A Class of Second-order Evolution Equations With Double Characteristics

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1. Notations

Let H be an abstract Hilbert space and A a linear operator, densely defined in H, which is unbounded, selfadjoint, positive definite and has a bounded in verse A^{-1}.

If J is an open set of the real line, let $\mathcal{Q}_A(J)$ denote the set of the series in the nonnegative powers of A^{-1}, with coefficients in $C^\infty(J)$, which converge in $L(H;H)$ (the Banach space of bounded linear operators on H), as well as each of their t-derivatives, uniformly with respect to t on compact subsets of J. We will consider differential operators of the form

$$P = \sum_{r+j\leq m} c_{r,j}(t,A)\,A^r\partial_t^j \qquad (\partial_t = \partial/\partial t) \tag{1}$$

where the r's are real numbers ≥ 0, the j's are integers ≥ 0, the sum is a finite one and each $c_{r,j}(t,A)$ belongs to $\mathcal{Q}_A(J)$. The operator P in (1) is said to be of order $\leq m$.

We may construct the scale of Sobolev spaces H^s for each $s\in\mathbb{R}$, in the following way: if $s\geq 0$, H^s is the space of elements u of H such that $A^s u\in H$, equipped with the norm $\| u \|_s = \| A^s u \|_0$, where $\| \circ \|_0$ denotes the norm of $H = H^0$; if $s<0$, H^s is the completion of H for the norm $\| u \|_s = \| A^s u \|_0$. By H^∞ we denote the intersection of all the H^s, equipped with the projective limit topology, and by $H^{-\infty}$ their union, with the inductive limit topology.

If J is an open set of the real line, we denote by $C^\infty(J, H^\infty)$ the space of C^∞ function in J valued in H^∞, which is the projective limit of $C^j(J, H^m)$ for j, $m\geq 0$ integers. If K is any compact subset of J, we denote by $C_C^\infty(K, H^\infty)$ the subspace of $C^\infty(J, H^\infty)$ consisting of those functions which vanish outside K, and by $C_C^\infty(J, H^\infty)$ the inductive limit of $C_C^\infty(K, H^\infty)$ as K ranges over all compact subsets of J. Then, $\mathcal{D}'(J, H^{-\infty})$ will be the (strong) dual of $C_C^\infty(J, H^\infty)$.

We may define another scale of Sobolev spaces: $\mathscr{H}^s$ for $s \in \mathbb{R}$, in which the t-variable also plays a role: the basic space is now $L^2(\mathbb{R}, H)$, which will be $\mathscr{H}^0$, and we consider the operator $(1-\partial_t^2+A^2)^{1/2}$ on $L^2(\mathbb{R}, H)$, which has the same properties that A has in relation to H, so that the scale $\mathscr{H}^s = \mathscr{H}^s(\mathbb{R})$ is constructed exactly as before.

We can then define $\mathscr{H}_C^S(K)$, $\mathscr{H}_C^{S'}(J)$ and $\mathscr{H}_{loc}^S(J)$ in the obvious way.

2. Statement of the main theorem

Let J be an open set of the real line, containing the origin. We will study operators of the form

(2) $$P = (\partial_t - a(t, A)A)\,(\partial_t - b(t, A)A) + c(t, A)A,$$

where $a(t, A)$, $b(t, A)$, and $c(t, A)$ belong to $\mathscr{Q}_A(J)$, and will systematically use the notation

(3) $$X = \partial_t - a(t, A)A; \quad Y = \partial_t - b(t, A)A; \quad \delta(t, A) = a(t, A) - b(t, A).$$

Let

$$a(t,A) = \sum_{i=0}^{\infty} a_i(t)A^{-i}\ ; \ b(t,A) = \sum_{i=0}^{\infty} b_i(t)A^{-i};\ c(t,A) = \sum_{i=0}^{\infty} c_i(t)A^{-i}.$$

When $a_0(0) = b_0(0) = 0$, we are in the case of double characteristics (at $t=0$). We will further assume that

(4) $$a_0(t) = at^k + t^{k+1}f(t), \quad b_0(t) = bt^k + t^{k+1}g(t), \quad c_0(t) = ct^{k-1} + t^k h(t),$$

where $f, g, h \in C^\infty(J)$ and a, b, c are complex numbers.

