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DEFINABILITY AND INVARIANCE IN FIRST ORDER STRUCTURES

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1 Introduction

This paper is about two basic notions of model theory namely, the notion of relation definable in a given language and the notion of relation invariant by automorphisms of a structure.

It was recognized long ago that these two notions are strongly related. M. Krasner [3], rightly considered a forerunner of the introduction of infinitary languages in model theory (Karp [2], Introduction), showed, in 1938, how to generate all invariant relations of a first order structure by means of set-theoretical operations applied to the primitive elements of the structure.

Although Krasner, in his paper [3], makes no reference to infinitary languages, his set-theoretical operations are the interpretations of the syntactic rules of definition of formulas of a suitable infinitary language, as himself recognized much latter (see [4]). However, his operations are neither easily described nor very manageable. This explain, perhaps, why his paper has not had wider circulation.

A significant feature of this work is the introduction of two natural operators (operators ξ^* and $(\xi^*)^{-1}$, section 2) on relations of a first order structure. These operators behave nicely with respect to composition and commute with the extensions to relations of bijections of the domain of the structure. Relying on these nice algebraic properties we substantially simplified and improved conceptually Krasner's arguments. At the same time, the consideration of relations whose arity are infinite ordinals allowed to give a smaller bound than Krasner's to the arity μ of the universe $\mathfrak{U}_\mu$ (section 2) where one has to operate in order to construct all invariant relations. If d is the power of the domain D of the structure, for Krasner μ is the first cardinal greater than d whereas for us it is the ordinal $d+2$ (theorem 3.3).

In sections 4 to 7, we treat the question of defining invariant relations of a first order structure E by means of formulas of an appropriated relational language $\mathfrak{L}^E_{\alpha\beta}$ associated to the structure.

In section 5, we give a simple and manageable definition of the relation $\|\varphi\|$ defined by a formula φ of the infinitary language $\mathfrak{L}^E_{\alpha\beta}$, by means of the operators $\xi^*, (\xi^*)^{-1}$ and the complement operator $\mathfrak{C}_\gamma$. Of course, our definition agrees with the usual notion of relation defined by a formula. In other words, it is easy to prove that if $x_i, i < \gamma$, are the free variable of φ where γ is, in general, an infinite ordinal, then $\|\varphi\|$ is the set of sequences $v(x_0), v(x_1), ..., v(x_i), i < \gamma$, where v is a valuation of variables true of φ in the structure E. Moreover, for a given choice of cardinals α and β, the operators $\xi^*, (\xi^*)^{-1}$ and $\mathfrak{C}_\gamma$ allow a simple set-theoretical description of the set $\hat{E}_{\alpha\beta}$ of relations definable by formulas of the language $\mathfrak{L}^E_{\alpha\beta}$ (theorem 6.2).

In a classical paper which is one of the cornerstones of model theory A. Tarski [8] showed how to characterize set-theoretically the sets of the real line definable in a finitary language of higher order. A relation of the universe of a structure E definable in a given language is invariant under automorphism of E, but the converse is not true in general. This poses the question of determining which languages are sufficiently strong to define all invariant relations of a given structure. In 1965, H. Rogers Jr. [6] showed how to define an invariant relation of finite arity by means of an explicit formula of a suitable infinitary language; he also proposed to call invariant relations absolutely definable since they are intrinsic to the structure. A straightforward consequence of theorems 3.3 and 6.2 is that any invariant relation of finite or infinite arity is definable in an infinitary language whose parameters α and β are determined by the power of the domain D (theorems 7.2 and 7.4).

A celebrated theorem of D. Scott [7] states that when the domain D is denumerable, all invariant relations are definable in the language $\mathfrak{L}^E_{\omega_1\omega}$. The present paper ends with a counter example to prove that when D is not denumerable, the language $\mathfrak{L}^E_{\alpha\omega}$ is not strong enough to define all invariant relations, no matter how great α is. In fact, we construct a structure $E = \langle D, \omega, \mathfrak{R}\rangle$ such that for any infinite regular cardinal α, no orbit of the automorphism group of E in $\mathfrak{U}_\omega$ is $\mathfrak{L}^E_{\alpha\beta}$-definable. The proof is based on theorems 3.3 and 6.2.

2 The closure of first order structures

Given a nonempty set D and an ordinal γ, a γ-tuple of elements of D is a map from γ into D. The set of all γ-tuples of elements of D will be denoted by D^γ. A *γ-ary relation* of elements of D is a subset of D^γ. Let μ be an infinite ordinal and let $\mathcal{U}_\mu(D)$ be the set of all γ-ary relations of D for all $\gamma < \mu$.

A *first order structure of arity* μ is a triple $E = \langle D, \mu, \mathcal{R}\rangle$ where $\mathcal{R}$ is a subset of $\mathcal{U}_\mu(D)$ which contains the diagonal Δ of D^2. The elements of $\mathcal{R}$ are the *primitive relations* of the structure and $\mathcal{U}_\mu(D)$ is the *universe* of E. We shall refer to elements of D^γ as *points of arity* γ of E.

Let A, B and D be sets and let $\xi : A \to B$ be any map. ξ induces in a natural way a map $\xi^* : D^B \to D^A$. By definition, if $f \in D^B$, then $\xi^*(f) = f \circ \xi$. Let $\wp(D^A)$ and $\wp(D^B)$ be the set of parts of D^A and D^B, respectively. The inverse map of ξ^* from $\wp(D^A)$ into $\wp(D^B)$ is denoted by $(\xi^*)^{-1}$ and the extension of ξ^* to these sets of parts is denoted with the same notation $\xi^* : \wp(D^A) \to \wp(D^B)$. If η maps B into C, then $(\eta \circ \xi)^* = \xi^* \circ \eta^*$ and if ξ is a bijection, then ξ^* is also a bijection and $(\xi^*)^{-1} = (\xi^{-1})^*$. In most cases, A and B will be ordinal numbers δ and γ and ξ^* will map points of D^γ into points of D^δ.

For each ordinal $\gamma < \mu$, let $\mathcal{C}_\gamma : \wp(D^\gamma) \to \wp(D^\gamma)$ be the map which maps $R \in \wp(D^\gamma)$ into its complement with respect to D^γ, i.e., $D^\gamma - R$.

Corresponding to all possible choices of $\delta, \gamma < \mu$, let $\mathcal{K}$ be the set of all maps $\xi^*, (\xi^*)^{-1}$ and $\mathcal{C}_\gamma$, where ξ is any map from δ into γ.

Definition 2.1 *We say that a set of relations $\mathcal{S} \subseteq \mathcal{U}_\mu(D)$ is E-closed if:*

1) $\mathcal{R} \subseteq \mathcal{S}$;

2) For all $R \in \mathcal{S}$ and $f \in \mathcal{K}$ such that $f(R)$ is defined, $f(R) \in \mathcal{S}$;

3) If $\mathcal{S}'$ is a subset of $\mathcal{S}$, then $\bigcap \mathcal{S}' \in \mathcal{S}$.

The set of subsets of $\mathcal{U}_\mu(D)$ which are E-closed is not empty because $\mathcal{U}_\mu(D)$ is E-closed. Hence, we may define the closure *of E, denoted by $\hat{E}$, to be the intersection of all E-closed subsets of $\mathcal{U}_\mu(D)$.*

If $\mathcal{S}$ is E-closed and $\mathcal{S}' \subseteq \mathcal{S}$ is a subset of relations of same arity γ, then $\mathcal{C}_\gamma \in \mathcal{K}$ and condition 3 above implie that $\bigcup \mathcal{S}' \in \mathcal{S}$.

Let $\gamma < \mu$ be an ordinal and let $\mathcal{P} = (\mathcal{P}_k)_{k \in I}$ be a partition of γ, that is, $\bigcup_{k\in I} \mathcal{P}_k = \gamma$ and $\mathcal{P}_k \cap \mathcal{P}_{k'} = \emptyset$, if $k \neq k'$. We denote by $\mathcal{D}(\mathcal{P})$ the set of points $p \in D^\gamma$ such that $p(i) = p(j)$ for all $i, j \in \mathcal{P}_k$ and for all $k \in I$. We say that $\mathcal{D}(\mathcal{P})$ is the *diagonal* of D^γ defined by the partition $\mathcal{P}$. Similarly, we define $\bar{\mathcal{D}}(\mathcal{P})$ to be the set of points $p \in D^\gamma$ such that $p(i) \neq p(j)$ for all $i \in \mathcal{P}_k$ and $j \in \mathcal{P}_{k'}$ and all $k, k' \in I, k \neq k'$.

