SEMI-PARALLEL IMMERSIONS
INTO SPACE FORMS

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SEMI-PARALLEL IMMERSIONS INTO SPACE FORMS

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ABSTRACT: In this paper we discuss some results on semi-parallel submanifolds of space forms, with particular emphasis to the case of surfaces. We prove that a semi-parallel surface of a space form is, under some additional conditions, either flat or parallel.

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1. INTRODUCTION:

Symmetric spaces are (locally) characterized by the condition $\nabla \cdot R = 0$, where ∇ is the riemannian connection (extended to act on tensors) and R is the curvature tensor. The integrability condition for $\nabla \cdot R = 0$ is $R \cdot R = 0$, where R is extended to act as a derivation on tensors. Spaces which satisfy the latter condition are called semi-symmetric. Semi-symmetric spaces where introduced by Cartan and a classification of such spaces was obtained by Szabo' (see [18],[19]).

In the theory of submanifolds the condition analogue to $\nabla \cdot R = 0$ is $\nabla \cdot \alpha = 0$, where α is the second fundamental form (see 2.1.). Such submanifolds, or isometric immersions, are called parallel or extrinsically symmetric and, in the case that the ambient manifold is a space form, have been classified by Ferus in [12], Backes and Reckziegel in [2] and Takeuchi in [20].

In the same context, the analogue of the semi-symmetric condition is $R \cdot \alpha = 0$ (see 2.2.) and submanifolds which verify this condition are called semi-parallel. Such submanifolds have been object of study in the past years by several authors, especially Deprez and Lumiste, but a classification is not yet available.

In this paper we will discuss some results on semi-parallel submanifolds of space forms with particular emphasis to the case of surfaces, where the condition $R \cdot R = 0$ is empty but the condition $R \cdot \alpha = 0$ is extremely restrictive. Our main results are the following:

1. *A connected semi-parallel surface in a 5-dimensional space form is either flat or totally umbilical or a piece of a Veronese surface in some totally umbilical 4-sphere.*

2. *A compact, connected, semi-parallel, not orientable surface in $\mathbb{R}^N$ is either flat or a Veronese surface in some totally umbilical 4-sphere.*

2. NOTATIONS AND KNOWN FACTS.

Let M^n be an n-dimensional riemannian manifold and

$Q^N(c)$ a simply connected, complete N-dimensional manifold of constant curvature c. Superscripts will, as usual, denote dimensions and will be dropped when clear from the context. We will consider isometric immersions $f: M \longrightarrow Q(c)$ and will use standard notations: ∇ will denote the riemannian connection of M, $\nu(M)$ the normal bundle, $\alpha: TM \times TM \longrightarrow \nu(M)$ the second fundamental form, $\nabla^{\perp}$ and $R^{\perp}$ the normal connection and it's curvature. Finally, if ξ is a normal vector we will denote by $A_\xi: T_xM \longrightarrow T_xM$ the Weingarten operator $\langle A_\xi X, Y\rangle = \langle \alpha(X,Y), \xi\rangle$.

We will say that the immersion f is *parallel* if for all tangent vectors X,Y and Z we have:

2.1. $(\nabla_X \alpha)(Y,Z) := \nabla^{\perp}_X[\alpha(Y,Z)] - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z) = 0.$

We will say tat the immersion f is *semi-parallel* if for all tangent vectors X and Y we have:

2.2. $R(X,Y)\alpha := \nabla_X \nabla_Y \alpha - \nabla_Y \nabla_X \alpha - \nabla_{[X,Y]} \alpha = 0,$

where ∇ acts on $\nu(M)$-valued forms by the analogue of 2.1..

Using the classical equations of Gauss, Codazzi-

Mainardi and Ricci, condition 2.2. may be written in the form:

2.3. $R^{\perp}(X,Y)[\alpha(Z,W)] = \alpha(R(X,Y)Z,W) + \alpha(Z,R(X,Y)W)$.

Again the basic equations imply that if $f: M \longrightarrow Q(c)$ is a semi-parallel immersion then M is semi-symmetric.