(5) $Re\ a > 0$, $Re\ b < 0$ and k is odd.

We will call $\delta = a - b$ and

(6) $$l_p = \frac{c}{\delta}.$$

Let J_1 be the greatest open interval containing the origin and such that $Re\,\dfrac{a_0(t)}{t^k}$ and $Re\,\dfrac{b_0(t)}{t^k}$ do not vanish on J_1, and let us consider the following hypoelliptic properties:

$(7)_s$ $$\forall u \in \mathscr{D}'(J, H^{-\infty}),\ Pu \in \mathscr{H}^s_{loc}(J) \Rightarrow u \in \mathscr{H}^{s+\frac{2}{k+1}}_{loc}(J).$$

(II) $\forall s \in R$, $\forall$ open set $J \subset J_1$, $(7)_s$ holds.

The following is then the main theorem of this work:

THEOREM 1. *Let P be given by* (2), *satisfying* (4) *and* (5), *and let P^* be the adjoint of P. Then, the following conditions are equivalent*:

a) *P satisfies* II

b) *P^* satisfies* II

c) *$\forall$ m integer ≥ 0, $l_p \neq m(k+1)$ and $l_p \neq m(k+1)+1$.*

In [2], the operators that are locally solvable and hypoelliptic at $t=0$ were completely characterized, when $k=1$. In [3], these operators were again studied, but under a pseudodifferential form, and Theorem 1 was proved for $k=1$.

3. The space ${}^k\mathscr{H}^{s,m}$ and their properties

In order to prove the theorem, we will need to construct and know some properties of auxiliary spaces ${}^k\mathscr{H}^{s,m}$, similarly to what was done in [3]. In the case $k>1$, however, there are some peculiarities that make the statement of the results a little more complicated.

Let us first assume that d and p are real numbers such that $k \cdot p$ is an integer ≥ 0. We define the space of operators ${}^kN^{d,p}(J)$ in the following way: $B \in {}^kN^{d,p}(J)$ if B is an operator like in (1), and may be written in the form:

$$B = \sum_{\substack{\frac{\alpha}{k}+\beta \leq p \\ \alpha,\beta \in \mathbb{Z}_+}} B_{\alpha\beta} t^\alpha \partial_t^\beta, \tag{8}$$

and $B_{\alpha\beta}$ is of the form (1), with order $\leq d - \dfrac{kp-\alpha+\beta}{k+1}$ on J.

Remark that $t^\alpha A^\gamma \partial_t^\beta \in {}^kN^{\beta+\gamma,\, \frac{\alpha}{k}+\beta}$ if $\alpha,\gamma,\beta \geq 0$; $\alpha,\beta \in \mathbb{Z}$ and $\gamma \in \mathbb{R}_+$.

Let now s and m be real numbers with $k \cdot m$ an integer ≥ 0. We define now ${}^k\mathscr{H}^{s,m}_{loc}(J)$: a distribution $u \in \mathscr{D}'(J, H^{-\infty})$ is in ${}^k\mathscr{H}^{s,m}_{loc}(J)$ if and only if $\forall B \in {}^kN^{d,p}(J)$, with $p \leq m$, we have $B u \in \mathscr{H}_{loc}^{s-d-\frac{k(m-p)}{k+1}}(J)$, or, what is equivalente, if and only if $t^\alpha \partial_t^\beta u \in \mathscr{H}_{loc}^{s-\frac{km-\alpha+\beta}{k+1}}(J)$, $\forall \alpha, \beta \in \mathbb{Z}_+$ with $\frac{\alpha}{k}+\beta \leq m$. We equip ${}^k\mathscr{H}_{loc}^{s,m}(J)$ with the coarsest locally convex topology which makes all the mappings B in the first definition, or $t^\alpha \partial_t^\beta$ in the second one, continuous from ${}^k\mathscr{H}_{loc}^{s,m}(J)$ to the corresponding spaces. Remark that ${}^k\mathscr{H}_{loc}^{s,o}(J) = \mathscr{H}_{loc}^s(J)$.