Proposition 2.2 $\mathcal{D}(\mathcal{P}), \bar{\mathcal{D}}(\mathcal{P}) \in \hat{E}$.

Proof. Let Δ be the diagonal of D^2 and let $\bar{\Delta} = \mathcal{C}_2(\Delta)$. For $i, j \in \gamma$, $i < j$ let $\xi_{ij} : 2 = \{0, 1\} \to \gamma$ be the map $\xi_{ij}(0) = i$, $\xi_{ij}(1) = j$. Then, $\mathcal{D}(\mathcal{P})$ is the intersection of the sets $(\xi^*_{ij})^{-1}(\Delta)$ for all $i, j \in \mathcal{P}_k$ and for all $k \in I$. Similarly, $\bar{\mathcal{D}}(\mathcal{P})$ is the intersection of all sets $(\xi^*_{ij})^{-1}(\bar{\Delta})$ for all $i \in \mathcal{P}_k$ and $j \in \mathcal{P}_{k'}$ and all $k, k' \in I, k \neq k'$. Since, by definition of E, $\Delta \in \mathcal{R}$, it follows from the definition of $\hat{E}$ that $\mathcal{D}(\mathcal{P}), \bar{\mathcal{D}}(\mathcal{P}) \in \hat{E}$. □

3 Invariant relations and the closure of E

Any map $g : D \to D$ extends naturally to a map $g^\gamma : D^\gamma \to D^\gamma$, $p \mapsto g^\gamma(p) = g \circ p$. This last map extends itself to a map from $\wp(D^\gamma)$ into $\wp(D^\gamma)$. We shall use the same notation to denote the extended map. If $h : D \to D$ is another map then, $(g \circ h)^\gamma = g^\gamma \circ h^\gamma$ and, if g is a bijection, $(g^{-1})^\gamma = (g^\gamma)^{-1}$.

An *automorphism* of E is a bijection $g : D \to D$ such that for any primitive relation $R \in \mathcal{R}$ of arity γ, $g^\gamma(R) = R$. Let G be the group of automorphisms of E. An *invariant relation* of E is a relation $R \in \mathcal{U}_\mu(D)$ which is kept fixed by G, that is, $g^\gamma(R) = R$ for all $g \in G$, γ being the arity of R. By definition, the primitive relations are invariant. We denote by $\mathcal{I}(E)$ the set of invariant relations of E.

Proposition 3.1 *For any maps $\xi : \delta \to \gamma$ and $g : D \to D$, we have that*

$$g^\delta \circ \xi^* = \xi^* \circ g^\gamma.$$

Proof. The proposition is an immediate consequence of definitions. □

Proposition 3.2 $\hat{E} \subseteq \mathcal{I}(E)$.

Proof. It suffices to show that $\mathcal{I}(E)$ is E-closed. By definition, $\mathcal{R} \subseteq \mathcal{I}(E)$, and it is trivial to verify that $\mathcal{I}(E)$ is closed under action of all maps $\mathcal{C}_\gamma$, $\gamma < \mu$, and also that it is closed under the intersection of subsets of $\mathcal{I}(E)$. That $\mathcal{I}(E)$ is closed under the actions of ξ^* and $(\xi^*)^{-1}$ follows from proposition 3.1. □

Our next theorem shows that, for sufficiently large μ, all invariant relations belong to the closure $\hat{E}$.

Theorem 3.3 *Let δ be the cardinal of D and assume $\mu \geq \delta + 2$. Then, $\hat{E} = \mathcal{I}(E)$.*

The proof of theorem 3.3 is made clearer introducing previously 3 lemmas.

Lemma 3.4 *Assume $\mu \geq \delta + 2$ and let N_1, N_2, N be the sets of points $p \in D^\delta$ such that the map $p : \delta \to D$ is respectively injective, surjective and bijective. Then, $N_1, N_2, N \in \hat{E}$ and $N \neq \emptyset$.*

Proof. Let $\mathcal{P} = (\mathcal{P}_i)_{i \in \delta}$ be partition of δ such that $\mathcal{P}_i = \{i\}$ for all $i < \delta$. Then, $N_1 = \bar{\mathcal{D}}(\mathcal{P})$. Hence, by proposition 2.2, $N_1 \in \hat{E}$.
Consider now the partition $\mathcal{P}' = (\mathcal{P}'_i)_{i<2}$ of $\delta + 1$ where $\mathcal{P}'_0 = \delta, \mathcal{P}'_1 = \{\delta\}$ and let ξ be the map $\xi : i \in \delta \mapsto i \in \delta + 1$. We shall prove that:

$$N_2 = \mathcal{C}_\delta \xi^*(\bar{\mathcal{D}}(\mathcal{P}')).$$

In fact, $p \in \bar{\mathcal{D}}(\mathcal{P}')$ if and only if $p(i) \neq p(\delta + 1)$ for all $i \in \delta$. Hence, $\xi^*(\bar{\mathcal{D}}(\mathcal{P}'))$ is the set of $p \in D^\delta$ for which there exists $a \in D$ and $i \in \delta$ with $p(i) \neq a$. Therefore, $\mathcal{C}_\delta \xi^*(\bar{\mathcal{D}}(\mathcal{P}'))$ is the set of $p \in D^\delta$ for which, for all $a \in D$, there exists $i \in \delta$ such that $p(i) = a$. This proves our assertion. It follows that $N_2 \in \hat{E}$. Finally, $N = N_1 \cap N_2 \in \hat{E}$. Since δ is the cardinal of D, there exists a bijection from δ onto D. Hence, $N \neq \emptyset$. □

Lemma 3.5 *Let $q \in N$ be a point and let $R \in \hat{E}$ be a relation of arity γ. There exists a relation $M_R \in \hat{E}$ of arity δ such that for every bijection $g : D \to D$, $g^\delta(q) \in M_R \Leftrightarrow g^\gamma(R) \subseteq R$.*

Proof. For every point $p \in R$ there exists a map $\xi : \gamma \to \delta$ such that $\xi^*(q) \in R$. In fact, it is enough to take $\xi = q^{-1} \circ p$. Let Θ_R be the set of all maps $\xi : \gamma \to \delta$ such that $\xi^*(q) \in R$ and define:

$$M_R = \bigcap_{\xi \in \Theta_R} (\xi^*)^{-1}(R).$$

By definition, $M_R \in \hat{E}$ and, by proposition 3.1,

$$\begin{aligned} g^\delta(q) \in M_R &\Leftrightarrow \forall \xi \in \Theta_R(g^\delta(q) \in (\xi^*)^{-1}(R)) \\ &\Leftrightarrow \forall \xi \in \Theta_R(\xi^*(g^\delta(q)) \in R) \\ &\Leftrightarrow \forall \xi \in \Theta_R(g^\gamma(\xi^*(q)) \in R) \\ &\Leftrightarrow g^\gamma(R) \subseteq R. \end{aligned}$$

This completes the proof. □

Lemma 3.6 *For every point $q \in N$, the orbit $\mathcal{O}_q$ of q in D^γ belongs to $\hat{E}$.*

Proof. If R is a relation of arity γ, let us denote by R' its complement in D^γ and consider the relation:

$$M = \bigcap_{R \in \mathfrak{R}} M_R \cap \bigcap_{R \in \mathfrak{R}} M_{R'} \cap N.$$

By lemmas 3.4 and 3.5, $M \in \hat{E}$. We shall show that $M = \mathcal{O}_q$. Let $g : D \to D$ be a bijection. Since $g^\delta(q)$ clearly belongs to N, by lemma 3.5, $g^\delta(q) \in M$ if and only if for all $R \in \mathfrak{R}$ of arity γ, $g^\gamma(R) \subseteq R$ and $g^\gamma(R') \subseteq R'$. Since $g^\gamma : D^\gamma \to D^\gamma$ is a bijection, the last assertion is equivalent to $g^\gamma(R) = R$ for all $R \in \mathfrak{R}$. Hence, $g^\delta(q) \in M$ if and only if $g \in G$. Moreover, for every $p \in M$, the bijection $g = p \circ q^{-1} : D \to D$ is such that $g \in G$ and $g^\delta(q) = p$. Therefore, M is the orbit of q in D^γ. □

Proof of theorem 3.3. Since every invariant relation R is the union of orbits of G, it is enough to prove the theorem when R is an orbit of G. Let R be an orbit of arity γ and let $q \in N$. Then, for $\xi \in \Theta_R$ and for $g \in G$, $\xi^*(g^\delta(q)) = g^\gamma(\xi^*(q))$. Hence, $\xi^*(\mathcal{O}_q) = R$. Theorem 3.3 follows now from lemma 3.6 and the definition of $\hat{E}$. □

4 The language $\mathcal{L}^E_{\alpha\beta}$

From now on we assume that μ is an infinite cardinal. For any set A, the cardinal number of A will be denoted by $|A|$.