2.4. REMARK: For an isometric immersion $f: M \longrightarrow Q(c)$ we have an associated triple system $L: T_xM \times T_xM \longrightarrow End(T_xM)$, $L(X,Y) = c<X,Y> + A_{\alpha(X,Y)} + R(X,Y)$, which is a Jordan triple system if the immersion is semi-parallel (see [1],[12]). The basic observation that leads to the classification of parallel immersions is the following: Given a point $x \in Q(c)$, a subspace $\mathbb{E} \subseteq T_xQ(c)$ and a Jordan triple system on $\mathbb{E}$, there exists a unique parallel immersion through x with those data at x. Therefore we can think of a semi-parallel immersion as a "2nd order envelope of parallel immersions" in the sense that we have at each point a parallel immersion with the same tangent space and second fundamental form (see [2]).

Let us now briefly comment the case of hypersurfaces. Let $f : M^n \longrightarrow Q^{n+1}(c)$ be a semi-parallel hypersurface and ξ a unit normal vector. Let $\{e_1,\ldots,e_n\}$ be an orthonormal basis for the tangent space of M which diagonalizes A_ξ and let $\lambda_i = \langle A_\xi e_i, e_i\rangle$ be the principal curvatures. In this situation 2.3. becomes:

2.5. $$(\lambda_i\lambda_j+c)(\lambda_i - \lambda_j) = 0.$$

It follows from the above that there are at most two distinct principal curvatures, say λ and μ. If $\lambda \neq \mu$ then $\lambda\mu = -c$ and the two eigenspaces T_λ and T_μ determine on that open set, two distributions. It is a standard consequence of the equations of Codazzi-Mainardi that those distributions are involutive and if the multiplicity of one of the principal curvatures, say λ, is at least two, then λ is constant along the leaves of T_λ and those leaves are totally umbilical. From those observations we get that if $c \neq 0$ and at one point there are two distinct principal curvatures, both of multiplicity bigger than one, then the hypersurface is isoparametric and therefore a tube around a totally geodesic submanifold, if M is connected. If one of the principal curvatures is simple then M is a 1-parameter envelope of umbilical submanifolds and those may be described quite explicitly (see [3]). If $c = 0$, besides such tubes we can have cylindrical immersions (i.e. rank $A_\xi \leq 1$), cones over spheres and products of such cones with Euclidean spaces (Deprez [9]).

2.6.REMARK: In [16] and [17] Nomizu and Ryan study the more general case of semi-symmetric hypersurfaces. In this case condition 2.3. becomes $(\lambda_i\lambda_j + c)(\lambda_i - \lambda_j)\lambda_k = 0$ and besides the above possibilities we have all the immersions with rank $A_\xi \leq 2$.

Strictly related to the case of semi-parallel hypersurfaces is the case of semi-parallel immersions with flat normal connection. If the normal curvature vanishes at a point $x \in M$ then there exists an orthonormal basis $\{e_1,\dots,e_n\}$ of T_xM such that if $i \neq j$, $\alpha(e_i,e_j) = 0$. In this case 2.3. is equivalent to:

2.7. $$K(e_i,e_j)\{\alpha(e_i,e_i) - \alpha(e_j,e_j)\} = 0,$$

where $K(e_i,e_j)$ is the sectional curvature of the plane spanned by e_i and e_j. If $c \geq 0$, the sectional curvature is non negative and, at least if M is complete, the topology is well understood: It's universal covering space is the riemannian product of manifolds homeomorphic to spheres and a manifold diffeomorphic to a Euclidean space (see [10]). An interesting geometric result, which we will use later in a weaker form, is the following:

2.8.PROPOSITION: *Let* f: M $\longrightarrow$ Q(c) *be a semi-parallel immersion with flat normal connection. If* $c \geq 0$, M *is connected and the Ricci curvature is positive at some point, then* f *is a product of umbilical immersions.*

2.9.REMARK: A scheme of classification of semi-parallel immersions in Euclidean space with flat normal connection may be found in [15], where the above proposition is attributed to Riives. We will sketch a proof of 2.8. since it is not easily found in the literature.