We may define ${}^k\mathscr{H}_c^{s,m}(K)$ and ${}^k\mathscr{H}_c^{s,m}(J)$ in the obvious way, and finally, if s,m are as above, we define ${}^k\mathscr{H}_{loc}^{s,-m}(J)$ as the (strong) dual of ${}^k\mathscr{H}_c^{-s,m}(J)$. The following are useful facts about these spaces:

PROPOSITION 1. a) *If $B \in {}^kN^{d,p}(J)$, then B: ${}^k\mathscr{H}_{loc}^{s,m}(J) \to {}^k\mathscr{H}_{loc}^{s-d,m-p}(J)$ is continuous, provided either $m-p \geq 0$ or $m \leq 0$ or $m \in \mathbb{Z}$.*

b) *t^α: ${}^k\mathscr{H}_{loc}^{s,m}(J) \to {}^k\mathscr{H}_{loc}^{s,m-\frac{\alpha}{k}}(J)$ is always continuous.*

PROPOSITION 2. *If $s' \leq s$ and $s' - \dfrac{km'}{k+1} \leq s - \dfrac{km}{k+1}$, then we have a continuous injection ${}^k\mathscr{H}_{loc}^{s,m}(J) \subsetneq {}^k\mathscr{H}_{loc}^{s',m'}(J)$.*

COROLLARY 1. a) *If $m \geq 0$, then $\mathscr{H}_{loc}^s(J) \subsetneq {}^k\mathscr{H}_{loc}^{s,m}(J) \subsetneq \mathscr{H}_{loc}^{s-\frac{km}{k+1}}(J)$*

b) *If $m \leq 0$, then $\mathscr{H}_{loc}^{s-\frac{km}{k+1}}(J) \subsetneq {}^k\mathscr{H}_{loc}^{s,m}(J) \subsetneq \mathscr{H}_{loc}^s J)$*

PROPOSITION 3. *Given a compact set K of the real line, $\varepsilon > 0$ and a real number s_1, there is $C > 0$ such that $\forall u \in {}^k\mathscr{H}_c^{s,m}(K)$, $\|u\|_{s',m'} \leq \varepsilon \|u\|_{s,m} + C \|u\|_{s_1}$.*

PROPOSITION 4. *Let $m \in \mathbb{Z}/k$. If $t^k A u$ and $\partial_t u$ both belong to ${}^k\mathscr{H}_{loc}^{s-1,m-1}(J)$, then $u \in {}^k\mathscr{H}_{loc}^{s,m}(J)$. The converse is true (Proposition 1) if $m \geq 1$ or $m \leq 0$.*

Let us suppose that $P \in {}^kN^{d,p}(\Omega)$, where $p \in \mathbb{Z}_+$, and let $J \subset \subset \Omega$, with $0 \in J$. Let S be the set of real numbers m such that $k \cdot m \in \mathbb{Z}$ and that: either $m \geq 0$ or $m+p \leq 0$ or $m \in \mathbb{Z}$, and let us consider the following conditions:

$(1)_{s,m}$: $\forall u \in \mathscr{D}'(J, H^{-\infty})$, $P u \in {}^k\mathscr{H}_{loc}^{s,m}(J) \Rightarrow u \in {}^k\mathscr{H}_{loc}^{s+d,m+p}(J)$.

$(2)_{s,m}$: $\forall \theta \in C_c^\infty(J)$, $\forall s' \in R$, $\forall K \subset \subset \Omega$, $\exists C > 0$ such that $\forall \phi \in C_c^\infty(K, H^\infty)$

$$\|\theta\phi\|_{s+d,m+p} \leq C(\|P\theta\phi\|_{s,m} + \|\phi\|_{s'})$$

$(3)_{s,m}$ $\forall \theta \in C_c^\infty(J)$, $\forall s'' \in \mathbb{R}$, $\forall g \in {}^k\mathscr{H}_{loc}^{-s-d,-m-p}(J)$,

$$\exists f \in {}^k\mathscr{H}_c^{-s,-m}(J) \text{ such that } \theta(P^*f-g) \in \mathscr{H}_c^{s''}(J).$$

