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ScienceDirect
Topology and its Applications 155 (2008) 1105–1110
Topology and its Applications
www.elsevier.com/locate/topol

A non-normal topology generated by a two-point selection ☆

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Received 21 November 2006; accepted 23 January 2008

Abstract

We construct a two-point selection $f : [\mathbb{P}]^2 \to \mathbb{P}$, where $\mathbb{P}$ is the set of the irrational numbers, such that the space $(\mathbb{P}, \tau_f)$ is not normal and it is not collectionwise Hausdorff either. Here, τ_f denotes the topology generated by the two-point selection f. This example answers a question posed by V. Gutev and T. Nogura. We also show that if $f : [X]^2 \to X$ is a two-point selection such that the topology τ_f has countable pseudocharacter, then τ_f is a Tychonoff topology.

MSC: primary 54B20, 54D80; secondary 54G20

Keywords: Two-point selection; Non-normal space

1. Introduction

In this paper, our spaces will be Tychonoff. The symbols $d(X)$ and $\psi(X)$ will stand for the density and the pseudocharacter of a space X, respectively. If X is a set, then $[X]^2 = \{A \subseteq X: |A| = 2\}$. The Euclidian order on the real line $\mathbb{R}$ is simply denoted by $<$.

For a space X, we let $\mathcal{F}(X)$ be the family of all nonempty closed subsets of X. If $\mathcal{V}$ is a finite family of nonempty open subsets of X, then we define

$$\langle \mathcal{V} \rangle = \left\{ F \in \mathcal{F}(X): F \subset \bigcup \mathcal{V} \text{ and } F \cap V \neq \emptyset \text{ for each } V \in \mathcal{V} \right\}.$$

The *Vietoris topology* on $\mathcal{F}(X)$ is the topology τ_V generated by the sets of the form $\langle \mathcal{V} \rangle$, where $\mathcal{V}$ is a finite family of nonempty open subsets of X.

In this paper, we will consider the following functions:

☆ Research supported by CONACYT grant No. 40057-F and PAPIIT grants No. IN106103. The research leading to this paper was essentially performed while the first-listed author was visiting the *Instituto de Matemática e Estatística* of the *University of São Paulo*, during August of 2006. He would like to gratefully acknowledge the hospitality received from this institution and the financial support received from FAPESP No. 2006/52282-6.

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doi:10.1016/j.topol.2008.01.013

Definition 1.1. Let X be a set. A function

$$f : [X]^2 \to X$$

is called a *two-point selection* if $f(F) \in F$, for all $F \in [X]^2$. For a topological space (X, τ), we say that a two-point selection $f : [X]^2 \to X$ is *continuous* if it is a continuous function from the space $([X]^2, \tau_V)$ to the space (X, τ).

Given a set X, the symbol $Sel_2(X)$ will denote the set of all two-point selections defined on X.

Following E. Michael [5], if $f \in Sel_2(X)$ is a two-point selection, then we say that $x <_f y$ if $f(\{x, y\}) = x$. Thus, every two-point selection defines an order-like relation. We write $x \leqslant_f y$ if either $x <_f y$ or $x = y$. It is evident that the relation $\leqslant_f$ is reflexive and antisymmetric, but, in general, $\leqslant_f$ is not transitive. As in the ordered spaces, an order-like relation defined by a two-point selection also induces a topology. Indeed, for a set X, $x \in X$ and a two-point selection $f \in Sel_2(X)$, we define

$$(-\infty, x)_f = \{y \in X\colon\ y <_f x\}$$

and

$$(x, +\infty)_f = \{y \in X\colon\ x <_f y\}.$$

For two-points $x, y \in X$, we define

$$(x, y)_f = (-\infty, y)_f \cap (x, +\infty)_f = \{z \in X\colon\ x <_f z \text{ and } z <_f y\}.$$

The topology on X generated by all open intervals $(-\infty, x)_f$ and $(x, +\infty)_f$, for $x \in X$, will be denoted by τ_f. Some topological properties of this topology are stated in the next two theorems.

Theorem 1.2. *(See [2].) If (X, τ) is a Hausdorff space and $f \in Sel_2(X)$ is a continuous two-point selection, then $\tau_f \subseteq \tau$.*

Theorem 1.3. *(See [3].) For a set X and a two-point selection $f \in Sel_2(X)$, the topology τ_f is Hausdorff and regular.*

This last result suggests the question whether or not τ_f is a Tychonoff topology on X, or maybe it could be a normal topology as V. Gutev and T. Nogura suggest in their paper [3, p. 903]. The following related question is posed in [3].