We shall define a *first order language* $\mathcal{L}^E_{\alpha\beta}$ associated to a given first order structure $E = \langle D, \mu, \mathfrak{R} \rangle$. It is assumed that α is a regular infinite cardinal and that $\beta \leq \alpha$ is an infinite cardinal greater than the arity of any primitive relation $R \in \mathfrak{R}$.

The symbols of $\mathcal{L}^E_{\alpha\beta}$ are: 1) symbols of variables; 2) logic symbols $\neg, \exists$ and $\bigwedge$; 3) a symbol of predicate $\bar{R}$ to represent each primitive relation $R \in \mathfrak{R}$; the symbol = representing the diagonal Δ of D^2; 4) parenthesis. The arity of a predicate symbol $\bar{R}$ is the arity of R. We denote the set of variables by V and assume that its cardinal is $\alpha \cup \beta \cup |D|$. Moreover, we assume that we have chosen an enumeration $\chi : i \in |V| \mapsto x_i \in V$ of the set of variables which will remain fixed throughout the paper. V is endowed with the order $x_i \preceq x_j$ if and only if $i \leq j$.

The set $\mathcal{F}^E_{\alpha\beta}$ of formulas of $\mathcal{L}^E_{\alpha\beta}$ is the intersection of all sets of sequences of symbols of length less than α satisfying conditions 1) to 4) below. We shall use freely the notion of concatenation of sequences without explicit mention. For a detailed treatment of this notion see Karp [2]. Since the structure E will remain fixed, we shall often write $\mathcal{L}_{\alpha\beta}$ and $\mathcal{F}_{\alpha\beta}$ instead of $\mathcal{L}^E_{\alpha\beta}$ and $\mathcal{F}^E_{\alpha\beta}$.

1. If $\bar{R}$ is a predicate symbol of arity γ and $\tau : \gamma \rightarrow V$ is a sequence of variables, then $\bar{R}\tau$ is a formula.

2. If φ is a formula, then $\neg\varphi$is a formula.

3. If $\gamma < \beta$, $\eta : \gamma \rightarrow V$ is a sequence of variables and φ is a formula, then $\exists\eta\varphi$ is a formula.

4. If $\gamma < \alpha$, $(\varphi_i)_{i<\gamma}$ is a sequence of formulas and the sequence of variables of $(\varphi_i)_{i<\gamma}$ has length less than μ, then $\bigwedge(\varphi_i)_{i<\gamma}$ is a formula. In case $\gamma = n$ is finite, we may use the notation $\varphi_0 \wedge \varphi_1 \wedge \ldots \wedge \varphi_{n-1}$, instead of $\bigwedge(\varphi_i)_{i<n}$.

The set $V(\varphi)$ of free variables of a formula φ is defined as usual. The arity of a formula φ is the ordinal of $V(\varphi)$, the order of $V(\varphi)$ being the order induced by the order of V. If δ is the arity of φ, we denote by $\sigma_\varphi : \delta \rightarrow V(\varphi)$ the order preserving bijection. We remark that the restriction on the length of the sequence of variables of $(\varphi_i)_{i<\gamma}$, in the definition of $\bigwedge(\varphi_i)_{i<\gamma}$, is imposed in order to eliminate the undesirable possibility of existence of formulas of arity $\geq \mu$.

Let $\lambda : V \rightarrow V$ be a permutation of V. We extend λ to a permutation $\lambda^\bullet$ of the set $\mathcal{F}^E_{\alpha\beta}$ in the following way:

1. $\lambda^\bullet(\bar{R}\tau) = \bar{R}(\lambda \circ \tau)$;

2. $\lambda^\bullet(\neg\varphi) = \neg\lambda^\bullet(\varphi)$;

3. $\lambda^\bullet(\exists\eta\varphi) = \exists(\lambda \circ \eta)\lambda^\bullet(\varphi)$;

4. $\lambda^\bullet(\bigwedge(\varphi_i)_{i<\gamma}) = \bigwedge(\lambda^\bullet(\varphi_i))_{i<\gamma}$.

It is easy to prove that $V(\lambda^\bullet(\varphi)) = \lambda(V(\varphi))$, $(\lambda_1 \circ \lambda_2)^\bullet = \lambda_1^\bullet \circ \lambda_2^\bullet$ and $(\lambda^{-1})^\bullet = (\lambda^\bullet)^{-1}$.

5 The semantic of $\mathcal{L}^E_{\alpha\beta}$

We shall define for each formula $\varphi \in \mathcal{F}^E_{\alpha\beta}$ of arity γ a relation $\|\varphi\| \subseteq D^\gamma$ called the *relation defined by* φ. We shall also say that a γ-tuple $p \in D^\gamma$ *satisfies* φ if and only if $p \in \|\varphi\|$. In the same way, we say that a relation $S \in \mathcal{U}_\mu(D)$ is $\mathcal{L}^E_{\alpha\beta}$*-definable* if and only if there exists a formula $\varphi \in \mathcal{F}^E_{\alpha\beta}$ such that $S = \|\varphi\|$.

Definition 5.1 *The definition of $\|\varphi\|$ is by induction on rules 1) to 4) of definition of formulas of $\mathcal{L}_{\alpha\beta}$.*

1) If φ is the formula $\bar{R}\tau$ with $\tau : \gamma \to V$, then $\|\varphi\| = ((\sigma_\varphi^{-1} \circ \tau)^*)^{-1}(R)$.

2) I f φ is $\neg\psi$ and ψ if of arity γ, then $\|\varphi\| = \mathfrak{C}_\gamma \|\psi\|$.

3) If φ is $\exists\eta\psi$, then $\|\varphi\| = (\sigma_\psi^{-1} \circ \sigma_\varphi)^* \|\psi\|$.

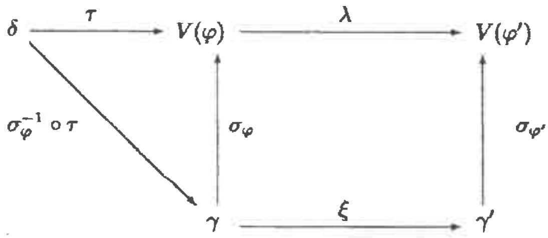
4) If φ is $\bigwedge(\varphi_i)_{i<\gamma}$, then $\|\varphi\| = \bigcap_{i\in\gamma} ((\sigma_\varphi^{-1} \circ \sigma_{\varphi_i})^*)^{-1} \|\varphi_i\|$.

We denote by $\left\|\mathcal{F}_{\alpha\beta}^E\right\|$ the set of all relations of arity γ, $\gamma < \mu$, that are defined by formulas in $\mathcal{F}_{\alpha\beta}^E$.

Let φ be a formula of arity γ and let $\lambda : V \to V$ be a bijection. Put $\varphi' = \lambda^\bullet(\varphi)$ and consider the order preserving bijections $\sigma_\varphi : \gamma \to V(\varphi)$ and $\sigma_{\varphi'} : \gamma' \to V(\varphi')$ where γ' is the arity of φ'. Let $\xi : \gamma \to \gamma'$ be a bijection.