THEOREM 2. *Let* $P \in {}^kN^{d,p}(\Omega)$, *with* $p \in \mathbb{Z}_+$. *Then:*
a) *If* $(\mathrm{j})_{s,m}$ *is true for some* $\mathrm{j} \in \{1, 2, 3\}$ *and some* $(s_0, m_0) \in \mathbb{R} \times S$, *then it is true for all such j's and all* (s, m_0) *with* $s \in \mathbb{R}$.
b) *If* $m_0 \in \mathbb{Z}$ *(and a) is satisfied), then* $(j)_{s,m}$ *is true for all j's and all* $(s,m) \in \mathbb{R} \times S$; *moreover,* $(1)_{s,m}$ *is true for all* $(s, m) \in \mathbb{R} \times \mathbb{Z}/k$ *(if* $p = 2$*).*

REMARK 1. If $(1)_{s,m}$ holds for a family of open sets that covers J, then of course it holds for J. If $(2)_{s,m}$ holds, then it holds for every $J' \subset J$. Hence, if $m \in S$, Theoren 2 implies that $(j)_{s,m}$ holds if and only if it holds for a family of open sets which covers J.

One of the steps in the proof of Theorem 2 uses the following

PROPOSITION 5. *If* $(2)_{s_0,m_0}$ *is true for* P, *then it is also true for* $P - R$, *if* $R \in {}^kN^{d-1,p-1}$ *(and if* $m \geq 0$ *or* $m+p-1 \leq 0$ *or* $m \in \mathbb{Z}$*).*

4. A stronger version of the main theorem and some partial results

We keep all the notations of section 2, including that of J_1, and we consider the following conditions ($(7)_s$ and II appear in pg. 2) and 3):

I) $\exists$ open neighborhood J of 0, with $J \subset J_1$, and $\exists s \in \mathbb{R}$ such that $(7)_s$ holds.

III) $\exists$open neighborhood J of 0, with $J \subset J_1$, $\exists\, (s, m) \in \mathbb{R} \times \mathbb{Z}$, and $\exists j \in \{1, 2, 3\}$ *such that* $(j)_{s,m}$ holds.

IV) $\begin{cases} 1)\ \forall J \subset\subset J_1,\ \forall\, (s, m) \in \mathbb{R} \times S,\ \forall j \in \{1, 2, 3\},\ (j)_{s,m} \text{ holds.} \\ 2)\ \forall J \subset J_1,\ \forall (s, m) \in \mathbb{R} \times \mathbb{Z}/k,\ (1)_{s,m} \text{ holds.} \end{cases}$

We will prove that I $\Leftrightarrow$ II $\Leftrightarrow$ III $\Leftrightarrow$ IV for an operator P, and then the following more precise version of Theorem 1:

THEOREM 3. *Let* P *be as in Theorem* 1. *Then, the following conditions are equivalent:*

a) I, II, III *or* IV *for* P;
b) I, II, III *or* IV *for* P^*;
c) $\forall m$ *integer* ≥ 0, $l_p \neq m(k+1)$ *and* $l_p \neq m\ (k+1) + 1$.

For the proof of Theorem 3, we need 2 partial results. The first is easily obtained, as in [2] and [3]:

PROPOSITION 6. *If P is as in Theorem 1 and Re $l_p \leq -\frac{k}{2}$, then $\exists C_0, C_1 > 0$, such that $\forall \phi \in C_c^\infty(J, H^\infty)$ (where $0 \in J$, J is bounded and contained in the domain of definition of P),*

$$\int (\|\phi_t\|_0^2 + \|t^k A\phi\|_0^2)\, dt \leq C_0 \left|\int (P\phi, \phi)_0 dt\right| + C_1 \int |t| (\|\phi_t\|_0^2 + \|t^k A\phi\|_0^2)\, dt,$$

where $\| \circ \|$ and $(\ ,\ )_0$ are those of $H^0 = H$.