Proof of 2.8.: We consider the case c = 0. The case c > 0 is an obvious consequence. A theorem of Erbacher (see [6], p. 139) states that f is the product of umbilical immersion if M has non negative sectional curvatures, flat normal connection, parallel mean curvature vector and constant scalar curvature. So it will be enough to show that the latter two conditions are satisfied.
Let $p \in M$ be a point where the Ricci curvature is positive and $\{e_1,\ldots,e_n\}$ be a basis of T_pM such that $\alpha(e_i,e_j) = 0$ if $i \neq j$. We will write α_{kl} for $\alpha(e_k,e_l)$. From 2.7. and the positivity of the Ricci curvature at p, we get:

i) $\alpha_{ii} = \alpha_{jj}$ or $K(e_i,e_j) = \langle\alpha_{ii},\alpha_{jj}\rangle = 0$;
ii) for all i, $\alpha_{ii} \neq 0$ and there exists $j \neq i$ with $\alpha_{ii} = \alpha_{jj}$

Since the condition $\mathrm{Ricci}_p > 0$ is open the above conditions hold in a neighborhood of p and is not difficult to see that we can choose a smooth orthonormal frame, $\{e_1,\ldots,e_n\}$ in a possibly smaller open set, such that the two conditions above hold true in this open set. From the Codazzi-Mainardi equations we get:

1. $\nabla^{\perp}_{e_j}\alpha_{ii} = (\nabla_{e_j}\alpha)(e_i,e_i) + 2\alpha(\nabla_{e_j}e_i,e_i) = (\nabla_{e_i}\alpha)(e_j,e_i) + 2\langle\nabla_{e_j}e_i,e_i\rangle\alpha_{ii} = \langle\nabla_{e_i}e_i,e_j)(\alpha_{ii} - \alpha_{jj})$.

Therefore:

2. If $i \neq j$ and $\alpha_{ii} = \alpha_{jj}$, then $\nabla^{\perp}_{e_i}\alpha_{jj} = 0$.
3. For all i, $\nabla^{\perp}_{e_i}\alpha_{ii} = 0$. (Take $j \neq i$ with $\alpha_{jj} = \alpha_{ii}$).
4. If $i \neq j$ and $\alpha_{ii} \neq \alpha_{jj}$ then $\nabla^{\perp}_{e_i}\alpha_{jj} = 0$. In fact $\langle\alpha_{ii},\alpha_{jj}\rangle = 0$ by *i*) and therefore $0 = e_j\langle\alpha_{ii},\alpha_{jj}\rangle = -\langle\nabla_{e_i}e_i,e_j\rangle\|\alpha_{jj}\|^2$. Since $\alpha_{jj} \neq 0$ the conclusion follows from 1.

From the above we conclude that the α_{ii}'s are parallel in the open set we are working on and therefore the mean curvature vector is parallel and the scalar curvature is constant in that open set. A simple connectness argument gives the desired conclusion.

2.10. Remark: Proposition 2.8 gives a classification of codimension two semi-parallel immersions into Euclidean space. In fact it is easily seen that $R^{\perp}(X,Y)\cdot H = 0$ for all tangent vectors X,Y. If $H \neq 0$, $R^{\perp}(X,Y)$ annihilates H and, by anti symmetry, the orthogonal complement of H, and therefore is zero if the codimension is two. If $H = 0$ then the point is a totally geodesic point by general properties of Jordan triple systems with $c \leq 0$, (see [1]), and therefore again $R^{\perp}(X,Y) = 0$.

3. SEMI-PARALLEL SURFACES.

In this section we will study semi-parallel immersions of a 2- dimensional manifold M into an N-dimensional space form $Q = Q^N(c)$.

Let $\{e_1, e_2\}$ be a local orthonormal tangent frame and we will set, as before, $\alpha_{ij} = \alpha(e_i, e_j)$. Also K will denote the Gaussian curvature and $R^{\perp} := R^{\perp}(e_1, e_2)$ the normal curvature operator. With these notations 2.3. becomes:

3.1. $R^\perp\alpha_{11} = -R^\perp\alpha_{22} = 2K\alpha_{12}$, $R^\perp\alpha_{12} = K(\alpha_{11}-\alpha_{22})$.