Proposition 5.2 *If $\lambda \circ \sigma_\varphi = \sigma_{\varphi'} \circ \xi$, then $\|\lambda^\bullet(\varphi)\| = (\xi^*)^{-1} \|\varphi\|$.*

Proof. The proof is by induction on rules 1) to 4) of definition of formulas.
1) If $\varphi = \bar{R}\tau$, then $\varphi' = \bar{R}(\lambda \circ \tau)$.



We have:

$$\begin{aligned}
(\xi^*)^{-1} \|\varphi\| &= ((\sigma_{\varphi'}^{-1} \circ \lambda \circ \sigma_\varphi)^*)^{-1}(\|\varphi\|) \\
&= ((\sigma_{\varphi'}^{-1} \circ \lambda \circ \sigma_\varphi)^*)^{-1}(((\sigma_\varphi^{-1} \circ \tau)^*)^{-1}(R)) \\
&= ((\sigma_{\varphi'}^{-1} \circ (\lambda \circ \tau))^*)^{-1}(R) \\
&= \|\lambda^\bullet(\varphi)\|.
\end{aligned}$$

Assume now that the proposition holds for a formula ψ of arity γ, that is, if $\psi' = \lambda^\bullet(\psi)$ and $\lambda \circ \sigma_\psi = \sigma_{\psi'} \circ \xi_\psi$, then $\|\lambda^\bullet(\psi)\| = (\xi_\psi^*)^{-1} \|\psi\|$. Let $\varphi' = \lambda^\bullet(\varphi)$ and let ξ_φ be a bijection such that $\lambda \circ \sigma_\varphi = \sigma_{\varphi'} \circ \xi_\varphi$. We shall prove that the proposition holds for $\varphi = \neg\psi$ and $\varphi = \exists\eta\psi$.
2) If $\varphi = \neg\psi$, then $\sigma_\varphi = \sigma_\psi$ and $\sigma_{\varphi'} = \sigma_{\psi'}$. Hence, $\xi_\varphi = \xi_\psi$. Therefore, we have that $(\xi_\varphi^*)^{-1} \|\varphi\| = (\xi_\varphi^*)^{-1} \|\neg\psi\| = (\xi_\varphi^*)^{-1}(\mathfrak{C}_{\gamma'} \|\psi\|) = \mathfrak{C}_{\gamma'}(\xi_\varphi^*)^{-1} \|\psi\| = \mathfrak{C}_{\gamma'}(\xi_\psi^*)^{-1} \|\psi\| = \mathfrak{C}_{\gamma'} \|\lambda^\bullet(\psi)\| = \|\neg\lambda^\bullet(\psi)\| = \|\lambda^\bullet(\neg\psi)\| = \|\lambda^\bullet(\varphi)\|$. We remark that, since ξ_φ is a bijection, the commutativity $(\xi_\varphi^*)^{-1}\mathfrak{C}_{\gamma'} = \mathfrak{C}_{\gamma'}(\xi_\varphi^*)^{-1}$ holds.
3) Let $\varphi = \exists\eta\psi$. Then, $\|\lambda^\bullet(\varphi)\| = \|\exists(\lambda \circ \eta)\lambda^\bullet(\psi)\| = (\sigma_{\psi'}^{-1} \circ \sigma_{\varphi'})^* \|\lambda^\bullet(\psi)\| = (\sigma_{\psi'}^{-1} \circ \sigma_{\varphi'})^*((\xi_\psi^*)^{-1} \|\psi)\|) = (\xi_\psi^{-1} \circ \sigma_{\psi'}^{-1} \circ \sigma_{\varphi'})^* \|\psi)\| = (\sigma_\psi^{-1} \circ \lambda^{-1} \circ \sigma_{\varphi'})^* \|\psi)\| = (\sigma_\psi^{-1} \circ \sigma_{\varphi'} \circ \xi_\varphi^{-1})^* \|\psi)\| = (\xi_\varphi^*)^{-1}((\sigma_\psi^{-1} \circ \sigma_{\varphi'})^* \|\psi)\|) = (\xi_\varphi^*)^{-1} \|\psi)\|)$.

4) To complete the proof of proposition 5.2, let $\varphi = \bigwedge(\varphi_i)_{i<\gamma}$, $\varphi_i' = \lambda^*(\varphi_i)$, $\varphi' = \lambda^*(\varphi)$ and let ξ_i and ξ_φ be bijections such that $\lambda \circ \sigma_{\varphi_i} = \sigma_{\varphi_i'} \circ \xi_i$ and $\lambda \circ \sigma_\varphi = \sigma_{\varphi'} \circ \xi_\varphi$. Assume that the proposition holds for each φ_i, that is, $\|\lambda^*(\varphi_i)\| = (\xi_i^*)^{-1} \|\varphi_i\|$, for $i < \gamma$. Then, we have that:

$$\begin{aligned}
\|\lambda^*(\varphi)\| &= \|\bigwedge(\lambda^*(\varphi_i))_{i<\gamma}\| \\
&= \bigcap_{i<\gamma} (((\sigma_{\varphi'}^{-1} \circ \sigma_{\varphi_i'})^*)^{-1} \|\varphi_i'\|) \\
&= \bigcap_{i<\gamma} (((\sigma_{\varphi'}^{-1} \circ \sigma_{\varphi_i'})^*)^{-1}((\xi_i^*)^{-1} \|\varphi_i\|)) \\
&= \bigcap_{i<\gamma} (((\sigma_{\varphi'}^{-1} \circ \sigma_{\varphi_i'} \circ \xi_i)^*)^{-1} \|\varphi_i\|) \\
&= \bigcap_{i<\gamma} (((\sigma_{\varphi'}^{-1} \circ \lambda \circ \sigma_{\varphi_i})^*)^{-1} \|\varphi_i\|) \\
&= \bigcap_{i<\gamma} (((\xi_\varphi \circ \sigma_\varphi^{-1} \circ \sigma_{\varphi_i})^*)^{-1} \|\varphi_i\|) \\
&= (\xi_\varphi^*)^{-1}(\bigcap_{i<\gamma} (\sigma_\varphi^{-1} \circ \sigma_{\varphi_i})^*) \|\varphi_i\|) \\
&= (\xi_\varphi^*)^{-1} \|\bigwedge(\varphi_i)_{i<\gamma}\| \\
&= (\xi_\varphi^*)^{-1} \|\varphi\|.
\end{aligned}$$

Hence, the proposition holds for φ. □

The following notation will be used throughout the rest of the paper. Let δ and γ be two ordinals and let $\xi : \delta \to \gamma$ be a map. Consider the following equivalence relation defined in δ: $j \in \delta$ is equivalent to $k \in \delta$ if and only if $\xi(j) = \xi(k)$. Let I be the set of first elements of the equivalence classes. Then, $\xi|I : I \to \gamma$ is injective and $\xi(I) = \xi(\delta)$. Let $\bar{\delta}$ and $\bar{\gamma}$ be the ordinals of I and $\xi(I)$ respectively and denote by $\xi_1 : \bar{\delta} \to I$ and $\xi_2 : \bar{\gamma} \to I$ the order preserving bijections. We denote by $\bar{\xi} : \bar{\delta} \to \bar{\gamma}$ the bijection $\bar{\xi} = \xi_2^{-1} \circ (\xi|I) \circ \xi_1$. The ordinals of $\gamma - I$ and $\gamma - \xi(I)$ will be denoted by δ_ξ' and γ_ξ' respectively. Where there is no danger of confusion, we shall write simply δ' and γ' instead of δ_ξ' and γ_ξ'. Finally, for each $i \in I$, let $\mathcal{P}_i$ be the equivalence class of i in the equivalence relation defined above. We denote by $\mathcal{P} = (\mathcal{P}_i)_{i\in I}$ the resulting enumeration of the equivalence classes.