COROLLARY 2. *If P is as in Proposition 6, then there is an open neighborhood J of 0 and $C > 0$ such that $\forall \phi \in C_c^\infty(J, H^\infty)$, we have $\|\phi\|_{1,1} \leq C\|P\phi\|_{-1,-1}$.*

The second result which we need is also easy, similar to what was done in [1]. We give below the result and the algebraic lemma in which it is based:

LEMMA. *Let E, F be two abelian groups, F a subgroup of E, and let P, Q, U, V be four endomorphisms of E which map F into itself. If $UP = QV$ and $V^{-1}(F) \cap P^{-1}(F) \subset F$, then $Q^{-1}(F) \subset F \Rightarrow P^{-1}(F) \subset F$.*

PROPOSITION 7. *Let $P = P(c) = (\partial_t - at^k A)\ (\partial_t - bt^k A) + ct^{k-1} A$. If, for some $m \in \mathbb{Z}_+$, we have $l_p = m(k+1)$ or $l_p = m(k+1) + 1$, then P is not hypoelliptic at $t = 0$.*

5. Proof of Theorem 3

Throughout this section, P will be of the form (2), satisfying (4) and (5). We associate to P the operator

$$\tilde{P} = (\partial_t - at^k A)\ (\partial_t - bt^k A) + ct^{k-1} A. \tag{9}$$

The following propositions reduce the proof of Theorem 3 for the general operator P to that of the simpler operator $\tilde{P}$:

PROPOSITION 8. *The following conditions are equivalent:*

a) III *holds for* P;

b) IV *holds for* P;

c) III *holds for* $\tilde{P}$;

d) IV *holds for* $\tilde{P}$ *(for* $\tilde{P}$, $J_1 = \mathbb{R}$).

PROOF. That b) $\Rightarrow$ a) and d) $\Rightarrow$ c) is evident. Suppose a) holds; on $J_1\backslash\{o\}$ P is elliptic, so $(1)_{s,m}$ holds for all $(s,m)\in\mathbb{R}\times\mathbb{Z}/k$ on $J_1\backslash\{o\}$, and by a) it also holds on some neighborhood of 0; now Remark 1 and Theoren 2 gives b). Similarly, c) $\Rightarrow$ d).

We use now the fact that $P-\tilde{P}=R_1+R_2$, where $R_2\in{}^kN^{1,1}(\Omega)$ and $R_1=tU$, with $U\in{}^kN^{2,2}(\Omega)$. Hence, Proposition 5 and Theorem 2 show that b) for P implies a) for $P-R_2$. To end the proof, it is enough to prove the following analog of Proposition 5:

(10) *If* $(2)_{0,0}$ *holds for* P *on the neighborhood* J *of* 0, *then it also holds for* $P-R$ *on a possibly smaller neighborhood* J' *of* 0, *if* $R=tU$, *with* $U\in{}^kN^{2,2}(\Omega)$.

Now, $\|t\phi\|_{0,0}\leq\varepsilon\|\phi\|_{0,0}$ if $\phi\in C_c^\infty((-\varepsilon,\varepsilon),H^\infty)$, and since $U\colon{}^k\mathscr{H}^{2,2}_{loc}(J)\longrightarrow{}^k\mathscr{H}^{0,0}_{loc}(J)$ is continuous, given a compact neighborhood K of 0, with $K\subset J$, there is $C>0$ such that

$$\|U\phi\|_{0,0}\leq C\|\phi\|_{2,2}\ \forall\phi\in C_c^\infty(K,H^\infty),$$

hence

$$\|R\phi\|_{0,0}=\|tU\phi\|_{0,0}\leq\frac{\varepsilon}{C}\|U\phi\|_{0,0}\leq\varepsilon\|\phi\|_{2,2},$$

$$\forall\phi\in C_c^\infty\left(\left(-\frac{\varepsilon}{C},\frac{\varepsilon}{C}\right)\cap K,H^\infty\right).$$

Since,

$$\forall\varepsilon>0,\ \exists\varepsilon'>0\ \text{ such that }\ \|R\phi\|_{0,0}\leq\varepsilon\|\phi\|_{2,2},\ \forall\phi\in C_c^\infty((-\varepsilon',\varepsilon'),H^\infty),$$

it is now easy to prove (10).