Question. If X is a set and $f \in Sel_2(X)$, must the space (X, τ_f) be collectionwise Hausdorff?

Our main purpose in this article is to define a two-point selection $f \in Sel_2(\mathbb{P})$ so that the space $(\mathbb{P}, \tau_f)$ is not collectionwise Hausdorff, and this space is also separable and contains a closed discrete subset of size $\mathfrak{c}$. Thus, by Jones' lemma [1, 2.1.10], the space $(\mathbb{P}, \tau_f)$ is not normal.

2. The example

For the construction of the example we will modify some ideas from the article [4]. We start our task with an easy lemma.

Lemma 2.1. *For every infinite set X there is a two-point selection $f \in Sel_2(X)$ such that τ_f is the discrete topology on X.*

Proof. Assume that $X = \bigcup_{\xi<\alpha} Z_\xi$, where $|X| = \alpha$, Z_ξ is a copy of the integers $\mathbb{Z}$, for all $\xi < \alpha$, and $Z_\xi \cap Z_\zeta = \emptyset$ whenever $\xi < \zeta < \alpha$. Let $x, y \in X$ and suppose that $x \in Z_\xi$ and $y \in Z_\zeta$, for some $\xi, \zeta < \alpha$. Then, we define

$$\hat{f}\big(\{x, y\}\big) = \begin{cases} x & \text{if } \xi < \zeta, \\ \min\{x, y\} & \text{if } \xi = \zeta. \end{cases}$$

It is clear that $f \in Sel_2(X)$ and τ_f is the discrete topology on X. □

The next lemma is crucial for the construction of the two-point selection which will be done by inductive steps.

Lemma 2.2. *Suppose that $A, D \subseteq \mathbb{P}$, $A \neq \emptyset$, D is dense in $\mathbb{P}$ and $A \cap D = \emptyset$. Then, every two-point selection $g : [A]^2 \to A$ extends to a two-point selection $\hat{g} : [A \cup D]^2 \to A \cup D$ such that if $\{x_0, \ldots, x_i\} \subseteq A$, $\{y_0, \ldots, y_j\} \subseteq A$ and $\{x_0, \ldots, x_i\} \cap \{y_0, \ldots, y_j\} = \emptyset$, then*

$$D \cap \left[\left(\bigcap_{l \leqslant i}(-\infty, x_l)_{\hat{g}}\right) \cap \left(\bigcap_{l \leqslant j}(y_l, +\infty)_{\hat{g}}\right)\right] \neq \emptyset.$$

Proof. Let $\mathcal{B}$ be a countable base for $\mathbb{P}$ consisting of bounded clopen nonempty subsets. Put

$$\mathcal{F} = \big\{F\colon\ F \text{ is a function which takes values on } \{0,1\}, dom(F) \in [\mathcal{B}]^{<\omega} \text{ and the elements of } dom(F) \text{ are pairwise disjoint}\big\}.$$

Enumerate $\mathcal{F}$ as $\{F_n\colon n \in \mathbb{N}\}$ and let $(k_n)_{n\in\mathbb{N}}$ be an increasing sequence of natural numbers such that $\bigcup dom(F_n) \subseteq (-\infty, k_n)$, for each $n \in \mathbb{N}$. Now, we proceed to extend the two-point selection g to a two-point selection $\hat{g} : [A \cup D]^2 \to A \cup D$. To do that we fix $x, y \in A \cup D$ and consider the following four cases:

Case I. $x, y \in A$. Then, we define $\hat{g}(\{x, y\}) = g(\{x, y\})$.

Case II. $x, y \in D$. So, we let $\hat{g}(\{x, y\}) = x$ provided that $x < y$.

Case III. $x \in A$, $y \in D$ and there is $n \in \mathbb{N}$ such that $x \in B$, for some $B \in dom(F_n)$, and $y \in [k_n, k_{n+1})$. Then, in this case, we define $\hat{g}(\{x, y\}) = x$ if $F_n(B) = 0$, and $\hat{g}(\{x, y\}) = y$ if $F_n(B) = 1$.

Case IV. $x \in A$, $y \in D$ and Case III does not hold. Then, we set $\hat{g}(\{x, y\}) = x$.