Proposition 5.3 *Let $\varphi \in \mathcal{F}_{\alpha\beta}^E$ be a formula of arity γ and let $\xi : \delta \to \gamma$, $\delta, \gamma < \mu$, be an order preserving injection. Assume $\gamma' < \beta$ and let $\eta = \chi \circ \xi_2$. Then, $\xi^* \|\varphi\| = \|\exists\eta\varphi\|$.*

Proof. If $\psi = \exists\eta\varphi$, then $\psi \in \mathcal{F}_{\alpha\beta}^E$ and $\|\exists\eta\varphi\| = (\sigma_\varphi^{-1} \circ \sigma_\psi)^* \|\varphi\|$. But, we have that $\sigma_\psi = \sigma_\varphi \circ \xi$. Hence, $\|\exists\eta\varphi\| = (\sigma_\varphi^{-1} \circ \sigma_\varphi \circ \xi)^* \|\varphi\| = \xi^* \|\varphi\|$. □

Proposition 5.4 *Let $\varphi \in \mathcal{F}_{\alpha\beta}^E$ be a formula of arity δ and let $\xi : \delta \to \gamma$, $\gamma < \mu$ be an order preserving injection. If $\gamma < \alpha$, there exists a formula $\psi \in \mathcal{F}_{\alpha\beta}^E$ such that $(\xi^*)^{-1} \|\varphi\| = \|\psi\|$.*

Proof. Let $\lambda = \chi \circ \xi \circ \sigma_\varphi^{-1}$ and extend λ to a permutation of V, also denoted by λ, defining $\lambda(x_i) = x_i$ for $i \in \gamma - \xi(\delta)$. Since $\lambda \circ \sigma_\varphi$ is an increasing bijection from δ onto $\lambda(V(\varphi))$, it follows that the arity of $\varphi' = \lambda^*(\varphi)$ is δ and $\sigma_{\varphi'} = \lambda \circ \sigma_\varphi$. Hence, $\sigma_{\varphi'}^{-1} \circ \lambda \circ \sigma_\varphi$ is the identity map of δ. Therefore, $\|\lambda^*(\varphi)\| = \|\varphi\|$.
Let φ_1 be the formula $\bigwedge(x_i = x_i)_{i\in\gamma-\xi(\delta)}$. Since $\gamma' \leq \gamma < \alpha$, $\varphi_1 \in \mathcal{F}_{\alpha\beta}^E$. On the other hand, $\sigma_{\varphi_1} = \chi \circ \sigma$, where σ is the increasing bijection from the cardinal of $\gamma - \xi(\delta)$ onto $\gamma - \xi(\delta)$. It follows that $\|\varphi_1\| = D^{\gamma'}$.

Let ψ be the formula $\lambda^*(\varphi) \bigwedge \varphi_1$. Then, $\psi \in \mathcal{F}^E_{\alpha\beta}$ and we have that $\sigma_\psi = \chi|\gamma$, $\sigma_\psi^{-1} \circ \sigma_{\varphi'} = \xi$ and $\sigma_\psi^{-1} \circ \sigma_{\varphi_1} = \sigma$. Hence, $\|\psi\| = \|\varphi' \bigwedge \varphi_1\| = (\xi^*)^{-1} \|\varphi'\| \cap (\sigma^*)^{-1}(D^{\gamma'}) = (\xi^*)^{-1} \|\varphi'\| \cap D^\gamma = (\xi^*)^{-1} \|\varphi\|$. □

Proposition 5.5 *Let $R \in \|\mathcal{F}_{\alpha\beta}\|$ be a relation of arity γ and let $\xi : \delta \to \gamma$ be a map such that $\delta < \mu$, $\delta < \alpha$ and $\delta' < \beta$. Then, $\xi^*(R) \in \|\mathcal{F}_{\alpha\beta}\|$.*

Proof. Let $\mathcal{D}(\mathcal{P})$ be the diagonal of D^γ corresponding to the partition $\mathcal{P} = (\mathcal{P}_i)_{i \in I}$ of δ. Then, for $p \in D^\delta$, we have:

$$\begin{aligned} p \in \xi^*(R) &\Leftrightarrow (p|I) \in (\xi|I)^*(R) \text{ and } p \in \mathcal{D}(\mathcal{P}) \\ &\Leftrightarrow (p|I) \in (\xi_2 \circ \bar{\xi} \circ \xi_1^{-1})^*(R) \text{ and } p \in \mathcal{D}(\mathcal{P}) \\ &\Leftrightarrow (p|I) \in (\xi_1^{-1})^*((\bar{\xi})^*(\xi_2^*(R))) \text{ and } p \in \mathcal{D}(\mathcal{P}) \\ &\Leftrightarrow (p|I) \circ \xi_1 \in (\bar{\xi})^*(\xi_2^*(R)) \text{ and } p \in \mathcal{D}(\mathcal{P}). \end{aligned}$$

But, $(p|I) \circ \xi_1 = p \circ \xi_1$. Hence, $\xi^*(R) = (\xi_1^{-1})^*((\bar{\xi})^*(\xi_2^*(R))) \bigcap \mathcal{D}(\mathcal{P})$.
By propositions 5.2, 5.3 and 5.4, we have $(\xi_1^{-1})^*((\bar{\xi})^*(\xi_2^*(R))) \in \|\mathcal{F}_{\alpha\beta}\|$.
Let δ_i be the ordinal of $\mathcal{P}_i \subseteq \delta$, $i \in \delta'$, and let $\sigma_i : \delta_i \to \mathcal{P}_i$ be the increasing bijetion. Then, we have that $\delta' \leq \delta < \alpha$ and $\delta_i' \leq \delta < \alpha$. Consequently, for $i \in \delta'$, $\Psi_i = \bigwedge(x_{\xi_1(i)} = x_{\sigma_i(j)})_{j \in \delta_i}$ and $\Psi = \bigwedge(\Psi_i)_{i \in \delta'}$ are both in $\mathcal{F}_{\alpha\beta}$. It is easy to show that $\|\Psi\| = \mathcal{D}(\mathcal{P})$. Hence, $\xi^*(R) \in \|\mathcal{F}_{\alpha\beta}\|$. □

Proposition 5.6 *Let $R \in \|\mathcal{F}_{\alpha\beta}\|$ be a relation of arity δ and let $\xi : \delta \to \gamma$ be a map such that $\gamma < \mu$, $\delta < \alpha$, $\gamma < \alpha$ and $\delta' < \beta$. Then, $(\xi^*)^{-1}(R) \in \|\mathcal{F}_{\alpha\beta}\|$.*

Proof. We have to show that:

$$(\xi^*)^{-1}(R) = (\xi_2^*)^{-1}((\bar{\xi}^{-1})^*(\xi_1^*(R \cap D(\mathcal{P})))).$$

Since $\delta < \alpha$, $\mathcal{D}(\mathcal{P}) \in \|\mathcal{F}_{\alpha\beta}\|$. Proposition 5.6 will follow then from propositions 5.2, 5.3 and 5.4.
For any relation $R \subseteq D^\delta$, let $R|I$ denote the set of restrictions $p|I$ for all $p \in R$. Then,

$$\begin{aligned} (\xi^*)^{-1}(R) &= (\xi^*)^{-1}(R \cap \mathcal{D}(\mathcal{P})) \\ &= ((\xi|I)^*)^{-1}((R \cap \mathcal{D}(\mathcal{P}))|I) \\ &= ((\xi_2 \circ \bar{\xi} \circ \xi_1)^*)^{-1}((R \cap \mathcal{D}(\mathcal{P}))|I) \\ &= (\xi_2^*)^{-1}((\bar{\xi}^{-1})^*(\xi_1^*((R \cap \mathcal{D}(\mathcal{P}))|I))) \\ &= (\xi_2^*)^{-1}((\bar{\xi}^{-1})^*(\xi_1^*(R \cap \mathcal{D}(\mathcal{P})))). \end{aligned}$$

This completes the proof. □

6 The closure $\hat{E}_{\alpha\beta}$ and the language $\mathcal{L}^E_{\alpha\beta}$

Let $E = (D, \mu, \mathcal{R})$ be a first order structure. We want to characterize the set $\left\|\mathcal{F}^E_{\alpha\beta}\right\|$ of relations definable in the language $\mathcal{L}^E_{\alpha\beta}$ as a set of relations obtained from the primitive relations $\mathcal{R}$ by means of suitable set theorctical operations. With this purpose in mind and corresponding to a choice of cardinals α and β, we introduce a subset $\mathcal{X}_{\alpha\beta}$ of the set $\mathcal{X}$ of maps defined in section 2.