PROPOSITION 9. a) *Conditions* I, II, III, IV *are equivalent.* b) *If* $Pu\in{}^k\mathscr{H}^{s,m}_{loc}(J)$ *but* $u\notin{}^k\mathscr{H}^{s+2,m+2}_{loc}(J)$, *then* $u\notin{}^k\mathscr{H}^{s+2-j,m+2-j\frac{(k+1)}{k}}_{loc}(J)$ *for* $j=0,1,2\ldots$

PROOF. a) We already know that III $\Leftrightarrow$ IV, and it is evident that II $\Rightarrow$ I. That IV $\Rightarrow$ II follows by remarking that we have a continuous injection ${}^k\mathscr{H}^{s+2,2}_{loc}(J)\subsetneq\mathscr{H}^{s+\frac{2}{k+1}}_{loc}(J)$ (Corollary 1).

Finally, I $\Rightarrow$ III: in fact, suppose that $(7)_s$ holds. Then, if $u\in\mathscr{D}'(J,H^{-\infty})$ is such that $Pu\in\mathscr{H}^s_{loc}(J)$, we get by $(7)_s$ that $u\in\mathscr{H}^{s+\frac{2}{k+1}}_{loc}(J)$, which is contained in ${}^k\mathscr{H}_{loc}^{s,-\frac{2}{k}}(J)$ (Corollary 1).

Hence, by b) case $j=2$, we have $u\in {}^k\mathscr{H}_{loc}^{s+2,2}(J)$, so that $(1)_{s,0}$, hence III holds.

b) We begin with the following claim (which is but b) in the case $j=1$):

(11) *If* $Pu\in {}^k\mathscr{H}_{loc}^{s,m}(J)$ *and* $u\in {}^k\mathscr{H}_{loc}^{s+1,m+2-\frac{k+1}{k}}(J)$, *then* $u\in {}^k\mathscr{H}_{loc}^{s+2,m+2}(J)$.

In fact, since $u\in {}^k\mathscr{H}_{loc}^{s+1,m+2-\frac{k+1}{k}}(J)$, we have $t^{k-1}Au\in {}^k\mathscr{H}_{loc}^{s,m}(J)$. Let now α be a sufficiently great positive real number. Then $P_1=P-\alpha\delta t^{k-1}A$ has $Re\ l_{p_1}\leq -\frac{k}{2}$ hence (by Corollary 2 and Theorem 2) P_1 satisfies $(1)_{s,m}$. Since $P_1u=Pu-\alpha\delta t^{k-1}Au\in {}^k\mathscr{H}_{loc}^{s,m}(J)$, we get $u\in {}^k\mathscr{H}_{loc}^{s+2,m+2}(J)$. Suppose now that b) case j was proved and let us prove b) case $j+1$. Let $Pu\in {}^k\mathscr{H}_{loc}^{s,m}(J)$, $u\notin {}^k\mathscr{H}_{loc}^{s+2,m+2}(J)$. By b)_j we have $u\notin {}^k\mathscr{H}_{loc}^{s+2-j,m+2-j\frac{(k+1)}{k}}(J)$. Since ${}^k\mathscr{H}_{loc}^{s,m}(J)\subsetneq {}^k\mathscr{H}_{loc}^{s-j,m-j\frac{(k+1)}{k}}(J)$. (Corollary 1), we have $Pu\in {}^k\mathscr{H}_{loc}^{s-j,m-j\frac{(k+1)}{k}}(J)$ and $u\notin {}^k\mathscr{H}_{loc}^{s+2-j,m+2-j\frac{(k+1)}{k}}(J)$, which by (11) (with $s-j$ and $m-j\dfrac{(k+1)}{k}$ substituded for s and m, respectively) gives b) $j+1$.

Corollary 3. *The following* 8 *conditions are equivalent*: I, II, III, IV *for* P *and* I, II, III, IV *for* $\tilde{P}$.