An immediate consequence of 3.1. is that if $R^\perp \equiv 0$, then either f is umbilical or M is flat. Moreover an immersion of a flat surface is semi-parallel if and only if $R^\perp \equiv 0$.

The above observation is of some interest also because it allows us to classify semi-parallel immersions into $Q^4(c)$. In fact, in this case, if $c \leq 0$ then $R^\perp \equiv 0$ (see 2.11.) and, if $c > 0$, either $R^\perp \equiv 0$ or f(M) is a piece of a Veronese surface (see 3.6. below).

Let ξ be a unit normal field. The Ricci equation:

3.2. $$R^\perp(X,Y)\xi = \alpha(A_\xi Y,X) - \alpha(A_\xi X,Y),$$

together with 3.1. gives:

3.3. $$\begin{cases} <\alpha_{ii},\alpha_{12}>(\alpha_{11}-\alpha_{22})+[<\alpha_{ii},\alpha_{22}-\alpha_{11}>-(-1)^i 2K]\alpha_{12} = 0. \\ (\|\alpha_{12}\|^2 - K)(\alpha_{11} - \alpha_{22}) + <\alpha_{12},\alpha_{22} - \alpha_{11}>\alpha_{12} = 0. \end{cases}$$

If $R^{\perp} \neq 0$, α_{12} and $\alpha_{11}-\alpha_{22}$ are linearly independent and in this case 3.3. (and the Gauss equation) gives:

$$3.4. \begin{cases} \|\alpha_{12}\|^2 = K \ , \ \|\alpha_{ii}\|^2 = 4K-c \quad , \quad \langle\alpha_{ii},\alpha_{12}\rangle = 0, \\ \|H\|^2 = 3K - c, \ \|\alpha_{11}-\alpha_{22}\|^2 = 4K, \ \langle\alpha_{11},\alpha_{22}\rangle = 2K-c. \end{cases}$$

From the above we deduce the following result of Deprez([8]):

3.5. THEOREM: *Let* f: $M^2 \longrightarrow Q^N(c)$ *be a semi-parallel immersion. Then there exists an open and dense set* $U \subseteq M^2$ *such that the connected components of* U *are of the following types:*

i) *Open parts of umbilical* $Q^2(K)$ *in* $Q^N(c)$, $K \geq c$;

ii) *Flat surfaces with* $R^{\perp} \equiv 0$;

iii) *Isotropic immersions with* $R^{\perp} \neq 0$, $\|H\|^2 = 3K-c$.

For further use we will give a look at the case $c = 1$. By composition with the inclusion $Q^N(1) = S^N \hookrightarrow R^{N+1}$ we get a semi-parallel immersion into Euclidean space. We will denote by $\tilde{\alpha}$, $\tilde{H}$ the 2nd fundamental form and the mean curvature vector of the new immersion. Since $\|f(x)\| = 1$, differentiating twice we get $\langle\tilde{\alpha}_{ii},f(x)\rangle =$

-1 and hence $\langle\tilde{H},f(x)\rangle = -1$. If $R^{\perp} \neq 0$, by 3.4. $K = (1/3)\|\tilde{H}\|^2$ and, since $\|\tilde{H}\| \geq 1$, $K \geq (1/3)$.

3.6. PROPOSITION: *Let* $f: M^2 \longrightarrow S^N$ *be a semi-parallel immersion of a connected surface with* $R^{\perp} \neq 0$ *somewhere. Then the following are equivalent:*

i) $K \equiv (1/3)$;

ii) f *is minimal;*

iii) $f(M^2)$ *is a piece of a Veronese surface;*

iv) $f(M^2)$ *is contained in a totally geodesic* S^4 *in* S^N.

Proof: Since $\langle\tilde{H},f(x)\rangle = -1$ and $K = (1/3)\|\tilde{H}\|^2 \geq (1/3)$ we have that $K \equiv (1/3)$ if and only if $\tilde{H}$ is parallel to $f(x)$ and therefore if and only if f is minimal in S^N. Moreover a minimal surface in S^N with constant curvature 1/3 is a piece of a Veronese surface, by a theorem of Bryant (see [5]). So the first three conditions are equivalent and clearly implies the fourth. Suppose now N = 4. Differentiating $\|f(x)\| = 1$ we get $\langle\tilde{\alpha}_{12},f(x)\rangle = 0 = \langle\tilde{\alpha}_{11}-\tilde{\alpha}_{22},f(x)\rangle$ and therefore $\tilde{H}$ is parallel to f(x) since e_1, e_2, $\tilde{\alpha}_{12}$, $\tilde{\alpha}_{11}-\tilde{\alpha}_{22}$ and $\tilde{H}$ are orthogonal. Therefore f is minimal in S^4.