Assume that $\{x_0, \ldots, x_i\} \subseteq A$, $\{y_0, \ldots, y_j\} \subseteq A$, and $\{x_0, \ldots, x_i\} \cap \{y_0, \ldots, y_j\} = \emptyset$. Choose $\{B_l\colon l \leqslant i + j\} \subseteq \mathcal{B}$ such that:

(1) $B_l \cap B_m = \emptyset$ whenever $l < m \leqslant i + j$.
(2) $|B_l \cap \{x_0, \ldots, x_i, y_0, \ldots, y_j\}| = 1$, for all $l \leqslant i + j$.

Now, we define $F : \{B_l\colon l \leqslant i + j\} \to \{0, 1\}$ by $F(B_l) = 1$ if $B_l \cap \{x_0, \ldots, x_i\} \neq \emptyset$ and $F(B_l) = 0$ if $B_l \cap \{y_0, \ldots, y_j\} \neq \emptyset$, for every $l \leqslant i + j$. Let $n \in \mathbb{N}$ be such that $F = F_n$. Choose $d \in (k_n, k_{n+1}]$. By definition, we have that $\hat{g}(\{x_l, d\}) = d$, for all $l \leqslant i$ and $\hat{g}(\{y_l, d\}) = y_l$, for all $l \leqslant j$. Thus, $d \in (\bigcap_{l \leqslant i}(-\infty, x_l)_{\hat{g}}) \cap (\bigcap_{l \leqslant j}(y_l, +\infty)_{\hat{g}})$. This shows the lemma. □

Example 2.3. There is a two-point selection $f \in Sel_2(\mathbb{P})$ such that the space $(\mathbb{P}, \tau_f)$ is not normal.

Proof. Write $\mathbb{P}$ as a disjoint union $A \cup (\bigcup_{n\in\mathbb{N}} C_n) \cup (\bigcup_{n\in\mathbb{N}} D_n)$ where C_n and D_n are countable dense subsets of $\mathbb{P}$, for each $n \in \mathbb{N}$. Observe that A has size $\mathfrak{c}$. Let $g : [A]^2 \to A$ be a two-point selection such that the space (A, τ_g) is discrete (this is possible by Lemma 2.1). We shall extend g to a two-point selection $f : [\mathbb{P}]^2 \to \mathbb{P}$ by transfinite induction. First, we define $f(\{x, y\}) = x$ whenever $x \in A$ and $y \in C_0$ and $f(\{x, y\}) = x$ if $x < y$ and $x, y \in C_0$. We extend f to $[A \cup C_0 \cup D_0]^2$ as in Lemma 2.2 by replacing A by $A \cup C_0$ and D by D_0. Suppose that f has been defined on $[A \cup (\bigcup_{i<n} C_i) \cup (\bigcup_{i<n} D_i)]^2$, for some $0 < n \in \mathbb{N}$. Take $x, y \in A \cup (\bigcup_{i \leqslant n} C_i) \cup (\bigcup_{i \leqslant n} D_i)$. We have to consider four possible cases.

Case I. $x \in A \cup (\bigcup_{i<n} C_i) \cup (\bigcup_{i<n-1} D_i)$ and $y \in C_n$. Then, we define $f(\{x, y\}) = x$.

Case II. $x \in D_{n-1}$ and $y \in C_n$. Then, we define $f(\{x, y\}) = x$ if and only if $x < y$.

Case III. $x, y \in C_n$. In this case, we put $f(\{x, y\}) = x$ if $x < y$.

Case IV. For the points which lie in D_n, we proceed to extend the two-point selection

$$f : \left[A \cup \left(\bigcup_{i \leqslant n} C_i\right) \cup \left(\bigcup_{i<n} D_i\right)\right]^2 \to A \cup \left(\bigcup_{i<n} C_i\right) \cup \left(\bigcup_{i<n} D_i\right)$$

to the set

$$\left[A \cup \left(\bigcup_{i \leqslant n} C_i\right) \cup \left(\bigcup_{i \leqslant n} D_i\right)\right]^2$$

as it is indicated in Lemma 2.2 by replacing A by $A \cup (\bigcup_{i \leqslant n} C_i) \cup (\bigcup_{i<n} D_i)$ and putting $D = D_n$.

Thus, g has been extended to a two-point selection $f \in Sel_2(\mathbb{P})$. Let us see that our two-point selection f satisfies the required conditions.

Claim 1. *A is a closed discrete subset of* $(\mathbb{P}, \tau_f)$.

