Chapter 5].

For a given algebra A over F, let $R_a \colon A \longrightarrow A$ be right multiplication by a, that is, $R_a(x) = xa$, $x \in A$. A finite dimensional commutative baric algebra (A, ω) is called genetic, in Schafer's sense, if the characteristic polynomial of a polynomial transformation $f(R_{x_1}, \ldots, R_{x_s})$ remains unchanged when the x_i are replaced by y_i, with $\omega(x_i) = \omega(y_i)$, $i = 1, \ldots, s$. Again from [12, Theorem 3.13], we see that if we extend a Schafer genetic algebra A to A_K, where K is a suitable extension of F, then A_K will have a canonical basis over K. Moreover, the identity $(A_1 \vee A_2)_K \cong (A_1)_K \vee (A_2)_K$ holds as the reader can easily verify. From these considerations, we obtain the following Theorem 6.

Theorem 7 *The class of genetic algebras in Schafer's sense, having idempotents of weight 1, is closed.*

Theorem 8 *The class of all special train algebras which have a nonzero idempotent is closed.*

5 On the duplication

We give now a sufficient condition to ensure that the duplicate of a indecomposable baric algebrais still indecomposable. Let (A, ω) be a commutative baric algebra, $e \in A$ an idempotent of weight 1, ω' the restriction of ω to A^2, $D(A)$ its commutative duplicate, $\psi: D(A) \longrightarrow A^2$ the homomorphism given by $\psi(x * y) = xy$. Then $D(A)$ is baric with the weight function $\omega_d = \omega' \circ \psi$.

Theorem 9 *For any commutative indecomposable baric algebra (A, ω) such that $A^2 = A$, the commutative duplicate $(D(A), \omega_d)$ is indecomposable.*

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