We assume that α and β satisfy the same restrictions imposed to define the language $\mathcal{L}^E_{\alpha\beta}$. That is, α and β are infinite cardinals, α is regular, $\beta \leq \alpha$ and β is greater than the arity of any primitive relation $R \in \mathcal{R}$. By definition, $\mathcal{X}_{\alpha\beta}$ is the set of all maps $\mathcal{C}_\gamma, \xi^*, (\xi^*)^{-1}$ satisfying the following conditions:

1. $\mathfrak{C}_\gamma \in \mathcal{K}_{\alpha\beta}$, for all $\gamma < \mu$;

2. If $\xi : \delta \to \gamma$, with $\delta, \gamma < \mu$, then $\xi^* \in \mathcal{K}_{\alpha\beta}$;

3. If $\xi : \delta \to \gamma$, then $(\xi^*)^{-1} \in \mathcal{K}_{\alpha\beta}$ if and only if $\delta, \gamma < \alpha$ and $\delta' < \beta$.

Definition 6.1 *A set of relations $\mathcal{S} \subseteq \mathcal{U}_\mu(D)$ is (α, β)-closed if:*
1) $\mathcal{R} \subseteq \mathcal{S}$;
2) For all $R \in \mathcal{S}$ and $f \in \mathcal{K}_{\alpha\beta}$ such that $f(R)$ is defined, $f(R) \in \mathcal{S}$;
3) If $\mathcal{S}'$ is a subset of $\mathcal{S}$ of cardinality less than α, then $\bigcap \mathcal{S}' \in \mathcal{S}$.
By definition, the (α, β)-closure of $\mathcal{R}$ is the intersection $\hat{E}_{\alpha\beta}$ of all (α, β)-closed subsets of $\mathcal{U}_\mu(D)$.

Theorem 6.2 $\hat{E}_{\alpha\beta} = \|\mathcal{F}_{\alpha\beta}\|$.

Proof. We shall prove that $\|\mathcal{F}_{\alpha\beta}\|$ is (α, β)-closed. It follows that $\hat{E}_{\alpha\beta} \subseteq \|\mathcal{F}_{\alpha\beta}\|$.
1) If $R \in \mathcal{R}$ is a primitive relation of arity γ and $\varphi = \bar{R}\tau$ where $\tau = \chi|\gamma$, then $\sigma_\varphi^{-1} \circ \tau$ is the identity map of γ. Hence, by definition of $\|\varphi\|$, $\|\varphi\| = R$. Consequently, $\mathcal{R} \subseteq \|\mathcal{F}_{\alpha\beta}\|$.
2) Let now $S = \|\varphi\| \in \|\mathcal{F}_{\alpha\beta}\|$ be a relation of arity γ. Then, $\mathfrak{C}_\gamma S = \|\neg\varphi\|$. Hence, $\mathfrak{C}_\gamma S \in \|\mathcal{F}_{\alpha\beta}\|$. If $\xi : \delta \to \gamma$ is such that ξ^*, respectively $(\xi^*)^{-1}$, belong to $\mathcal{K}_{\alpha\beta}$, then, by propositions 5.2, 5.5 and 5.6, $(\xi^*)S$ and $(\xi^*)^{-1}S$ also belong to $\|\mathcal{F}_{\alpha\beta}\|$.
3) To prove that if $S_i \in \|\mathcal{F}_{\alpha\beta}\|$ for all $i < \gamma < \alpha$, then $\bigcap_{i<\gamma} S_i \in \|\mathcal{F}_{\alpha\beta}\|$, we begin remarking that if $\varphi \in \mathcal{F}_{\alpha\beta}$ is a formula of arity δ, then there exists a formula $\varphi' \in \mathcal{F}_{\alpha\beta}$ such that $\|\varphi\| = \|\varphi'\|$ and $\sigma_{\varphi'} = \chi|\delta$. In fact, it is enough to consider the map $\lambda : V \to V$ such that $\lambda|V(\varphi) = \chi \circ \sigma_\varphi^{-1}$ and $\lambda(x) = x$ for all $x \notin V(\varphi)$. Then, if $\varphi' = \lambda^*(\varphi)$, $\sigma_{\varphi'} = \chi|\delta$ and, by proposition 5.2, $\|\varphi\| = \|\varphi'\|$. Since for $S \subseteq D^\delta$, $S' \subseteq D^{\delta'}$ and $\delta \neq \delta'$, $S \cap S' = \emptyset$ and for any $\varphi \in \mathcal{F}_{\alpha\beta}$, $\emptyset = \|\varphi \wedge \neg\varphi\| \in \|\mathcal{F}_{\alpha\beta}\|$, it is suffice to assume that all S_i have the same arity δ. Moreover, we may also assume that $S_i = \|\varphi_i\|$ and $\sigma_{\varphi_i} = \chi|\delta$. Then, $\bigwedge(\varphi_i)_{i<\gamma} \in \|\mathcal{F}_{\alpha\beta}\|$ and, by definition of $\|\bigwedge(\varphi_i)_{i<\gamma}\|$, we have that:

$$\left\|\bigwedge(\varphi_i)_{i<\gamma}\right\| = \bigcap_{i<\gamma} \|\varphi_i\| = \bigcap_{i<\gamma} S_i.$$

This completes the proof that $\hat{E}_{\alpha\beta} \subseteq \|\mathcal{F}_{\alpha\beta}\|$.
Let $\mathcal{F}$ be the set of formulas $\varphi \in \mathcal{F}_{\alpha\beta}$ such that $\|\varphi\| \in \hat{E}_{\alpha\beta}$. We shall show that $\mathcal{F}$ satisfies the conditions 1) to 4) of the definition of $\mathcal{F}_{\alpha\beta}$. This proves that $\mathcal{F} = \mathcal{F}_{\alpha\beta}$ and $\|\mathcal{F}_{\alpha\beta}\| \subseteq \hat{E}_{\alpha\beta}$, completing the proof that $\hat{E}_{\alpha\beta} = \|\mathcal{F}_{\alpha\beta}\|$.
1) If $\varphi = \bar{R}\tau$, $\tau : \delta \to V$, is an atomic formula of arity δ, then $\|\varphi\| = (\xi^*)^{-1}R$ where $\xi = \sigma_\varphi^{-1} \circ \tau : \delta \to \gamma$ is a surjective map. Then, $\gamma \leq \delta < \beta \leq \alpha$ and the ordinal γ' of $\gamma - \xi(\delta)$ is less than or equal to γ. Hence, $\xi \in \mathcal{K}_{\alpha\beta}$ and $\|\varphi\| = (\xi^*)^{-1}R \in \hat{E}_{\alpha\beta}$. Therefore, $\mathcal{F}$ contains the atomic formulas.
2) If $\varphi \in \mathcal{F}$ is a formula of arity $\gamma < \mu$, then $\|\neg\varphi\| = \mathfrak{C}_\gamma \|\varphi\| \in \hat{E}_{\alpha\beta}$. Hence, $\neg\varphi \in \mathcal{F}$.
3) If $\psi \in \mathcal{F}$ is a formula of arity γ, $\gamma_0 < \beta$, $\eta : \gamma_0 \to V$ and $\varphi = \exists\eta\psi$ has arity δ, then $\|\varphi\| = (\xi^*)\|\psi\|$ where $\xi = \sigma_\psi^{-1} \circ \sigma_\varphi : \delta \to \gamma$ is an order preserving injection. σ_ψ maps bijectively and increasingly $\gamma - \xi(\delta)$ onto the complement $V(\varphi) \cap \eta(\gamma_0)$ of $V(\varphi)$ in $V(\psi)$. Therefore, the ordinalγ' of $\gamma - \xi(\delta)$ is less than β. Hence, by proposition 5.2, $\|\varphi\| \in \hat{E}_{\alpha\beta}$ and $\varphi \in \mathcal{F}$.