Proof. Since (A, τ_g) is discrete and f is an extension of g, then A is a discrete subset of $(\mathbb{P}, \tau_f)$. Take $x \in \mathbb{P} \setminus A$. Then, there is $n \in \mathbb{N}$ such that $x \in C_n \cup D_n$. If $x \in C_n$, then choose $a, b \in C_n$ so that $a < x < b$. From the definition we can easily see that $x \in (a, b)_f$ and $A \cap (a, b)_f = \emptyset$. Assume that $x \in D_n$. It is then possible to choose $a, b \in C_{n+1}$ so that $a < x < b$. By the second case, we obtain that $x \in (a, b)_f$ and, by construction, $A \cap (a, b)_f = \emptyset$. □

Claim 2. $\bigcup_{n\in\mathbb{N}} D_n$ *is a dense subset of* $(\mathbb{P}, \tau_f)$.

Proof. Let $\{x_0, \dots, x_i\} \subseteq \mathbb{P}$ and $\{y_0, \dots, y_j\} \subseteq \mathbb{P}$ be such that $\{x_0, \dots, x_i\} \cap \{y_0, \dots, y_j\} = \emptyset$. Then, we can find $m \in \mathbb{N}$ such that

$$\{x_0, \dots, x_i\} \cup \{y_0, \dots, y_j\} \subseteq A \cup \left(\bigcup_{n\leqslant m} C_n\right) \cup \left(\bigcup_{n<m} D_n\right).$$

According to the construction based on Lemma 2.2, there exists $d \in D_m$ such that

$$d \in \left(\bigcap_{l\leqslant i}(-\infty, x_l)_{\hat{g}}\right) \cap \left(\bigcap_{l\leqslant j}(y_l, +\infty)_{\hat{g}}\right).$$

By construction, $\bigcup_{n\in\mathbb{N}} D_n$ is a dense subset of the space $(\mathbb{P}, \tau_f)$. Hence, $(\mathbb{P}, \tau_f)$ is separable and since A is a closed discrete subset of $(\mathbb{P}, \tau_f)$ of size $\mathfrak{c}$, by Jones' lemma, we obtain that the space $(\mathbb{P}, \tau_f)$ cannot be normal. □

Observe that the space $(\mathbb{P}, \tau_f)$ from the previous example contains a discrete subset inside of A whose points cannot be separated by pairwise disjoint open subsets. This answers Question 1 from [3] in the negative.

Let τ be the Euclidian topology on $\mathbb{P}$. If the two-point selection $f : ([\mathbb{P}]^2, \tau_V) \to (\mathbb{P}, \tau)$ is continuous, by Theorem 1.2, then we know that $\tau_f \subseteq \tau$. Hence, $(\mathbb{P}, \tau_f)$ is a Lindelöf topology and so it is normal. Thus, the two-point selection defined in Example 2.3 cannot be continuous.

Theorem 1.3 asserts that the topology τ_f is always Hausdorff and regular, for every $f \in Sel_2(X)$. This suggests the following question.

Question 2.4. Are there a set X and a two-point selection $f \in Sel_2(X)$ such that the space (X, τ_f) is not Tychonoff?

The authors were unable to answer this question, but if there is a non-Tychonoff example, it should have uncountable pseudocharacter and be non-separable.

Theorem 2.5. *If $f \in Sel_2(X)$ satisfies that the topology τ_f has countable pseudocharacter, then τ_f is a Tychonoff topology.*

Proof. Let $x \in X$ and let $F \subseteq X$ be a τ_f-closed subset of X such that $x \notin F$. Then, choose points $a_0, \dots, a_n, b_0, \dots, b_k \in X$ so that the open set $U = \bigcap_{i\leqslant n}(a_i, +\infty)_f \cap \bigcap_{j\leqslant k}(-\infty, b_j)_f$ satisfies that $x \in U \subseteq X \setminus F$. Since the topology τ_f has countable pseudocharacter, we can find a countable subset C of X such that:

(1) $\{x, a_0, \dots, a_n, b_0, \dots, b_k\} \subseteq C$,
(2) for each $c \in C$, $\{c\} = (\bigcap_{C\ni a<_f c}(a, +\infty)_f) \cap (\bigcap_{c<_f b\in C}(-\infty, b)_f)$, and
(3) for each pair of distinct points $c, d \in C$, if $(c, d)_f \neq \emptyset$ then $(c, d)_f \cap C \neq \emptyset$.