We will prove now the main result of this section:

3.7. THEOREM: *Let* $f: M^2 \longrightarrow Q^5(c)$ *be a semi-parallel immersion of a connected surface. If* $R^{\perp} \neq 0$ *at some point, then* $f(M^2)$ *is a piece of a Veronese surface in some* $Q^4(\tilde{c})$, $\tilde{c} > 0$, *totally umbilical in* $Q^5(c)$.

Proof: The main point of the proof is to show that the Gaussian curvature has to be constant. We will use the moving frame method. Let $\{e_1, e_2\}$ be a local smooth orthonormal tangent frame in a non void open set U where $R^{\perp} \neq 0$. We want to show that the function $\varphi = 3K - c$ is constant. Taking a possible smaller open set we can assume $\varphi \neq 0$. Set $k = K^{1/2}$ and define:

3.7.1. $e_3 = H/\sqrt{\varphi}$; $e_4 = (\alpha_{11}-\alpha_{22})/2k$; $e_5 = \alpha_{12}/k$.

Let $\{\omega_A\}$, $\{\omega_{AB}\}$, $A,B = 1,\ldots,5$ be the dual frame and the connection forms, respectively. As usual capital Latin indices will run from 1 to 5, small Latin indices from 1 to 2 and Greek indices from 3 to 5. Since $\omega_{\lambda} = 0$ along M we have $\omega_{i\lambda} = \sum_{j=1}^{2} \langle \alpha_{ij}, e_{\lambda} \rangle \omega_j$, and therefore, by our choice of the e_A's we have:

$$3.7.2.\begin{cases} \omega_{13} = \sqrt{\varphi}\omega_1 , & \omega_{14} = k\omega_1 , & \omega_{15} = k\omega_2, \\ \omega_{23} = \sqrt{\varphi}\omega_2 , & \omega_{24} = -k\omega_2 , & \omega_{25} = k\omega_1 . \end{cases}$$

(see 3.4.). The structure equations give:

$$3.7.3. \; d\omega_A = \sum_B \omega_{AB}\wedge\omega_B , \; d\omega_{12} = -K\omega_1\wedge\omega_2 , \; d\omega_{A\lambda} = \sum_B \omega_{AB}\wedge\omega_B$$

Set now:

$$3.7.4. \qquad d\varphi = a\omega_1 + b\omega_2 , \; \omega_{AB} = a_{AB}\omega_1 + b_{AB}\omega_2 .$$

We want to show that $a = 0 = b$. For this we compute $d\omega_{i\lambda}$ using 3.7.2 and 3.7.3.. For $(i,\lambda) = (1,3)$ we have:

$$d\omega_{13} = d(\sqrt{\varphi}\omega_1) = (a_{12}\sqrt{\varphi} - b/2\sqrt{\varphi})\, \omega_1\wedge\omega_2 ,$$

$$d\omega_{13} = \sum_B \omega_{1B}\wedge\omega_{B3} = (a_{12}\sqrt{\varphi} - b_{34}k + a_{35}k)\omega_1\wedge\omega_2 .$$

This and the analogue calculations for the other pairs of indices give:

$$k(a_{35} - b_{34}) = -b/2\sqrt{\varphi} ,$$

$$k(a_{34} + b_{35}) = -a/2\sqrt{\varphi} \; ,$$
$$2a_{12} - a_{45} - \varphi b_{34}/k = b/2(\varphi + c) \; ,$$
$$2a_{12} - a_{45} + \varphi a_{35}/k = b/2(\varphi + c),$$
$$b_{45} - 2b_{12} + \varphi a_{34}/k = a/2(\varphi + c) \; ,$$
$$b_{45} - 2b_{12} + \varphi b_{35}/k = a/2(\varphi + c) \; .$$