4) To complete the proof, let $(\varphi_i)_{i<\gamma_0}$, $\gamma_0 < \alpha$, be a sequence of formulas of $\mathcal{F}$ of arity δ_i, $i < \gamma_0$, and let $\varphi = \bigwedge(\varphi_i)_{i<\gamma_0}$. Assume that $\varphi \in \mathcal{F}_{\alpha\beta}$. The arity γ of φ is the ordinal of $V(\varphi) = \bigcup_{i<\gamma_0} V(\varphi_i)$. Since $\delta_i < \alpha$ for all $i < \gamma_0$ and α is a regular cardinal, it follows that $\gamma < \alpha$. Let $\xi_i = \sigma_\varphi^{-1} \circ \sigma_{\varphi_i} : \delta_i \to \gamma$. Then, the ordinal γ' of $\gamma - \xi_i(\delta_i)$ is less than α and, since ξ_i is an increasing injection, also $\delta_i < \alpha$. Therefore, $(\xi_i^*)^{-1} \in \mathcal{K}_{\alpha\beta}$, $(\xi_i^*)^{-1} \|\varphi_i\| \in \hat{E}_{\alpha\beta}$ and $\|\varphi\| = \bigcap_{i<\gamma_0} \|\varphi_i\| \in \hat{E}_{\alpha\beta}$. This shows that $\varphi \in \mathcal{F}$, completing the proof of theorem 6.2. □

Butz and Moerdijk [1] has proved a theorem of definability by means of Boolean valued models similar to theorem 6.2.

7 Definability and invariance

Proposition 7.1 *Let $\mathcal{L}^E_{\alpha\beta}$ be the first order language associated to a first order structure E. Then, $\left\|\mathcal{F}^E_{\alpha\beta}\right\| \subseteq \mathcal{I}(E)$.*

Proof. The proof is by induction on rules 1) to 4) of definition of formulas of $\mathcal{L}^E_{\alpha\beta}$, taking into consideration that, if $\xi : \delta \to \gamma$, with $\delta, \gamma < \mu$, is any map and $g : D \to D$ is a bijection, then $g^\delta \circ \xi^* = \xi^* \circ g^\gamma$, $g^\gamma \circ (\xi^*)^{-1} = (\xi^*)^{-1} \circ g^\delta$ and $\mathcal{C}_\gamma \circ g^\gamma = g^\gamma \circ \mathcal{C}_\gamma$. □

Proposition 7.1 follows also from proposition 3.2 and theorem 6.2 with the remark that $\hat{E}_{\alpha\beta} \subseteq \hat{E}$.

In this section, we want to determine conditions on α and β in order that the equality $\left\|\mathcal{F}^E_{\alpha\beta}\right\| = \mathcal{I}(E)$ holds. Keeping the notation as in sections 2 and 6 and assuming that μ is an infinite cardinal, we start remarking that if α and β are both greater or equal to μ, then the sets of operators $\mathcal{K}$ and $\mathcal{K}_{\alpha\beta}$ of the structure $E = \langle D, \mu, \mathcal{R}\rangle$ coincide. Clearly, $|\mathcal{I}(E)| \leq |\mathcal{U}^\mu(D)|$ and $\mu \leq |\mathcal{U}^\mu(D)|$. It follows that if $\alpha \geq |\mathcal{U}_\mu(D)|$ and $\beta \geq \mu$, then $\hat{E} = \hat{E}_{\alpha\beta}$.

Assume now that D is infinite. Let $d = |D|$ and let d^+ be the first cardinal greater than d. We know (theorem 3.1) that if $\mu \geq d+2$, then $\hat{E} = \mathcal{I}(E)$. Assume $\mu = d^+$. If γ is an ordinal less than μ, then $|\gamma| \leq d$. Hence, $|D^\gamma| \leq d^d = 2^d$ and

$$|\mathcal{U}_\mu(D)| = |\bigcup_{\gamma<\mu} \wp(D^\gamma)| \leq 2^{(2^d)} \times \mu = 2^{(2^d)}.$$

If D is finite and $\mu = \omega$, the first infinite cardinal, then $\hat{E} = \hat{E}_{\omega\omega} = \mathcal{I}(E)$. Taking into consideration theorem 6.2, we have

Theorem 7.2 *If D is an infinite set of cardinal d, $\mu = d^+$, $\alpha = (2^{(2^d)})^+$, $\beta = d^+$, then $\left\|\mathcal{F}^E_{\alpha\beta}\right\| = \mathcal{I}(E)$. If D is finite, $\left\|\mathcal{F}^E_{\omega\omega}\right\| = \mathcal{I}(E)$.*

Imposing mild restrictions on the set of primitive relations $\mathcal{R}$ one can lower the cardinal α in the statement of theorem 7.2. Assume that the arity of every primitive relation $R \in \mathcal{R}$ is less than $d = |D|$ an assume also that $|\mathcal{R}| \leq d$. The notation being as in theorem 3.3, we have the following limitations on the cardinality of intersections used in the proves of lemmas 3.4, 3.5, 3.6 and theorem 3.3:

1. N is the intersection of a subset of $\mathcal{U}_\mu(D)$ of power d.

2. If R is a relation of arity γ, then M_R is the intersection of a subset of $\mathcal{U}_\mu(D)$ of power less than or equal to $|\Theta_R| = d^\gamma$. Hence, if $\gamma < d$, R is the intersection of a set of power $\leq d$.

3. $\mathcal{O}_q$ is the intersection of a subset of $\mathcal{U}_\mu(D)$ of power $|\mathcal{R}|$.

4. If R is an invariant relation of arity γ and $\gamma < d$, then R is the union of a set of power $|\Theta_R| = d^\gamma = d$.

We have proved the following special case of theorem 3.3.

Theorem 7.3 *Let $E = \langle D, \mu, \mathcal{R}\rangle$ be a first order structure. Assume that D is infinite, $|\mathcal{R}| \leq |D|$ and $\mu \geq |D| + 2$. Then, $\hat{E}$ contains all invariant relations of arity less than $|D|$.*

The next theorem follows from theorem 6.2 and 7.3.

Theorem 7.4 *Let $E = \langle D, \mu, \mathcal{R}\rangle$ be a first order structure and let d be the cardinal of D. Assume that D is infinite, $|\mathcal{R}| \leq d$ and $\mu = d^+$. Then, all invariant relations of arity less than d are in $\|\mathcal{F}^E_{d^+d^+}\|$.*

Theorem 7.4 is a generalization to relations whose arity are infinite ordinals of a theorem of Rogers [6].

8 A counter example

By a well known theorem of D. Scott (see [7] and [5]), given a structure $E = \langle D, \omega, \mathcal{R}\rangle$, assuming that $\mathcal{R}$ is finite and the domain D is denumerable, every invariant relation $S \in \mathcal{U}_\mu(D)$ is $\mathcal{L}^E_{\omega_1\omega}$-definable. We shall show by a counter example that, when D is not denumerable, for every regular cardinal α, there may exist invariant relations $S \in \mathcal{U}_\omega$which are not $\mathcal{L}^E_{\alpha\omega}$-definable.

Let $d_2 \neq d_3$ be infinite cardinals and let D_i, $i < 4$, be disjoint sets satisfaying the following conditions: D_0 and D_1 are unitary sets, $D_0 = \{a_0\}$, $D_1 = \{a_1\}$, $a_0 \neq a_1$, $|D_2| = d_2$ and $|D_3| = d_3$. Define $D = \bigcup_{i<4} D_i$, $R_0 = D_0 \times D_2$, $R_1 = D_1 \times D_3$, $R = R_0 \cup R_1$ and consider the first order structure $E = \langle D, \omega, \mathcal{R}\rangle$ such that $\mathcal{R} = \{R, \Delta\}$ and Δ is the diagonal of D^2.

Let G be the automorphism group of E. For any $g \in G$, we must have $g(a_0) = a_0$ or $g(a_0) = a_1$, but $g(a_0) = a_1$ is impossible for then, we should have $g(D_2) = D_3$ against the hypothesis on the cardinal numbers of D_2 and D_3. It follows that a bijetion $g : D \to D$ is an automorphism of E if and only if $g(D_i) = D_i$, for $i < 4$.

For $n < \omega$, define an equivalence relation $\sim_n$ in D^n in the following way: two points $p, q \in D^n$ are equivalent if and only if, for $i, j < n$, we have

$$p_i = p_j \Leftrightarrow q_i = q_j.$$

The equivalence classes of D^n are clearly invariant under G.

Let T^n be the set of maps from $n = \{0, 1, ..., n-1\}$ into $4 = \{0, 1, 2, 3\}$ and, for $\tau \in T^n$, denote the product $D_{\tau(0)} \times D_{\tau(1)} \times ... \times D_{\tau(n-1)}$ by $\Pi_\tau D$. D^n is the disjoint union of all sets $\Pi_\tau D$ when τ varies in T^n and each set $\Pi_\tau D$ is also invariant under the action of G.