Let τ be the topology on X generated by the half intervals $\{(-\infty, a)_f\colon a \in C\}$ and $\{(b, +\infty)_f\colon b \in C\}$. We shall show that τ is a normal topology. For the proof of it, we need two preliminary results:

Claim 1. *If $c \in C$ and $d \in X$ satisfy that $(c, d)_f = \emptyset = (d, c)_f$, then $d \in C$.*

Proof. By the choice of C, there exists $e \in C$ such that either

(a) $c \in (-\infty, e)_f$ and $d \notin (-\infty, e)_f$, or
(b) $c \in (e, +\infty)_f$ and $d \notin (e, +\infty)_f$.

If (a) holds, then $c <_f e$ and either $e = d$ or $e <_f d$. Since $(c, d)_f = \emptyset$, we must have that $d = e \in C$.
If (b) holds, then $e <_f c$ and either $e = d$ or $d <_f e$. As $(d, c)_f = \emptyset$, then $d = e \in C$. □

Claim 2. *Let $c \in C$. For each $y \in (c, +\infty)_f$ there exists a τ-neighborhood V of y such that $y \in V \subseteq cl_\tau(V) \subseteq (c, +\infty)_f$.*

Proof. Let $c \in C$ and $y \in (c, +\infty)_f$. The proof will be divided in two cases:

Case I. The open intervals $(c, y)_f$ and $(y, c)_f$ are empty. By Claim 1, we know that $y \in C$ and then, as in the proof of the regularity of τ_f in [3, Lemma 2.2], the interval $(c, +\infty)_f$ is clopen.
Case II. $(c, y)_f \cup (y, c)_f \neq \emptyset$. Without loss of generality, we assume that $(y, c)_f$ is not empty.

If $y \notin C$, then there exists $d \in C \setminus \{y\}$ such that either

(a) $c \in (d, +\infty)_f$ and $y \notin (d, +\infty)_f$ or
(b) $c \in (-\infty, d)_f$ and $y \notin (-\infty, d)_f$.

If (a) holds, then $y \in (-\infty, d)_f$ and then

$$y \in (c, +\infty)_f \cap (-\infty, d)_f \subseteq [c, +\infty)_f \cap (-\infty, d]_f \subset (c, +\infty)_f.$$

If (b) holds, then $y \in (d, +\infty)_f$. So,

$$y \in (c, +\infty)_f \cap (d, +\infty)_f \subset [c, +\infty)_f \cap [d, +\infty)_f \subset (c, +\infty)_f.$$

If $y \in C$, then there exists $d \in C$ such that $d \in (y, c)_f$. Then, $c \in (d, +\infty)_f$. Thus,

$$y \in (c, +\infty)_f \cap (-\infty, d)_f \subset [c, +\infty)_f \cap (-\infty, d]_f \subset (c, +\infty)_f. \qquad \square$$

Claim 3. *Let $c \in C$. For each $y \in (-\infty, c)_f$ there exists a τ-neighborhood V of y such that $y \in V \subseteq Cl_\tau(V) \subseteq (-\infty, c)_f$.*

The proof of this claim is completely similar to the one of Claim 2.
It is evident that Claims 2 and 3 guarantee that the topology τ is regular (this topology is not necessarily T_1). Since τ is second countable, we obtain that the topology τ is normal. From the definition of C we can see that $U \in \tau$. Thus, by the regularity of τ, we can find $W \in \tau$ such that $x \in W$ and $cl_\tau(W) \subseteq U$. By Urysohn's lemma, there is a τ-continuous function $g : X \to [0, 1]$ such that $g(y) = 1$, for all $y \in cl_\tau(W)$, and $g(y) = 0$, for each $y \in X \setminus U$. Observe that $F \subseteq X \setminus U$ and g is τ_f continuous. Therefore, x and F are separated by a τ_f-continuous function. This shows that the topology τ_f is Tychonoff. □

Theorem 2.6. *If $f \in Sel_2(X)$, then $\psi(X, \tau_f) \leqslant d(X, \tau_f)$.*

Proof. Fix $x \in X$. If x is an isolated point, we are done. Suppose that x is not isolated and let D be a τ_f-dense subset of X such that $x \notin D$ and $|D| = d(X, \tau_f)$. We shall consider two cases:

Case I. Assume that there exists $z \in X$ such that $(x, z)_f = \emptyset = (z, x)_f$. In this case, we claim that

$$\{x\} = \big(X \setminus \{z\}\big) \cap \left(\bigcap_{D \ni d <_f x} (d, +\infty)_f \right) \cap \left(\bigcap_{x <_f d \in D} (-\infty, d)_f \right).$$

To establish this equality we will prove that either $(x, y)_f \neq \emptyset$ or $(y, x)_f \neq \emptyset$ for each $y \in X \setminus \{x, z\}$. Indeed, let $y \in X \setminus \{x, z\}$.