Algebraic manipulations of the above give:

3.7.5.
$$\begin{cases} \omega_{34} = [(3/16\varphi(\varphi + c)]^{1/2}(-a\omega_1 + b\omega_2), \\ \omega_{35} = -[(3/16\varphi(\varphi + c)]^{1/2}(b\omega_1 + a\omega_2), \\ \omega_{45} = 2\omega_{12} + [5/4(\varphi + c)](-b\omega_1 + a\omega_2). \end{cases}$$

Differentiating ω_{34} using the expression in 3.7.5. we get:

$$d\omega_{34} = \{-[(2\varphi+c)/\varphi(\varphi+c)]ab\omega_1\wedge\omega_2 - a\omega_{12}\wedge\omega_2 - b\omega_{12}\wedge\omega_1 + db\wedge\omega_2 + \omega_1\wedge da\}\{3/16\varphi(\varphi+c)\}^{1/2}.$$

The structure equations, together with 3.7.2. and 3.7.5. give:

$$d\omega_{34} = \omega_{35}\wedge\omega_{54} = \{2b\omega_1\wedge\omega_{12} + 2a\omega_2\wedge\omega_{12} + [10ab/4(\varphi+c)]\omega_1\wedge\omega_2\}\{3/16\varphi(\varphi+c)\}^{1/2}.$$

Comparing the two expressions we get:

$$\omega_1\wedge da+db\wedge\omega_2=[(9\varphi+2c)/2\varphi(\varphi+c)]ab\omega_1\wedge\omega_2+b\omega_1\wedge\omega_{12}-a\omega_{12}\wedge\omega_2.$$

The same calculation for ω_{35} and ω_{45} gives:

$$\omega_1\wedge db-da\wedge\omega_2 = [(9\varphi+2c)(b^2-a^2)/4\varphi(\varphi+c)]\omega_1\wedge\omega_2 -a\omega_1\wedge\omega_{12}+b\omega_2\wedge\omega_{12},$$

$$\omega_1\wedge db+da\wedge\omega_2 = [(17\varphi-3c)(a^2+b^2)/20\varphi(\varphi+c)]\omega_1\wedge\omega_2 -a\omega_1\wedge\omega_{12}+b\omega_{12}\wedge\omega_2.$$

Combining with $0 = d(d\varphi) = -\omega_1\wedge da+db\wedge\omega_2+a\omega_{12}\wedge\omega_2+b\omega_1\wedge\omega_{12}$ we get:

3.7.6.
$$\begin{cases} da=b\omega_{12}+\{[(62\varphi^2+7c)a^2-(28\varphi+13c)b^2]/40\varphi(\varphi+c)\}\omega_1+ [(9\varphi+2c)/4\varphi(\varphi+c)]ab\omega_2 \\ db=-a\omega_{12}+\{[(62\varphi^2+7c)b^2-(28\varphi+13c)a^2]/40\varphi(\varphi+c)\}\omega_2+ [(9\varphi+2c)/4\varphi(\varphi+c)]ab\omega_1 \end{cases}$$

Differentiating again the last two equations we get:

$$3.7.7.\begin{cases} 0=b[2(\varphi+c)]^{-1}\{K+(a^2+b^2)(84\varphi+299c)[800\varphi^2(\varphi+c)]^{-1}\} \\ \\ 0=a[2(\varphi+c)]^{-1}\{K+(a^2+b^2)(84\varphi+299c)[800\varphi^2(\varphi+c)]^{-1}\} \end{cases}$$

If $c \geq 0$, 3.7.7. implies $a = 0 = b$. We will treat now the case $c < 0$. We shall suppose $c = -1$, $\nabla\varphi \neq 0$ and use a classical result of Beltrami (see [4] and also [11] p. 161) on the differential parameters on a surface. Beltrami's theorem states that if the ratio $\Delta\varphi/\|\nabla\varphi\|^2$ is a function of φ alone, there exists a function ψ on M^2 such that (φ,ψ) gives a system of local coordinates and in those coordinates the metric takes the form:

$$ds^2 = \|\nabla\varphi\|^{-2}(d\varphi^2 + \exp.(2\int(\Delta\varphi/\|\nabla\varphi\|^2)d\varphi)d\psi^2).$$