D^n is also the disjoint union of the invariant sets $\Pi_\tau D \cap A$ when τ varies in T^n and A varies in the set of equivalence classes of $\sim_n$. Hence, given an orbit $\mathcal{O}$ of G in D^n, there exists an unique $\tau \in T^n$ and an unique A such that $\mathcal{O} \subseteq \Pi_\tau D \cap A$. An easy argument shows that $\mathcal{O} = \Pi_\tau D \cap A$. Therefore, the orbits of G in D^n are the non empty intersection $\Pi_\tau D \cap A$ when τ and A vary as above.

Let σ be the involutive permutation of $4 = \{0,1,2,3\}$ defined by: $\sigma(0) = 1, \sigma(1) = 0, \sigma(2) = 3, \sigma(3) = 2$. We want to prove that $\Pi_\tau D \cap A = \emptyset$ if and only if $\Pi_{\sigma\circ\tau} D \cap A = \emptyset$. Since σ is involutive, it is suffices to show that $\Pi_\tau D \cap A = \emptyset$ implies $\Pi_{\sigma\circ\tau} D \cap A = \emptyset$. Assume $\Pi_\tau D \cap A = \emptyset$ and let q be a point of A. Define an equivalence relation $\sim$ in $n = \{0, 1, ..., n-1\}$ by the condition: $j \in n$ is equivalent to $k \in n$ if and only if $q_j = q_k$.

Let $J \subseteq n$ be the set of least elements of the equivalence classes and let C_j be the equivalence class of $j \in J$. If $\Pi_{\sigma\circ\tau} D \cap A \neq \emptyset$, there exists $p \in \Pi_{\sigma\circ\tau} D$ such that $p \sim_n q$. Then, for all $j \in J$ and $k < n$, $p_j = p_k$ if and only if $k \in C_j$. Since the sets D_i, $i < 4$, are disjoints, $p_j = p_k$ implies $\sigma(\tau(j)) = \sigma(\tau(k))$ and, consequently, $\tau(j) = \tau(k)$. For each $j \in J$ choose a point $\bar{p}_j \in D_{\tau(j)}$. Put $\bar{p}_k = \bar{p}_j$ for all $k \in C_j$ and let $\bar{p} \in D^n$ be the point defined in this way. Then, $\bar{p} \in \Pi_\tau D$ and $\bar{p}_j = \bar{p}_k \Leftrightarrow q_j = q_k, j \in J, k \in C_j$. Hence, $\bar{p} \sim q$. Therefore, $\Pi_\tau D \cap A \neq \emptyset$ which is absurd, proving that $\Pi_{\sigma\circ\tau} D \cap A = \emptyset$.

Since for any $\xi : m \to n$, $m, n < \omega$, ξ^* commutes with the action of G in $\mathcal{U}_\omega$, the image $\xi^*(\mathcal{O})$ of an orbit $\mathcal{O}$ of G is also an orbit of G. Moreover, if $\mathcal{O} = \Pi_\tau D \cap A$, then $\xi^*(\mathcal{O}) = \Pi_{\tau\circ\xi} D \cap \xi^*(A)$. In fact, $\xi^*(\mathcal{O}) \subseteq \xi^*(\Pi_\tau D) \cap \xi^*(A)$, $\xi^*(\Pi_\tau D) = \Pi_{\tau\circ\xi} D$ and $\xi^*(A)$ is contained in some equivalence class A' of $\sim_m$. Since $\xi^*(\mathcal{O})$ and $\Pi_{\sigma\circ\tau} D \cap A'$ are orbits, we must have $\xi^*(\mathcal{O}) = \Pi_{\sigma\circ\tau} D \cap A' = \Pi_{\tau\circ\xi} D \cap \xi^*(A)$.

For each orbit $\mathcal{O} = \Pi_\tau D \cap A$, denote by $\sigma^*(\mathcal{O})$ the orbit $\Pi_{\sigma\circ\tau} D \cap A$. Then,

$$\begin{aligned} \xi^*\sigma^*(\mathcal{O}) &= \xi^*(\Pi_{\sigma\circ\tau} D \cap A) \\ &= \Pi_{\sigma\circ(\tau\circ\xi)} D \cap \xi^*(A) \\ &= \sigma^*(\Pi_{\tau\circ\xi} D \cap \xi^*(A)) \\ &= \sigma^*\xi^*(\mathcal{O}) \end{aligned}$$

Assume that α is an infinite regular cardinal and let $\bar{E}$ be the set of all relations $S \in \hat{E}_{\alpha\omega}$ for which the following property holds.

> **Property P**: If S contains an orbit $\mathcal{O}$ of G, then S contains also $\sigma^*(\mathcal{O})$.

We shall prove that $\bar{E} = \hat{E}_{\alpha\omega}$. For this purpose, it is enough to show that $\bar{E}$ is closed under the rules of definition of $\hat{E}_{\alpha\omega}$.

1) $\mathfrak{R} \subseteq \bar{E}$. This follows from $\sigma^*(R_0) = R_1$ and the fact that R_0 and R_1 are the only orbits in D^2.

2) Assume that $S \in \bar{E}$ is a relation of arity $n < \omega$ and $\mathcal{O} \subseteq \mathfrak{C}_n S$ is an orbit of G. If $\sigma^*(\mathcal{O}) \subseteq S$, then, since S has the property P, $\sigma^*(\sigma^*(\mathcal{O})) = \mathcal{O} \subseteq S$. This is absurd for $\mathcal{O}$ is not empty and $\mathcal{O} \subseteq \mathfrak{C}_n S$. Hence, $\sigma^*(\mathcal{O}) \cap \mathfrak{C}_n S \neq \emptyset$. Since $\mathfrak{C}_n S$ is invariant under G, and $\sigma^*(\mathcal{O})$ is an orbit, $\sigma^*(\mathcal{O}) \subseteq \mathfrak{C}_n S$, proving that $\mathfrak{C}_n S \in \bar{E}$.

3) Assume that $S = \xi^*(S')$ where $S' \in \bar{E}$ and let $\xi : m \to n$ be a map, $m, n < \omega$. Let $\mathcal{O} \subseteq S$ be an orbit of G. Since S' is invariant under G, $S' = \bigcup_{i \in I} \mathcal{O}_i$. Consequently, there exists $i_0 \in I$ such that $\xi^*(\mathcal{O}_{i_0}) = \mathcal{O}$. Hence,

$\sigma^*(\mathcal{O}) = \xi^*(\sigma^*(\mathcal{O}_{i_0}))$. Since, by hypothesis on S, $\sigma^*(\mathcal{O}_{i_0}) \subseteq S'$, it follows that $\sigma^*(\mathcal{O}) \subseteq S$, proving that $S \in \bar{E}$.

4) Assume $S = (\xi^*)^{-1}S'$, $S' \in \bar{E}$ and $\mathcal{O} \subseteq S$ is an orbit. Then, $\mathcal{O}' = \xi^*\mathcal{O}$ is also an orbit. Hence, $\sigma^*(\mathcal{O}') = \sigma^*\xi^*(\mathcal{O}) = \xi^*\sigma^*(\mathcal{O}) \subseteq S'$. This shows that $\sigma^*(\mathcal{O}) \in S$, proving that $S \in \bar{E}$.

5) $\bar{E}$ is clearly closed under intersections of subsets of $\bar{E}$ of any cardinality.

We have proved that $\bar{E} = \hat{E}_{\alpha\omega}$. Therefore, if $\mathcal{O} \in \hat{E}_{\alpha\omega}$ is an orbit, $\sigma^*(\mathcal{O}) \subseteq \mathcal{O}$ which is absurd for $\sigma^*(\mathcal{O}) \neq \emptyset$ and $\sigma^*(\mathcal{O}) \cap \mathcal{O} = \emptyset$. Hence, $\hat{E}_{\alpha\omega}$ does not contain any orbit of G. Therefore, by theorem 6.2, for any infinite regular cardinal α, no orbit of G in $\mathcal{U}_\omega$ is $\mathcal{L}^E_{\alpha\omega}$-definable.

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