Proof of Theorem 3. The remark that $l_{p^*}=\overline{l_p}$ immediately shows that, if we prove a) $\Rightarrow$ c) $\Rightarrow$ a), then we will also have b) $\Rightarrow$ c) $\Rightarrow$ b).

a) $\Rightarrow$ c): a) $\Rightarrow$ II for $\tilde{P}\Rightarrow\tilde{P}$ hypoelliptic $\Rightarrow$ c). The last implication is Proposition 7.

c) $\Rightarrow$ a): Since $l_{\tilde{p}}=l_p$ it is enough to prove c) $\Rightarrow$ a) for $\tilde{P}$, given by (9). Let us call $P(c)=(\partial_t-at^kA)(\partial_t-bt^kA)+ct^{k-1}A$, $X=\partial_t-at^kA$, $Y=\partial_t-bt^kA$. It is enough to prove, for $j=0,1,2,\ldots$

$(12)_j$ c) $\Rightarrow$ a) when $Re\ e_{p(c)}\leq (j-1)\ (k+1)$.

By Corollary 2 (and Theorem 2), $(12)_0$ is true. For the inductive step, it is enough to prove:

(13) if $c \neq 0$, $c \neq \delta$, and $P(c-(k+1)\delta)$ satisfies IV, then $P(c)$ satisfies $(1)_{s,m}$ (for $m \geq 5$). For this, we use the following identity, which concatenates $P(c)$ and $P(c-(k+1)\delta)$:

(14) $P(c-(k+1)\delta)\ (tY - \frac{c}{\delta}) = (tY - \frac{c}{\delta} + 2)\ P(c)$

Let $u \in \mathscr{D}'(J, H^{-\infty})$ be such that

(15) $P(c)u \in {}^k\mathscr{H}^{s,m}_{loc}(J)$ (hence, by Corollary 1,

(15′) $P(c)u \in {}^k\mathscr{H}^{s-1,m-1-\frac{1}{k}}_{loc}(J)$).

Since $tY - \frac{c}{\delta} + 2 \in {}^kN^{1,1+\frac{1}{k}}$, we have $(tY - \frac{c}{\delta} + 2)\ P(c)u \in {}^k\mathscr{H}^{s-1,m-\frac{k+1}{k}}_{loc}(J)$, and since by hypothesis $P(c-(k+1)\delta)$ satisfies $(1)_{s,m}$ for all $(s,m) \in \mathbb{R} \times \mathbb{Z}/k$, we get from (14):

(16) $(tY - \frac{c}{\delta})u \in {}^k\mathscr{H}^{s+1,m+1-\frac{1}{k}}_{loc}(J)$.

We also have

(17) $(tX + \frac{c}{\delta} - 1)\ (tY - \frac{c}{\delta}) - t^2P(c) = \frac{c}{\delta}(1 - \frac{c}{\delta})$.

Since $tX + \frac{c}{\delta} - 1 \in {}^kN^{1,1+\frac{1}{k}}$ and $t^2 \in {}^kN^{0,\frac{2}{k}}$, (15), (16) and (17) give: $\frac{c}{\delta}(1 - \frac{c}{\delta})u \in {}^k\mathscr{H}^{s,m-\frac{2}{k}}_{loc}(J)$, and since $c \neq 0$, $c \neq \delta$ we get

(18) $u \in {}^k\mathscr{H}^{s,m-\frac{2}{k}}_{loc}(J)$.

Now we remark that

(19) $P(c-(k+1)\delta) = P(c) - (k+1)\delta t^{k-1}A$

From (15′), (18) and (19) we get $P(c-(k+1)\delta)u \in {}^k\mathscr{H}^{s-1,m-1-\frac{1}{k}}_{loc}(J)$, *hence*

(20) $u \in {}^k\mathscr{H}^{s+1,m+1-\frac{1}{k}}_{loc}(J)$.

From (15), (19) and (20) we get $P(c-(k+1)\delta)u \in {}^k\mathscr{H}^{s,m}_{loc}(J)$, hence $u \in {}^k\mathscr{H}^{s+2,m+2}_{loc}(J)$. Q.E.D.

FINAL REMARK. After this, it is not difficult to prove that a), b), c) are also equivalent to:

d) $\forall (s,m) \in R \times S$, $\forall$ open $J' \subset\subset J_1$, P defines an isomorphism from

$$ {}^k\mathscr{H}^{s+2,m+2}_{loc}(J') \Big/ C^\infty(J',H^\infty) \quad \text{onto} \quad {}^k\mathscr{H}^{s,m}_{loc}(J') \Big/ C^\infty(J',H^\infty). $$

e) Same as d), but for P^* instead of P.

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