In our case, from 3.7.6.and 3.7.7., we get:

$$\|\nabla\varphi\|^2=800\varphi^2(\varphi-1)^2/3(299-84\varphi),$$
$$\Delta\varphi = \|\nabla\varphi\|^2(17\varphi+3)/20\varphi(\varphi-1).$$

Therefore $ds^2 = Ed\varphi^2 + Gd\psi^2$ with

$$E = 3(299-84\varphi)/800\varphi^2(\varphi-1)^2 \ , \ G = 3h(299-84\varphi)/800\varphi^{23/10}.$$

where h is a constant. The Gaussian curvature K satisfies the equation:

$$2\partial^2 G/\partial\varphi^2 - G^{-1}(\partial G/\partial\varphi)^2 - E^{-1}(\partial G/\partial\varphi)(\partial E/\partial\varphi) + 4EGK = 0.$$

Substituting $K = (\varphi-1)/3$ we find that the function φ is solution of the polynomial equation:

$$42336\varphi^3+13361712\varphi^2-41446782\varphi+67855359 = 0 \qquad \text{(sic!)}$$

and this is absurd since we have supposed that φ is not constant.

Therefore even in the case $c < 0$ we have $a = 0 = b$.

We will show now that the mean curvature is parallel in the normal connection. In fact, from 3.7.5, $\omega_{34} = 0 = \omega_{35}$, and this, together with the constancy of φ, gives:

$$\nabla^{\perp}_X H = \sqrt{\varphi} \sum_{\lambda} \langle\nabla^{\perp}_X e_3, e_\lambda\rangle e_\lambda = \sqrt{\varphi} \sum_{\lambda} \omega_{3\lambda}(X) e_\lambda = 0.$$

It follows from results of Chen and Yau (see [6], p.

106), that f embeds a neighborhood of the point where $R^{\perp} \neq 0$ as a minimal surface in a totally umbilical $Q^4(\tilde{c})$ in $Q^5(c)$. By minimality $\tilde{c} \geq K$ and the conclusion of the theorem follows from 3.6. and a simple connectness argument.

3.8. REMARK: For $c = 0$ the above result is due to Lumiste (see [14]). The essential difference here is the use of Beltrami's theorem to treat the case $c < 0$.

4. COMPACT SEMI-PARALLEL SURFACES IN $\mathbb{R}^N$.

Let $f : M^2 \longrightarrow \mathbb{R}^N$ be a semi-parallel immersion of a compact, non flat surface. Since the Gaussian curvature is non negative, M^2 is diffeomorphic to a sphere or to a projective plane. In this section we will estimate the total absolute curvature of f, $\tau(f)$, and prove the following:

4.1. THEOREM: *In the above hypothesis* $\chi(M) \leq \tau(f) \leq 3\chi(M)$.

In particular we deduce the following:

4.2. COROLLARY:

i) If M *is not orientable then* f *embeds* M *as a Veronese surface in some 4-sphere of* $\mathbb{R}^N$.

ii) If M *is orientable and* f *is tight,* f *is totally umbilical.*

Proof of 4.2. : If M is not orientable then $\chi(M) = 1$ and $\tau(f) \leq 3$ by 4.1.. On the other hand $\tau(f) \geq 3$ by Morse inequalities and therefore $\tau(f) = 3$ and f is tight. By a theorem of Kuiper and Pohl (see [13]), f is then projectively equivalent to a Veronese surface, in particular f(M) is contained in some 5-dimensional affine subspace of $\mathbb{R}^N$ and *i*) follows from 3.7..

If M is orientable and f is tight then $\tau(f) = 2$ and by a theorem of Chern and Lashof f(M) is contained in some 3-dimensional affine subspace of $\mathbb{R}^N$ (see [7]). The conclusion then follows from 3.6.