First, let us suppose that $x <_f z$. If $x <_f y$, then it follows from $(x,z)_f = \emptyset$ that $z <_f y$. Thus, $z \in (x,y)_f$. If $y <_f x$ then $x \in (y,z)_f$. Choose $d \in D \cap (y,z)_f$. Since $(x,z)_f = \emptyset$, we must have that $y <_f d <_f x$.

Now, assume that $z <_f x$. If $x <_f y$, then $x \in (z,y)_f$. Pick $d \in D \cap (z,y)_f$. From $(z,x)_f = \emptyset$ we obtain that $x <_f d <_f y$. If $y <_f x$, then $y <_f z$ and hence $z \in (y,x)_f$.

Hence, we deduce that

$$\left(\bigcap_{D \ni d <_f x} (d,+\infty)_f\right) \cap \left(\bigcap_{x <_f d \in D} (-\infty,d)_f\right) \subseteq \{x,z\}.$$

This proves the claim.

Case II. For each $y \in X \setminus \{x\}$, $(x,y)_f \cup (y,x)_f \neq \emptyset$. It then follows that

$$\{x\} = \left(\bigcap_{D \ni d <_f x} (d,+\infty)_f\right) \cap \left(\bigcap_{x <_f d \in D} (-\infty,d)_f\right). \qquad \square$$

The next corollary follows directly from Theorems 2.5 and 2.6.

Corollary 2.7. *If $f \in Sel_2(X)$ is a two-point selection such that τ_f is separable, then τ_f is a Tychonoff topology.*

Let X be a set and let $f \in Sel_2(X)$. Suppose that $\{x_n\colon n \in \mathbb{N}\} \subseteq (-\infty,x)_f$ and $\{y_n\colon n \in \mathbb{N}\} \subseteq (x,+\infty)_f$ satisfy that $x \in cl_{\tau_f}(\{x_n\colon n \in \mathbb{N}\}) \cap cl_{\tau_f}(\{y_n\colon n \in \mathbb{N}\})$. Then, $\psi(x,\tau_f) = \omega$. Indeed, we claim that

$$\{x\} = \left(\bigcap_{n \in \mathbb{N}} (x_n,+\infty)_f\right) \cap \left(\bigcap_{n \in \mathbb{N}} (-\infty,y_n)_f\right).$$

Let $y \in X \setminus \{x\}$. If $x <_f y$, then there is $m \in \mathbb{N}$ such that $y_m \in (x,y)_f$ and then $y \notin (-\infty,y_m)_f$. If $y <_f x$, then we can find $m \in \mathbb{N}$ so that $x_m \in (y,x)_f$ and hence $y \notin (x_m,+\infty)_f$. From these observations and Theorems 2.5 and 2.6 we can established the following corollary.

Corollary 2.8. *If $f \in Sel_2(X)$ is a two-point selection such that τ_f is sequential, then τ_f is a Tychonoff topology.*

Recall that a space X is said to be *functionally Hausdorff* if whenever for two distinct points $x, y \in X$ there is a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 0$ and $f(x) = 1$.

Question 2.9. Are there a set X and a two-point selection $f \in Sel_2(X)$ such that the space (X,τ_f) is functionally Hausdorff and it is not Tychonoff?

Question 2.10. Are there a set X and a two-point selection $f \in Sel_2(X)$ such that every real-valued continuous function on (X,τ_f) is constant?

Acknowledgement

We are grateful to the referee for correcting some typos.

References

[1] R. Engelking, General Topology, Sigma Ser. Pure Math., vol. 6, Heldermann, 1989.
[2] V. Gutev, T. Nogura, Selections and order-like relations, Appl. Gen. Topol. 2 (2001) 205–218.
[3] V. Gutev, T. Nogura, A topology generated by selections, Topology Appl. 153 (2005) 900–911.
[4] V. Gutev, A.H. Tomita, Selections generating new topologies, Publ. Mat. 51 (1) (2007) 3–15.
[5] E. Michael, Topologies on spaces of subsets, Trans. Amer. Math. Soc. 71 (1951) 152–182.