Proof of 4.1.: We recall that the total absolute curvature is given by:

$$\tau(f) = (c_{N-1})^{-1}\int_M \Big(\int_{\|\xi\|=1} |\det(A_\xi)|\, d\sigma_{N-3}\Big) dM \ ,$$

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ii) If M *is orientable and* f *is tight,* f *is totally umbilical.*

Proof of 4.2. : If M is not orientable then $\chi(M) = 1$ and $\tau(f) \leq 3$ by 4.1.. On the other hand $\tau(f) \geq 3$ by Morse inequalities and therefore $\tau(f) = 3$ and f is tight. By a theorem of Kuiper and Pohl (see [13]), f is then projectively equivalent to a Veronese surface, in particular f(M) is contained in some 5-dimensional affine subspace of $\mathbb{R}^N$ and *i*) follows from 3.7..

If M is orientable and f is tight then $\tau(f) = 2$ and by a theorem of Chern and Lashof f(M) is contained in some 3-dimensional affine subspace of $\mathbb{R}^N$ (see [7]). The conclusion then follows from 3.6.

Proof of 4.1.: We recall that the total absolute curvature is given by:

$$\tau(f) = (c_{N-1})^{-1}\int_M \left(\int_{\|\xi\|=1} |\det(A_\xi)|\, d\sigma_{N-3}\right) dM ,$$

where $d\sigma_n$ is the volume density of the n-sphere, $c_n = \int_{S^n} d\sigma_n$, and dM is the volume density of M.

If the mean curvature vector at x, H(x), is zero then x is a totally geodesic point (see remark 2.10.) and so $A_\xi = 0$ for all ξ's normal to M at x. Suppose $H(x) \neq 0$ and choose an adapted frame $\{e_1,\ldots,e_N\}$ with $e_3 = H/\|H\|$ and, if $R^\perp \neq 0$, $e_4 = (\alpha_{11}-\alpha_{22})/\|\alpha_{11}-\alpha_{22}\|$ and $e_5 = \alpha_{12}/\|\alpha_{12}\|$. Set $A_\lambda = A_{e_\lambda}$, $\lambda \geq 3$,. From 3.4. we have:

1.If $R^\perp=0$, $A_3 = \begin{bmatrix} \sqrt{K} & 0 \\ 0 & \sqrt{K} \end{bmatrix}$, $A_\lambda = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ if $\lambda \geq 4$.

2.If $R^\perp\neq0$, $A_3 = \begin{bmatrix} \sqrt{3K} & 0 \\ 0 & \sqrt{3K} \end{bmatrix}$, $A_4 = \begin{bmatrix} \sqrt{K} & 0 \\ 0 & -\sqrt{K} \end{bmatrix}$,

$A_5 = \begin{bmatrix} 0 & \sqrt{K} \\ \sqrt{K} & 0 \end{bmatrix}$, $A_\lambda = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ if $\lambda \geq 6$.

Let ξ be a unit normal vector at x. Write $\xi = \sum_{\lambda\geq3} \xi_\lambda e_\lambda$.

Then $\det(A_\xi) = K\varphi(\xi_3,\ldots,\xi_N)$, where K is the gaussian curvature and φ is given by:

$$\varphi(\xi_3,\ldots,\xi_N) = \begin{cases} (\xi_3)^2 & \text{if } R^\perp = 0 \\ |3(\xi_3)^2-(\xi_4)^2-(\xi_5)^2| & \text{if } R^\perp \neq 0 \end{cases}$$

Observe now that φ is the integrand that appears in the expression of the total absolute curvature of the immersion of the unit sphere in $\mathbb{R}^N$ as totally umbilical surface, in the first case, and as a Veronese surface in the second. Therefore the integral of φ on the unit N-3 sphere is $(c_{N-1})/2\pi$ in the first case and $3(c_{N-1})/2\pi$ in the second case. From the above and the Gauss-Bonnet Theorem we get:

$$\chi(M) = (c_{N-1})^{-1}\int_M K \, dM \int_{S^{N-3}} (\xi_3)^2 \, d\sigma_{N-3} \leq \tau(f) \leq$$

$$(c_{N-1})^{-1}\int_M K \, dM \int_{S^{N-3}} |3(\xi_3)^2-(\xi_4)^2-(\xi_5)^2| \, d\sigma_{N-3} = 3\chi(M).$$

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